Isiah Zaplana Agut
Instituto Universitario de Matemática Pura y Aplicada
Universitat Politècnica de València
isiah.zaplana@gmail.com

Supervised by:
Jose Alberto Conejero Casares

DYNAMICAL PROPERTIES OF SKEW-PRODUCTS OF OPERATORS

Contents

Summary

In recent years the field of linear dynamics has been taken an important part in the theory of functional analysis and dynamical systems. The study of the dynamical properties of operators and semigroups of operators in spaces of infinite dimension has awaken the interest of many researches. The main idea behind this area of the mathematics is to study the orbits of certain operators in the space where they are defined. Some of these properties will be defined in this work (like hypercyclicity, weakly-mixing, mixing or linear chaos). For more information about this field see [8, 29].

The aim of this work is not only point out different dynamical properties of operators but also consider these properties in a more general context. The idea is that, rather that consider just a Banach space X and a linear and continuous operator $T : X \to X$, consider the product space $A \times X$ where A is a probability space and a map P over $A \times X$. This map is what we call later the skew-product of the operator T. P is defined using an ergodic map $f : A \to A$ and an integrable function $h : A \to \mathbb{C}$ and has not linearity in general. For that reason, the results of linear dynamics cannot been applied to P. In this work we study how some of this properties also hold for the skew-products.

The structure of this work is the following:

In the first chapter we present the general concepts of functional analysis, dynamical systems and ergodic theory to introduce the main part of this work and to understand better the tools we will use. The purpose of this first chapter is been a self-contained introduction to the field.

The second chapter introduces the concept of skew-product and analyze the linear and dynamical properties that, under certain conditions, these maps can exhibit. The final part of the chapter presents some examples of skew-products of classical operators. Also it is studied under which conditions this examples has also the behavior described at the beginning of the chapter.

Chapter 1

Preliminaries

1.1 General Topology and Functional Analysis Theory

In this section we set up basic definitions, theorems and some tools that will be helpful in this work. Firstable, we will give some basic definitions to point out the general framework in where we will work. After that, we will give some concrete results of the fields of mathematics that are connected with the main purpose of this work. The main references can be found in the books [32, 40, 43, 45].

1.1.1 Metric, Banach, Frèchet and Hilbert spaces

We can start with the notion of metric, Banach and Hilbert spaces and their properties:

Definition 1.1.1 (Metric space)**.** A real-valued function $d : X \times X \to \mathbb{R}$, where X is a general set, defined for each pair of elements $x, y \in X$ is called a **metric** if it satisfies:

(i) $d(x, y) \geq 0$, $d(x, x) = 0$ and $d(x, y) > 0$ if $x \neq y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$, the triangle inequality.

A set X provided with a metric is called a **metric space** and $d(x, y)$ is called the distance between x and y.

We will understand by a **neighborhood** of a point $p \in X$ a set $\mathcal{U} \subset X$, which contains an open set $\mathcal{V}$ containing p, where the open sets $\mathcal{V}$ are unions of open balls in the metric space, where the open balls of center $x \in X$ and radius $r > 0$ are the sets defined as follows: $\{y \in X \ : \ d(x, y) < r\}$

A point x in a metric space X is called **isolated** if some neighbourhood of x contains no other point in X.

A **compact set** A is a set in which every sequence defined using elements of A has a convergent subsequence and the limit of that sequence lies in A.

A metric space is said to be **locally compact** if each point has a compact neighbourhood.

A sequence $(x_n)_n$ is called a **Cauchy Sequence in** X if:

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ :\ (n, m \geq n_0 \Rightarrow |x_n - x_m| < \varepsilon)$$

Finally, we say that a metric space is **complete** if every Cauchy sequence in X converges to an element of X.

The next theorem will be one of the most used theorems throughout the main part of this work:

Theorem 1.1.2 (Baire category theorem)**.** *Let (X, d) be a complete metric space and $\{G_n\}_n$ a sequence of nonempty dense open sets. Then*

$$G := \bigcap_{n=1}^{\infty} G_n,$$

is a dense G_δ-set[1] *in X.*

Now, we will give some concrete definitions that allows us to know where exactly we are working:

Definition 1.1.3. A functional $p : X \to \mathbb{R}^+$ on a vector space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is called a **seminorm** if for all $x, y \in X$ and $\lambda \in \mathbb{K}$:

(i) $p(x + y) \leq p(x) + p(y)$

(ii) $p(\lambda x) = |\lambda| p(x)$.

If, in addition,

(iii) $p(x) = 0$ implies that $x = 0$ then p is called a **norm**.

Definition 1.1.4 (Frèchet space)**.** A Frèchet space is a vector space X endowed with a separating increasing sequence of seminorms $(p_n)_n$, which is complete in the metric given by:

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} min(1, p_n(x - y)), x, y \in X.$$

Definition 1.1.5 (Normed space)**.** The pair $(X, ||\cdot||)$ is called a **normed space** where X is a vector space endowed with a norm $||\cdot||$.

Every normed linear space may be regarded as a metric space, being $\|x - y\|$ the distance between x and y. A **Banach space** is a normed linear space which is complete with the metric defined by its norm.

[1] Where we understand a G_δ-set as a countable intersection of open sets

Definition 1.1.6 (Hilbert space)**.** A **Hilbert space** H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product. So H is a complex vector space on which there is an inner product $\langle x, y\rangle$ associating a complex number to each pair of elements x, y of H that satisfies the following properties:

(i) $\langle y, x\rangle = \overline{\langle x, y\rangle}$.

(ii) It is linear in its first argument. For all complex numbers:

$$\langle ax_1 + bx_2, y\rangle = a\langle x_1, y\rangle + b\langle x_2, y\rangle.$$

(iii) It is positive definite: $\langle x, y\rangle \geq 0$ and it's equal to 0 if and only if $x = 0$.

The norm defined by the inner product $\langle\cdot,\cdot\rangle$ is the real-valued function:

$$\|x\| = \sqrt{\langle x, x\rangle}$$

and the distance between two points x,y in H is defined in terms of the norm by:

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}.$$

As in the case of sets, where we use maps, we need to connect two different normed spaces. The maps between normed spaces are called **operators**.

Proposition 1.1.7. *Let X and Y be Banach spaces and let $T : X \to Y$ be a linear operator. The following four statements are equivalent:*

(i) T is continuous at 0.

(ii) T is continuous.

(iii) T is uniformly continuous.

(iv) T is bounded, i.e., there exists a constant $C > 0$ such that $\|Tx\|_Y \leq C\|x\|_X$ for all $x \in X$.

Definition 1.1.8. Let X and Y be Banach spaces and $T : X \to Y$ be a continuous linear operator. We define

$$\|T\| := \inf\{C > 0 : \|Tx\|_Y \leq C\|x\|_X \text{ for all } x \in X\},$$

and we refer to $\|T\|$ as the operator norm of T.

Some equivalent formulations are the following:

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|_Y = \sup_{\|x\|=1} \|Tx\|_Y.$$

Definition 1.1.9. Let X, Y be Banach spaces. A map $T : X \to Y$ is said to have **closed graph** when for any sequence $(x_n)_n \in X$ with $\lim_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} Ax_n = y \in Y$ then $x \in X$ and $Ax = y$.

Theorem 1.1.10 ((Closed Graph Theorem))**.** *Let X, Y be Banach spaces and let $T : X \to Y$ be a linear map that has closed graph. Then T is continuous.*

Definition 1.1.11. Let X and Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the space of all continuous linear operators $T: X \to Y$ endowed with the operator norm. This space turns into a Banach space whenever Y is a Banach space. If $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$, the **dual** $X^* = \mathcal{L}(X,\mathbb{K})$ of a Banach space X is the space of all continuous linear functionals on X. If $x^* \in X^*$ then we write,

$$x^*(x) = \langle x, x^* \rangle, \quad x \in X.$$

The **adjoint** $T^*: X^* \to X^*$ of an operator T on X is defined by $T^*x^* = x^* \circ T$; that is,

$$\langle x, T^*x^* \rangle = \langle Tx, x^* \rangle, \quad x \in X, \ x^* \in X^*.$$

Now we will show some general and fundamental results of functional analysis that will be necessary throughout this work, such as:

Theorem 1.1.12 (Hahn-Banach)**.** *Let X be a vector space, M a subspace of X, p a seminorm on X and $u: M \to \mathbb{K}$ a linear functional such that $|u(x)| \leq p(x)$ for all $x \in M$. Then u has a linear extension $\widetilde{u}$ defined in X such that $|\widetilde{u}(x)| \leq p(x)$ for all $x \in X$.*

Theorem 1.1.13 (Banach-Steinhaus)**.** *Let X and Y be Banach or Frèchet spaces and $T_j: X \to Y$ with $j \in J$ a family of operators. If, for every $x \in X$, the set $\{T_j x \ : \ j \in J\}$ is bounded in Y, then the family $(T_j)_{j \in J}$ is equicontinuous, i.e.,* $\sup\{\|T_j\| \ : \ j \in J\} < \infty$

1.1.2 Spectral Theory

In this work we will use, at some point, concepts and results based on the spectral theory of operators. Some basic results of functional analysis that will be useful in understanding some of these concepts are the following:

Definition 1.1.14. Let X be a complex Banach space X, and let T be an operator on X. The *spectrum* $\sigma(T)$ of T is defined as:

$$\sigma(T) = \{\lambda \in \mathbb{C} \ ; \ \lambda I - T \text{ is not invertible}\}.$$

Moreover, each $0 \neq x \in X$ satisfying $Tx = \lambda x$ is an **eigenvector for** T corresponding to λ, that is an **eigenvalue for** T.

The *point spectrum* $\sigma_p(T)$ is the set of eigenvalues of T.
The number

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

is called the *spectral radius* of T.

Proposition 1.1.15. *The spectrum $\sigma(T)$ is a nonempty compact set for an operator $T: X \to X$. Moreover, $|\lambda| \leq \|T\|$ for any $\lambda \in \sigma(T)$.*

Proposition 1.1.16. *If $T: X \to X$ is an invertible operator on X, then $\sigma(T^{-1}) = \sigma(T)^{-1}$*

Theorem 1.1.17 (Riesz decomposition theorem). *If $\sigma(T) = \sigma_1(T) \cup \sigma_2(T)$, where σ_1 and σ_2 are two disjoint non-empty closed sets, there exist non-trivial T-invariant closed subspaces M_1 and M_2 of X such that $X = M_1 \oplus M_2$, and*

$$\sigma(T|_{M_1}) = \sigma_1 \quad \textit{and} \quad \sigma(T|_{M_2}) = \sigma_2.$$

Theorem 1.1.18 (Spectral Radius Formula). *For the spectral radius of an operator $T : X \to X$ we have that:*

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = r(T)$$

1.1.3 Classical Banach and Frechet spaces

Now we introduce the classical sequence and function spaces that we will use in this work. Here, $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$. The symbol X will always stand for a Banach space over $\mathbb{K}$.

If $p \in [1, \infty)$, we define $\ell^p := \ell^p(X)$ and $L^p := L^p(X)$ as follows:

- $\ell^p(X) = \{x = (x_n)_n \in X^{\mathbb{N}} : \|x\|_p < \infty\}$ where $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$
- $L^p(X) = \{f \in \mathcal{M}(X) : \|f\|_p < \infty\}$ where $\|f\|_p = \left(\int_X |f(x)|^p dx\right)^{\frac{1}{p}}$ and $\mathcal{M}(X)$ denotes the set of the measurable functions $f : X \to X$

If $p = \infty$, we define $\ell^\infty := \ell^\infty(X)$ and $L^\infty := L^\infty(X)$ as follows:

- $\ell^\infty(X) = \{x = (x_n)_n \in X^{\mathbb{N}} : \|x\|_\infty < \infty\}$ where $\|x\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$
- $L^\infty(X) = \{f \in \mathcal{M}(X) : \|f\|_\infty < \infty\}$ where $\|f\|_\infty = \inf\{M > 0 : |f(x)| < M\}$

A particular case of that spaces that will be very used throughout this work will be $L^p_\rho(R^+)$ where $\rho : \mathbb{R}^+ \to \mathbb{R}$ is a strictly positive locally integrable function (i.e. $\int_0^b \rho(t)dt < \infty$ for all $b > 0$). These spaces are defined as follows:

$$L^p_\rho(R^+) := \{f : \mathbb{R}^+ \to \mathbb{K},\ f \text{ is measurable on } \mathbb{R}^+ \text{ and } \|f\|_{p,\rho} < \infty\},$$

where $\|f\|_{p,\rho} = \left(\int_0^\infty |f(t)|^p \rho(t)dt\right)^{\frac{1}{p}}$

The last example we will see is of a Frechet space. We define the following space:

$$H(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C} : f \text{ is holomorphic}\},$$

that is, the space of all entire functions (that is, complex-valued functions that are complex differentiable in a neighborhood of every point in its domain ($\mathbb{C}$)).

The natural concept of convergence for entire functions is that of local uniform convergence, that is, the uniform convergence on all compact sets. In contrast to Banach spaces, convergence is described here by a countably infinite collection of conditions. More precisely, we have that $f_k \to f$ in $H(\mathbb{C})$ if and only if, for all $n \in \mathbb{N}$, $p_n(f_k - f) \to 0$ as $k \to \infty$, where $p_n(f) := \sup\{|f(z)| : |z| \le n\}$. Here, $(p_n)_n$ is an increasing sequence of seminorms.

1.1.4 Basic Ergodic Theory

In this section will introduce some rudiments of ergodic theory that will be necessary throughout this work. The main reference for this section will be [49].

We start with some basic and general definitions:

Definition 1.1.19. Let A an arbitrary set. A collection $\mathcal{A}$ of subsets of A is called a **σ-algebra** of A if it satisfies the following conditions:

(i) $A \in \mathcal{A}$

(ii) If $B \in \mathcal{A}$ then $A \setminus B \in \mathcal{A}$

(iii) If $B_n \in \mathcal{A}$ for every $n \geq 1$ then $\bigcup\limits_{n=1}^{\infty} B_n \in \mathcal{A}$ (that is, that is closed under countable unions)

Definition 1.1.20. A pair $(A, \mathcal{A})$ where A is an arbitrary set and $\mathcal{A}$ is a σ-algebra is called a **measurable space** (that is, it admits a measure).

Definition 1.1.21. Let $(A, \mathcal{A})$ be a measurable space. A function $\mu : \mathcal{A} \to \mathbb{R}^+$ is said to be a **measure** if the following conditions holds:

(i) $\mu(\emptyset) = 0$

(ii) $\mu(\bigcup\limits_{n=1}^{\infty} B_n) = \sum\limits_{n=1}^{\infty} \mu(B_n)$ whenever $\{B_n\}_n$ is a sequence of elements of $\mathcal{A}$ that are pairwise disjoint

Definition 1.1.22. Let $(A, \mathcal{A})$ be a measurable space and let $\mu : \mathcal{A} \to \mathbb{R}^+$ be a measure. We say that μ has **full support** if $\mu(U) > 0$ for every non-empty open set $U \in \mathcal{A}$

Definition 1.1.23. A triple $(A, \mathcal{A}, \mu)$ where A is an arbitrary set, $\mathcal{A}$ is a σ-algebra and $\mu : \mathcal{A} \to \mathbb{R}^+$ is a measure is called a **measure space**. When, in addition, we have that $\mu(A) = 1$ we said that $(A, \mathcal{A}, \mu)$ is a **probability space**.

Definition 1.1.24. Let $(A, \mathcal{A})$ and $(B, \mathcal{B})$ be two measurable spaces. A function $f : (A, \mathcal{A}) \to (B, \mathcal{B})$ is said to be a **measurable function** when $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$.

Definition 1.1.25. Let $(A, \mathcal{A})$ be a measurable space, $f : A \to A$ a measurable function and μ a probability measure. The function f is said to be **measure-preserving with respect to** μ if for every $B \in \mathcal{A}$ we have that $\mu(f^{-1}(B)) = \mu(B)$

Definition 1.1.26. Let $(A, \mathcal{A})$ be a measurable space, $f : A \to A$ a measurable function and μ a probability measure. The function f is said to be **ergodic with respect to** μ if f is measure-preserving with respect to μ and for every $B \in \mathcal{A}$ that verifies that $f^{-1}(B) \subset B$, $\mu(B) \in \{0, 1\}$.

