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Between strong continuity and almost continuity

J. K. Kohli and D. Singh

Abstract. As embodied in the title of the paper strong and weak variants of continuity that lie strictly between strong continuity of Levine and almost continuity due to Singal and Singal are considered. Basic properties of almost completely continuous functions ($\equiv R$-maps) and δ-continuous functions are studied. Direct and inverse transfer of topological properties under almost completely continuous functions and δ-continuous functions are investigated and their place in the hierarchy of variants of continuity that already exist in the literature is outlined. The class of almost completely continuous functions lies strictly between the class of completely continuous functions studied by Arya and Gupta (Kyungpook Math. J. 14 (1974), 131-143) and δ-continuous functions defined by Noiri (J. Korean Math. Soc. 16, (1980), 161-166). The class of almost completely continuous functions properly contains each of the classes of (1) completely continuous functions, and (2) almost perfectly continuous ($\equiv$ regular set connected) functions defined by Dontchev, Ganster and Reilly (Indian J. Math. 41 (1999), 139-146) and further studied by Singh (Quaestiones Mathematicae 33(2)(2010), 1–11) which in turn include all δ-perfectly continuous functions initiated by Kohli and Singh (Demonstratio Math. 42(1), (2009), 221-231) and so include all perfectly continuous functions introduced by Noiri (Indian J. Pure Appl. Math. 15(3) (1984), 241-250).

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1. Introduction

Several weak, strong and other variants of continuity occur in the lore of mathematical literature and arise in diverse situations in mathematics and applications of mathematics. In 1960, Levine [18] introduced the concept of a strongly continuous function. Ever since then several strong variants of continuity have been introduced and studied by host of authors, which in general are stronger than continuity but weaker than strong continuity of Levine. One such variant of continuity is complete continuity due to Arya and Gupta [1]. In this paper we elaborate on a generalization of complete continuity called 'almost complete continuity' (≡ R-maps [3]) which is independent of continuity but stronger than 'δ-continuity' initiated by Noiri [23]'. We study basic properties of almost completely continuous functions and δ-continuous functions and discuss their interplay and interrelations with other variants of continuity that already exist in the mathematical literature. We reflect upon their place in the hierarchy of variants of continuity that lie strictly between strong continuity and almost continuity [32]. It turns out that the class of almost completely continuous functions properly contains the class of almost perfectly continuous (≡ regular set connected) functions defined by Dontchev, Ganster and Reilly ([4] [36]), and so includes the class of δ-perfectly continuous functions [13]; and is strictly contained in the class of δ-continuous functions, which in turn is properly contained in the class of almost continuous functions introduced by Singal and Singal [32].
The paper is organized as follows. Section 2 is devoted to basic definitions and preliminaries. In Section 3, we elaborate on the place of almost complete continuity in the hierarchy of variants of continuity that already exist in the literature. Therein examples are given to reflect upon the distinctiveness of the variants of continuity so discussed. Basic properties of almost completely continuous functions and δ-continuous functions are discussed in Section 4, and Section 5 is devoted to the study of preservance / interplay of topological properties under almost completely continuous functions and δ-continuous functions.

2. Preliminaries and basic definitions

A subset A of a space X is said to be **regular open** if it is the interior of its closure, i.e., $A = \overline{A}^0$. The complement of a regular open set is referred to as a **regular closed** set. A union of regular open sets is called **δ-open** [39]. The complement of a δ-open set is referred to as a **δ-closed** set. A subset A of a space X is called a **regular G_δ-set** [19] if A is the intersection of a sequence of closed sets whose interiors contain A, i.e., if $A = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} F_n^o$, where each F_n is a closed subset of X. The complement of a regular G_δ-set is called a **regular F_σ-set**. A subset A of a space X is said to be **semiopen** if $A^0 \subset A \subset \overline{A^0}$. A subset A of a space X is said to be **cl-open** [35] if for each $x \in A$ there exists a clopen set H such that $x \in H \subset A$, or equivalently,

A is expressible as a union of clopen sets. The complement of a *cl*-open set is referred to as a **cl-closed** set.

Definitions 2.1. *A function $f : X \to Y$ from a topological space X into a topological space Y is said to be*

