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Remarks on the rings of functions which have a finite number of discontinuities

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Abstract

Let X be an arbitrary topological space. $F(X)$ denotes the set of all real-valued functions on X and $C(X)_F$ denotes the set of all $f \in F(X)$ such that f is discontinuous at most on a finite set. It is proved that if r is a positive real number, then for any $f \in C(X)_F$ which is not a unit of $C(X)_F$ there exists $g \in C(X)_F$ such that $g \neq 1$ and $f = g^r f$. We show that every member of $C(X)_F$ is continuous on a dense open subset of X if and only if every non-isolated point of X is nowhere dense. It is shown that $C(X)_F$ is an Artinian ring if and only if the space X is finite. We also provide examples to illustrate the results presented herein.

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1. Introduction

Let X be a nonempty topological space and $I(X)$ denote the set of isolated points of X. The ring of all real-valued functions on X with pointwise addition and multiplication is denoted by $F(X)$, continuous members of $F(X)$ is denoted by $C(X)$, the set of points at which $f \in F(X)$ is continuous is denoted by $C(f)$ and bounded members of $C(X)$ is denoted by $C^*(X)$. For any $f \in F(X)$, it is well known and easy to prove that $X \setminus C(f)$ is a countable union of closed sets. The set of all $f \in F(X)$ such that $X \setminus C(f)$ is finite is a subring of $F(X)$

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and it is denoted by $C(X)_F$. The ring $C(X)_F$ where X is T_1 was introduced and studied in [5]. In this paper, topological spaces don't have to satisfy any separation axioms unless otherwise stated. Recall that lattice-ordered rings are subdirect sums of totally ordered rings. Let $f, g \in C(X)_F$, then

$$f \vee g = \frac{1}{2}(f + g + |f - g|) \in C(X)_F,$$

and $f \wedge g = -(-f \vee -g)$ [5]. Thus, for any topological space X we observe that $C(X)_F$ is a lattice-ordered subring of $F(X)$ and $C(X)$ is a lattice-ordered subring of $C(X)_F$. Also $C^*(X)_F$, consisting of all functions in $C(X)_F$ which are bounded, is a lattice-ordered subring of $C(X)_F$. Let S be a subset of X. The characteristic function of S is denoted by χ_S. Let R be a commutative ring. A nonzero ideal I in R is an essential ideal if I intersects every nonzero ideal of R nontrivially. The socle of R denoted by $Soc(R)$ is the sum of all minimal ideals of R, or the intersection of all essential ideals of R. In [5], it is shown that $Soc(C(X)_F)$ consists of all functions which vanish everywhere except on a finite subset of X. A subset of X is called a G_δ-set if it is a countable intersection of open sets. As usual, clA and $intA$ will denote the closure and interior of a subset A in a space X, respectively.

In Section 2, we show that for any $f \in C(X)_F$ which is not a unit of $C(X)_F$ there exists $g \in C(X)_F$ such that $g \neq 1$ and $f = g^r f$ where r is a positive number. Also, it is proved that if X is a T_1-space, $f \in C(X)_F$ and $\mathcal{Z}(f) \subseteq C(f)$, then $\mathcal{Z}(f)$ is G_δ. In section 3, It is shown that $C(X)_F$ is an Artinian ring if and only if the space X is finite. We prove that every point of $X \setminus I(X)$ is nowhere dense and every dense open subset of X has finite complement if and only if $C(X)_F = \{f \in F(X) | f|_D \in C(D)$ for some dense open subset of X $\}$. For undefined notations, the reader is referred to [4] and [6].

2. C_F-embedded subsets of X

Let $f \in C(X)_F$. The set $f^{-1}(0) = \{x \in X | f(x) = 0\}$ denoted by $\mathcal{Z}(f)$ is called the $\mathcal{Z}ero$-set of f and $Co\mathcal{Z}(f) = X \setminus \mathcal{Z}(f)$ is called a Co-$\mathcal{Z}ero$-set in X [5]. The collection of all $\mathcal{Z}$ero-sets in X is denoted by $\mathcal{Z}[C(X)_F]$ or $\mathcal{Z}(X)$.