Moreover, if there exists just one probability measure on $(A, \mathcal{A})$ with respect to which f is ergodic, then we say that f is **uniquely ergodic with respect to** μ.

Definition 1.1.27. Let A be a compact metric space and let $f : A \to A$ be a homeomorphism. We say that f is **minimal** if $\mathrm{orb}(x, f)$ is dense in A for every nonzero $x \in A$.

Proposition 1.1.28. *Let $(A, \mathcal{A}, \mu)$ be a compact metric probability space with μ having full support and let $f : A \to A$ to be a uniquely ergodic homeomorphism. Then f is minimal.*

Now we will give two important results that, with the Baire's Category Theorem (see 1.1.2), will play an important role in this work.

Theorem 1.1.29 (Birkhoff Ergodic Theorem)**.** *Let $(A, \mathcal{A}, \mu)$ be a probability space and let $f : A \to A$ be an ergodic function with respect to μ. Then for every $\phi \in L^1(\mu)$ (the space of all integrable functions with respect to the measure μ) we have that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \phi(f^n(a)) = \int_A \phi d\mu,$$

for μ-almost every $a \in A$.

Theorem 1.1.30 (Oxtoby's Theorem)**.** *Let $(A, \mathcal{A}, \mu)$ be a probability space and let $f : A \to A$ be an ergodic function with respect to μ. If A is a compact metric space, $\mathcal{A}$ the σ-algebra of Borel subsets of A and f is uniquely ergodic with respect to μ then for every $\phi \in \mathcal{C}(A)$ (the space of all continuous functions over A) we have that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \phi(f^n(a)) = \int_A \phi d\mu,$$

for every $a \in A$.

1.2 Discrete Linear Dynamics: Iterates of an Operator

In this section we will introduce some basic definitions and results of linear dynamical systems. The results that are shown in this chapter can be found in [8] and [29]. Dynamical systems play a important role in this work because our main purpose is to study the dynamical properties of a general maps called skew-products, that we will define later.

1.2.1 General Concepts

Dynamical systems are used to study the behavior of evolving systems. Let X be a set of elements that describes the different acceptable states of a system. If $x_n \in X$ is the state of the system at time $n \geq 0$, then its evolution will be given by a linear map $T : X \to X$ such that $x_{n+1} = T(x_n)$. We will work in a Banach space X and a continuous map T.

Definition 1.2.1 (Discrete dynamical system)**.** Let X be an metric space and let T be a continuous map $T : X \to X$. A **discrete dynamical system** is a pair (X, T). We define the **orbit** of a point $x \in X$ as the set $Orb(x, T) = \{T^n(x) : n \in \mathbb{N}\}$, where T^n denotes the n-th iterate of a map T. We will often simply say that T or $(T; X)$ is a dynamical system.

Definition 1.2.2. Let $(S; Y)$ and $(T; X)$ be dynamical systems.

1. Then T **is called quasi-conjugate to** S if there exists a continuous map $\varphi : Y \to X$ with dense range such that $T \circ \varphi = \varphi \circ S$; that is, the following diagram commutes.

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \downarrow\varphi & & \downarrow\varphi \\ X & \xrightarrow{T} & X \end{array}$$

2. If φ can be chosen to be a homeomorphism, then S and T are called **conjugates**.

3. φ is called a **quasi-conjugation** or a **conjugation** depending on in which case we are.

Definition 1.2.3. We say that a property $\wp$ for dynamical systems is preserved under (quasi-)conjugacy if the following holds: if a dynamical system $S : Y \to Y$ has property $\wp$ then every dynamical system $T : X \to X$ that is (quasi-)conjugate to S also has property $\wp$.

Definition 1.2.4. Let $T : X \to X$ be a dynamical system. Then $Y \subset X$ is called T-invariant or invariant under T if $T(Y) \subset Y$.

Definition 1.2.5. We say that $x \in X$ is a **fixed point for the dynamical system** $T : X \to X$ if $Tx = x$, and we say that $x \in X$ is a **periodic point for the dynamical system** T if $T^n x = x$ for some $n \in \mathbb{N}$. The set of all periodic points is denoted by $Per(T)$. If $x \in Per(T)$ then the smallest positive integer n such that $T^n x = x$ is called a **primary period** of x.

Definition 1.2.6. A dynamical system $T : X \to X$ is:

(i) **topologically transitive** if for any pair of nonempty open sets $U,V \subset X$ there exists an $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$;

(ii) **weakly mixing** if the map $T \times T$ is topologically transitive on $X \times X$;

(iii) **mixing** if for any pair of nonempty open sets $U,V \subset X$ there exists some $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for every integer $n \geq n_0$;

Note 1.2.7. For any linear dynamical system we have that:

$$\textit{mixing} \Longrightarrow \text{weak mixing} \Longrightarrow \text{topological transitivity}$$

In 1989 Robert L. Devaney proposed the definition of chaos that is the most used in Linear Dynamics; see [23]. This concept reflects the unpredictability of chaotic systems because the definition contains the **sensitive dependence on initial conditions**, i.e.:

Definition 1.2.8. Let (X, d) be a metric space without isolated points. Then the dynamical system $T : X \to X$ is said to have **sensitive dependence on initial conditions** if there exists some $\delta > 0$ such that, for every $x \in X$ and $\varepsilon > 0$, there exists some $y \in X$ with $d(x, y) < \varepsilon$ such that, for some $n \geq 0$, $d(T^n x, T^n y) > \delta$. The number δ is called a **sensitivity constant** for T.

Definition 1.2.9 (Devaney chaos)**.** A dynamical system $T : X \to X$ is called chaotic in the sense of **Devaney** if it satisfies the following properties:

(i) T is topologically transitive,

(ii) $Per(T)$ is dense in X,

(iii) T has sensitive dependence on initial conditions.

However, Banks, Brooks, Cairns, Davis and Stacey proved in 1992, through their work in [4], that if X is an infinite set, the sensitivity is a consequence of transitivity and dense periodicity.

Theorem 1.2.10 (Banks, Brooks, Cairns, Davis & Stacey)**.** *Let X be a non-finite metric space. If a dynamical system $T : X \to X$ is topologically transitive and has a dense set of periodic points then T has sensitive dependence on initial conditions with respect to any metric defining the topology of X.*

A link between chaos theory and linear operator theory was established by by Birkhoff's Transitivity Theorem in 1920. In this theorem, he showed that the topological transitivity was equivalent to the existence of an element with dense orbit. This last condition was coined as hypercyclicity by Beauzamy in 1986:

Definition 1.2.11 (Beauzamy)**.** Let X be a topological vector space.
An operator $T : X \to X$ is said to be **hypercyclic** if there is some $x \in X$ whose orbit $Orb(x, T)$ is dense in X. In that case, x is called a **hypercyclic vector** for T. The set of hypercyclic vectors is denoted by $HC(T)$.

Theorem 1.2.12 (Birkhoff Transitivity Theorem)**.** *Let X be a separable complete metric space without isolated points, and let $T : X \to X$ be a continuous map. Then the following assertions are equivalent:*

(i) T is topologically transitive;

(ii) T is hypercyclic operator.

If one of these conditions holds then, using Theorem 1.1.2, one can see that the set $HC(T)$ of hypercyclic vectors is a dense G_δ-set.

In 1991 Godefroy and Shapiro also adopted Devaney's definition of chaos for linear chaos.

Definition 1.2.13 (Godefroy & Shapiro)**.** Let X be a complete metric vector space. An operator $T : X \to X$ is called chaotic in the sense of Devaney, if:

(i) T is hypercyclic.

(ii) $Per(T)$ is dense in X.

One can prove that Devaney chaos implies weakly-mixing, which means that the precedent schedule changes for this one:

$$
\begin{array}{c}
\text{Chaos} \\
\Downarrow \\
\text{Mixing} \Longrightarrow \text{Weakly-Mixing} \Longrightarrow \text{Hypercyclic}
\end{array}
$$

Proposition 1.2.14. *The converses of these implications do not hold in general:*

- *Hypercyclicity does not imply weakly-mixing (see [21])*
- *Hypercyclicity does not imply mixing (if it does hypercyclicity will imply also weakly-mixing, which is a contradiction with the previous point)*
- *Hypercyclicity does not imply chaos (if it does hypercyclicity will imply also weakly-mixing)*
- *Mixing does not imply chaos (see [29] page 47)*
- *Chaos does not imply mixing (see [2])*

Proposition 1.2.15. *The following properties are preserved by quasi-conjugacy:*

(i) *Topological transitivity.*

(ii) *The property of having a dense orbit.*

(iii) *The property of having a dense set of periodic points.*

(iv) *Devaney Chaos.*

(v) *The mixing property.*

(vi) *The weak-mixing property.*

The reader can find the proofs of the following results in [8]. Additionally, the original proofs of some of these results can be found in [35]:

Proposition 1.2.16. *Let T be a hypercyclic operator on a (real or complex) Banach space X. Then we have:*

(i) *T^* has no eigenvalues, that is, $\sigma_p(T^*) = \emptyset$;*

(ii) *the orbit of every $x^* \neq 0$ in X^* under T^* is unbounded.*

Theorem 1.2.17 (Kitai)**.** *Let T be a hypercyclic operator on a complex Banach space X. Then every connected component of $\sigma(T)$ meets the unit circle S^1, i.e., $\sigma(T) \cap S^1 \neq \emptyset$.*

Proposition 1.2.18. *Let T be a linear map on a complex vector space X. Then the set $Per(T)$ of periodic points of T is given by*

$$span\{x \in X \ ; \ Tx = \lambda x \text{ for some } \lambda \in \mathbb{C} \text{ with } \lambda^n = 1 \text{ for some } n \in \mathbb{N}\}.$$

Proposition 1.2.19. *Let T be a chaotic operator on a complex Banach space X. Then its spectrum has no isolated points and it contains infinitely many roots of unity; in particular, $\sigma(T) \cap S^1$ is infinite.*

Theorem 1.2.20. *No compact operator is hypercyclic.*

1.2.2 Hypercyclicity criterion

There are some criteria under which an operator is chaotic, mixing or weakly mixing. These criteria are the following:

Theorem 1.2.21 (Kitai's Criterion)**.** *Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$ and a map $S : Y_0 \to Y_0$ such that, for any $x \in X_0$, $y \in Y_0$:*

(i) $T^n x \to 0$,

(ii) $S^n y \to 0$,

(iii) $TSy \to y$,

then T is mixing.

Theorem 1.2.22 (Godefroy-Shapiro Criterion)**.** *Let $T \in L(X)$ with X a Banach space. Suppose that the subspaces*

$$X_0 := span\{x \in X; \quad Tx = \lambda x \quad \text{for some} \quad \lambda \in \mathbb{K} \quad \text{with} \quad |\lambda| < 1\}$$

$$Y_0 := span\{x \in X; \quad Tx = \lambda x \quad \text{for some} \quad \lambda \in \mathbb{K} \quad \text{with} \quad |\lambda| > 1\}$$

are dense in X.
Then T is mixing, and in particular hypercyclic.

If, moreover, X is a complex space and the subspace

$$Z_0 := span\{x \in X; \quad Tx = \lambda x \quad \text{for some} \quad \lambda \in \mathbb{C} \quad \text{with} \quad |\lambda|^n = 1 \quad \text{for some} \quad n \in \mathbb{N}\}$$

is dense in X, then T is chaotic.

Theorem 1.2.23 (Gethner-Shapiro Criterion)**.** *Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$, an increasing sequence $(n_k)_k$ of positive integers, and a map $S : Y_0 \to Y_0$ such that, for any $x \in X_0$, $y \in Y_0$:*

(i) $T^{n_k} x \to 0$,

(ii) $S^{n_k} y \to 0$,

(iii) $TSy \to y$,

then T is weakly mixing.

Theorem 1.2.24 (Bes-Peris Hypercyclicity Criterion)**.** *Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$, an increasing sequence $(n_k)_k$ of positive integers, and maps $S_{n_k} : Y_0 \to X$, $k \geq 1$ such that, for any $x \in X_0$, $y \in Y_0$:*

(i) $T^{n_k} x \to 0$,

(ii) $S_{n_k} y \to 0$,

(iii) $T^{n_k} S^{n_k} y \to y$,

then T is weakly mixing, and in particular hypercyclic.

In the last theorem, we can consider a general family of operators $(T_n)_n$ and the result still holds.

1.2.3 Examples of Operators

In this section we will study some examples of operators that one can see that are chaotic, mixing, weakly-mixing or hypercyclic using the different criteria we showed in the previous section.

Rolewicz's Operators

This kind of operators are defined on $X := \ell^p$, $1 \le p < \infty$, or $X := c_0$ (this space is the space of all sequences that have null limit, i.e., $\{(x_n)_n \ : \ \lim_n |x_n| = 0\}$). We consider in these spaces the backward-shift operator, defined by:

$$B(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots),$$

and consider:

$$T = \lambda B : X \to X,$$

$$T(x_1, x_2, x_3, \dots) \to \lambda(x_2, x_3, x_4, \dots)$$

where $\lambda \in \mathbb{K}$.

First, if $|\lambda| \le 1$ then $\|T^n x\| = |\lambda|^n \|B^n x\| \le \|x\|$ for all $x \in X$ and $n \ge 0$. Thus T cannot be hypercyclic.

On the other hand, T is hypercyclic whenever $|\lambda| > 1$. In fact, it is mixing and chaotic (that is deduced from the Godefroy-Shapiro Criterion using Hahn-Banach Theorem).

Weighted Shifts Operators

We consider the space ℓ^p or c_0 and the backward-shift B, that gives the **weighted backward shift**, defined as follows:

$$B_w(x_1, x_2, x_3, \ldots) = (w_2 x_2, w_3 x_3, w_4 x_4, \ldots),$$

where $w = (w_n)_n$ is called a **weight sequence**. The weights w_n will always be assumed to be non-zero.

Firstable, in order to have a well-defined and continuous operator, the weights must hold:

$$\sup_n \frac{w_n}{w_{n+1}} < \infty$$

If this previous condition holds, we have the following result:

Theorem 1.2.25. *In ℓ^p with $1 \le p < \infty$ or in c_0 we have that:*

1. *The following assertions are equivalent:*
   (i) B_w *is hypercyclic*
   (ii) B_w *is weakly mixing*
   (iii) $\sup_{n\ge 1} \prod_{\nu=1}^{n} |w_\nu| = \infty$
2. *The following assertions are equivalent:*
   (i) B_w *is mixing*
   (ii) $\lim_{n\to\infty} \prod_{\nu=1}^{n} |w_\nu| = \infty$

3. *Suppose that the basis $(e_n)_n$ is unconditional (that is, for every $x \in \ell^p$ or $x \in c_0$, the representation $x = \sum_{n=1}^{\infty} a_n e_n$ converges unconditionally). Then the following assertions are equivalent:*

   (i) B_w *is chaotic;*

   (ii) The series
$$\sum_{n=1}^{\infty} \frac{1}{\prod_{\nu=1}^{n} |w_\nu|^p}$$
   converges in ℓ^p or c_0

   (iii) The sequence
$$\left(\frac{1}{\prod_{\nu=1}^{n} |w_\nu|^p}\right)_n$$
   belongs to ℓ^p or c_0

MacLane's Operators

We consider the Frèchet space of all holomorphic functions defined in the whole complex plane, $\mathcal{H}(\mathbb{C})$. We next consider the differentiation operator $D : f \to f'$ on $\mathcal{H}(\mathbb{C})$. Through the Kitai's Criterion or the Godefroy-Shapiro Criterion one can deduce that this kind of operators are also mixing and chaotic and for that hypercyclic.