(a) ***strongly continuous*** [18] *if $f(\overline{A}) \subset A$ for all $A \subset X$.*
(b) ***perfectly continuous*** ([24], [16]) *if $f^{-1}(V)$ is clopen in X for every open set $V \subset Y$.*
(c) ***δ-perfectly continuous*** [13] *if for each δ-open set V in $Y, f^{-1}(V)$ is a clopen set in X.*
(d) ***almost perfectly continuous*** [36] ($\equiv$ ***regular set connected*** [4]) *if $f^{-1}(V)$ is clopen in X for every regular open set V in Y.*
(e) ***cl-supercontinuous*** [35] ($\equiv$ ***clopen continuous*** [27]) *if for each $x \in X$ and each open set V containing $f(x)$ there is a clopen set U containing x such that $f(U) \subset V$.*
(f) ***almost* cl*-supercontinuous*** [12] ($\equiv$ ***almost clopen continuous*** [7]) *if for each $x \in X$ and each regular open set V containing $f(x)$ there is a clopen set U containing x such that $f(U) \subset V$.*
(g) **z*-supercontinuous*** [9] *if for each $x \in X$ and for each open set V containing $f(x)$, there exists a cozero set U containing x such that $f(U) \subset V$.*
(h) ***almost* z*-supercontinuous*** [17] *if for each $x \in X$ and each regular open set V containing $f(x)$, there exists a cozero set U containing x such that $f(U) \subset V$.*
(i) **$\mathbf{D}_\delta$*-supercontinuous*** [11] *if for each $x \in X$ and for each open set V containing $f(x)$, there exists a regular F_σ-set U containing x such that $f(U) \subset V$.*
(j) ***almost* $\mathbf{D}_\delta$*-supercontinuous*** [17] *if for each $x \in X$ and each regular open set V containing $f(x)$, there exists a regular F_σ-set U containing x such that $f(U) \subset V$.*
(k) **D*-supercontinuous*** [10] *if for each $x \in X$ and each open set V containing $f(x)$ there exists an open F_σ-set U containing x such that $f(U) \subset V$.*
(l) **$\mathbf{D}^*$*-supercontinuous*** [34] *if for each $x \in X$ and each open set V containing $f(x)$ there exists a strongly open F_σ-set U containing x such that $f(U) \subset V$.*
(m) ***strongly θ-continuous*** [23] *if for each $x \in X$ and for each open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U}) \subset V$.*
(n) ***supercontinuous*** [22] *if for each $x \in X$ and for each open set V containing $f(x)$, there exists a regular open set U containing x such that $f(U) \subset V$.*
(o) ***almost strongly θ- continuous*** [26] *if for each $x \in X$ and for each regular open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U}) \subset V$.*
(p) ***δ-continuous*** [23] *if for each $x \in X$ and for each regular open set V containing $f(x)$, there exists a regular open set U containing x such that $f(U) \subset V$.*

(q) ***almost continuous*** [32] *if for each $x \in X$ and for each regular open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset V$.*
(r) ***completely continuous*** [1] *if $f^{-1}(V)$ is a regular open set in X for every open set $V \subset Y$.*

Definitions 2.2. *A space X is said to be endowed with a/an*

(a) ***partition topology*** [37] *if every open set in X is closed.*
(b) ***δ-partition topology*** [13] *if every δ-open set in X is closed or equivalently every δ-closed set in X is open.*
(c) ***almost partition topology*** [36] *if every regular open set in X is closed.*
(d) ***extremally disconnected topology*** *if the closure of every open set in X is open in X.*

It turns out that the notions of almost partition topology and extremally disconnected topology are identical notions. Moreover,
partition topology $\Rightarrow$ δ-partition topology $\Rightarrow$ almost partition topology ($\equiv$ extremally disconnected topology)
However, none of the above implications is reversible. For, let X be an infinite (uncountable) set equipped with a cofinite (cocountable) topology. Then the topology of X is a δ-partition topology which is not a partition topology. For an example of an almost partition topology which is not a δ-partition topology consider a Hausdorff extremally disconnected crowded space (i.e., a space with no isolated points) X (see for example Eric K. Van Douwen [5, Example 3.3]). Then for each $x \in X$, the set $X - \{x\}$ is a *cl*-open set and so δ-open but not clopen. Thus the topology of X is an almost partition topology which is not a δ-partition topology.

3. Almost completely continuous functions ($\equiv$ R-maps) [1]

A function $f : X \to Y$ from a topological space X into a topological space Y is said to be an **almost completely continuous function** if $f^{-1}(V)$ is a regular open set in X for every regular open set V in Y or equivalently $f^{-1}(F)$ is a regular closed set in X for every regular closed set F in Y.
The following two diagrams well illustrate the place of almost complete continuity and δ-continuity in the hierarchy of variants of continuity that already exist in the mathematical literature and are related to the theme of the present paper.
However, none of the implications is reversible as is shown by examples in ([13] [14] [15] and [17]) or follow from the definitions or observations/examples outlined in the following paragraphs.

[1] Carnahan [2] in his doctoral dissertation referred to almost completely continuous functions as R-maps (also see Noiri [25], Kohli [8]). The present nomenclature appears to be more appropriate as it represents a weak variant of complete continuity [1].