Proposition 2.1 ([5])**.** *For a topological space X, $f \in C(X)_F$ is a unit element if and only if $\mathcal{Z}(f) = \varnothing$.*

Units in $C^*(X)_F$ is characterized by the following result.

Lemma 2.2. *A function g in $C^*(X)_F$ is a unit of $C^*(X)_F$ if and only if g is bounded away from zero.*

Proof. Necessity. If there is a function h in $C^*(X)_F$ such that $gh = 1$, then $|h| < n$ for some $n \in \mathbb{N}$ and so $|g| > \frac{1}{n}$. Hence g is bounded away from zero.
Sufficiency. Suppose that g is bounded away from zero. So there is $r > 0$ such that $|g(x)| > r$ for every $x \in X$. Thus for any $x \in X$, $0 < \frac{1}{|g(x)|} < \frac{1}{r}$ and so $\frac{1}{g} \in C^*(X)_F$, i.e., g is a unit of $C^*(X)_F$. □

Proposition 2.3. *If X is an arbitrary topological space and r is a positive real number, then, for any $f \in C(X)_F$ which is not a unit of $C(X)_F$ there exists $g \in C(X)_F$ such that $g \neq 1$ and $f = g^r f$.*

Proof. Since f is not a unit of $C(X)_F$, Proposition 2.1 implies that there exists $a \in \mathcal{Z}(f)$. If $g = 1 - \chi_{\{a\}}$, then $1 \neq g \in C(X)_F$ and $f = g^r f$. □

Definition 2.4 ([5])**.** A nonempty subset $\mathcal{F}$ of $\mathcal{Z}(X)$ is said to be a $\mathcal{Z}$-filter on X, if it satisfies the following conditions:

(1) $\varnothing \notin \mathcal{F}$;
(2) if $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathcal{F}$, then $\mathcal{Z}_1 \cap \mathcal{Z}_2 \in \mathcal{F}$; and
(3) if $\mathcal{Z} \in \mathcal{F}, \mathcal{Z}' \in \mathcal{Z}(X)$ and $\mathcal{Z}' \supseteq \mathcal{Z}$, then $\mathcal{Z}' \in \mathcal{F}$.

Let I be a proper ideal in $C(X)_F$. Then $\mathcal{Z}[I] = \{\mathcal{Z}(f) | f \in I\}$ is a $\mathcal{Z}$-filter on X [5].

Definition 2.5. A $\mathcal{Z}$-filter $\mathcal{U}$ on X is called a $\mathcal{Z}$-ultrafilter on X, if there isn't any $\mathcal{Z}$- filter $\mathcal{F}$ on X, such that $\mathcal{U} \subsetneq \mathcal{F}$.

Definition 2.6 ([5])**.** An ideal I in a subring of $C(X)_F$ is called fixed (resp., free), if the intersection of all members of $\mathcal{Z}[I]$ is non-empty (resp., empty).

In [5], it is noted that every $\mathcal{Z}$-filter $\mathcal{F}$ is of the form $\mathcal{Z}[I_{\mathcal{F}}]$ for some ideal $I_{\mathcal{F}}$ in $C(X)_F$ and $\mathcal{F}$ is called fixed (resp., free), if $I_{\mathcal{F}}$ is fixed (resp., free).

Definition 2.7. A $\mathcal{Z}$-filter $\mathcal{F}$ on X is called a prime $\mathcal{Z}$-filter, if the union of two $\mathcal{Z}ero$-sets belongs to $\mathcal{F}$, then at least one of them belongs to $\mathcal{F}$.