The last example we will show in this section is the concerning with the **Birkhoff's Operators**. These operators are also defined in $\mathcal{H}(\mathbb{C})$. These are a kind of translation operators given by

$$T_a f(z) = f(z + a) \ , \ a \neq 0,$$

where $f \in \mathcal{H}(\mathbb{C})$. Using the Kitai's Criterion we can show that this operators and mixing and using the Godefroy-Shapiro Criterion, that are also chaotic and hence, hypercyclic.

1.3 Continuous Linear Dynamics: Semigroups of Operators

In this section we will introduce the continuous case for a linear dynamical system. The results can be found in [6, 11, 18, 19, 29].

1.3.1 General Concepts

From now on, X will denote a separable Banach space. The aim of this section is to provide an analogous of the concepts and results studied in the discrete case but in the continuous framework. For that, we must start defining the concept of semigroup and C_0-semigroup:

Definition 1.3.1. A one-parameter family of operators $\{T_t : X \to X \ : \ t \geq 0\}$ is called a **semigroup of operators** if the following two conditions are satisfied:

(i) $T_0 = I$

(ii) $T_{t+s} = T_t T_s$ for all $s, t \geq 0$

Definition 1.3.2. A one-parameter family of operators $\{T_t : X \to X \,:\, t \geq 0\}$ is called a **C_0-semigroup of operators** or a **strongly continuous semigroup of operators** if the following three conditions are satisfied:

(i) $T_0 = I$

(ii) $T_{t+s} = T_t T_s$ for all $s, t \geq 0$

(iii) $\lim_{s \to t} T_s x = T_t x$ for all $x \in X$ and for all $t \geq 0$ (that is nothing more that the convergence in the Strong Operator Topology)

This new condition expresses the pointwise continuity of the semigroup. If this condition is replaced by:

(iii)' $\lim_{s \to t} T_s = T_t$ for all $t \geq 0$ in the bounded sets of X.

we say that the semigroup is an **uniformly continuous semigroup of operators**.

The Banach-Steinhaus Theorem (see 1.1.13) yields that C_0-semigroups are locally equicontinuous, that is, for any $b > 0$ we have that:

$$\sup_{t \in [0,b]} \|T_t\| < \infty,$$

or equivalently, there exists some $M > 0$ such that

$$\|T_t x\| \leq M\|x\| \text{ for all } t \in [0, b],\ x \in X$$

Remark 1.3.3. Local equicontinuity implies that $T_{t_n} x_n$ converges to zero whenever$(t_n)_n$ is bounded and x_n converges to zero

We can refine the estimation over the operator norm of the C_0-semigroup.

Proposition 1.3.4. *If $\tau = (T_t)_{t \geq 0}$ is a C_0-semigroup, then there exist $M \geq 1$ and $w \in \mathbb{R}$ such that $\|T_t\| \leq Me^{wt}$ for all $t \geq 0$*

One concept that is fundamental in the study of C_0-semigroups is the concept of infinitesimal generator of the semigroup. If $\tau = (T_t)_{t \geq 0}$ is a C_0-semigroup on X the set

$$D(A) := \{x \in X \ , \ \text{ existe } \lim_{t \to 0} \frac{1}{t}(T_t x - x)\}$$

is a dense subset of X. That set is called **domain of the infinitesimal generator** A. We define a map $A : D(A) \to X$ that acts as follows:

$$Ax := \lim_{t \to 0} \frac{1}{t}(T_t x - x)$$

This map has the following properties:

- It is linear
- It has closed graph and by Closed Graph Theorem we deduce that A is an operator

- $T_t(D(A)) \subset D(A)$ with $AT_tx = T_tAx$ for every $t \geq 0$ and $x \in D(A)$

In that conditions we say that $(A, D(A))$ is the **infinitesimal generator** of the C_0-semigroup τ

Proposition 1.3.5. *The infinitesimal generator $(A, D(A))$ of a C_0-semigroup τ determines the semigroup uniquely.*

Using the following general theorem of Spectral Theory:

Theorem 1.3.6 (Point Spectral Mapping Theorem for Semigroups)**.** *Let $(A, D(A))$ be the generator of a C_0-semigroup $\tau = (T_t)_{t\geq 0}$ defined on a complex Banach space X. Then we have the following identity:*

$$\sigma_p(T_t)\backslash 0 = e^{t\sigma_p(A)},$$

for $t \geq 0$

We can deduce another property of the infinitesimal generator of a C_0-semigroup:

Proposition 1.3.7. *If X is a complex Banach space and $\tau = (T_t)_{t\geq 0}$ is a C_0-semigroup with $(A, D(A))$ as infinitesimal generator then, for every $x \in X$ and $\lambda \in \mathbb{C}$, if*

$$Ax = \lambda x$$

then

$$T_tx = e^{\lambda t}x$$

for every $t \geq 0$, which means that the eigenvectors of A are also eigenvectors of T_t for every $t \geq 0$. In fact, the eigenvalues of A became eigenvalues of T_t in the form $e^{\lambda t}$ for every $t \geq 0$.

1.3.2 Hypercyclicity, weakly-mixing, mixing and chaoticity for C_0-semigroups

Now we are in condition of translating the different dynamical properties studied in the discrete case to the continuous case.

Definition 1.3.8. Let $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X. Then for any $x \in X$ we call

$$\text{orb}(x, (T_t)) = \{T_tx \ : \ t \geq 0\}$$

to be the **orbit of x under τ**

Definition 1.3.9. Let $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X. If there exists $x \in X$ such that $\text{orb}(x, (T_t))$ is dense in X then we say that x is a **hypercyclic vector for τ**. In that case we say that the semigroup is **hypercyclic**

Definition 1.3.10. Let $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X. If, for any pair of nonempty open subsets $U, V \subset X$, there exists some $t \geq 0$ such that $T_t(U) \cap V \neq \emptyset$ we say that the semigroup is **topologically transitive**

An analogous of the Birkhoff Transitive Theorem (see 1.2.12) holds in this case also:

Theorem 1.3.11 (Birkhoff Transitive Theorem for C_0-semigroups)**.** *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. Then $(T_t)_{t\geq 0}$ is hypercyclic if and only if it is topologically transitive. In that case, the set of hypercyclic vectors for $(T_t)_{t\geq 0}$ is a dense G_δ-set*

We now introduce a notion that allow us to characterize the concept of being hypercyclic for C_0-semigroups in a very useful way.

Definition 1.3.12. A **discretization of** $\tau = (T_t)_{t\geq 0}$ is a sequence of operators $(T_{t_n})_n$ with $t_n \to \infty$. In addition, if there is some $t_0 > 0$ such that $t_n = nt_0$ for $n \in \mathbb{N}$, then $(T_{t_n})_n = (T_{nt_0})_n$ is called an **autonomous discretization of** τ

Using also the separability of X one can deduce that the notion of hypercyclicity for C_0-semigroups is equivalent to the hypercyclicity of a discretization of the C_0-semigroup.

Definition 1.3.13. Let $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X. Then we say that τ is **mixing** if, for any pair of nonempty open subsets $U, V \subset X$, there exists some $t_0 \geq 0$ such that $T_t(U) \cap V \neq \emptyset$ for all $t \geq t_0$

Definition 1.3.14. Let $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X. Then we say that τ is **weakly-mixing** if $(T_t \oplus T_t)_{t\geq 0}$ is topologically transitive on $X \oplus X$

Remark 1.3.15. Note that if $\tau_1 = (T_t)_{t\geq 0}$ is a C_0-semigroup on X and $\tau_2 = (S_t)_{t\geq 0}$ is a C_0-semigroup on Y then $\tau_1 \oplus \tau_2 = (T_t \oplus S_t)_{t\geq 0}$ is a C_0-semigroup on $X \oplus Y$. Besides, the direct sum of a mixing semigroup with a hypercyclic semigroup is hypercyclic.

Definition 1.3.16. Let $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X. Then:

- A point $x \in X$ is called a **periodic point of** τ if there is some $t > 0$ such that $T_t x = x$. The set of periodic points for τ is denoted by $Per((T_t))$
- τ is said to be **chaotic** if it is hypercyclic and its set of periodic points is dense in X.

As in the discrete case, we have that:

$$\begin{array}{c} \text{Chaos} \\ \Downarrow \\ \text{Mixing} \Longrightarrow \text{Weakly-Mixing} \Longrightarrow \text{Hypercyclic} \end{array}$$

Definition 1.3.17. Let $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ two C_0-semigroups.

1. Then $(T_t)_{t\geq 0}$ is **quasi-conjugate** to $(S_t)_{t\geq 0}$ if there exists a continuous map $\varphi : Y \to X$ with dense range such that $T_t \circ \varphi = \varphi \circ S_t$ for all $t \geq 0$; that is, for every $t \geq 0$, the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{S_t} & Y \\ \downarrow\varphi & & \downarrow\varphi \\ X & \xrightarrow{T_t} & X \end{array}$$

2. If φ can be chosen to be a homeomorphism, then $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are **conjugates**.

3. φ is called a **quasi-conjugation** or a **conjugation** depending on the case in which we are.

Proposition 1.3.18. *Hypercyclicity, mixing, weakly-mixing and chaos for a C_0-semigroups are preserved under quasiconjugacy.*

Concluding the preliminaries about the behavior of C_0-semigroups, we now will give some criteria to determine when a C_0-semigroup is hypercyclic or mixing or even chaotic. For understand better how these criteria work it is necessary to introduce some results concerning to the discretization of semigroups (concept that has been given before).

Proposition 1.3.19. *Let $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X. Then the following assertions are equivalent:*

(i) τ is weakly-mixing

(ii) some discretization of τ is mixing

(iii) some discretization of τ is weakly-mixing

(iv) every autonomous discretization of τ is weakly-mixing

Proposition 1.3.20. *Let $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X. Then the following assertions are equivalent:*

(i) τ is mixing

(ii) every discretization of τ is mixing

(iii) every discretization of τ is weakly-mixing

(iv) every discretization of τ is hypercyclic

(v) every autonomous discretization of τ is mixing

(vi) some autonomous discretization of τ is mixing

Proposition 1.3.21. *Let $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X and $x \in X$. Then the following assertions are equivalent:*

(i) x is hypercyclic for τ

(ii) x is hypercyclic for some discretization of τ

(iii) x is hypercyclic for some autonomous discretization of τ

(iv) x is hypercyclic for every autonomous discretization of τ

The following are two results about the heritability of some of the dynamic properties studied throughout this section and it can be found in [6], [18] and [19]:

Theorem 1.3.22 ((Conejero-Müller-Peris))**.** *Let $\tau = (T_t)_{t\geq 0}$ be a hypercyclic C_0-semigroup on X and let $x \in HC(\tau)$. Then $x \in HC(T_t)$ for every $t > 0$*

That means that being hypercyclic for a C_0-semigroup can be inherit by every single operator of the semigroup except for the identity operator.

However, the same is not true when we talk about chaoticity.

Theorem 1.3.23 ((Bayart-Bermúdez)). *There exists a C_0-semigroup $(T_t)_{t\geq 0}$ on a separable Hilbert space H such that there exists $t_0 \neq t_1$ with T_{t_0} being chaotic but with T_{t_1} not being chaotic.*

Now we are in conditions of give the different criteria for C_0-semigroups that can be found in [11].

Theorem 1.3.24 (Hypercyclicity Criterion for Semigroups). *Let $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X. If there are dense subsets $X_0, Y_0 \subset X$, a discretization of τ, and maps $S_{t_n} : Y_0 \to X$, $n \in \mathbb{N}$, such that, for any $x \in X_0$, $y \in Y_0$,*

(i) $T_{t_n} x \to 0$,

(ii) $S_{t_n} y \to 0$,

(iii) $T_{t_n} S_{t_n} y \to y$,

then τ is weakly-mixing, and in particular hypercyclic

Theorem 1.3.25 (Mixing Criterion for Semigroups). *Let $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X. If there are dense subsets $X_0, Y_0 \subset X$, and maps $S_t : Y_0 \to X$, $t \geq 0$, such that, for any $x \in X_0$, $y \in Y_0$,*

(i) $T_t x \to 0$,

(ii) $S_t y \to 0$,

(iii) $T_t S_t y \to y$,

then τ is mixing

And finally a criteria for chaotic behavior that uses the spectral theory applied to C_0-semigroups that can be found in [22].

Before to give this criteria we need to introduce the concept of weakly holomorphic function:

Definition 1.3.26. A **weakly holomorphic function** $f : U \to X$ on an open set $U \subset \mathbb{C}$ is an X-valued function such that, for every $x^* \in X^*$, the complex-valued function $z \to < f(z), x^* >$ is holomorphic on U.

Theorem 1.3.27 ((Desch-Schappacher-Web)). *Let X be a complex separable Banach space, and $\tau = (T_t)_{t\geq 0}$ a C_0-semigroup on X with generator $(A, D(A))$. Assume that there exists an open connected subset U and weakly holomorphic functions $f_j : U \to X$, $j \in J$, such that:*

(i) $U \cap i\mathbb{R} \neq \emptyset$

(ii) $f_j(\lambda) \in \ker(\lambda I - A)$ *for every* $\lambda \in U$, $j \in J$

(iii) for any $x^* \in X^*$, if $< f_j(\lambda), x^* >= 0$ for all $\lambda \in U$ and $j \in J$ then $x^* = 0$

Then τ is mixing and chaotic.

1.3.3 Examples of C_0-semigroups

We now introduce the main examples of C_0-semigroups along with the different linear properties that they have.

- **Translation Semigroup**:
  On $(L^p_\rho(\mathbb{R}^+), \|\cdot\|)$, introduce in the section of "Classical Banach spaces" (see 1.1.3), we can define a translation semigroup. For every $f \in (L^p_\rho(\mathbb{R}^+), \|\cdot\|)$ we define a one-parameter family of operators as

  $$T_t(f(x)) = f(x+t),$$

  with $t, x \geq 0$.

  It is easy to see that this family of operators satisfies the conditions for being a semigroup. Moreover, $(T_t)_{t\geq 0}$ is a C_0-semigroup if and only if there exist $M \geq 1$ and $w \in \mathbb{R}$ such that, for all $t \geq 0$,

  $$\rho(x) \leq Me^{wt}\rho(x+t),$$

  for almost all $x \geq 0$. This identity is used in the following way:

  $$\rho(x) \leq Me^{w(y-x)}\rho(y),$$

  whenever $y \geq x \geq 0$.

  A function ρ fulfilling that condition is usually called an **admissible weight function**. This C_0-semigroup has the following properties:

  - The infinitesimal generator is $(A, D(A))$ with A being the differentiation operator and $D(A) = \{f \in L^p_\rho(\mathbb{R}^+) \ : \ \exists f' \text{ and } f' \in L^p_\rho(\mathbb{R}^+)\}$
  - The following assertions are equivalent:
    (i) $(T_t)_{t\geq 0}$ is hypercyclic
    (ii) $(T_t)_{t\geq 0}$ is weakly-mixing
    (iii) $\liminf\limits_{x\to\infty} \rho(x) = 0$
  - The following assertions are equivalent:
    (i) $(T_t)_{t\geq 0}$ is mixing
    (ii) $\lim\limits_{x\to\infty} \rho(x) = 0$
  - The following assertions are equivalent:
    (i) $(T_t)_{t\geq 0}$ is chaotic
    (ii) $\int_0^\infty \rho(x)dx < \infty$ (that is, that ρ is integrable on $\mathbb{R}^+$)

- **Exponential Semigroup**
  Given a Banach space X and an operator $A : X \to X$, we can construct a one-parameter family of operators given by the identity:

  $$T_t x = e^{tAx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x,$$

  for any $t \geq 0$. Is easy to see that is well-define (as A is an opertator, we know that is bounded and due to that $\| \sum_{n=0}^{\infty} \dfrac{t^n}{n!} A^n \| \leq \sum_{n=0}^{\infty} \dfrac{t^n}{n!} M^n =$

$e^{Mt} < \infty$ that proves that the operator T_t is well-defined for every $t \geq 0$) and that satisfies the conditions of C_0-semigroup.