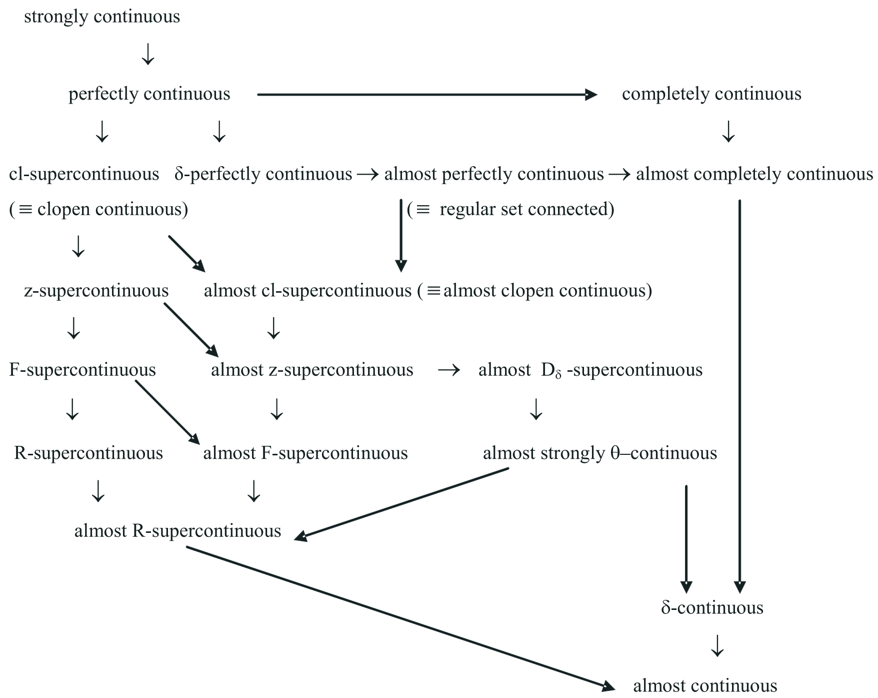

Diagram 1

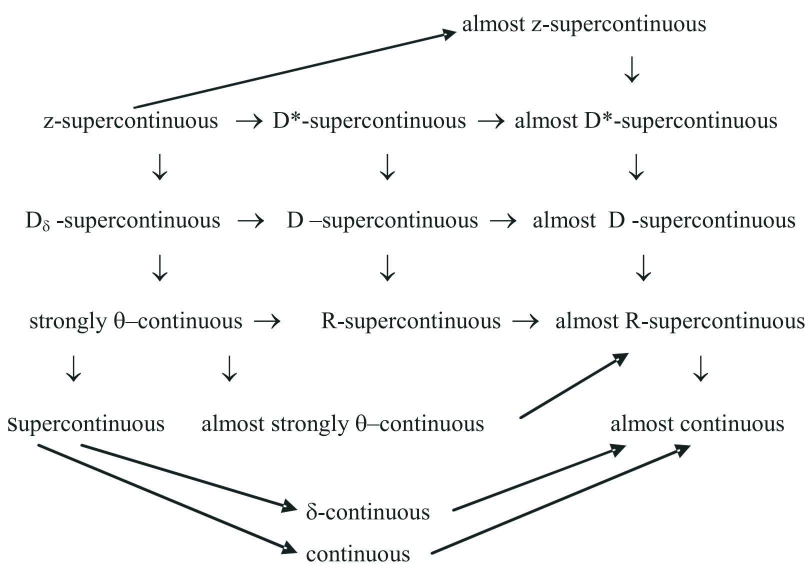

Diagram 2

Observations and Examples:

3.1 Let X be endowed with a partition topology. Then every continuous function $f : X \to Y$ is perfectly continuous and hence completely continuous.

3.2 Let X be endowed with a partition topology. Then every almost continuous function $f : X \to Y$ is almost perfectly continuous ($\equiv$ regular set connected) and so almost completely continuous.

3.3 Let X be endowed with an almost partition topology. If $f : X \to Y$ is almost completely continuous, then f is almost perfectly continuous.

3.4 Let X be endowed with a δ-partition topology. If $f : X \to Y$ is δ-continuous, then it is δ-perfectly continuous and so almost perfectly continuous and hence almost completely continuous.

3.5 If X is equipped with a δ-partition topology and if $f : X \to Y$ is supercontinuous, then f is perfectly continuous.

3.6 If X is a zero dimensional space, then every almost continuous function $f : X \to Y$ is almost cl-supercontinuous but not necessarily almost completely continuous.

3.7 Let $X = \{a, b, c, d\}$ and let the topology τ on X be given by $\tau = \{\phi, X, \{a, b\}\}$. Let Y be the two points Sierpinski space $\{0, 1\}$ with $\{0\}$ as the only non empty proper open subset of Y and let $f : X \to Y$ be defined by $f(a) = f(b) = 0$ and $f(c) = f(d) = 1$. Then f is a continuous function which is almost completely continuous but not completely continuous.

3.8 The function $f : R \to R$ given by $f(x) = x^2$ is a z-supercontinuous function which is not almost completely continuous, since $V = (0, 1)$ is a regular open set but is not a regular open set.

3.9 Let X be the real line endowed with the usual topology and let Y be the real line with cofinite topology. Then the identity function from X onto Y is δ-perfectly continuous and so almost completely continuous but not completely continuous.

3.10 Let $X = Y$ be the real line equipped with the usual topology. Then the identity function defined on X is z-supercontinuous, almost completely continuous but neither completely continuous nor almost perfectly continuous.

3.11 Let X denote the set of rationals endowed with usual topology and f denote the identity mapping defined on X. Then f is cl-supercontinuous but not almost completely continuous. So f is δ-continuous but not almost completely continuous.

3.12 Let X denote the space considered by Douwen [5, Example 3.3] which is a Hausdorff extremally disconnected crowded space whose topology is not a δ-partition topology. Then the identity function defined on X is almost perfectly continuous but not δ-perfectly continuous.

We may recall that a space X is almost locally connected [20] if for each $x \in X$ and each regular open set U containing x there exists an open connected set V containing x such that $V \subset U$. Vincent J. Mancuso in his studies on almost locally connected spaces proved that an open, almost continuous function is almost completely continuous (see [20, Lemma 3.17]). In consequence, almost local connectedness is preserved under open almost continuous surjections [20, Theorem 3.18]. Moreover, since such functions map connected sets to connected sets the assumption of connectedness of map in Theorem 3.18 of Mancuso [20] is superfluous.

