Characterizations and perturbation analysis of a class of matrices related to core-EP inverses

Mengmeng Zhou,[*] Jianlong Chen,[†] Néstor Thome[‡]

Abstract

Let $A, B \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ and $\mathrm{ind}(B) = s$ and let $L_B = B^2 B^{\textcircled{\dagger}}$. A new condition $(C_{s,*})$: $R(A^k) \cap N((B^s)^*) = \{0\}$ and $R(B^s) \cap N((A^k)^*) = \{0\}$, is defined. Some new characterizations related to core-EP inverses are obtained when B satisfies condition $(C_{s,*})$. Explicit expressions of $B^{\textcircled{\dagger}}$ and $BB^{\textcircled{\dagger}}$ are also given. In addition, equivalent conditions, which guarantee that B satisfies condition $(C_{s,*})$, are investigated. We proved that B satisfies condition $(C_{s,*})$ if and only if L_B has a fixed matrix form. As an application, upper bounds for the errors $\| B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} \| / \| A^{\textcircled{\dagger}} \|$ and $\| BB^{\textcircled{\dagger}} - AA^{\textcircled{\dagger}} \|$ are studied.

Key words: Core inverse; Core-EP inverse; Eigenprojection; Perturbation.
AMS subject classifications: 15A09; 15A23; 65F35.

1 Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ complex matrices. For $A \in \mathbb{C}^{m\times n}$, the notations A^*, $\mathrm{rk}(A)$, $R(A)$ and $N(A)$ stand for the conjugate transpose, the rank, the range space and the null space of matrix A, respectively. The symbols I and $\| \cdot \|$ denote the identity matrix of an appropriate order and spectral norm, respectively. The unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following equations:

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA,$$

is called the Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ [25] and denoted by $A^\dagger$. It is well known that the Moore-Penrose inverse of a matrix solves the optimization problem of computing least-squares minimum-norm and it has important applications in real situations [1].

Let $A \in \mathbb{C}^{n\times n}$. The unique matrix $X \in \mathbb{C}^{n\times n}$ is called the Drazin inverse of A and denoted by A^D [7] if there exist $X \in \mathbb{C}^{n\times n}$ and positive integer k such that the following equations hold:

$$XA^{k+1} = A^k, \quad XAX = X, \quad XA = AX.$$

If k is the smallest positive integer such that $\mathrm{rk}(A^k) = \mathrm{rk}(A^{k+1})$, then k is called the index of A and denoted by $\mathrm{ind}(A)$. When $k = 1$, the Drazin inverse of A is the group inverse of A

[*]Mengmeng Zhou (E-mail: mmz9209@163.com): School of Mathematics, Southeast University, Nanjing 210096, China
[†]Jianlong Chen (Corresponding author E-mail: jlchen@seu.edu.cn): School of Mathematics, Southeast University, Nanjing 210096, China
[‡]Néstor Thome (Email: njthome@mat.upv.es): Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Valencia 46022, Spain

and denoted by $A^{\#}$. It is well known that the Drazin inverse is useful to solve differential linear equations and difference linear equations [3] because of its eigenstructure properties.

In 2010, the core inverse of a complex matrix was introduced by Baksalary et al. [2]. In 2017, Xu et al. [34] characterized the core inverse by three equations. Let $A \in \mathbb{C}^{n\times n}$. The unique matrix $X \in \mathbb{C}^{n\times n}$ is the core inverse of A and denoted by $A^{\textcircled{\#}}$ if and only if X satisfies the following three equations:

$$(AX)^* = AX, \quad AX^2 = X, \quad XA^2 = A.$$

In 2014, Manjunatha Prasad et al. [19] generalized the core inverse of a complex matrix to the core-EP inverse of a complex matrix. In 2018, Gao et al. [9] extended the core-EP inverse of a complex matrix to a ring. In rings, the core-EP inverse was characterized as the unique solution of a system of three equations. Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. The unique matrix $X \in \mathbb{C}^{n\times n}$ such that the following three equations hold:

$$XA^{k+1} = A^k, \quad AX^2 = X, \quad (AX)^* = AX,$$

is called the core-EP inverse of A and denoted by $A^{\textcircled{\dagger}}$. The core-EP inverse of A is the core inverse of A when $k = 1$. More details of the core-EP inverse can be found in [8, 20–22, 30]. Recently, Wang et al. [32] solved the constrained matrix approximation problem by using core inverses. Later, Mosić et al. [24] generalized this result and obtained the unique solution to the constrained matrix minimization problem in the Euclidean norm by applying the core-EP inverse. Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ and any $b \in \mathbb{C}^n$. Ji et al. [11] proved that the constrained problem $\min\limits_{x\in R(A^k)} \parallel b - Ax \parallel_2$ has a unique least squares solution $A^{\textcircled{\dagger}}b$, where $\|\cdot\|_2$ is the 2-norm in $\mathbb{C}^n$. The previous discussion highlights the crucial role that the core-EP inverse plays in solving the constrained system of linear equations.