In fact, we can show that, for any $t \geq 0$, $\lim_{s \to t} T_s = T_t$ that is the definition of being an uniformly continuous semigroup of operators.

Moreover, one have this strong result:

Theorem 1.3.28. *Let $(T_t)_{t\geq 0}$ a C_0-semigroup on X. The following assertions are equivalent:*

(i) $(T_t)_{t\geq 0}$ *is uniformly continuous*

(ii) The generator $(A, D(A))$ of $(T_t)_{t\geq 0}$ is defined everywhere (that is, $D(A) = X$)

(iii) There is an operator A on X such that $T_t = e^{tA}$ for every $t \geq 0$

That provides a characterization of the uniformly continuous semigroups (that are no more no less than the exponential of an operator).

Chapter 2

Discrete Dynamics on Skew-Products of Operators

The aim of this chapter is to present the concept of the skew-product of operators and to relate this concept with some of the dynamical properties of operators presented in the previous chapters.

2.1 General Definitions

We first give the general definition of skew-products in ergodic theory:

Definition 2.1.1. A **skew-product** is an automorphism P of a measure space E such that E is the direct product of two measure spaces $A \times X$ and the action of P in E is defined by:

$$P(\alpha, x) = (R(\alpha), S(\alpha, x)),$$

where R is an automorphism of A (the "base") and $S(\alpha, -)$, with α fixed, is an automorphism of X (the "fibre").

The particular case of skew-products we are working with is the following:

Definition 2.1.2. Let X be a separable complex Banach space, (A, μ) a probability space, $T : X \to X$ a linear and continuous operator, $f : A \to A$ an ergodic map with respect to the measure μ and $h : A \to \mathbb{C}$ a $L^1(\mu)$ function. The map

$$P : A \times X \to A \times X$$

defined by:

$$P(\alpha, x) = (f(\alpha), h(\alpha)Tx)$$

is said to be a **skew-product of the operator** T.

Skew-products provide a rich source of dynamical systems whose dynamics vary as the state of the system evolves. One may think of a skew-product as a dynamical system dependent on a parameter that is perturbed as the system evolves in a particular way.

In connection with linear dynamics the first thing we notice about skew-products is that we have a lack of linear structure so P is not an operator and we cannot apply the known results of linear dynamics making necessary to redefine the concepts and prove the results. We first focus on the concept of topological transitivity of the skew-product.

2.2 Topologically Transitive Skew-Products of Operators

This section is based in the papers of Bayart, Costakis and Hadjiloucas [20, 7]. In that papers the authors study the dynamical behavior of the skew-products of backward shift operators and the dynamics.

We recall that the definition of being topologically transitive is that for every non-empty open sets $U, V \in A \times X$ there exists $n \geq 0$ such that $P^n(U) \cap V \neq \emptyset$. Now, the iterates of a skew-product of operators are of the form:

$$P^n(\alpha, x) = (f^n(\alpha), h(f^{n-1}(\alpha))h(f^{n-2}(\alpha)) \ldots h(\alpha)T^n x)$$

If we denote $h^n(\alpha) = h(f^{n-1}(\alpha))h(f^{n-2}(\alpha)) \ldots h(\alpha)$, then

$$P^n(\alpha, x) = (f^n(\alpha), h^n(\alpha)T^n x)$$

In order to work better with skew-products in this general framework we will suppose that X is a separable Banach space, A is a compact metric space and $f : A \to A$ and $h : A \to \mathbb{C}$ are continuous functions unless it was explicitly mentioned.

The first important result is a characterization of the concept of topological transitivity for general skew-products.

Proposition 2.2.1. *Let P be a skew-product of the operator T over $A \times X$ where (A,μ) is a probability compact metric space and X is a Banach space. Then P is topologically transitive if and only if for every $a \in A$, $x \in X$ and $\delta > 0$ the set*

$$E(a, x, \delta) = \{(b, y) \in A \times X \ : \ \exists n \geq 0 \ , \ d(a, f^n(b)) < \delta, \|h^n(b)T^n y - x\| < \delta\}$$

is dense in $A \times X$.

Proof. $\boxed{\Longrightarrow}$

Suppose that $E(a, x, \delta)$ is not dense in $A \times X$ for some $a \in A$, $x \in X$ and $\delta > 0$. By definition of being dense, there exist $(b, y) \in A \times X$ and $\varepsilon > 0$ such that $B_D((b, y), \varepsilon) \cap E(a, x, \delta) = \emptyset$, where $B_D(-,-)$ is the ball in the distance D define in the product of two metric spaces (A and X).

The first thing we are going to see is that for every $a \in A$, $x \in X$ and $\delta > 0$, $E(a, x, \delta)$ is open. To see this we just need to prove that for every $(b, y) \in E(a, x, \delta)$ there exists an open set V such that $(b, y) \in V \subset E(a, x, \delta)$. For doing that, we will use the continuity of f and h. For an arbitrary $a \in A$,$x \in X$ and $\delta > 0$, as f and h are continuous functions there exists $\varepsilon > 0$ such that if

$$d(b, \beta) < \varepsilon \qquad \text{then} \qquad d(f^n(b), f^n(\beta)) < \delta - d(a, f^n(b))$$

where $\delta - d(a, f^n(b)) > 0$ because $(b, y) \in E(a, x, \delta)$.

$$d(b, \beta) < \varepsilon \qquad \text{then} \qquad |h^n(b) - h^n(\beta)| < \frac{\delta - \|h^n(b)T^n y - x\|}{\|T^n y\| + 1}$$

where $\dfrac{\delta - \|h^n(b)T^n y - x\|}{\|T^n y\| + 1} > 0$ because $(b, y) \in E(a, x, \delta)$, which implies that the numerator is strictly positive and, by definition of norm, the denominator is positive so the entire fraction is strictly positive.

Now, we define $V := B_d(b, \varepsilon) \times B(y, \varepsilon)$. It is clear that $(b, y) \in V$ and for seeing that $V \subset E(a, x, \delta)$ we just must see that for every $(c, z) \in V$ it holds that $(c, z) \in E(a, x, \delta)$. As $(c, z) \in V$ we have that $d(b, c) < \varepsilon$ which implies that $d(f^n(b), f^n(c)) < \delta - d(a, f^n(b))$ and as a consequence of that we have that

$$d(a, f^n(c)) \leq d(a, f^n(b)) + d(f^n(b), f^n(c)) < d(a, f^n(b)) + \delta - d(a, f^n(b)) = \delta.$$

In other hand we have that $d(b, c) < \varepsilon$ implies also that $|h^n(b) - h^n(c)| < \dfrac{\delta - \|h^n(b)T^n y - x\|}{\|T^n y\| + 1}$ and, as a result, we obtain that

$$\begin{aligned} \|h^n(c)T^n y - x\| &\leq \|h^n(b)T^n y - x\| + |h^n(b) - h^n(c)|\|T^n y\| < \\ &< \|h^n(b)T^n y - x\| + \frac{\delta - \|h^n(b)T^n y - x\|}{\|T^n y\| + 1}\|T^n y\| \\ &< \|h^n(b)T^n y - x\|\left(1 - \frac{\|T^n y\|}{\|T^n y\| + 1}\right) + \frac{\delta\|T^n y\|}{\|T^n y\| + 1} \overset{(1)}{<} \\ &\overset{(1)}{<} \delta\left(1 - \frac{\|T^n y\|}{\|T^n y\| + 1}\right) + \frac{\delta\|T^n y\|}{\|T^n y\| + 1} = \frac{\delta}{\|T^n\| + 1} + \frac{\delta\|T^n y\|}{\|T^n y\| + 1} = \delta, \end{aligned}$$

where (1) is that as $(b, y) \in E(a, x, \delta)$ by hypothesis we have that $\|h^n(b)T^n y - x\| < \delta$ who allows us to consider the inequality. Then, we have shown that $V \subset E(a, x, \delta)$ and for that we can conclude that $E(a, x, \delta)$ is open. Besides, is non-empty because (a, x) belongs always to $E(a, x, \delta)$ for any $\delta > 0$.

Moreover, $B_D((b, y), \varepsilon)$ is also a non-empty open set. So, we have two non-empty open sets, $E(a, x, \delta)$ and $B_D((b, y), \varepsilon)$. As we have by hypothesis that P is topologically transitive, there exists ≥ 0 such that $P^n(B_D((b, y), \varepsilon)) \cap E(a, x, \delta) \neq \emptyset$. That means that there exists $(c, z) \in B_D((b, y), \varepsilon)$ such that $P^n(c, z) \in E(a, x, \delta)$.

$$\begin{gathered} P^n(c, z) \in E(a, x, \delta) \iff (f^n(c), h^n(c)T^n z) \in E(a, x, \delta) \iff \\ \iff d(a, f^n(f^n(c))) < \delta \quad \|h^n(f^n(c))T^n(h^n(c)T^n z)\| < \delta \iff^{1} \\ \iff d(a, f^{2n}(c)) < \delta \quad \|h^{2n}(c)T^{2n} z\| < \delta, \end{gathered}$$

which means that (c, z) belongs also to $E(a, x, \delta)$ which is a contradiction with the identity suppose before ($B_D((b, y), \varepsilon) \cap E(a, x, \delta) = \emptyset$), so $E(a, x, \delta)$ is dense for every $a \in A$, $x \in X$ and $\delta > 0$.

$\boxed{\Longleftarrow}$

[1] $h^n(f^n(c))T^n(h^n(c)T^n z) = h^n(f^n(c))h^n(c)T^{2n} z =$
$= h(f^{2n-1}(c))h(f^{2n-2}(c)) \ldots h(f^n(c))h(f^{n-1}(c)) \ldots h(c)T^{2n} z = h^{2n}(c)T^n z$

Suppose now that P is not topologically transitive, i.e., that there exist two non-empty open sets $U, V \subset A \times X$ such that $P^n(U) \cap V = \emptyset$ for every $n \geq 0$.

Every $(b, y) \in U$ verifies that $P^n(b, y) \neq (a, x)$ for every $(a, x) \in V$ and for every $n \geq 0$, which is equivalent to $(f^n(b), h^n(b)T^n y) \neq (a, x)$ for every $(a, x) \in V$ and for every $n \geq 0$. So that for an arbitrary (b, y) there exists $\varepsilon > 0$ such that $d(a, f^n(b)) > \varepsilon$ and $\|h^n(b)T^n y - x\| > \varepsilon$ for every $n \geq 0$ and for every $(a, x) \in V$. But that implies that $(b, y) \notin E(a, x, \varepsilon)$. As, by hypothesis, $\overline{E(a, x, \delta)}$ is dense for every $a \in A$, $x \in X$ and $\delta > 0$, we have that $(b, y) \in \overline{E(a, x, \varepsilon)} \setminus E(a, x, \varepsilon)$. As a consequence, $(b, y) = \lim\limits_{n\to\infty}(b_n, y_n)$ with $(b_n, y_n) \in E(a, x, \varepsilon)$. But, if $(b_n, y_n) \in E(a, x, \varepsilon)$ then $\|h^m(b_n)T^m y_n - x\| < \frac{\varepsilon}{2}$ for some $m \geq 0$. As the norm is continuous we have:

$$\lim_{n\to\infty} \|h^m(b_n)T^m y_n - x\| = \| \lim_{n\to\infty} h^m(b_n)T^m y_n - x\| = \|h^m(b)T^m y\|,$$

so as $\|h^m(b_n)T^m y_n - x\| < \frac{\varepsilon}{2}$ we have that $\|h^m(b)T^m y - x\| \leq \frac{\varepsilon}{2} < \varepsilon$. But now we have a contradiction because we had that $\|h^n(b)T^n y - x\| > \varepsilon$ for any $n \geq 0$, so P is topologically transitive. $\square$

This result allows us to state a criterion for topological transitivity of skew-products. For now on, when we talk about a skew-product P of the operator T we will understand a skew-product $P : A \times X \to A \times X$ with (A,μ) a compact probability space and X a Banach space, defined by the expression:

$$P(a, x) = (f(a), h(a)Tx),$$

where $f : A \to A$ is ergodic with respect to measure μ, $h : A \to \mathbb{C}$ is a $L^1(\mu)$ function (or a continuous one if it is necessary) and $T : X \to X$ a linear and continuous operator.

Theorem 2.2.2. *Let P be a skew-product of the operator T and μ is an ergodic probability measure on A for f giving non-zero measure to every non-empty open set. Suppose that*

$$\gamma := \int_A \log |h| d\mu$$

is finite. Assume that there exist two dense subsets D_1, D_2 of X and a sequence of maps $S_n : D_2 \to X$ such that the following hold:

(i) $\liminf_n \|T^n x\|^{1/n} < e^{-\gamma}$ *for every* $x \in D_1$

(ii) $\limsup_n \|S_n y\|^{1/n} < e^{\gamma}$ *for every* $y \in D_2$

(iii) $\lim\limits_{n\to\infty} \|T^n S_n y - y\| = 0$ *for every* $y \in D_2$

Then P is topologically transitive.

Proof. We want to see that $E(a, x, \delta)$ is dense for every $a \in A$, $x \in X$ and $\delta > 0$ and use the precedent result to conclude that P is topologically transitive. For doing that we should consider an arbitrary element $(c, z) \in A \times X$ and an arbitrary ε and find $(b, y) \in E(a, x, \delta)$ such that $d(b, c) < \varepsilon$ and $\|y - z\| < \varepsilon$. Let us take U as the open ball $B_d(a, \delta)$ and let us define A_1 and A_2 as:

$$A_1 := \{b \in A \ : \ \frac{1}{n}\sum_{j=0}^{n-1} 1_U(f^j(b)) \to \mu(U)\},$$

$$A_2 := \{b \in A \ : \ \frac{1}{n}\sum_{j=0}^{n-1} \log|h(f^j(b))| \to \int_A \log|h| d\mu\}$$

Now, using the Birkhoff ergodic theorem (1.1.29), we have that $\mu(A_1) = 1$ and $\mu(A_2) = 1$, and as a consequence, $\mu(A_1 \cap A_2) = 1$[2]. Since μ has full support, $\mu(B_d(c,\varepsilon)) > 0$ and then $\mu(A_1 \cap A_2 \cap B_d(c,\varepsilon)) > 0$. We can find $b \in A_1 \cap A_2 \cap B_d(c,\varepsilon)$. As D_1 and D_2 are dense, we can take $u \in D_1$ and $v \in D_2$ with $\|u - z\| < \frac{\varepsilon}{2}$ and $\|v - x\| < \frac{\delta}{2}$. From condition (i), we have that

$$\inf\{\|T^{n_k}u\|^{\frac{1}{n_k}}\} < e^{-\gamma},$$

so there exist $\eta > 0$ such that

$$\|T^{n_k}u\| < e^{n_k(-\gamma-\eta)} \tag{2.1}$$

Let ω be any positive number satisfying that $(\max(1, \|T\|)e^{\gamma+\frac{\eta}{2}})^\omega \leq e^{\frac{\eta}{2}}$. We claim that for any $p \in [n_k, (1+\omega)n_k]$, one has

$$\|T^p u\| \leq e^{p(-\gamma-\frac{\eta}{2})} \tag{2.2}$$

This follows from:

-
$$\boxed{\|T^p u\|} = \|T^{p-n_k+n_k}u\| \leq \|T^{p-n_k}\|\|T^{n_k}u\| \leq \boxed{\max(1,\|T\|)^{\omega n_k} e^{n_k(-\gamma-\eta)}}, \tag{2.3}$$
where the last inequality comes from the inequality 2.1 and from the fact that if $p \in [n_k, (\omega+1)n_k]$ then $p - n_k \leq \omega n_k$ which implies that $\|T^{p-n_k}\| \leq \|T\|^{p-n_k} \leq \|T\|^{\omega n_k}$.