Theorem 2.8. *The following statements are correct.*

(1) *If I is a prime ideal in $C(X)_F$, then $\mathcal{Z}[I] = \{\mathcal{Z}(f) | f \in I\}$ is a prime $\mathcal{Z}$-filter on X.*
(2) *If $\mathcal{F}$ is a prime $\mathcal{Z}$-filter on X, then $\mathcal{Z}^{-1}[\mathcal{F}] = \{f \in C(X)_F | \mathcal{Z}(f) \in \mathcal{F}\}$ is a prime ideal in $C(X)_F$.*

Proof. The proof is similar to [6, Theorem 2.12]. □

Lemma 2.9. *Let $f \in C(X)_F$. Then there is a positive unit u of $C(X)_F$ such that $(-1 \vee f) \wedge 1 = uf$.*

Proof. It's straightforward. □

Theorem 2.10. *Let X be a topological space which is T_1. Then the following statements are equivalent.*

(1) *For any $f \in C(X)_F$, there is a unit u of $C(X)_F$ such that $f = u|f|$.*
(2) *For any $g \in C^*(X)_F$, there exists a unit v of $C^*(X)_F$ such that $g = v|g|$.*

Proof. (1) $\Rightarrow$ (2). Let $g \in C^*(X)_F$. Then by (1), there exists a unit u of $C(X)_F$ such that $g = u|g|$, and so $|u(x)| = 1$ for any $x \in X$ such that $g(x) \neq 0$. Now, Proposition 2.1 implies that $\mathcal{Z}(u) = \varnothing$. Let $A = \{x \in X | u(x) > 0\}$ and $B = \{x \in X | u(x) < 0\}$. Then X is a disjoint union of A and B. If $u = 1$ or

$u = -1$, then $u \in C^*(X)$. Let $u \neq 1$ and $u \neq -1$, then there is a finite subset F of X such that $C(u) = X \setminus F$. Since X is a T_1-space, $C(u)$ is open in X. $A \setminus F$ and $B \setminus F$ are disjoint open sets in $C(u)$ and so $A \setminus F$ and $B \setminus F$ are disjoint open sets in X. Therefore, there is a unite v of $C^*(X)_F$ such that $v|_A = 1$ and $v|_B = -1$. Clearly, $g = v|g|$ and so (1) implies (2).
(2) $\Rightarrow$ (1). Let $f \in C(X)_F$. Then, there is $g \in C^*(X)_F$ such that $g = (f \wedge 1) \vee -1$, so according to (2) there is a unit v of $C^*(X)_F$ such that $v|g| = g$. From $\mathcal{Z}(f) = \mathcal{Z}(g)$ it follows that $v|f| = f$ and hence, (2) implies (1). □

Theorem 2.11. *Let X be an arbitrary topological space, f be in $F(X)$ and H be a finite closed subset of X such that $f \in C(X \setminus H)$. If $H \cap f^{-1}(0) = \varnothing$, then $f^{-1}(0)$ is a G_δ-set in X.*

Proof. Since by our assumption $f^{-1}(0) \cap H = \varnothing$, there is $m \in \mathbb{N}$ such that $f^{-1}(-\frac{1}{m}, \frac{1}{m}) \cap H = \varnothing$. Thus for every $n \geq m$, $f^{-1}(-\frac{1}{n}, \frac{1}{n})$ is open in $X \setminus H$ and so it is open in X. Hence, $f^{-1}(0) = \bigcap_{n=m}^{\infty} f^{-1}(-\frac{1}{n}, \frac{1}{n})$ is a G_δ-set in X. □

By the above theorem we note that if X is a T_1-space, $f \in C(X)_F$ and $\mathcal{Z}(f) \subseteq C(f)$, then $\mathcal{Z}(f)$ is G_δ. The following example shows that Theorem 2.11 is not true in general case.

Example 2.12. Let $\beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$ and $y \in \beta\mathbb{N} \setminus \mathbb{N}$. It is well known that $\{y\}$ is not a G_δ-set in $\beta\mathbb{N}$. Thus, $f = 1 - \chi_{\{y\}} \in C(\beta\mathbb{N})_F$ and $\mathcal{Z}(f) = \{y\}$ is not a G_δ-set.