The core inverse arises as an inverse having some common properties satisfied by the Moore-Penrose and the Drazin inverse. Roughly speaking, it can be seen as an intermediate inverse between both of them. In consequence, for instance, it is useful when optimization properties and eigenstructure of matrices must be combined. However, the core inverse was defined only on the class of index one matrices. In order to exploit this kind of properties for matrices of arbitrary index (not necessarily at most one index), it arises the core-EP inverse for offering the corresponding advantages and further applications. Some numerical methods for computing the core-EP inverse and to analyze perturbations can be found in [17, 18]. Some extensions to Minkowski spaces appear in [31] and to tensors in [27]. Weighted core-EP inverses were analyzed, for example, in [28] and determinantal applications can be seen in [14].

From a numerical point of view, a study of perturbation bounds for the Drazin inverse was published in [4, 6, 12, 13, 15, 26, 33]. Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$, and let $B \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(B) = s$. Castro-González et al. [5] characterized the Drazin inverse of a class of singular matrices, which satisfy conditions:

$$R(A^k) \cap N(B^s) = \{0\} \text{ and } R(B^s) \cap N(A^k) = \{0\}.$$

They also considered the perturbation of the Drazin inverse under these conditions. In [16], the author studied the closed form and perturbation bounds for the core inverse under certain assumptions. Later, Ma et al. [17] generalized the perturbation results for the core inverse in [16] to the core-EP inverse. Gao et al. [10] investigated the continuity of the core-EP inverse by two methods. Moreover, they considered perturbation bounds for the core-EP inverse under prescribed conditions. Mosić [23] investigated the perturbation for the weighted core-EP inverse.

Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Motivated by above discussion, we will consider matrices $B \in \mathbb{C}^{n\times n}$, which satisfy the following condition for some positive integer s:

$$(C_{s,*}) \qquad R(A^k) \cap N((B^s)^*) = \{0\} \text{ and } R(B^s) \cap N((A^k)^*) = \{0\}.$$

Then, we investigate necessary and sufficient conditions which ensure that matrices $B \in \mathbb{C}^{n\times n}$ satisfy the condition $(C_{s,*})$. Furthermore, we consider the perturbation bounds for the core-EP inverse. It is worth noting that this perturbation result for the core-EP inverse is different from the perturbation results given in [10, 17].

The rest of this paper is organized as follows. In Section 2, some auxiliary lemmas are presented. In Section 3, we present expressions of B^π under the condition $(C_{s,*})$, Moreover, we prove that B^π is similar to A^π, that is, there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that $B^\pi = PA^\pi P^{-1}$. In Section 4, we present new equivalent characterizations for a class of matrices which satisfy the condition $(C_{s,*})$. Then we investigate representations of a class of matrices satisfying condition $(C_{s,*})$. In Section 5, we give the explicit expression of $B^{\textcircled{\dagger}}$ and obtain the perturbation bounds for the core-EP inverse, where $B \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(B) = s$ satisfies the condition $(C_{s,*})$. In addition, a numerical example is presented to show that the perturbation bounds is efficient.

2 Preliminaries

Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. It is well known that the orthogonal projector $I - AA^{\textcircled{\dagger}}$ corresponding to the zero eigenvalue of A is called the eigenprojection at zero of A, and we will denote by A^π. That is, $A^\pi = I - AA^{\textcircled{\dagger}}$ satisfies $(A^\pi)^2 = A^\pi = (A^\pi)^*$. Moreover, by [8], $R(A^\pi) = N((A^k)^*)$ and $N(A^\pi) = R(A^k)$. Let $B \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(B) = s$, we denote $L_B = B^2B^{\textcircled{\dagger}}$. We observe that $R(L_B) = R(B^s)$ and $N(L_B) = N((B^s)^*)$.

Lemma 2.1. [30] *(Core-EP decomposition) Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then A can be uniquely written as $A = A_1 + A_2$, where*

(i) $\mathrm{ind}(A_1) \le 1$;

(ii) $A_2^k = 0$;

(iii) $A_1^*A_2 = A_2A_1 = 0$.