4. Basic properties of almost completely continuous and δ-continuous functions.

Theorem 4.1. *If $f : X \to Y$ and $g : Y \to Z$ are almost completely continuous functions, then so is their composition.*

Theorem 4.2. *If $f : X \to Y$ is an almost completely continuous function and $g : Y \to Z$ is a completely continuous function, then their composition $g \circ f$ is completely continuous.*

Proof. Let W be an open set in Z. Since g is completely continuous, $g^{-1}(W)$ is a regular open set in Y. In view of almost complete continuity of f, $f^{-1}(g^{-1}(W))$ $= (g \circ f)^{-1}(W)$ is a regular open set and so $g \circ f$ is completely continuous. □

We may recall that a function $f : X \to Y$ is **almost open** [32] if the image of every regular open set in X is open in Y.

Theorem 4.3. *If $f : X \to Y$ is an almost open surjection and $g : Y \to Z$ is a function such that $g \circ f$ is almost completely continuous, then g is almost continuous. Further, if in addition f maps regular open sets to regular open sets, then g is an almost completely continuous function.*

Proof. Let V be any regular open set in Z. Since $g \circ f$ is almost completely continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is regular open set in X. Again, since f is an almost open surjection, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is open in Y and so g is almost continuous. The last assertion is immediate, since in this case $g^{-1}(V)$ is a regular open set and so g is almost completely continuous. □

The following lemma is due to Singal and Singal [32] and will be useful in the sequel to follow.

Lemma 4.4 ([32]). *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of spaces and let $X = \prod X_\alpha$ be the product space. If $x = (x_\alpha) \in X$ and V is a regular open set containing x, then there exists a basic regular open set $\prod V_\alpha$ such that $x \in \prod V_\alpha \subset V$, where V_α is a regular open set in X_α for each $\alpha \in \Lambda$ and $V_\alpha = X_\alpha$ for all except finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$.*

The next result shows that if a function into a product space is almost completely continuous, then its composition with each projection map is almost completely continuous.

Theorem 4.5. *Let $\{f_\alpha : X \to X_\alpha : \alpha \in \Lambda\}$ be a family of functions and let $f : X \to \prod_{\alpha\in\Lambda} X_\alpha$ be defined by $f(x) = (f_\alpha(x))$ for each $x \in X$. If f is almost completely continuous, then each f_α is almost completely continuous. Further, if each f_α is almost completely continuous, then f is δ-continuous.*

Proof. For each $\alpha \in \Lambda$, let $\Pi_\alpha : \prod X_\alpha \to X_\alpha$ denote the projection map. To show that each f_β is almost completely continuous, let V be any regular open set in X_β. Then for each $\beta \in \Lambda$, $\Pi_\beta^{-1}(V) = \prod\limits_{\alpha \neq \beta} X_\alpha \times V$ is a regular open set in $\prod X_\alpha$ and so each Π_β is almost completely continuous. Now, since for each $\beta \in \Lambda$, $f_\beta = \Pi_\beta \circ f$, the result is immediate in view of Theorem 4.1.
Further, suppose that each f_α is almost completely continuous. To show that f is δ-continuous, it suffices to show that $f^{-1}(V)$ is δ-open for every regular open set V in the product space $\prod X_\alpha$. In view of Lemma 4.4, V is expressible as a union of basic regular open sets of the form $\prod V_\alpha$, where each V_α is a regular open set in X_α and $V_\alpha = X_\alpha$ for all but finitely many $\alpha_1, \alpha_2, \ldots, a_n \in \Lambda$. So $f^{-1}(V) = f^{-1}(\cup \prod V_\alpha) = \cup f^{-1}(\prod V_\alpha) = \cup(\bigcap\limits_{i=1}^{n} f_{\alpha_i}^{-1}(V_{\alpha_i}))$. Since each f_α is almost completely continuous, each $\bigcap\limits_{i=1}^{n} f_{\alpha_i}^{-1}(V_{\alpha_i})$ is regular open and so $f^{-1}(V)$ being a union of regular open sets is δ-open. □

Theorem 4.6. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, be the graph function. If g is almost completely continuous, then f is almost completely continuous. Further, if f is almost completely continuous, then g is δ-continuous.*

Proof. Suppose that g is almost completely continuous. First we observe that the projection map $p_y : X \times Y \to Y$ is almost completely continuous. For if V is a regular open set in Y, then $p_y^{-1}(V) = X \times V$ is a regular open set in $X \times Y$ and so the projection p_y is almost completely continuous. Hence by Theorem 4.1 the composition $p_y \circ g = f$ is almost completely continuous.
Now suppose that f is almost completely continuous. Then in view of Lemma 4.4, every regular open set V in the product space $X \times Y$ is a union of basic regular open sets of the form $U_\alpha \times V_\alpha$, where each and U_α and V_α are regular open sets in X and Y, respectively. Then $g^{-1}(V) = g^{-1}(\cup(U_\alpha \times V_\alpha)) = \cup\, g^{-1}(U_\alpha \times V_\alpha) = \cup(U_\alpha \cap f^{-1}(V_\alpha))$. Since f is almost completely continuous, each $U_\alpha \cap\ f^{-1}(V_\alpha)$ is regular open and so $g^{-1}(V)$ being a union of regular open sets is δ-open. □