-
$$\boxed{e^{-p(\gamma+\frac{\eta}{2})}} \geq e^{-n_k\gamma - n_k(\frac{\eta}{2}) - \omega n_k \gamma - \omega n_k(\frac{\eta}{2})} = \boxed{e^{-(\gamma+\eta)n_k} e^{n_k(\frac{\eta}{2}-\omega\gamma-\omega\frac{\eta}{2})}}, \tag{2.4}$$
where the first inequality comes from the fact that if $p \in [n_k, (\omega+1)n_k]$ then $-p \in [-(\omega+1)n_k, -n_k]$ which means that $-p \geq -(1+\omega)n_k$.

From 2.3 and 2.4 we obtain the inequality 2.2 as follows:

$\|T^p u\| \leq \max(1,\|T\|)^{\omega n_k} e^{n_k(-\gamma-\eta)}$ from the first deduction. On the other hand we have that $(\max(1,\|T\|)e^{\gamma+\frac{\eta}{2}})^\omega \leq e^{\frac{\eta}{2}}$ or equivalently, $(\max(1,\|T\|)^\omega \leq e^{-\omega(\gamma+\frac{\eta}{2})}e^{\frac{\eta}{2}}$, so $(\max(1,\|T\|)^{\omega n_k} \leq e^{-\omega n_k(\gamma+\frac{\eta}{2})}e^{n_k\frac{\eta}{2}}$. To sum up we obtain that:

$$\begin{aligned}\|T^p u\| &\leq \max(1,\|T\|)^{\omega n_k} e^{n_k(-\gamma-\eta)} \leq e^{-\omega n_k(\gamma+\frac{\eta}{2})}e^{n_k\frac{\eta}{2}}e^{n_k(-\gamma-\eta)} \\ &\leq e^{-p(\gamma+\frac{\eta}{2})}\end{aligned}$$

Now, we fix ω and consider $\alpha > 0$ with $\dfrac{\mu(U)}{1+\omega} + \alpha < \mu(U) - \alpha$, and we set $E = \{j \in \mathbb{N} \ : \ f^j(b) \in U\}$. Since $b \in A_1$ we have that, by the Birkhoff Ergodic Theorem (1.1.29), there exists $N \in \mathbb{N}$, such that, for any $n \geq N$,one has that:

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} 1_U(f^j(b)) - \mu(U)\right| < \alpha$$

[2]Since $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2)$ the $1 \geq \mu(A_1 \cup A_2) \geq \mu(A_i) = 1$ for $i = 1, 2$ we deduce that $\mu(A_1 \cap A_2) = 2 - 1 = 1$

which is equivalent to

$$\left|\frac{|E\cap[1,n]|}{n}\right| < \alpha,$$

by the definition of the set E. This gives:

$$\mu(U)-\alpha < \frac{|E\cap[1,n]|}{n} < \mu(U)+\alpha$$

Suppose now, that for k large enough, the set $E\cap[n_k,(1+\omega)n_k]=\emptyset$. Then:

$$\frac{|E\cap[n_k,(1+\omega)n_k]|}{(1+\omega)n_k} \overset{(1)}{=} \frac{|E\cap[1,n_k]|}{(1+\omega)n_k} \overset{(2)}{\leq} \frac{\mu(U)+\alpha}{(1+\omega)} < \frac{\mu(U)}{1+\omega}+\alpha \overset{(3)}{<} \mu(U)-\alpha,$$

where (1) is because $E\cap[n_k,(1+\omega)n_k]=\emptyset$ for large values of k, (2) is because of the inequality deduce from the Birkhoff Ergodic Theorem and (3) comes from the condition that we set for α. This leads us to contradiction since:

$$\mu(U)-\alpha < \frac{|E\cap[n_k,(1+\omega)n_k]|}{(1+\omega)n_k} < \mu(U)-\alpha.$$

So $E\cap[n_k,(1+\omega)n_k]\neq\emptyset$, so we obtain a sequence $(p_k)_k$, each of them in $[n_k,(1+\omega)n_k]$, which means that $\|T^{p_k}u\|\leq e^{p_k(-\gamma-\frac{\eta}{2})}$, and also in E which means that $f^{p_k}(b)\in U=B_d(a,\delta)$.

Now, we use condition (ii) to ensure that:

$$\|S_{p_k}v\|\leq e^{p_k(\gamma-\frac{\eta}{2})}$$

for k large enough (we need that because if not $\frac{\eta}{2}$ could not be the number that verifies the inequality). On the other hand, since $b\in A_2$, for $\varepsilon=\frac{\eta}{4}$ there exists $N_1\in\mathbb{N}$ such that for every $n\geq N_2$ the following expression holds:

$$|\frac{1}{n}\sum_{j=0}^{n-1}\log|h(f^j(b))|-\gamma| = |\frac{1}{n}\log|\prod_{j=0}^{n-1}h(f^j(b))|-\gamma| = |\frac{1}{n}\log|h^n(b)|-\gamma| < \frac{\eta}{4}$$

That implies that

$$\gamma-\frac{\eta}{4} < \frac{1}{n}\log|h^n(b)| < \frac{\eta}{4}+\gamma \Longleftrightarrow$$

$$\Longleftrightarrow n(\gamma-\frac{\eta}{4}) < \log|h^n(b)| < n(\frac{\eta}{4}+\gamma) \Longleftrightarrow e^{n(\gamma-\frac{\eta}{4})} < |h^n(b)| < e^{n(\frac{\eta}{4}+\gamma)}.$$

Thus,

$$h^{p_k}(b)\|T^{p_k}u\| \leq e^{p_k(\gamma+\frac{\eta}{4})}e^{p_k(-\gamma-\frac{\eta}{2})} = e^{p_k(-\frac{\eta}{4})},$$

so when k tends to infinity $h^{p_k}(b)\|T^{p_k}u\|$ converges to zero. On the other hand,

$$|h^{p_k}(b)|^{-1}\|S_{p_k}v\| \leq e^{p_k(-\gamma+\frac{\eta}{4})}e^{p_k(\gamma-\frac{\eta}{2})} = e^{p_k(-\frac{\eta}{4})},$$

so $|h^{p_k}(b)|^{-1}\|S_{p_k}v\|$ converges also to zero. Now, we can conclude the proof because if we set $y=u+(h^{p_k}(b))^{-1}S_{p_k}v$ we have for k large enough that:

1. $d(b,c)<\varepsilon$ because $b\in B_d(c,\varepsilon)$ by definition.

2. $\|y-z\| < \varepsilon$ because $\|y-z\| = \|u + (h^{p_k}(b))^{-1}S_{p_k}v - z\| \leq \|u-z\| + \|(h^{p_k}(b))^{-1}S_{p_k}v\| \leq \|u-z\| + |h^{p_k}(b)|^{-1}\|S_{p_k}v\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ because of how we have chosen u and because $|h^{p_k}(b)|^{-1}\|S_{p_k}v\|$ converges to zero.

3. $(b,y) \in E(a,x,\delta)$ because $U = B_d(a,\delta)$ and as $b \in A_1$ we know that $f^{p_k}(b) \in U$ which means $d(a, f^{p_k}(b)) < \delta$. On the other hand,

$$\begin{aligned} h^{p_k}(b)\|T^{p_k}y - x\| &= h^{p_k}(b)\|T^{p_k}(u + (h^{p_k}(b))^{-1}S_{p_k}v) - x\| \leq \\ &\leq h^{p_k}(b)\|T^{p_k}u\| + h^{p_k}(b)(h^{p_k}(b))^{-1}\|T^{p_k}S_{p_k}v - x\| = \\ &= h^{p_k}(b)\|T^{p_k}u\| + \|T^{p_k}S_{p_k}v - v + (v-x)\| \leq \\ &\leq h^{p_k}(b)\|T^{p_k}u\| + \|T^{p_k}S_{p_k}v - v\| + \|v-x\| \leq \\ &\leq \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{2} = \delta \end{aligned}$$

because $h^{p_k}(b)\|T^{p_k}u\|$ and $\|T^{p_k}S_{p_k}v - v\|$ converges to zero.

□

One of the most interesting things that we can deduce from the proof of this theorem is that it does not depend on f, as long as if was continuous and ergodic. Another way of presenting this criterion is shown below (we omit this proof because is completely analogous to the proof presented for the previous theorem).

Theorem 2.2.3. *Let P be a skew-product of the operator T and μ is an ergodic probability measure on A for f giving non-zero measure to every non-empty open set. Suppose that*

$$\gamma := \int_A \log|h|d\mu$$

is finite. Assume that there exist two dense subsets D_1, D_2 of X, a sequence of integers $(n_k)_k$ and a bounded linear operator $S : X \to X$ such that the following conditions hold:

(i) $\liminf_{n_k} \|T^{n_k}x\|^{1/n_k} < e^{-\gamma}$ *for every $x \in D_1$*

(ii) $\limsup_n \|S^{n_k}y\|^{1/n_k} < e^{\gamma}$ *for every $y \in D_2$*

(iii) $TSy = y$ *for every $y \in D_2$*

Then P is topologically transitive.

The interest of having right inverse comes from the fact that we can estimate the norm of $S^{p_k}v$ knowing the norm of $S^{n_k}v$. That was not possible in the proof of theorem 2.2.2.

Now, we will study the topologically transitivity of skew-products from a spectral theory point of view. We first introduce a theorem from [41] that will be very useful in this section:

Theorem 2.2.4 (Müller)**.** *If X is a Banach space, then the set $\{x \in X : \liminf_{n\to\infty} \|T^n x\|^{1/n} = r(T)\}$ is dense in X with $r(T)$ the spectral radius of T.*

Therefore, if $r(T) < e^{-\gamma}$ (with γ as we have defined before) then $D_1 := \{x \in X \;:\; \liminf_{n\to\infty} \|T^n x\|^{1/n} = r(T)\}$ is dense and then we have the first condition of theorem 2.2.2.

If, in addition, we suppose that T is invertible, we can deduce from the following inequalities:

$$1 \leq r(T)r(T^{-1})$$

$$r(T) < e^{-\gamma}$$

that $r(T^{-1}) > e^{\gamma}$. So that, by Müller's result we have that taking

$$D_2 := \{x \in X \;:\; \liminf_{n\to\infty} \|T^{-n}x\|^{1/n} = r(T^{-1})\},$$

this set is dense in X. Moreover, this implies that for every $x \in D_2$ it holds that

$$\liminf_{n\to\infty} \|T^{-n}x\|^{1/n} > e^{\gamma}.$$

This contradicts the second condition of theorem 2.2.3.

To sum up, it is impossible to follow an argument that gives both conditions using the spectral radius. However, the existence of sufficiently many eigenvectors will be enough for our purposes.

Theorem 2.2.5. *Let P be a skew-product of the operator T and μ is an ergodic probability measure on A for f giving non-zero measure to every non-empty open set. Suppose that*

$$\gamma := \int_A \log|h| d\mu$$

is finite and that the following two vector spaces

$$H_+^\gamma(T) = span\{\ker(T-\lambda I) \;:\; |\lambda| > e^{-\gamma}\} \quad \text{and} \quad H_-^\gamma(T) = span\{\ker(T-\lambda I) \;:\; |\lambda| < e^{-\gamma}\}$$

are dense. Then P is topologically transitive.

Proof. This result is, in fact, a corollary of Theorem 2.2.2. We just should choose $D_1 := H_-^\gamma(T) = \mathrm{span}\{\ker(T-\lambda I) \;:\; |\lambda| < e^{-\gamma}\}$ and $D_2 := H_+^\gamma(T) = \mathrm{span}\{\ker(T-\lambda I) \;:\; |\lambda| > e^{-\gamma}\}$.

Now, any $y \in D_2$ y will be of the form:

$$y = \sum_{i=1}^{r} a_i y_i$$

with $a_i \in \mathbb{C}$ with $1 \leq i \leq r$ and $y_i \in \ker(T - \lambda_i I)$, which means that $Ty_i = \lambda_i y_i$. Now, defining the operators S_n as:

$$S_n y := \sum_{i=1}^{r} a_i \frac{1}{\lambda_i^n} y_i$$

is easy that the Theorem 2.2.2 holds because:

- $\liminf_n \|T^n x\|^{1/n} < e^{-\gamma}$ for every $x \in D_1$:
  Since $x \in D_1$ we have that $x = \sum_{i=1}^{r} b_i x_i$ for some $b_i \in \mathbb{C}$ with $Tx_i = \lambda_i x_i$ with $|\lambda_i| < e^{-\gamma}$. For that:

$$\begin{aligned}\liminf_n \|T^n x\|^{1/n} &= \liminf_n \|T^n \left(\sum_{i=1}^{r} b_i x_i\right)\|^{1/n} = \liminf_n \|\sum_{i=1}^{r} b_i T^n x_i\|^{1/n} = \\ &= \liminf_n \|\sum_{i=1}^{r} b_i \lambda_i^n x_i\|^{1/n} \leq \liminf_n \left(\sum_{i=1}^{r} \|b_i \lambda_i^n x_i\|\right)^{1/n} = \\ &= \liminf_n \left(\sum_{i=1}^{r} |b_i||\lambda_i|^n \|x_i\|\right)^{1/n} < \liminf_n \left(\sum_{i=1}^{r} |b_i| e^{-n\gamma} \|x_i\|\right)^{1/n} = \\ &= \liminf_n e^{-\gamma} \left(\sum_{i=1}^{r} |b_i|\|x_i\|\right)^{1/n} = e^{-\gamma},\end{aligned}$$

  where the last identity is due the fact that $\sum_{i=1}^{r} |b_i|\|x_i\|$ is a real constant that converges to one when we consider $\lim_n (\sum_{i=1}^{r} |b_i|\|x_i\|)^{1/n}$

- $\limsup_n \|S_n y\|^{1/n} < e^{\gamma}$ for every $y \in D_2$:

$$\limsup_n \|S_n y\|^{1/n} = \limsup_n \left\|\sum_{i=1}^{r} a_i \frac{1}{\lambda_i^n} y_i\right\|^{1/n} \leq$$

$$\leq \limsup_n \left(\sum_{i=1}^{r} |a_i| |\frac{1}{\lambda_i^n}| \|y_i\|\right)^{1/n} < \limsup_n e^{\gamma} \left(\sum_{i=1}^{r} |a_i|\|y_i\|\right)^{1/n} = e^{\gamma},$$

  in the same way as before.

- $\lim_n \|T^n S_n y - y\| = 0$ for every $y \in D_2$:

$$\lim_n \|T^n S_n y - y\| = \lim_n \left\|T^n \left(\sum_{i=1}^{r} a_i \frac{1}{\lambda_i^n} yi\right) - y\right\| =$$

$$= \lim_n \left\|\sum_{i=1}^{r} a_i \frac{1}{\lambda_i^n} \lambda_i^n y_i - y\right\| = \lim_n \left\|\sum_{i=1}^{r} a_i y_i - y\right\| = 0,$$

  by the definition of $y \in D_2$

□

In case that f was uniquely ergodic we have the following result:

Proposition 2.2.6. *Let P be a skew-product of the operator T and μ is an ergodic probability measure on A for f giving non-zero measure to every non-empty open set. Suppose that, in addition, f is uniquely ergodic with respect to μ and P is topologically transitive. Then every component of $\sigma(T)$ (the spectrum of T) meets the circle of radius $e^{-\gamma}$ centered at 0.*

Proof. First, we suppose that $\sigma(T)$ is connected. By contradiction, suppose that $\sigma(T)\cap\{z\in\mathbb{C} \ : \ |z|=e^{-\gamma}\}=\emptyset$. Since $\sigma(T)$ is connected there exists $\eta>0$ such that $\sigma(T)$ is either contained in $D(0,e^{-\gamma-\eta})$ or outside of $D(0,e^{-\gamma+\eta})$. Now we will prove that either $\|T^n\|\leq e^{n(-\gamma-\eta)}$ or $\|T^nx\|\geq e^{n(-\gamma+\eta)}$ for every $x\in X$ and for n large enough. To see this we proceed as follows:

By the compactness of $\sigma(T)$ and the spectral radius formula (1.1.18), we have that $\lim\limits_{n\to\infty}\|T^n\|^{1/n}=r(T)<e^{-\gamma-\eta}$. Hence,

$$\lim_{n\to\infty}\|T^nx\|^{1/n}\leq\lim_{n\to\infty}\|T^n\|^{1/n}\|x\|^{1/n}<e^{-\gamma-\eta}$$

or equivalently,

$$\|T^nx\|<e^{n(-\gamma-\eta)}$$

for n large enough, that is one of the desirable inequalities. The other can be obtained in an analogous way taking in count that, as $0\notin\sigma(T)$, T is invertible and since $\sigma(T^{-1})=\sigma(T)^{-1}$ we can refer to the proof of the previous case.