Moreover, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A_1 = U\begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix}U^*, \quad A_2 = U\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}U^*,$$

where $T \in \mathbb{C}^{r\times r}$ is nonsingular, N is nilpotent with index k and $\mathrm{rk}(A^k) = r$.

For $A \in \mathbb{C}^{n\times n}$ being as in Lemma 2.1, it is known [30] that

$$A^{\textcircled{\dagger}} = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*, \quad A^{\pi} = U\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}U^*.$$

Lemma 2.2. [9] *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then the following statements hold:*

(i) $AA^{\textcircled{\dagger}} = A^m(A^{\textcircled{\dagger}})^m$, *for arbitrary positive integer m;*

(ii) $A^{\textcircled{\dagger}} = A^D A^k (A^k)^{\dagger}$;

(iii) $(A^{\textcircled{\dagger}})^{\textcircled{\dagger}} = (A^{\textcircled{\dagger}})^{\#} = A^2 A^{\textcircled{\dagger}}$;

(iv) $((A^{\textcircled{\dagger}})^{\textcircled{\dagger}})^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$.

Lemma 2.3. [3] *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a complex square matrix with $A \in \mathbb{C}^{r\times r}$ nonsingular and denote $Z = I + A^{-1}BCA^{-1}$. Then*

(i) $\operatorname{rk}(M) = \operatorname{rk}(A)$ *if and only if* $D = CA^{-1}B$;

(ii) *If* $\operatorname{rk}(M) = \operatorname{rk}(A)$, *then* $\operatorname{ind}(M) = 1$ *if and only if Z is nonsingular.*

Lemma 2.4. *Let $B_1 \in \mathbb{C}^{m\times m}$ be nonsingular and let $P \in \mathbb{C}^{m\times n}$ and $Q \in \mathbb{C}^{n\times m}$ be arbitrary matrices. Define $W = \begin{pmatrix} B_1 & B_1P \\ QB_1 & QB_1P \end{pmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}$. Then W is core invertible if and only if $I + PQ$ is nonsingular. In this case,*

$$W^{\#} = \begin{pmatrix} ((I+Q^*Q)B_1(I+PQ))^{-1} & ((I+Q^*Q)B_1(I+PQ))^{-1}Q^* \\ Q((I+Q^*Q)B_1(I+PQ))^{-1} & Q((I+Q^*Q)B_1(I+PQ))^{-1}Q^* \end{pmatrix}. \tag{2.1}$$

Proof. We observe that $\operatorname{rk}(W) = \operatorname{rk}(B_1)$. By Lemma 2.3 (ii), we have W is core invertible if and only if $I + PQ$ is nonsingular. By setting X as the matrix on the right-hand side of the equality (2.1), it is easy to check that $(WX)^* = WX$, $WX^2 = X$ and $XW^2 = W$. That is, $X = W^{\#}$. □

Lemma 2.5. [29, 35] *Let $A, B, C \in \mathbb{C}^{n\times n}$. Then*

(i) $\operatorname{rk}(AB) = \operatorname{rk}(B) - \dim(R(B) \cap N(A))$;

(ii) $\operatorname{rk}(ABC) \geq \operatorname{rk}(AB) + \operatorname{rk}(BC) - \operatorname{rk}(B)$.

A slight modification in Lemma 2.1 in [5] yields the following result.

Lemma 2.6. [5] *Let $A, U \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and U is a unitary matrix. Then*

$I - A^{\pi} + UA^{\pi}U^*A^{\pi}$ is nonsingular if and only if $I - A^{\pi} + U^*A^{\pi}UA^{\pi}$ is nonsingular.

3 An expression for the eigenprojection at zero of B

In this section, we give expressions of B^{π} when $B \in \mathbb{C}^{n\times n}$ satisfies $(C_{s,*})$.

Firstly, we give an auxiliary lemma.

Lemma 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. If $B \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(B) = s$ satisfies condition $(C_{s,*})$, then the following statements hold:*

(i) $I + (L_B^* - A)A^{\textcircled{\dagger}}$ *is nonsingular;*

(ii) $I + (L_B - A)A^{\textcircled{\dagger}}$ *is nonsingular;*

(iii) $I + (A^{\textcircled{\dagger}})^*(L_B - A^*)$ *is nonsingular;*

(iv) $I - (I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}A^{\pi} - A^{\pi}(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}$ *is nonsingular.*