Theorem 4.7. *Let $\{f_\alpha : X_\alpha \to Y_\alpha : \alpha \in \Lambda\}$ be a family of functions. Let $f : \prod X_\alpha \to \prod Y_\alpha$ be defined by $f((x_\alpha)) = (f_\alpha(x_\alpha))$ for each (x_α) in $\prod X_\alpha$. If f is almost completely continuous, then each f_α is almost completely continuous. Further, if each f_α is almost completely continuous, then f is δ-continuous.*

Proof. For each $\alpha \in \Lambda$, let $p_\alpha : \prod X_\alpha \to X_\alpha$ and $q_\alpha : \prod Y_\alpha \to Y_\alpha$ denote the projection maps. Then in view of definition of f, it follows that $q_\alpha \circ f = f_\alpha \circ p_\alpha$ for each $\alpha \in \Lambda$. To show that f_α is almost completely continuous, let F be a regular closed set in Y_α. Then $q_\alpha^{-1}(F) = (\prod\limits_{\beta \neq \alpha} Y_\beta) \times F$ is a regular closed set in $\prod Y_\alpha$. Since f is almost completely continuous $f^{-1}(q_\alpha^{-1}(F))$ is a regular closed set in $\prod X_\alpha$. But $f^{-1}(q_\alpha^{-1}(F)) = (q_\alpha \circ f)^{-1}(F) = (f_\alpha \circ p_\alpha)^{-1}(F) =$

$p_\alpha^{-1}(f_\alpha^{-1}(F)) = (\prod_{\beta\neq\alpha} X_\beta) \times f_\alpha^{-1}(F)$ and so $f_\alpha^{-1}(F)$ is a regular closed set in X_α and thus f_α is almost completely continuous.
To prove the last part of the theorem, we observe that in view of Lemma 4.4 every regular open set V in the product space $\prod Y_\alpha$ is the union of basic regular open sets of the form $\prod V_\alpha$, where each $V_\alpha \subset Y_\alpha$ is regularly open and $V_\alpha = Y_\alpha$ for all except finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$. Thus $f^{-1}(V)$ is the union of sets of the form $\prod f_\alpha^{-1}(V_\alpha)$, where each $f_\alpha^{-1}(V_\alpha)$ is a regular open set in X_α and $f_\alpha^{-1}(V_\alpha) = X_\alpha$ for all α except $\alpha_1, \alpha_2, \ldots, \alpha_n$ and so $\prod_\alpha f_\alpha^{-1}(V_\alpha)$ is a basic regular open set in the product space $\prod X_\alpha$. Thus $f^{-1}(V)$ being the union of regular open sets is a δ-open set. □

Theorem 4.8. *Let $f, g : X \to Y$ be δ-continuous functions from a space X into a Hausdorff space Y. Then the equalizer $E = \{x \in X : f(x) = g(x)\}$ of the functions f and g is a δ-closed set in X.*

Proof. To show that E is δ-closed, we shall show that its complement $X \setminus E$ is a δ-open subset of X. To this end, let $x \in X \setminus E$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist disjoint open sets V and W containing $f(x)$ and $g(x)$, respectively. Then $V_1 = \overline{V}^0$ and $W_1 = \overline{W}^0$ are disjoint regular open sets containing $f(x)$ and $g(x)$, respectively. Since f and g are δ-continuous functions, $U = f^{-1}(V_1) \cap g^{-1}(W_1)$ is a δ-open set containing x which is contained in $X \setminus E$ and so $X \setminus E$ is δ-open. □

Corollary 4.9. *Let $f, g : X \to Y$ be an almost completely continuous functions from X into a Hausdorff space Y. Then the equalizer $E = \{x \in X : f(x) = g(x)\}$ of the functions f and g is a δ-closed set in X.*

Theorem 4.10. *(Noiri [23, Theorem 5.2]): Let $f : X \to Y$ be a δ-continuous function into a Hausdorff space Y. Then $G(f)$ the graph of f is a δ-closed subset of $X \times Y$.*

Corollary 4.11. *Let $f : X \to Y$ be an almost completely continuous function into a Hausdorff space Y. Then $G(f)$ the graph of f is a δ-closed subset of $X \times Y$.*

5. Preservation/interplay of topological properties

Connectedness is preserved by functions satisfying fairly mild continuity conditions (see [12, p. 9]) and so it is preserved under δ-continuous functions and hence under almost completely continuous functions. Next we consider the transfer of separation properties under δ-continuous functions and almost completely continuous functions. First we quote the following definitions.

Definition 5.1. *A topological space X is said to be a*

(i) δT_1*-space* ([7][2], [12]) *if for each pair of distinct points x and y in X there exist regular open sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$.*
(ii) δT_0*-space [12] if for each pair of distinct points x and y in X there exists a regular open set containing one of the points x and y but not the other.*

The following implications are either well known (see [40]) or immediate from definitions.

$$\begin{array}{ccccc} \text{Hausdorff space} & \Rightarrow & \delta T_1\text{-space} & \Rightarrow & \delta T_0\text{-space} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{KC} \Rightarrow \text{US} & \Rightarrow & T_1\text{-space} & \Rightarrow & T_0\text{-space} \end{array}$$

However, none of the above implications is reversible (see [12], [40]).