Definition 2.13 ([5])**.** Two nonempty subsets A and B of a topological space X are said to be $\mathcal{F}$-completely separated in X if there is a function $f \in C^*(X)_F$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$.

Theorem 2.14 ([5])**.** *Let X be a topological space. Then two nonempty subsets A and B are $\mathcal{F}$-completely separated in X if and only if they are contained in disjoint $\mathcal{Z}$ero-sets.*

Definition 2.15. Let S be a nonempty subset of X. We say that S is C_F-embedded in X if for any function $f \in C(S)_F$, there exists $g \in C(X)_F$ called an extension of f such that $g|_S = f$.
In the same way, S is called C_F^*-embedded if every $f \in C^*(S)_F$ can be extended to $g \in C^*(X)_F$, i.e.,

$$\exists g \in C^*(X)_F \text{ such that } g|_S = f.$$

Example 2.16. Let $X = \mathbb{R}$ and $S_1 = \mathbb{R} \setminus \{0\}$. Then, $f \in C^*(S_1)$ with value 1 for all positive r, and -1 for negative r, has no continuous extension, i.e., S_1 is not C^*-embedded in X. Clearly, S_1 is C_F^*-embedded and C_F-embedded. Let $S = \mathbb{R} \setminus \mathbb{Z}$ and h be the restriction of the bracket function to S. Then, $g \in C(S)_F$ since it is in $C(S)$. Since there is no $g \in C(\mathbb{R})_F$ such that $g|_S = h$, S is not C_F-embedded in X.

We observe that if S is a subset of a topological space X with finite complement, then S is C_F^* -embedded and C_F-embedded in X.

Proposition 2.17. *Let $\eta : S \to X$ be a one-one mapping and K be the set of discontinuity points of η. If K is a finite set , then $\phi : C(X)_F \to C(S)_F$ defined by $\phi(g) = g \circ \eta$ is a ring homomorphism.*

Proof. If $g \in C(X)_F$, then $H = X \setminus C(g)$ is a finite set. Thus, $S \setminus C(g \circ \eta)$ is a finite set since it is a subset of $\eta^{-1}(H) \cup K$ and η is continuous at every point of $X \setminus K$ and one-one so $\phi(g) = g \circ \eta \in C(S)_F$, i.e., ϕ is well defined. Clearly, ϕ is a ring homomorphism. □

The following result is an immediate consequence of Proposition 2.17.

Corollary 2.18. *Let T be a nonempty subset of a topological space X. Then, the restriction function $\phi : C(X)_F \to C(T)_F$ (resp. $\phi : C^*(X)_F \to C^*(T)_F$) defined by $\phi(f) = f|_T$ is an onto ring homomorphism if and only if T is C_F-embedded (resp. C^*_F-embedded).*

Theorem 2.19. *Each C_F-embedded subset is a C^*_F-embedded subset of X.*

Proof. Let X be a topological space and $S \subseteq X$ be a C_F-embedded subset of X. Now suppose that $f \in C^*(S)_F$. Since $C^*(S)_F \subseteq C(S)_F$, there exists an extension $g \in C(X)_F$ such that $g|_S = f$. Thus there is a positive integer m such that for every $x \in S$, $|f(x)| \leq m$. We put

$$h := (g \vee -m) \wedge m,$$

obviously $h \in C^*(X)_F$ and for each $s \in S$, we have

$$h(s) = (g(s) \vee -m) \wedge m = (f(s) \vee -m) \wedge m = f(s).$$

Hence, S is a C^*_F-embedded subset of X. □

Theorem 2.20. *Let S and X be two subsets of a topological space Y such that $\varnothing \neq S \subseteq X \subseteq Y$. If X is a C_F-embedded in Y then S is a C_F-embedded in X if and only if S is C_F-embedded in Y.*