Proof. (i) : Suppose that $(I + (L_B^* - A)A^{\textcircled{\dagger}})x = 0$, for $x \in \mathbb{C}^n$. Then $A^{\pi}x = -L_B^*A^{\textcircled{\dagger}}x$. From $L_B = B^2B^{\textcircled{\dagger}} = B^{s+1}(B^s)^{\dagger}$, we obtain $R(L_B^*) = R(B^s(B^s)^{\dagger}B^*) \subseteq R(B^s)$ and

$$\mathrm{rk}(B^s) = \mathrm{rk}(B^2B^{\textcircled{\dagger}}B^{\textcircled{\dagger}}) \leq \mathrm{rk}(L_B) = \mathrm{rk}(L_B^*) \leq \mathrm{rk}(B^s),$$

from which $R(L_B^*) = R(B^s)$. Then, we have $A^{\pi}x \in R(A^{\pi}) \cap R(L_B^*) = N((A^k)^*) \cap R(B^s) = \{0\}$. So, $A^{\pi}x = 0$, we get that $x \in N(A^{\pi}) = R(A^k)$. From $L_B^*A^{\textcircled{\dagger}}x = 0$, we get $A^{\textcircled{\dagger}}x \in N(L_B^*) \cap R(A^k) = N((B^s)^*) \cap R(A^k) = \{0\}$. Thus, we obtain $A^{\textcircled{\dagger}}x = 0$. Since $x \in N(A^{\textcircled{\dagger}}) \cap R(A^k) = N((A^k)^*) \cap R(A^k) = \{0\}$, we know that $x = 0$. Hence, $I + (L_B^* - A)A^{\textcircled{\dagger}}$ is nonsingular.

(ii) and (iii) : Similar to the proof of (i).

(iv) : Let $x \in \mathbb{C}^n$ such that $(I-(I+(A^{\textcircled{\dagger}})^*(L_B^*-A^*))^{-1}A^{\pi}-A^{\pi}(I+(L_B-A)A^{\textcircled{\dagger}})^{-1})x = 0$. Since

$$(I - (I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}A^{\pi})x = A^{\pi}(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}x,$$

after doing some algebraic computations, we have

$$(I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}(A^{\textcircled{\dagger}})^*L_B^*x = A^{\pi}(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}x.$$

Since, by definition of $A^{\textcircled{\dagger}}$ and $(A^{\textcircled{\dagger}})^*(I+(L_B^*-A)A^{\textcircled{\dagger}}) = (I+(A^{\textcircled{\dagger}})^*(L_B^*-A^*))A^{\textcircled{\dagger}}$ holds, we obtain that

$$(I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}(A^{\textcircled{\dagger}})^* = A^{\textcircled{\dagger}}(I + (L_B^* - A)A^{\textcircled{\dagger}})^{-1}.$$

Then

$$(I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}(A^{\textcircled{\dagger}})^*L_B^*x \in R(A^{\textcircled{\dagger}}) \cap R(A^{\pi}) = R(A^k) \cap N((A^k)^*) = \{0\}.$$

So,

$$(A^{\textcircled{\dagger}})^*L_B^*x = 0 = A^{\pi}(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}x.$$

Then $L_B^*x = (B^2B^{\textcircled{\dagger}})^*x \in N((A^{\textcircled{\dagger}})^*) \cap R(B^s) = N((A^k)^*) \cap R(B^s) = \{0\}$. Thus, $x \in N(L_B^*) = N((B^s)^*)$. Since $(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}x \in N(A^{\pi}) = R(A^k)$, we have $(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}x = A^ky$ for some $y \in \mathbb{C}^n$. Since $AA^{\textcircled{\dagger}}$ is an orthogonal projector onto $R(A^k)$ [8], we have $x = L_BA^{\textcircled{\dagger}}A^ky = B^2B^{\textcircled{\dagger}}A^{\textcircled{\dagger}}A^ky$. Hence, $x \in N((B^s)^*) \cap R(B^s) = \{0\}$. This completes the proof. $\square$

Now, we present an expression for B^π by using Lemma 3.1.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. If $B \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(B) = s$ satisfies condition $(C_{s,*})$, then B^π is similar to A^π. Moreover,*

$$\begin{aligned} B^\pi &= -(I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}A^\pi X^{-1} = -X^{-1}A^\pi(I + (L_B - A)A^{\textcircled{\dagger}})^{-1} \\ &= XA^\pi X^{-1} = X^{-1}A^\pi X, \end{aligned}$$

where $X = I - (I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}A^\pi - A^\pi(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}$. As a consequence, $BB^{\textcircled{\dagger}}$ is similar to $AA^{\textcircled{\dagger}}$.