Proposition 5.2. *Let $f : X \to Y$ be a δ-continuous injection. If Y is a δT_0-space or a δT_1-space or a Hausdorff space, then so is X.*

Definition 5.3. *A topological space X is said to be*

(a) ***almost regular*** [30] *if every regular closed set and a point outside it are contained in disjoint open sets.*
(b) ***almost completely regular*** [31] *if for every δ-closed set F in X and a point $x \notin F$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$.*
(c) ***mildly normal*** [33] *if every pair of disjoint regular closed sets are contained in disjoint open sets.*
(d) ***nearly paracompact*** [29] *if every regular open cover of X has a locally finite open refinement.*
(e) ***nearly compact*** [28] *if every open cover of X admits a finite subcollection the interiors of the closures of whose members cover X.*
(f) **S-*closed*** [38] *if every semi open cover of X has a finite subcollection whose closures cover X or equivalently, every regular closed cover of X has a finite subcover* (*see* [2]).

Proposition 5.4. *Every (almost) completely continuous function defined on a Hausdorff (or almost regular) S-closed space is (almost) perfectly continuous.*

Proof. : Let $f : X \to Y$ be an (almost) completely continuous function from a Hausdorff (or almost regular) S-closed space X into a space Y and let V be any (regular) open subset of Y. Since f is (almost) completely continuous, $f^{-1}(V)$ is a regular open set in X. Now, since a Hausdorff (or almost regular) S-closed space is extremally disconnected (see [38, Theorem 7] and [21, Theorem 3]) and since in an extremally disconnected space every regular open set is clopen, $f^{-1}(V)$ is a clopen set in X and so f is (almost) perfectly continuous. □

[2]Ekici calls δT_1-spaces as r-T_1-space in [7].

Theorem 5.5. *Let $f : X \to Y$ be a δ-continuous closed surjection defined on an almost regular space X. If either f is open or $f^{-1}(y)$ is compact for each $y \in Y$, then Y is an almost regular space. If in addition Y is a semiregular space, then Y is regular.*

Proof. Case I: f is open. Let F be a regularly closed set in Y such that $y \notin F$. Then $f^{-1}(F) \cap f^{-1}(y) = \phi$ and in view of δ-continuity of $f, f^{-1}(F)$ is a δ-closed set. Let $x \in f^{-1}(y)$. In view of almost regularity of X, there exist disjoint open sets U and V containing x and $f^{-1}(F)$, respectively. Then since f is a closed surjection, $f(U)$ and $Y \setminus f(X \setminus V)$ are disjoint open sets containing y and F, respectively.
Case II: $f^{-1}(y)$ is compact for each $y \in Y$. Since X is almost regular, there exist disjoint open sets U and V containing $f^{-1}(y)$ and $f^{-1}(F)$, respectively. Since f is closed, the sets $Y \setminus f(X \setminus U)$ and $Y \setminus f(X \setminus V)$ are disjoint open sets containing y and F, respectively. The last assertion is immediate in view of the fact that a space is regular if and only if it is semiregular and almost regular [30]. □

Corollary 5.6. *Let $f : X \to Y$ be an almost completely continuous closed surjection defined on an almost regular space X. If either f is open or $f^{-1}(y)$ is compact for each $y \in Y$, then Y is an almost regular space.*

Theorem 5.7. *Let $f : X \to Y$ be an open, closed, δ-continuous surjection. If X is an almost completely regular space, then so is Y. Further, if Y is a semiregular space, then Y is completely regular.*

Proof. To prove that Y is almost completely regular, let F be a regular closed set in Y such that $y \notin F$. Since f is δ-continuous, $f^{-1}(F)$ is a δ-closed set in X. Let $x \in f^{-1}(y)$. In view of almost complete regularity of X, There exists a continuous real valued function $\varphi : X \to [0, 1]$ such that $\varphi : X \to [0, 1]$ such that $\varphi(f^{-1}(F)) = 0$ and $\varphi(x) = 1$. Define $\hat{\varphi} : Y \to [0, 1]$ by taking $\hat{\varphi}(y) = \sup\{\varphi(x) : x \in f^{-1}(y)\}$ for each $y \in Y$. Then $\hat{\varphi}(y) = 1, \hat{\varphi}(F) = 0$ and by [6, p.96, Exercise 16] $\hat{\varphi}$ is continuous. Hence Y is almost completely regular. The last assertion is immediate in view of the fact that a space is completely regular if and only if it is semiregular and almost completely regular [31]. □

Corollary 5.8. *Let $f : X \to Y$ be an open, closed, almost completely continuous surjection. If X is an almost completely regular space, then so is Y.*

Theorem 5.9. *Let $f : X \to Y$ be an almost completely continuous closed surjection. If X is a mildly normal space, then so is Y.*

Proof. Let A and B be any two disjoint regular closed subsets of Y. In view of almost complete continuity of $f, f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular closed subsets of X. Since X is mildly normal, there exist disjoint open sets U and V containing $f^{-1}(A)$ and $f^{-1}(B)$, respectively. Again, since f is closed, the sets $f(X \setminus U)$ and $f(X \setminus V)$ are closed sets. It is easily verified that the sets $Y \setminus (X \setminus U)$ and $Y \setminus f(X \setminus V)$ are disjoint open sets containing A and B, respectively and so Y is mildly normal. □