Proof. Necessity, assume that S is a C_F-embedded in X and $f \in C(S)$. So f has an extension $g \in C(X)_F$ such that $g|_S = f$ and according to assumption, X is C_F-embedded in Y, so there is $h \in C(Y)_F$ such that $h|_X = g$. Hence, S is C_F-embedded in Y.
Sufficiency, assume that S is C_F-embedded in Y and put $f \in C(S)_F$. Since S is C_F-embedded in Y, there is an extension $g \in C(Y)_F$ such that $g|_S = f$ and so $g|_X \in C(X)_F$ is an extension of f which completes the proof. □

Theorem 2.21. *A C^*_F-embedded subset S of X is C_F-embedded in X if and only if it is $\mathcal{F}$-completely separated from every $\mathcal{Z}$ero-set disjoint from it.*

Proof. Let S be a C^*_F-embedded in X. Assume that $h \in C(X)_F$ and $\mathcal{Z}(h) \cap S = \varnothing$ and let $f(s) = \frac{1}{h(s)}$ for any $s \in S$, then $f \in C(S)_F$. Thus by the hypothesis, there is $g \in C(X)_F$, such that $g|_S = f$. Therefore, $gh \in C(X)_F$. From $gh[S] = \{1\}$ and $gh[\mathcal{Z}(h)] = \{0\}$ it follows that $\mathcal{Z}(h)$ and S are $\mathcal{F}$-completely separated in X.
Conversely, suppose that $f \in C(S)_F$. So $\arctan \circ f \in C^*(X)_F$ and there is

$g \in C(X)_F$ such that $g|_S = \arctan \circ f$. Therefore $\mathcal{Z} = \{x \in X : |g(x)| \geq \frac{\pi}{2}\}$ belongs to $\mathcal{Z}(X)$ and $\mathcal{Z} \cap S = \varnothing$. So according to the assumption there exists $h \in C^*_F(X)$ such that $h[S] = \{1\}$ and $h[\mathcal{Z}] = \{0\}$. It's obvious that for every $x \in X$, $|g(x)h(x)| < \frac{\pi}{2}$ and $gh|_S = \arctan \circ f$. Hence $\tan \circ (gh) \in C(X)_F$ is an extension of f and so S is C_F-embedded in X. □

By the above theorem and Theorem 2.14 we have the following result.

Theorem 2.22. *Each C^*_F-embedded $\mathcal{Z}$ero-set is C_F-embedded.*

Proof. It is straigthforward. □

3. Some algebraic aspects of $C(X)_F$

Recall that $T^{'}(X)$ is the ring of all $f \in F(X)$ where for each f there is an open dense subset D of X such that $f|_D \in C(D)$ [1]. Let X be a topological space which is T_1. Then $C(X)_F$ is a subring of $T^{'}(X)$. The following example shows that it is not true in general.

Example 3.1. Let $X = \{0, 1, 2\}$ and $\tau = \{\varnothing, \{0\}, \{1, 2\}, X\}$. Then, $C(X)_F = F(X)$ and (X, τ) is not a discrete space and so $C(X) \neq F(X)$. Since X is the only dense open subset of the space, $C(X) = T^{'}(X)$. Thus, $C(X)_F$ is not a subring of $T^{'}(X)$.

In the following result we show that every non-isolated point of a topological space X is nowhere dense if and only if $C(X)_F$ is a subring of $T^{'}(X)$.

Lemma 3.2. *Let X be an arbitrary topological space. Then, $C(X)_F$ is a subring of $T^{'}(X)$ if and only if for any $x \in X \setminus I(X)$, $intcl\{x\} = \varnothing$, i.e., $\{x\}$ is nowhere dense.*

Proof. Let $C(X)_F$ be a subring of $T^{'}(X)$. If $x \in X \setminus I(X)$, then $\chi_{\{x\}} \in T^{'}(X)$ since $\chi_{\{x\}} \in C(X)_F$. Thus, there is a dense open subset D of X such that $\chi_{\{x\}} \in C(D)$. Clearly, $x \notin D$ and so $int(cl\{x\}) = \varnothing$. Conversely, let $int(cl\{x\}) = \varnothing$ for every $x \in X \setminus I(X)$. If $f \in C(X)_F$, then $F = X \setminus C(f)$ is a finite subset of $X \setminus I(X)$ and $X \setminus clF = X \setminus (\cup_{x \in F} cl\{x\}) = \cap_{x \in F}(X \setminus cl\{x\})$ is an open dense subset of X contained in $C(f)$. Thus, $f \in T'(X)$, i.e., $C(X)_F \subseteq T^{'}(X)$. □