Now, since f is uniquely ergodic and h is continuous we can use the Oxtoby's Theorem (1.1.30), that states that:

$$\lim_n\frac{1}{n}\sum_{j=0}^{n-1}\log|h(f^j(a))|=\gamma,$$

uniformly for every $a\in A$, which means that, taking $\varepsilon=\eta$, there exists $N\in\mathbb{N}$ such that, for every $n\geq N$ one has:

$$\left|\frac{1}{n}\sum_{j=0}^{n-1}\log|h(f^j(a))|-\gamma\right|=\left|\frac{1}{n}\log\left|\prod_{j=0}^{n-1}h(f^j(a))\right|-\gamma\right|=$$

$$=\left|\frac{1}{n}\log|h^n(a)|-\gamma\right|<\eta$$

which implies that

$$n(\gamma-\eta)<\log|h^n(a)|<n(\gamma+\eta)\Longleftrightarrow e^{n(\gamma-\eta)}<|h^n(a)|<e^{n(\gamma+\eta)}$$

Hence, either

$$|h^n(a)|\|T^nx\|\leq e^{n(\gamma+\eta)}e^{n(-\gamma-\eta)}=1$$

or

$$|h^n(a)|\|T^nx\|\geq e^{n(\gamma-\eta)}e^{n(-\gamma+\eta)}=1$$

for n large enough, which implies that (a,x) cannot be a transitive because these inequalities would imply that $E(a,x,\delta)$ is not dense for every $a\in A$, $x\in X$ and $\delta>0$ so by the characterization of the transitivity gave in the introduction of the section is not transitive. But that contradicts the hypothesis that we have so $\sigma(T)$ meets the disk centered in 0 and of radio $e^{-\gamma}$.

For the general case, it is enough to consider every connected component of $\sigma(T)$ and use the Riesz Decomposition Theorem (1.1.17) in order to have that, if M is a invariant subspace of X for T, then the operator induced by P on $A\times X/M$ is topologically transitive too. Restricted to every single connected component of the spectrum of the operator T we can use the first part of the proof. $\square$

Remark 2.2.7. If we do not assume that f is uniquely ergodic, then we obtain that for μ-almost every $a \in A$, (a, x) is not transitive for every $x \in X$, which is not the uniformly convergence that provides the Oxtoby's Theorem (1.1.30) and that is fundamental for the construction of the precedent proof (summarizing, that the hypothesis of uniquely ergodicity for f cannot be ignored).

The case where f is uniquely ergodic with respect to the measure μ is interesting because it allows us to change the two above mentioned critera in a stronger one.

Theorem 2.2.8. *Let P be a skew-product of the operator T and μ is an ergodic probability measure on A for f giving non-zero measure to every non-empty open set. Suppose that f is a uniquely ergodic homeomorphism with respect to μ and that*

$$\gamma := \int_A \log |h| d\mu$$

is finite. Assume that there exist two dense subsets D_1, D_2 of X, a sequence of maps $S_n : D_2 \to X$ such that the following hold:

(i) $\limsup_n \|T^n x\|^{1/n} < e^{-\gamma}$ *for every* $x \in D_1$

(ii) $\limsup_n \|S_n y\|^{1/n} < e^{\gamma}$ *for every* $y \in D_2$

(iii) $\lim_{n\to\infty} \|T^n S_n y - y\| = 0$ *for every* $y \in D_2$

Then for any $b \in A$ there exists $x \in X$ such that (b, x) has dense orbit under P.

Proof. Let $b \in A$ fixed and let $(a, x, \delta) \in A \times X \times (0, +\infty)$ be an arbitrary element. As before, we just have to prove that $E(a, x, \delta)$ is dense in $A \times X$. As we have fixed $b \in A$ we have to prove that

$$E(a, x, \delta) = \{y \in X \ : \ \exists n \geq 0 \ , \ d(a, f^n(b)) < \delta \ , \ \|h^n(b)T^n y - x\| < \delta\}$$

is dense in X. To see that we consider an arbitrary element $z \in X$ and an arbitrary $\varepsilon > 0$ and see that $B(z, \varepsilon) \cap E(a, x, \delta) \neq \emptyset$, i.e., that exists $y \in X$ such that $y \in E(a, x, \delta)$ and $\|y - z\| < \varepsilon$.

Since f is uniquely ergodic homeomorphism and μ has full support we have that f is minimal (see result 1.1.28), which means that every orbit of an element of A is dense in A. Therefore, there will be $n_0 > 0$ such that $d(a, f^{n_0}(b)) < \delta$. As D_1 and D_2 are dense subsets of X we can choose $u \in D_1$ and $v \in D_2$ such that $\|u - z\| < \frac{\varepsilon}{2}$ and $\|v - x\| < \frac{\delta}{2}$.

Using now conditions (i) and (ii) we have that there exists $\eta > 0$ and $N_1 \in \mathbb{N}$ such that, for every $k \geq N_1$:

$$\|T^{n_k} u\| \leq e^{n_k(-\gamma-\eta)} \quad \text{ and } \quad \|S_{n_k} v\| \leq e^{n_k(\gamma-\eta)},$$

because of the the definition of lim sup. Now, we use Oxtoby's Theorem (1.1.30) with $\varepsilon' = \frac{\eta}{2}$ and we obtain that there exists $N_2 \in \mathbb{N}$ such that, for every $n \geq N_2$ we have that:

$$|\frac{1}{n} \sum_{j=0}^{n-1} \log |h(f^j(b)| - \gamma| < \frac{\eta}{2},$$

where the convergence is uniformly. With a similar argument this last inequality is equivalent to:

$$e^{n(\gamma-\dfrac{\eta}{2})} < |h^n(b)| < e^{n(\gamma+\dfrac{\eta}{2})}$$

which implies also that:

$$e^{n_k(\gamma-\dfrac{\eta}{2})} < |h^{n_k}(b)| < e^{n_k(\gamma+\dfrac{\eta}{2})}$$

for every subsequence $(n_k)_k$ of $(n)_n$.

So, as reasoning in the precedent results, we have that both of $h^{n_k}(b)\|T^{n_k}u\|$ and $(h^{n_k}(b))^{-1}\|S_{n_k}v\|$ tend to zero in general for $k \geq N_1$ and $n \geq N_2$ (remember that $h^{n_k}(b)\|T^{n_k}u\| \leq e^{n_k(\gamma+\frac{\eta}{2})}e^{n_k(-\gamma-\eta)} = e^{n_k(-\frac{\eta}{2})}$ and $(h^{n_k}(b))^{-1}\|S_{n_k}v\| \leq e^{n_k(\gamma-\frac{\eta}{2})}e^{n_k(\gamma-\eta)} = e^{n_k(-\frac{3\eta}{2})}$). If we set $y = u + (h^{n_k}(b))^{-1}S_{n_k}v$, we have that

$$\|z-y\| = \|z-u-(h^{n_k}(b))^{-1}S_{n_k}v\| \leq \|z-u\|+\|(h^{n_k}(b))^{-1}S_{n_k}v\| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,$$

because of the inequality gave at the beginning of the proof and because the convergence of $(h^{n_k}(b))^{-1}S_{n_k}v$ to zero. That gives the first of the things that we want. For the second:

$$\begin{aligned}
\|h^{n_k}(b)T^{n_k}y - x\| &= \|h^{n_k}(b)T^{n_k}(u + (h^{n_k}(b))^{-1}S_{n_k}v) - x\| = \\
&= \|h^{n_k}(b)T^{n_k}u + h^{n_k}(b)T^{n_k}(h^{n_k}(b))^{-1}S_{n_k}v - x\| \leq \\
&\leq \|h^{n_k}(b)T^{n_k}u\| + \|h^{n_k}(b)T^{n_k}(h^{n_k}(b))^{-1}S_{n_k}v - x\| = \\
&= \|h^{n_k}(b)T^{n_k}u\| + \|T^{n_k}S_{n_k}v - v + v - x\| \leq \\
&\leq \|h^{n_k}(b)T^{n_k}u\| + \|T^{n_k}S_{n_k}v - v\| + \|v - x\| \leq \\
&\leq \frac{\delta}{4}+\frac{\delta}{4}+\frac{\delta}{2} = \delta,
\end{aligned}$$

because $\|h^{n_k}(b)T^{n_k}u\|$ and $\|T^nS_nv - v\|$ go to zero (the second one due to condition (iii)) and the inequality of the beginning of the proof. □

Notice that if we have a uniquely ergodic homeomorphism we obtain more than transitivity, because we obtain that every point of A gives a dense orbit under P.

2.3 Examples of Transitive Skew-Products

2.3.1 Backward Shift Operators

We consider the Banach space of sequences $X = \ell^p$ or $X = c_0$ with $1 \leq p < \infty$. The **Backward Shift Operator** defined on this space will be $B : \ell^p \to \ell^p$ defined by:

$$B((x_1, x_2, \dots, x_n, \dots)) = (x_2, x_3, \dots, x_n, \dots).$$

However, we will work with multiples of the backward shift, i.e.,

$$(\lambda B)((x_1, x_2, \dots, x_n, \dots)) = \lambda(x_2, x_3, \dots, x_n, \dots),$$

for $\lambda \in \mathbb{C}$.

In the preliminaries we have pointed out that λB is hypercyclic for $|\lambda| > 1$. The skew-product that arises when we consider an operator of this kind is of the form:

$$P_\lambda(a, x) = (f(a), h(a)(\lambda B)x)$$

To set sufficient conditions for this skew-product to be topologically transitive we need to point out some consideration.

The first is that there is a conceptual difference between the concept of being topologically transitive in this framework and the concept for the rest of cases. Here, we understand that the skew-product will be topologically transitive if there exists some point $(a, x) \in A^p$ whose orbit under P is dense. In order to follow the proof given by Costakis and Hadjiloucas (that is easier to understand) we will consider this alternative definition in this section. We need also a complementary result:

Proposition 2.3.1. *Let A be a compact metric space, $f : A \to A$ continuous with $f(A) = A$, and μ a probability measure on the Borel subsets of A (that is, the σ-algebra generated by the open sets) that has full support. If f is an ergodic measure-preserving transformation with respect to μ, then*

$$\mu(\{a \in A \, : \, \{f^n(a) \, : \, n = 1, 2, \dots\} \textit{ is dense }\}) = 1.$$

Proof. Since A is compact, there exists a finite subcover of open subsets for every cover of A. Let $U_0, \dots, U_N$ be a fixed subcover of A. The first thing we notice is that

$$\{a \in A \, : \, \{f^n(a) \, : \, n = 1, 2, \dots\} \text{ is dense }\} \equiv \{\bigcap_{m=0}^{N} \bigcup_{j=1}^{\infty} f^{-j}(U_m)\}$$

Now, $\bigcup\limits_{j=1}^{\infty} f^{-j}(U_m)$ is invariant under f for every $m = 0, \dots, N$ because for each one of these m's we have:

$$f^{-1}(\bigcup_{j=1}^{\infty} f^{-j}(U_m)) \subset \bigcup_{j=1}^{\infty} f^{-1}(f^{-j}(U_m)) = \bigcup_{j=1}^{\infty} f^{-j-1}(U_m) \subset \bigcup_{j=1}^{\infty} f^{-j}(U_m).$$

Since f is ergodic, we have that $\mu(\bigcup\limits_{j=1}^{\infty} f^{-j}(U_m)) \in \{0, 1\}$. As every U_m is open and f is continuous, $f^{-j}(U_m)$ remains open and $\bigcup\limits_{j=1}^{\infty} f^{-j}(U_m)$ is also open.

As a consequence, $\mu(\bigcup\limits_{j=1}^{\infty} f^{-j}(U_m)) = 1$ because μ has full support and also $\mu(\bigcap\limits_{m=0}^{N} \bigcup\limits_{j=1}^{\infty} f^{-j}(U_m)) = 1$ because if $\mu(A_i) = 1$ for every $i = 1, \dots, S$ then $\mu(\cap A_i) = 1$ (the proof is made for two sets in 2.2.2 and the general case is obtained by induction). So, by the equality of sets gave at the beginning we have that $\mu(\{a \in A \, : \, \{f^n(a) \, : \, n = 1, 2, \dots\} \text{ is dense }\} = 1$ as we want. □

Now we can give the main result of this section, that is based in the work of Costakis and Hadjiloucas [20]:

Theorem 2.3.2. *Let A be a compact metric space, $f : A \to A$ a continuous map with $f(A) = A$, μ an ergodic probability measure on A for f giving non-zero measure to every non-empty open set, and $h : A \to \mathbb{C}$ a continuous function. For every complex number λ consider the skew-product $P_\lambda : A \times \ell^p \to A \times \ell^p$ defined by:*

$$P_\lambda(a, x) = (f(a), h(a)(\lambda B)x)$$

Suppose that:

$$\gamma := \int_A \log |h(a)| d\mu$$

is finite. Then:

(i) If $|\lambda| > e^{-\gamma}$, P_λ is topologically transitive.

(ii) If $|\lambda| < e^{-\gamma}$, for μ-almost every $a \in A$, $\{P^n_\lambda(a, x) \ : \ n = 0, 1, \dots\}$ is not dense in $A \times \ell^p$ for every $x \in \ell^p$.

(iii) If f is uniquely ergodic and $|\lambda| < e^{-\gamma}$, then P_λ is not topologically transitive.

Proof. If we set

$$A_1 = \{a \in A \ : \ \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \log |h(f^j(a))| = \gamma\},$$

then, by Birkhoff's Ergodic Theorem (1.1.29) we have that $\mu(A_1) = 1$. Taking now any λ such that $|\lambda| > e^{-\gamma}$ we have that there exists $0 < \rho < 1$ such that

$$|\lambda| e^{(1-\rho)\gamma} > 1,$$

To prove assertion (i) we should consider three different cases:

$\boxed{\textbf{Case 1} \Longrightarrow \gamma > 0}$

For every $a \in A_1$ and for $\varepsilon = \rho\gamma$ there exists N_a (depends on a because for every $a \in A_1$ we have a different limit) such that for any $n \geq N_a$ we have that

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \log |h(f^j(a))| - \gamma \right| = \left| \frac{1}{n} \log | \prod_{j=0}^{n-1} h(f^j(a))| - \gamma \right| =$$

$$= \left| \frac{1}{n} \log |h^n(a)| - \gamma \right| < \rho\gamma$$

which is equivalent to

$$-\rho\gamma < \frac{1}{n} \log |h^n(a)| - \gamma < \rho\gamma$$

$$n(\gamma - \rho\gamma) < \log |h^n(a)| < n(\gamma - \rho\gamma)$$

$$e^{n(\gamma-\rho\gamma)} < |h^n(a)| < e^{n(\gamma-\rho\gamma)}$$

Now, let $X_1 = \{x \in \ell^p \ : \ x_n = 0 \text{ from some } n \text{ onwards}\}$, which is separable and dense in ℓ^p. Let (a_m, x_j) with $m, j \in \mathbb{N}$ be a countable dense set in $A \times \ell^p$. This can be done thanks to the fact of the separability of ℓ^p and the compactness of A.

Let now

$$E(m, j, \delta, n) = \{(a, x) \ : \ d(a_m, f^n(a)) < \frac{1}{\delta} \text{ and } \|h^n(a)(\lambda B)^n x - x_j\| < \frac{1}{\delta}\}$$

for $m, j, \delta, n \in \mathbb{N}$. Our interest now is to show that

$$\bigcup_{n=1}^{\infty} E(m, j, \delta, n)$$

is open and dense in $A \times \ell^p$ for every $m, j, \delta \in \mathbb{N}$.