Theorem 5.10. *Let $f : X \to Y$ be a δ-continuous surjection from a nearly compact space X onto Y. Then Y is nearly compact. Further, if in addition Y is semiregular, then Y is compact.*

Proof. Let $\mathbf{U} = \{U_\alpha | \alpha \in \Delta\}$ be a regular open cover of Y. Since f is δ-continuous, the collection $\mathbf{V} = \{f^{-1}(U_\alpha) | \alpha \in \Delta\}$ is a δ-open cover of X. In view of near compactness of X there exists a finite subcollection $\{f^{-1}(U_{\alpha_i}) : i = 1, \ldots, n\}$ of $\mathbf{V}$ which covers X. Since f is a surjection, the finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ of $\mathbf{U}$ covers Y and so Y is nearly compact. A semiregular nearly compact space is compact [28]. □

Theorem 5.11 ([21, Theorems 3 and 4])**.** *An almost regular (or Hausdorff) space is S-closed if and only if it is nearly compact and extremally disconnected.*

Theorem 5.12. *Let $f : X \to Y$ be an almost completely continuous closed surjection from a nearly compact (S-closed) space X. Then Y is nearly compact (S-closed). Further, if in addition Y is almost regular (or Hausdorff), then Y is nearly compact and extremally disconnected.*

Proof. Near compactness of Y is immediate in view of Theorem 5.10. To prove that Y is S-closed whenever X is S-closed, let $\mathbf{U} = \{U_\alpha | \alpha \in \Delta\}$ be a regular closed cover of Y. Since f is almost completely continuous, the collection $\mathbf{V} = \{f^{-1}(U_\alpha) | \alpha \in \Delta\}$ is a regular closed cover of X. Again, since X is S-closed there exists a finite subcollection $\{f^{-1}(U_{\alpha_i}) : i = 1, \ldots, n\}$ of $\mathbf{V}$ which covers X. Since f is a surjection, the finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ of $\mathbf{U}$ covers Y and so Y is S-closed. Moreover, if Y is almost regular (or Hausdorff), then in view of Theorem 5.11, Y is an extremally disconnected space. □

Theorem 5.13. *Let $f : X \to Y$ be a closed, δ-continuous, almost open surjection such that $f^{-1}(y)$ is compact for each $y \in Y$. If X is a nearly paracompact space, then so is Y. Moreover, if Y is semiregular, then Y is paracompact.*

Proof. Let $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ be a regular open cover of Y. In view of δ-continuity of $f, \mathbf{A} = \{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is a δ open cover of X. Let $\mathbf{B} = \{U_\beta : \beta \in \Gamma\}$ be the natural regular open refinement of $\mathbf{A}$ covering X. Since X is nearly paracompact, there exists a locally finite open refinement $\{W_\delta : \delta \in \Omega\}$ of $\mathbf{B}$. Since each U_β is regularly open, it is easily verified that each W_δ may be chosen to be regularly open and so in view of almost openness of f, each $f(W_\delta)$ is open. Again, since f is a closed function such that $f^{-1}(y)$ is compact for each $y \in Y$, it maps every locally finite collection to a locally finite collection and hence $\{f(W_\delta) : \delta \in \Omega\}$ is a locally finite open refinement of $\mathcal{V}$. Thus Y is nearly paracompact. The last assertion is immediate, since a semiregular almost paracompact space is paracompact [29]. □

Corollary 5.14. *Let $f : X \to Y$ be a closed, almost completely continuous, almost open surjection such that $f^{-1}(y)$ is compact for each $y \in Y$. If X is a nearly paracompact space, then so is Y.*

Theorem 5.15. *Let $f : X \to Y$ be an almost completely continuous surjection which maps clopen sets to clopen sets. If X is an extremally disconnected space, then so is Y. Further, if in addition Y is nearly compact, then Y is S-closed and almost regular.*

Proof. Suppose X is extremally disconnected. To show that Y is extremally disconnected, it suffices to prove that every regular open set in Y is clopen. To this end, let V be a regular open set in Y. In view of almost complete continuity of $f, f^{-1}(V)$ is a regular open in X. Since X is extremally disconnected, $f^{-1}(V)$ is a clopen set in X. Again, since f is a surjection which maps clopen sets to clopen sets, $V = f(f^{-1}(V))$ is a clopen set in Y and so Y is extremally disconnected. For what remains we need only note that every extremally disconnected space is almost regular and so the space Y is S-closed in view of Theorem 5.11. □

References

[1] S. P. Arya and R. Gupta, *On strongly continuous mappings*, Kyungpook Math. J. **14** (1974), 131–143.

[2] D. E. Cameron, *Properties of S-closed spaces*, Proc. Amer. Math. Soc. **72**, no. 3 (1978), 581–586.

[3] D. Carnahan, *Some properties related to compactness in topological spaces*, Ph.D Thesis, Univ. of Arkansas, 1973.

[4] J. Dontchev, M. Ganster and I. Reilly, *More on almost s-continuity*, Indian J. Math. **41** (1999), 139–146.

[5] E. K. Van Douwen, *Applications of maximal topologies*, Topology Appl. **51**, no. 2 (1993), 125–139.

[6] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1966.

[7] E. Ekici, *Generalization of perfectly continuous, regular set-connected and clopen functions*, Acta. Math. Hungar. **107**, no. 3 (2005), 193–206.