Remark 3.3. Recall that a subset A of a topological space X is said to be a generalized closed (briefly g-closed) set if $clA \subseteq U$ whenever $A \subseteq U$ and U is open in X. If the set of all closed sets and g-closed sets in X are coincide, then X is called a $T_{\frac{1}{2}}$-space [7]. It is well known that X is a $T_{\frac{1}{2}}$-space if and only if for each $x \in X$, $\{x\}$ is either closed or open [3]. Thus, if X is a $T_{\frac{1}{2}}$-space then by Lemma 3.2, $C(X)_F$ is a subring of $T'(X)$. It is noted that all topological spaces in the paper [5] is assume to be T_1 but some of its results hold in the class of $T_{\frac{1}{2}}$-topological spaces, for example if we borrow the proof of [5, Proposition 3.1], word-by-word, then we observe that Proposition 3.1 in [5] holds for $T_{\frac{1}{2}}$-spaces.

Lemma 3.4. *Let X be an arbitrary topological space. Then, $T'(X) \subseteq C(X)_F$ if and only if every dense open subset of X has finite complement.*

Proof. $\Rightarrow$. Let D be a dense open subset of X and $T'(X) \subseteq C(X)_F$. Then, $\chi_D \in C(X)_F$ since $\chi_D \in T'(X)$. If $X \setminus D$ is infinite, then χ_D is not continuous at x for any $x \in X \setminus D$ and so $X \setminus D \subset X \setminus C(\chi_D)$. Thus, $\chi_D \notin C(X)_F$ which is a contradiction.
$\Leftarrow$. It is obvious. □

The following results is a direct consequence of the above lemma and Lemma 3.2.

Theorem 3.5. *Let X be an arbitrary topological space. Then, $T'(X) = C(X)_F$ if and only if every point of $X \setminus I(X)$ is nowhere dense and every dense open subset of X has finite complement.*

Corollary 3.6. *Let X be a topological space which is $T_{\frac{1}{2}}$. Then, $T'(X) = C(X)_F$ if and only if every dense open subset of X has finite complement.*

The above corollary shows that [5, Proposition 5.4] holds for topological spaces which are $T_{\frac{1}{2}}$.

Remark 3.7. Recall that if the intersection of any two nonempty open sets in a topological space X is nonempty, then X is called hyperconnected. Thus, if $f \in C(X)_F$ and X is T_1, then there is a finite closed subset H of X such that $f \in C(X \setminus H)$. Since X is a hyperconnected T_1-space and H is a finite subset of X, we have $X \setminus H$ is hyperconnected. Therefore $C(X \setminus H)$ consists of constant functions. Hence, the image of the function f is a finite set. Also, we have the following:

$$\mathbb{R} \simeq C^*(X) = C(X) \subsetneq C(X)_F \subseteq T'(X) \subsetneq F(X).$$

In particular if the topology of X is cofinite topology and Y is the direct sum of X with itself, then by Theorem 3.5 we have the following:

$$C(Y) \subsetneq C(Y)_F = T'(Y) \subsetneq F(Y).$$

Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$. The set $\mathbb{N}$ with topology $\mathcal{F} \cup \{\varnothing\}$ become a topological space denoted by X. From [1, Example 3.4] it follows that

$$C(X) \subsetneq T'(X) \subsetneq F(X).$$

Clearly, $C(X) \neq C(X)_F$ and there is a dense open subset D in X such that $X \setminus D$ is infinite and so $\chi_D \in T'(X) \setminus C(X)_F$.