$\boxed{\textit{Open}}$

Fix $m, j, \delta, n \in \mathbb{N}$ and let (a, x) be an arbitrary element of $E(m, j, \delta, n)$ (we suppose that is nonempty because if not it will be direct). We have to show that there exists an open set V such that $(a, x) \in V \subset E(m, j, \delta, n)$. Since f and h are continuous there exists $\eta > 0$ such that:

$$\text{If } d(a, b) < \eta \quad \text{then} \quad d(f^n(a), f^n(b)) < \frac{1}{\delta} - d(a_m, f^n(a))$$

$$\text{If } d(a, b) < \eta \quad \text{then} \quad |h^n(a) - h^n(b)| < \frac{\frac{1}{\delta} - \|h^n(a)(\lambda B)^n x - x_j\|}{\|(\lambda B)^n x\| + 1}$$

Now, we define $V = B_d(a, \eta) \times B(x, \eta)$. It is obvious that $(a, x) \in V$ so to conclude this part of the proof we just need to show that $V \subset E(m, j, \delta, n)$. For doing that, we take $(\beta, y) \in V$ and using the previous inequalities we have that:

$$\begin{aligned} d(a_m, f^n(b)) &\le d(a_m, f^n(a)) + d(f^n(a), f^n(b)) \le \\ &\le d(a_m, f^n(a)) + \frac{1}{\delta} - d(a_m, f^n(a)) \\ &= \frac{1}{\delta} \end{aligned}$$

$$\begin{aligned} \|h^n(b)(\lambda B)^n x - x_j\| &\le \|h^n(a)(\lambda B)^n x - x_j\| + |h^n(a) - h^n(b)| \|(\lambda B)^n x\| < \\ &< \|h^n(a)(\lambda B)^n x - x_j\| + \frac{\frac{1}{\delta} - \|h^n(a)(\lambda B)^n x - x_j\|}{\|(\lambda B)^n x\| + 1} \|(\lambda B)^n x\| = \\ &= \|h^n(a)(\lambda B)^n x - x_j\| \left(1 - \frac{\|(\lambda B)^n x\|}{\|(\lambda B)^n x\| + 1}\right) + \frac{1}{\delta} \frac{\|(\lambda B)^n x\|}{\|(\lambda B)^n x\| + 1} < \\ &< \frac{1}{\delta} \frac{1}{\|(\lambda B)^n x\| + 1} + \frac{1}{\delta} \frac{\|(\lambda B)^n x\|}{\|(\lambda B)^n x\| + 1} = \frac{1}{\delta}, \end{aligned}$$

where the last inequality comes from the fact that $(a, x) \in E(m, j, \delta, n)$, which means that $\|h^n(a)(\lambda B)^n x - x_j\| < \dfrac{1}{\delta}$. So we have that $E(m, j, \delta, n)$ is open and that $\bigcup\limits_{n=1}^{\infty} E(m, j, \delta, n)$ is also open.

$\boxed{\textit{Dense}}$

To see the density of this set we pick an arbitrary point $(b, y) \in A \times \ell^p$ and an arbitrary $\varepsilon > 0$ and we want to find an element in $B_d((b, y), \varepsilon)$ (i.e, $d(b, a) < \varepsilon$

and $\|y-x\|<\varepsilon$) such that it also belongs to $\bigcup_{n=1}^{\infty} E(m,j,\delta,n)$. Without loss of generality we can consider that $0<\varepsilon<\frac{1}{\delta}$. Now we take $z\in X_1$ such that there exists $N_1\in\mathbb{N}$ verifying that for every $n\geq N_1$ it holds that:

$$z_n=0\ \ \forall n\geq N_1 \quad \text{and} \quad \|z-y\|<\frac{\varepsilon}{2}.$$

Now we set

$$D:=\Big\{a\in A\ :\ \{f^n(a)\ :\ n=0,1,2,\dots\}\text{ is dense in }A\Big\}.$$

By the previous result we have that $\mu(D)=1$. Since $\mu(A_1)=1$ also, we have that $\mu(A_1\cap D)=1$. Since $B_d(b,\varepsilon)$ is an non-empty open set and μ has full support we can deduce that $\mu(B_d(b,\varepsilon))>0$ and for that, $\mu(A_1\cap D\cap B_d(b,\varepsilon))>0$.

Therefore there exists $a\in A_1\cap D\cap B_d(b,\varepsilon)$. We recall that, for every $a\in A_1$, we have:

$$|h^n(a)|>e^{n\gamma(1-\rho)}\Longleftrightarrow|\lambda^n h^n(a)|>|\lambda|^n e^{n\gamma(1-\rho)}, \tag{2.5}$$

because, by hypothesis, $|\lambda|>e^{-\gamma}>0$. On the other hand, we have that

$$|\lambda^n h^n(a)|>|\lambda|^n e^{n\gamma(1-\rho)}>e^{-n\gamma}e^{n\gamma(1-\rho)}=e^{-n\rho\gamma},$$

and then

$$\frac{1}{|\lambda^n h^n(a)|}\leq\frac{1}{e^{-n\rho\gamma}},$$

where the right part of the inequality tends to zero when n tends to infinity, i.e. for every $\varepsilon'>0$ there exists $N_2\in\mathbb{N}$ and $N_2\geq N_1$ such that

$$|h^n(a)\lambda^n|^{-1}<\varepsilon'$$

. Taking $\varepsilon'=\frac{\varepsilon}{2\|x_j\|}$ we obtain, that for the $a\in A_1\cap D\cap B_d(b,\varepsilon)$ chosen by 2.5 there exists $N_2\geq N_1$ such that:

$$|h^n(a)\lambda^n|^{-1}<\frac{\varepsilon}{2\|x_j\|}.$$

As $a\in D$ also, we have that the orbit of a under f is dense, so there exists $n_0\in\mathbb{N}$ such that $n_0\geq N_2$ and

$$d(a_m,f^{n_0}(a))<\varepsilon<\frac{1}{\delta},$$

so we have the a of the pair (a,x) that belongs to $\bigcup_{n=0}^{\infty} E(m,j,\delta,n)$ and it also verifies that $d(b,a)<\varepsilon$ because $a\in B_d(b,\varepsilon)$.

On the other hand we take

$$x=z+(h^{n_0}(a))^{-1}\left(\frac{1}{\lambda}S\right)^{n_0}x_j,$$

where $S:\ell^p\to\ell^p$ is the forward shift operator defined as $S(x_1,\dots)=(0,x_1,\dots)$. So we must see that $\|x-y\|<\varepsilon$ and $\|h^{n_0}(a)(\lambda B)^{n_0}x-x_j\|<\frac{1}{\delta}$. For the first inequality we proceed as follows:

$$\|x-z\|=|h^{n_0}(a)\lambda^{n_0}|^{-1}\|S^{n_0}x_j\|<\frac{\varepsilon}{2\|x_j\|}\|x_j\|=\frac{\varepsilon}{2},$$

because of the estimation that we have obtain before and because of the fact that the forward shift maintains the norm of any element.

$$\|x - y\| \leq \|x - z\| + \|z - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

since we have picked z such that $\|z - y\| < \dfrac{\varepsilon}{2}$.

For the second inequality we have that:

$$\|h^{n_0}(a)(\lambda B)^{n_0} x - x_j\| = \|h^{n_0}(a)(\lambda B)^{n_0}(z + (h^{n_0}(a))^{-1}\left(\frac{1}{\lambda} S\right)^{n_0} x_j) - x_j\| =$$

$$= \|h^{n_0}(a)(\lambda B)^{n_0} z + (h^{n_0}(a))^{-1} h^{n_0}(a)(\lambda B)^{n_0}(\left(\frac{1}{\lambda} S\right)^{n_0} x_j) - x_j\| = 0 + x_j - x_j = 0 < \frac{1}{\delta},$$

because $z_{n_0} = 0$ because $z_n = 0$ for every $n \geq N_1$ and because S is the right inverse of B (i.e., $BS = I$).

So that $(a, x) \in E(m, j, \delta, n_0)$ and, as a consequence, $(a, x) \in \bigcup_{n=1}^{\infty} E(m, j, \delta, n)$.

As

$$\bigcup_{n=1}^{\infty} E(m, j, \delta, n)$$

is open and dense we have that

$$\bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{\delta=1}^{\infty} \bigcup_{n=1}^{\infty} E(m, j, \delta, n)$$

is a dense G_δ set in $A \times \ell^p$ by the Baire's Category Theorem (1.1.2). As in the result proved before, this set is precisely the set of points in $A \times \ell^p$ whose orbit under P is dense. This concludes this part of the proof.

$\boxed{\textbf{Case 2} \Longrightarrow \gamma < 0}$

It is completely analogous to the precedent case just considering $\varepsilon = -\rho\gamma$ in the limit that gives the Birkhoff's Ergodic Theorem.

$\boxed{\textbf{Case 3} \Longrightarrow \gamma = 0}$

It is also completely analogous to the precedent two cases just considering that there exists $\rho > 0$ such that $|\lambda| e^{-\rho} > 1$. So, taking $\varepsilon = \rho$ in the limit that gives the Birkhoff's Ergodic Theorem and following the steps followed in the previous cases we obtain the result.

To prove now assertion (ii) we proceed as follows:

$\boxed{\textbf{Case 1} \Longrightarrow \gamma > 0}$

Recall that we have that $|\lambda| < e^{-\gamma} \Longleftrightarrow |\lambda| e^{\gamma} < 1$. Now, we take $\varepsilon > 0$ such that $|\lambda| e^{\gamma + \varepsilon} < 1$. The Birkhoff's Ergodic Theorem provides that the set

$$A_1 = \{a \in A \ : \ \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \log |h(f^j(a))| = \gamma\}$$

has full measure. So, for every $\varepsilon' > 0$ and for every $a \in A_1$ there exists N_a(it depends on a because for every $a \in A_1$ we have the limit) such that for every

$n \geq N_a$ it is verified that:

$$\left|\frac{1}{n}\sum_{j=0}^{n-1}\log|h(f^j(a))| - \gamma\right| = \left|\frac{1}{n}\log\prod_{j=0}^{n-1}|h(f^j(a))| - \gamma\right| =$$
$$= |\frac{1}{n}\log|h^n(a)| - \gamma| < \varepsilon'$$

which is equivalent to

$$\gamma - \varepsilon' < \frac{1}{n}\log|h^n(a)| < \gamma + \varepsilon'$$

$$e^{n(\gamma-\varepsilon')} < |h^n(a)| < e^{n(\gamma+\varepsilon')}$$

Now, we take $\varepsilon' = \varepsilon$, and we have that for every $a \in A_1$ and for every $n \geq N_a$:

$$e^{n(\gamma-\varepsilon)} < |h^n(a)| < e^{n(\gamma+\varepsilon)} \iff |\lambda|^n e^{n(\gamma-\varepsilon)} < |\lambda|^n|h^n(a)| < |\lambda|^n e^{n(\gamma+\varepsilon)}$$

Now, as we have that $|\lambda|e^{\gamma+\varepsilon} < 1$, then $\lim_n |\lambda|^n e^{n(\gamma+\varepsilon)} = 0$, so for every $a \in A_1$ we have that $|h^n(a)|\|(\lambda B)^n x\|$ converges also to zero for every $x \in \ell^p$. This means that for every $a \in A_1$ the orbit of (a, x) under P_λ is not dense for every $x \in \ell^p$ which concludes this part of the proof (If part of the orbit goes to zero as we increase the number of iterates it cannot be dense). Since A_1 has full measure, the condition we have obtained can be reformulated as follows:
For μ-almost every $a \in A$ the orbit of (a, x) under P_λ is not dense in $A \times \ell^p$ for every $x \in \ell^p$.

$\boxed{\textbf{Case 2} \implies \gamma \leq 0}$

It is completely analogous, taking $\varepsilon > 0$ such that $|\lambda|e^{\gamma-\varepsilon} < 1$ and $\varepsilon' = -\varepsilon$.
Finally, to prove assertion (iii) we proceed as follows:

$\boxed{\textbf{Case 1} \implies \gamma > 0}$

Recall that we have that $|\lambda| < e^{-\gamma} \iff |\lambda|e^{\gamma} < 1$. Now, we take $\varepsilon > 0$ such that $|\lambda|e^{\gamma+\varepsilon} < 1$. The Oxtoby's Theorem (1.1.30) provides that the set

$$A_2 = \{a \in A \ : \ \lim_n \frac{1}{n}\sum_{j=0}^{n-1}\log|h(f^j(a))| = \gamma\}$$

where the convergence is uniformly in a, has full measure. So, for every $\varepsilon' > 0$ there exists N, that does not depend of a, such that for every $n \geq N$ it is verified that:

$$\left|\frac{1}{n}\sum_{j=0}^{n-1}\log|h(f^j(a))| - \gamma\right| = \left|\frac{1}{n}\log\prod_{j=0}^{n-1}|h(f^j(a))| - \gamma\right| =$$
$$= |\frac{1}{n}\log|h^n(a)| - \gamma| < \varepsilon'$$

which is equivalent to

$$\gamma - \varepsilon' < \frac{1}{n}\log|h^n(a)| < \gamma + \varepsilon'$$

$$e^{n(\gamma-\varepsilon')} < |h^n(a)| < e^{n(\gamma+\varepsilon')}$$

for every $a \in A_2$.

Now, we take $\varepsilon' = \varepsilon$, and we have that for every $n \geq N$:

$$e^{n(\gamma-\varepsilon)} < |h^n(a)| < e^{n(\gamma+\varepsilon)} \iff |\lambda|^n e^{n(\gamma-\varepsilon)} < |\lambda|^n |h^n(a)| < |\lambda|^n e^{n(\gamma+\varepsilon)}$$

for every $a \in A_2$. Now, as we have that $|\lambda| e^{\gamma+\varepsilon} < 1$, $\lim_n |\lambda|^n e^{n(\gamma+\varepsilon)} = 0$, so we have that $|h^n(a)| \|(\lambda B)^n x\|$ converges also to zero for every $(a,x) \in A \times \ell^p$. That means that the orbit of (a,x) under P_λ is not dense for every $(a,x) \in A \times \ell^p$ which contradicts the definition of topological transitivity that we had handled throughout this section.

$\boxed{\textbf{Case 2} \Longrightarrow \gamma \leq 0}$

It is completely analogous, taking $\varepsilon > 0$ such that $|\lambda| e^{\gamma-\varepsilon} < 1$ and $\varepsilon' = -\varepsilon$. □

2.3.2 Translation Operators

We have introduced in the preliminaries the Birkhoff's Operators. A generalization of the Birkhoff's Operator are the **Translation Operators**.