[8] J. K. Kohli, *A unified approach to continuous and non-continuous functions II*, Bull. Aust. Math. Soc. **41** (1990), 57–74.

[9] J. K. Kohli and R. Kumar, *z-supercontinuous functions*, Indian J. Pure Appl. Math. **33**, no. 7 (2002), 1097–1108.

[10] J. K. Kohli and D. Singh, *D-supercontinuous functions*, Indian J. Pure Appl. Math. **32**, no. 2 (2001), 227–235.

[11] J. K. Kohli and D. Singh, *D_δ-supercontinuous functions*, Indian J. Pure Appl. Math. **34**, no. 7 (2003), 1089–1100.

[12] J. K. Kohli and D. Singh, *Almost cl-supercontinuous functions*, Appl. Gen. Topol. **10**, no. 1 (2009), 1–12.

[13] J. K. Kohli and D. Singh, *δ-perfectly continuous functions*, Demonstratio Math. **42**, no. 1 (2009), 221–231.

[14] J. K. Kohli, D. Singh and J. Aggarwal, *F-supercontinuous functions*, Appl. Gen. Topol. **10**, no. 1 (2009), 69–83.

[15] J. K. Kohli, D. Singh and J. Aggarwal, *R-supercontinuous functions*, Demonstratio Mat. **43**, no. 3-4 (2010), to appear.

[16] J. K. Kohli, D. Singh and C. P. Arya, *Perfectly continuous functions*, Stud. Cercet. Stiint. Ser. Mat. Univ. Bacău **18** (2008), 99–110.

[17] J. K. Kohli, D. Singh and R. Kumar, *Generalizations of z-supercontinuous functions and D-supercontinuous functions*, Appl. Gen. Topol. **9**, no. 2 (2008), 239–251.

[18] N. Levine, *Strong continuity in topological spaces*, Amer. Math. Monthly **67** (1960), 269.

[19] J. Mack, *Countable paracompactness and weak normality properties*, Trans. Amer. Math. Soc. **148** (1970), 265–272.
[20] V. J. Mancuso, *Almost locally connected spaces*, J. Austral. Math. Soc. Ser. A **31** (1981), 421–428.
[21] A. Mathur, *A note on S-closed spaces*, Proc. Amer. Math. Soc. **74**, no. 2 (1979), 350–352.
[22] B. M. Munshi and D. S. Bassan, *Supercontinuous mappings*, Indian J. Pure Appl. Math. **13** (1982), 229-236.
[23] T. Noiri, On *δ-continuous functions*, J. Korean Math. Soc. **16** (1980), 161–166.
[24] T. Noiri, *Supercontinuity and some strong forms of continuity*, Indian J. Pure. Appl. Math. **15**, no. 3 (1984), 241–250.
[25] T. Noiri, *Strong forms of continuity in topological spaces*, Rend. Circ. Mat. Palermo (2) Suppl. **12** (1986), 107–113.
[26] T. Noiri and S. M. Kang, *On almost strongly θ-continuous functions*, Indian J. Pure Appl. Math. **15**, no. 1 (1984), 1–8.
[27] I. L. Reilly and M. K. Vamanamurthy, *On super-continuous mappings*, Indian J. Pure. Appl. Math. **14**, no. 6 (1983), 767–772.
[28] M. K. Singal and Asha Mathur, *On nearly compact spaces*, Boll. Un. Mat. Ital. (4) **2** (1969), 702–710.
[29] M. K. Singal and S. P. Arya, *On nearly paracompact spaces*, Mat. Vesnik **6** (21) (1969), 3–16.
[30] M. K. Singal and S. P. Arya, *On almost regular spaces*, Glasnik Mat. Ser. III **4** (24) (1969), 89–99.
[31] M. K. Singal and S. P. Arya, *Almost normal and almost completely regular spaces*, Glasnik Mat. Ser. III **5** (25) (1970), 141–152.
[32] M. K. Singal and A. R. Singal, *Almost continuous mappings*, Yokohama Math. J. **16** (1968), 63–73.
[33] M. K. Singal and A. R. Singal, *Mildly normal spaces*, Kyungpook Math. J. **13** (1973), 27–31.
[34] D. Singh, *D^*-supercontinuous functions*, Bull. Cal. Math. Soc. **94**, no. 2 (2002), 67–76.
[35] D. Singh, *cl-supercontinuous functions*, Appl. Gen. Topol. **8**, no. 2 (2007), 293–300.
[36] D. Singh, *Almost perfectly continuous functions*, Quaest. Math. **33** (2010), 1–11.
[37] L. A. Steen and J. A. Seebach, Jr., *Counter Examples in Topology*, Springer Verlag, New York, 1978.
[38] T. Thompson, *S-closed spaces*, Proc. Amer. Math. Soc. **60** (1976), 335–338.
[39] N. K. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl. **2** (78) (1968), 103–118.
[40] A. Wilansky, *Between T_1 and T_2*, Amer. Math. Monthly **74** (1967), 261–266.

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J. K. Kohli (jk_kohli@yahoo.com)
Department of Mathematics, Hindu College, University of Delhi, Delhi 110 007, India

D. Singh (dstopology@rediffmail.com)
Department of Mathematics, Sri Aurobindo College, University of Delhi - South Campus, Delhi 110 017, India