Let X be an arbitrary topological space and R be a subring of $F(X)$ such that $\chi_{\{x\}} \in R$ for some $x \in X$. If I is the ideal generated by $\chi_{\{x\}}$ in R, then I is ring isomorphic with the field of real numbers and so I is a minimal ideal of R. Note that for any $x \in X$ we have $\chi_{\{x\}} \in C(X)_F$. If an ideal of $C(X)_F$ contains a nonzero function f and $r = f(x) \neq 0$ for some $x \in X$, then it contains $\chi_{\{x\}} = \frac{1}{r} f \chi_{\{x\}}$ and so we have the following well known result.

Lemma 3.8 ([5]). *Let X be an arbitrary topological space, then $Soc(C(X)_F)$ which is equal to the ideal generated by $\chi_{\{x\}}$'s is a free ideal both in $C(X)_F$ and in $C^*(X)_F$.*

Proposition 3.9. *Let X be an arbitrary topological space. Then, $Soc(C(X)_F)$ is the intersection of all the free ideals in $C(X)_F$, and of all the free ideals in $C^*(X)_F$.*

Proof. If I is a free ideal in $C(X)_F$ or $C^*(X)_F$ and $x \in X$, then there exists $g \in I$ such that $a = g(x) \neq 0$ and so by the comment before Lemma 3.8, $\chi_{\{x\}} \in I$. Thus, $Soc(C(X)_F) \subseteq I$ and so by Lemma 3.8 the proof is complete. □

An algebraic characterization of finite topological spaces is given in the following result.

Theorem 3.10. *Let X be a topological space. Then the following are equivalent.*

(1) *X is a finite set.*
(2) *$C(X)_F = Soc(C(X)_F)$.*

Proof. (1) ⇒ (2). Let X be a finite set. Then $1 = \sum_{x \in X} \chi_{\{x\}} \in Soc(C(X)_F)$ by Lemma 3.8 . Thus, $C(X)_F = Soc(C(X)_F)$.
(2) ⇒ (1). If $C(X)_F = Soc(C(X)_F)$, then $1 \in Soc(C(X)_F)$ and so by Lemma 3.8 , there are $x_i \in X$ and $c_i \in \mathbb{R}$ where $1 \leq i \leq n$ and $n \in \mathbb{N}$ such that $1 = \sum_{i=1}^{n} c_i \chi_{\{x_i\}}$. Thus, X is a finite set. □

Recall that a commutative ring is called Artinian if it satisfies the descending chain condition on ideals and it is called semisimple if the intersection of all the maximal ideals called Jacobson radical is zero. It is well known that a ring is Artinian semisimple if and only if it is equal to the sum of its minimal ideals [2]. Let X be a topological space, then $C(X)_F$ is semisimple since by [5] its Jacobson radical is zero. Thus by the above theorem we have the following result.

Corollary 3.11. *A topological space X is finite if and only if $C(X)_F$ is an Artinian ring.*

Theorem 3.12. *Let $\Phi : C(X)_F \to C(X)_F$ be a $C(X)_F$-module homomorphism which is one to one. Then Φ is a $C(X)_F$-module isomorphism.*

Proof. First we claim that $\Phi(1)$ is a unit of $C(X)_F$. Assume to the contrary that $\Phi(1)$ is not a unit of $C(X)_F$. Thus by Proposition 2.3, there is $1 \neq g \in C(X)_F$ such that $\Phi(1) = g\Phi(1)$ and so

$$\Phi(1 - g) = (1 - g)\Phi(1) = 0,$$

which is a contradiction since Φ is one to one. Therefore, there exists $h \in C(X)_F$ such that $h\Phi(1) = 1$. Now for any $f \in C(X)_F$, we have $f = f\Phi(h) = \Phi(fh) \in \Phi(C(X)_F)$ which completes the proof. □

Corollary 3.13. *Let $f : X \to X$ be a bijection. If f is continuous, then $\Phi : C(X)_F \to C(X)_F$ defined by $\Phi(g) = g \circ f$ is a $C(X)_F$-module isomorphism.*

Proof. It is straightforward. □

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