First of all we should point out that the definitions of a skew-product and a topologically transitive skew-product have a natural generalization if we consider a Frèchet space X and an operator T on X. In this section X will denote the space of entire functions $\mathcal{H}(\mathbb{C})$ and $T : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ will be the translation operator, defined by:

$$T(u(z)) = u(z+1).$$

As we have seen in the preliminaries, this kind of operators are hypercyclic. If we consider a skew-product of a translation operator, the condition for being topologically transitive is very weak. Before starting the main theorem we enunciate an important approximation theorem that will be needed:

Theorem 2.3.3 (Mergelyan's Approximation Theorem)**.** *Let K be a compact subset of the complex plane $\mathbb{C}$ such that $C \setminus K$ is connected. Then, every continuous function $f : K \to \mathbb{C}$, such that the restriction $f|_{int(K)}$ is holomorphic, can be approximated uniformly on K with polynomials*

The main result is:

Theorem 2.3.4. *Let A be a compact metric space, let $f : A \to A$ be a continuous map, let μ be an ergodic probability measure on A for f giving non-zero measure to every non-empty open set and let $h : A \to \mathbb{C}$ be a continuous function. Let T be the translation operator defined on $\mathcal{H}(\mathbb{C})$. Suppose that h is μ-almost everywhere non-zero. Then the skew-product $P(a,u) = (f(a), h(a)T(u))$ is topologically transitive.*

Proof. As always we want to prove that, given an arbitrary $R > 0$ and an arbitrary $a, u, \delta \in A \times \mathcal{H}(\mathbb{C}) \times (0, +\infty)$, $E(a,u,\delta)$ is dense in $A \times \mathcal{H}(\mathbb{C})$, where

$$E(a,u,\delta) = \{(b,v) \in A \times \mathcal{H}(\mathbb{C}) \ : \ \exists n \geq 0 \ , \ d(a, f^n(b)) < \delta \quad \text{and}$$

$$\|h^n(b) T^n(v) - u\|_{C(\overline{D}(0,R))} < \delta\},$$

where $\overline{D}(0,R)$ is the closed disk centered in 0 and with radius R and where $\| \cdot \|_{C(K)}$ is the sup-norm on K. So, we take an arbitrary $(c,w) \in A \times \mathcal{H}(\mathbb{C})$

and an arbitrary $\varepsilon, \rho > 0$ and we want to find a $(b,v) \in B_d((c,w),\varepsilon)$ and in $E(a,u,\delta)$. Let $Z = \{b \in A \ : \ h(b) = 0\}$. By hypothesis, h is μ-almost everywhere non-zero, so $\mu(Z) = 0$. Since f is ergodic and $\mu(Z) = 0$ we have that $A_1 = \{b \in A \ : \ \forall n \geq 1 \ , \ h^n(b) \neq 0\} = \{b \in A \ : \ \forall n \geq 0 \ , \ f^n(b) \notin Z\}$ because if $f^n(b) \notin Z$ for every $n \geq 0$ then $h(f^n(b)) \neq 0$ for every $n \geq 0$ so by the definition of $h^n(b)$ we have that $h^n(b) \neq 0$ for every $n \geq 1$, which gives the equality between the two sets. This set has full measure, ($\mu(A_1) = 1$). That is because ergodicity implies by Birkhoff Ergodic Theorem that $A = \{b \in A \ : \ \frac{1}{n} \log|h^n(b)| \to \gamma\}$ has full measure. But this equivalent to the fact that $h^n(b) \neq 0$ for every $n \geq 1$, which together with the condition of $\mu(Z) = 0$ gives that the set defined (A_1) has also full measure.

Now, using the Birkhoff's Ergodic Theorem (1.1.29) we have that we can find $b \in A_1$ and a sequence $(n_k)_k$ tending to infinity such that $d(b,c) < \varepsilon$ and

$$d(a, f^{n_k}(b)) < \delta \tag{2.6}$$

(to see the proof of the existence of that sequence and the way to construct it see the proof of the main theorem 2.2.2). On the other hand, for k large enough we have that $\overline{D}(0,R) + n_k$ and $\overline{D}(0,\rho)$ are disjoint (trivially). As $\overline{D}(0,R) + n_k$ and $\overline{D}(0,\rho)$ are compacts with connected complement in $\mathbb{C}$ we can use Mergelyan's Approximation Theorem to affirm that there exists a function $v \in \mathcal{H}(\mathbb{C})$ (polynomial) satisfying:

$$\|v - w\|_{C(\overline{D}(0,\rho))} < \varepsilon \quad \text{and} \quad \|h^{n_k}(b)v - T^{-n_k}(u)\|_{C(\overline{D}(0,R)+n_k)} < \delta,$$

where the last inequality implies

$$\|h^{n_k(b)} T^{n_k}(v) - u\|_{C(\overline{D}(0,R))} < \delta \tag{2.7}$$

Joining the inequalities 2.6 and 2.7 we have that $(b,v) \in E(a,u,\delta)$. Since $\|v - w\|_{C(\overline{D}(0,\rho))} < \varepsilon$ and $d(b,c) < \varepsilon$, we have that $(b,v) \in B_d((c,w),\varepsilon)$ which completes the proof. $\square$

2.3.3 Differentiation Operators

In the preliminaries we have introduced the Maclane's Operator, which now we call **Differentiation Operators**. If we remember, we considered the space of entire functions $\mathcal{H}(\mathbb{C})$ and the operator $D : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ defined by:

$$D(u) = u'.$$

As we have seen in the preliminaries this operator is hypercyclic. If we consider the skew-product $P(a,u) = (f(a), h(a)D(u))$ a very weak condition is require for having that it is topologically transitive:

Theorem 2.3.5. *Let A be a compact metric space, let $f : A \to A$ be a continuous map, let μ be an ergodic probability measure on A for f giving non-zero measure to every non-empty open set and let $h : A \to \mathbb{C}$ be a continuous function. Let D be the differentiation operator defined on $\mathcal{H}(\mathbb{C})$. Suppose $\gamma := \int_A \log|h| d\mu$ is finite. Then the skew-product $P(a,u) = (f(a), h(a)T(u))$ is topologically transitive.*

Proof. Like in the other cases we want to prove that, given an arbitrary $R > 0$ and an arbitrary $a, u, \delta \in A \times \mathcal{H}(\mathbb{C}) \times (0, +\infty)$, $E(a, u, \delta)$ is dense in $A \times \mathcal{H}(\mathbb{C})$, where

$$E(a,u,\delta) = \{(b,v) \in A \times \mathcal{H}(\mathbb{C}) \ : \ \exists n \geq 0 \ , \ d(a, f^n(b)) < \delta \quad \text{and}$$

$$\|h^n(b)T^n(v) - u\|_{C(\overline{D}(0,R))} < \delta\}.$$

So, we take an arbitrary $(c, w) \in A \times \mathcal{H}(\mathbb{C})$ and an arbitrary $\varepsilon, \rho > 0$. We want to find $(b, v) \in B_d((c, w), \varepsilon) \cap E(a, u, \delta)$. As in the precedent theorem, and using the reasoning followed in 2.2.2 we obtain, by the Birkhoff's Ergodic Theorem, that we can find $b \in A_1$ and a sequence $(n_k)_k$ tending to infinity such that $d(b, c) < \varepsilon$, $d(a, f^{n_k}(b)) < \delta$ and $|h^{n_k}(b)| \geq e^{n_k(\gamma - \varepsilon')}$, where ε' can be chosen to be smaller than γ because it comes from the limit of the Birkhoff's Theorem.

Now, we take a polynomial P and construct the sequence $v_k = P + (h^{n_k})^{-1} I^{n_k}(u)$, where I is the integration operator (defined by the expression $I\phi(z) = \int\limits_0^z \phi(w)dw$). Besides, $(v_k)_k$ converges to P as k goes to infinity because $\lim\limits_k v_k = \lim\limits_k (P + (h^{n_k})^{-1} I^{n_k}(u)) = P + \lim\limits_k (h^{n_k})^{-1} I^{n_k}(u) = P + 0 = P$ because $|(h^{n_k})^{-1}| \|I^{n_k}(u)\| \leq e^{-n_k(\gamma - \eta/2)} \frac{C_{K,\phi}^{n_k}}{n_k!}$, where $C_{K,\phi}$ is a constant that only depends on the compact K and ϕ. So we can conclude that $|(h^{n_k})^{-1}| \|I^{n_k}(u)\|$ converges to zero and v_k converges to P.

To conclude the proof we just have to point out that taking $v = v_k$ for some k large enough we will have that $\|v - w\|_{C(\overline{D}(0,R))} < \varepsilon$ by Mergelyan's Theorem. In addition, since $d(b, c) < \varepsilon$ we have found $(b, v) \in B_d((c, w), \varepsilon)$. We also obtain that $\|h^{n_k}(b) D^{n_k}(v) - u\| < \delta$ because for n_k greater than the degree of P,

$$\begin{aligned}
\|h^{n_k}(b) D^{n_k}(v_k) - u\| &= \|h^{n_k}(b) D^{n_k}(P + (h^{n_k})^{-1} I^{n_k}(u)) - u\| = \\
&= \|h^{n_k}(b) D^{n_k}(P) + h^{n_k}(b)(h^{n_k})^{-1} D^{n_k}(I^{n_k}(u)) - u\| \\
&= \|D^{n_k}(I^{n_k}(u)) - u\| < \delta
\end{aligned}$$

which, together with the inequality obtained before $(d(a, f^{n_k}(b)) < \delta)$ show that $(b, v) \in E(a, u, \delta)$ $\square$

Bibliography

[1] Albanese, A.A.; Barrachina, X. & Mangino, E. M. *Distributional chaos for strongly continuous semigroups of operators.* Commun. Pure Appl. Anal. (2013) To appear

[2] Badea, C. & Grivaux, S. *Unimodular eigenvalues, uniformly distributed sequences and linear dynamics.* Adv. Math, 211:766-793, 2007.

[3] Balibrea, F.; Smítal, J. & Stefánková, M. *The three versions of distributional chaos.* Chaos Solitons Fractals, 23(5):1581-1583, 2005.

[4] Banks, J.; Brooks, J.; Cairns, G.; Davis, G. & Stacey, P. *On Devaney's definition of chaos.* Amer. Math. Monthly 99 (1992), no. 4, 332–334.

[5] Barrachina, X. & Peris, A. *Distributional chaotic translation semigrouos.* Journal of Difference Equations and Applications 18(4): 751-761, 2012.

[6] Bayart, F. & Bermudez, T. *Semigroups of chaotic operators.* Bull. Lond. Math. Soc., 41(5):823-830, 2009.

[7] Bayart, F.; Costakis, C. & Hadjiloucas, D. *Topologically Transitive Skew-Products of Operators.* Ergodic Theory and Dinamical Systems 30 (2010), no. 1, 33–49

[8] Bayart, F. & Matheron, E. *Dynamics of linear operators.* Cambridge Tracts in Mathematics, Cambridge University Press. 179 (2009).

[9] Beauzamy, B. *Un opérateur, sur l'espace de Hilbert, dont tous les polynômes sont hypercycliques.* C. R. Acad. Sci. Paris Sér. I Math. 303 (1986), no. 18, 923-925.

[10] Beauzamy, B. *Introduction to operator theory and invariant subspaces.* North-Holland Mathematical Library. Amsterdam: North-Holland Publishing Co, 1988.

[11] Bermúdez, T.; Bonilla, A; Conejero, J.A. & Peris, A. *Hypercyclic, topologically mixing and chaotic semigroups on Banach spaces.* Studia Mathematica, 170(1):57-75, 2005.

[12] Bermúdez, T.; Bonilla, A; Martínez-Giménez, F. & Peris, A. *Li-Yorke and distributionally chaotic operators.* J. Math. Anal. Appl. 373(1): 83-93, 2011.

[13] Bermúdez, T.; Bonilla, A; Müller, V. & Peris, A. *Distributional chaos for linear operators.* Prepint

[14] Birkhoff, G. *Surface transformations and their dynamical applications.* Acta Math. 43 (1920), 1-119.

[15] Bonet, J.; Martínez-Giménez, F. & Peris, A. *Linear chaos on Fréchet spaces.* Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13(7):1649-1655, 2003.

[16] Bourdon, P.S. & Shapiro, J.H. *Cyclic Phenomena for Composition Operators.* Memoirs of the AMS, 125, AMS, Providence, RI, 1997.

[17] Chan, K.C. *The Density of Hypercyclic Operators on a Hilbert Space.* J. Operator Theory.

[18] Conejero, J.A.; Müller, V. & Peris, A. *Hypercyclic behavior of operators in a hypercyclic C_0-semigroup.* Journal of Functional Analysis, 244(1):342-348, 2007.

[19] Conejero, J.A. & Peris, A. *Hypercyclic translation C_0-semigroups on complex sectors.* Discrete Contin. Dyn. Syst. 25(4):1195-1208, 2009.

[20] Costakis, C. & Hadjiloucas, D. *Topologically Transitive Skew-Products of Backward Shift Operators and Hypercyclicity.* Proceedings of the American Mathematical Society 136 (2008), no. 3, 937–946.

[21] De la Rosa, M. & Read, C. *A hypercyclic operator whose direct sum $T \oplus T$ is not hypercyclic.* J. Operator Theory, 61:369-380, 2009.

[22] Desch, G.; Schappacher, W. & Webb, G.F. *Hypercyclic and chaotic semigroups of linear operators.* Ergodic Theory Dynamical Systems, 17(4):793-819, 1997.

[23] Devaney, R. L. *An introduction to chaotic dynamical systems.* Second edition. Addison-Wesley Studies in Nonlinearity. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.

[24] Engel, K.J. & Nagel, R. *One-parameter semigroups for linear evolution equations.* Vol. 194. Graduate Texts in Mathematics. New York: Springer-Verlag, 2000.

[25] Engel, K.J. & Nagel, R. *A short course on operator semigroups.* Universitext. New York: Springer, 2006.

[26] Feldman, N. S. *Linear Chaos.* Washington Lee University.

[27] Gethner, R. M. & Shapiro, J. H. *Universal vectors for operators on spaces of holomorphic functions.* Proc. AMS, 100 (1987), 281-288.

[28] Godefroy, G. & Shapiro, J. H. *Operators with dense, invariant, cyclic vector manifolds.* J. Funct. Anal. 98 (1991), no. 2, 229-269.

[29] Grosse-Erdmann, K.G. & Peris, A. *Linear chaos.* Universitext, Springer-Verlag.

[30] Halmos, P.R. *A Hilbert Space Problem Book.* Springer,New York 1982

[31] Herrero, D. *Hypercyclic operators and chaos.* J. Operator Th., 28 (1992),93-103.

[32] Hille, E. & Phillips, R. S. *Functional analysis and semi-groups.* American Mathematical Society Colloquium Publications, 31, 1957.

[33] Jiménez Munguía, R. R. *Vectores hipercíclicos para múltiplos del desplazamiento hacia atrás.* B.Sc.Universidad Autónoma del Estado de Hidalgo,Pachuca,2007

[34] Kalmes, T. *On chaotic C_0-semigroups and infinitely regular hypercyclic vectors.* Proc. Amer. Math. Soc., 134(10): 2997-3002, 2006.

[35] Kitai, C. *Invariant closed sets for linear operators.* Thesis (Ph.D.)-University of Toronto, Toronto, Canada, 1982.

[36] deLaubenfels, R. & Emamirad, H. *Chaos for functions of discrete and continuous weighted shift operators.* Ergodic Theory Dynamical Systems, 21(5):1411-1427, 2001.

[37] Li, T. Y. & Yorke, J. A. *Period three implies chaos.* American Mathematical Monthly, 82(10):985-992, 1975.

[38] Martínez-Giménez, F.; Oprocha, P. & Peris, A. *Distributional chaos for backward shifts.* J. Math. Anal. Appl., 351(2):607-615, 2009.

[39] Matsui, M.; Yamada, M. & Takeo, F. *Supercyclic and chaotic translation semigroups.* Proc. Amer. Math. Soc., 131(11): 3535-3546, 2003.

[40] Meise, R. & Vogt, D. *Introduction to functional analysis.* Oxford Graduate Texts in Mathematics, 2. The Clarendon Press, Oxford University Press, New York, 1997.

[41] Müller, V. *Orbits, weak orbits and local capacity of operators.* Integral Equations and Operator Theory, 41(2): 230-253, 2001.

[42] Oprocha, P. *Distributional chaos revisited.* Trans. Amer. Math. Soc., 361(9):4901-4925, 2009.

[43] Robertson, A.P. *Topological vector spaces.* Cambridge University Press 1964.

[44] Rolewicz, S. *On orbits of elements.* Studia Math.32(1969),17-22.

[45] Rudin, W. *Real and complex analysis.* Second edition. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1974.

[46] Salas, H. N. *Hypercyclic weighted shifts.* Trans. Amer. Math. Soc., 347(1995), 993-1004.

[47] Schweizer, B & Smítal, J *Measures of chaos and a spectral descomposition of dynamical systems on the interval.* Trans. Amer. Math. Soc., 344(2): 737-754, 1994.

[48] Smítal, J & Stefánková, M. *Distributional Chaos for triangular maps.* Chaos Solitons Fractals, 21(5):1125-1128, 2004.

[49] Walters, P. *An Introduction to Ergodic Theory.* Springer-Verlag New York Berlin Heidelberg, 1981.