Proof. By Lemma 3.1, we know that the expressions $I+(A^{\textcircled{\dagger}})^*(L_B^*-A^*)$, $I+(L_B-A)A^{\textcircled{\dagger}}$ and X are nonsingular. Set $H = (I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}$ and $W = (I + (L_B - A)A^{\textcircled{\dagger}})^{-1}$. Then $X = I - HA^\pi - A^\pi W$. Since

$$A^\pi(I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*)) = A^\pi = (I + (L_B - A)A^{\textcircled{\dagger}})A^\pi,$$

we have $A^\pi H = A^\pi = WA^\pi$. Since

$$\begin{aligned} XHA^\pi &= (I - HA^\pi - A^\pi W)HA^\pi = HA^\pi - HA^\pi HA^\pi - A^\pi WHA^\pi \\ &= -A^\pi WHA^\pi = A^\pi W(I - HA^\pi - A^\pi W) = A^\pi WX, \end{aligned}$$

we have $HA^\pi X^{-1} = X^{-1}A^\pi W$. Setting $Q = -HA^\pi X^{-1}$. It is easy to see that $XA^\pi = -HA^\pi$ and $A^\pi X = -A^\pi W$. So, we get $XA^\pi X^{-1} = X^{-1}A^\pi X$. We have $Q = XA^\pi X^{-1}$, which is obviously idempotent. Next, we will prove that $R(Q) = N((B^s)^*)$ and $N(Q) = R(B^s)$. Let $x \in N((B^s)^*)$, by Lemma 2.2, we have

$$A^\pi x + (A^{\textcircled{\dagger}})^*L_B^* x = A^\pi x + (A^{\textcircled{\dagger}})^*((B^s)^\dagger)^*(B^{s+1})^* x = A^\pi x.$$

From the definition of H, we have $x = HA^\pi x$. So, $x \in R(Q)$. Suppose that $x \in R(Q)$. There exists $y \in \mathbb{C}^n$ such that $x = HA^\pi y$. We have $(A^\pi + (A^{\textcircled{\dagger}})^*L_B^*)x = A^\pi y$. We easily see $A^\pi x = (A^\pi H)A^\pi y = A^\pi y$. Then $(A^{\textcircled{\dagger}})^*L_B^* x = 0$. We have $L_B^* x \in N((A^{\textcircled{\dagger}})^*)\cap R(B^s) = N((A^k)^*) \cap R(B^s) = \{0\}$. Thus, $x \in N(L_B^*) = N((B^s)^*)$. Hence, $R(Q) = N((B^s)^*)$.

Since X is nonsingular, $N(Q) = N(A^\pi W)$. Let $x \in N(Q)$. Then

$$A^\pi W x = (W^{-1} - L_B A^{\textcircled{\dagger}})Wx = (I - L_B A^{\textcircled{\dagger}} W)x = 0.$$

We obtain that $x = L_B A^{\textcircled{\dagger}} W x$. So, $x \in R(L_B) = R(B^s)$. That is, $N(Q) \subseteq R(B^s)$. Since $\mathrm{ind}(B) = s$ and $\mathbb{C}^n = R(Q) \oplus N(Q) = R(B^s) \oplus N((B^s)^*)$, we have $N(Q) = R(B^s)$. Thus, $Q = B^\pi$ and B^π is similar to A^π. As a consequence, $BB^{\textcircled{\dagger}}$ is similar to $AA^{\textcircled{\dagger}}$. □

4 Characterizations of matrices satisfying condition $(C_{s,*})$

In this section, we characterize matrices $B \in \mathbb{C}^{n\times n}$ satisfying the condition $(C_{s,*})$ with $\mathrm{ind}(B) = s$. We prove that the matrix $B \in \mathbb{C}^{n\times n}$ satisfies $(C_{s,*})$ if and only if $I - A^\pi - B^\pi$ is nonsingular. Then we present the representation of matrix L_B with respect to the core-EP decomposition of matrix $A \in \mathbb{C}^{n\times n}$.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then the following statements on $B \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(B) = s$ are equivalent:*

(i) *B satisfies condition $(C_{s,*})$;*

(ii) $\operatorname{rk}(B^s) = \operatorname{rk}(A^k) = \operatorname{rk}((A^k)^* L_B) = \operatorname{rk}(L_B A^k)$*;*

(iii) $\operatorname{rk}(B^s) = \operatorname{rk}(A^k) = \operatorname{rk}((A^k)^* L_B A^k)$*;*

(iv) $\operatorname{rk}(B^s) = \operatorname{rk}(A^k)$ *and $I - (I - B^\pi)A^\pi$ is nonsingular;*

(v) *$I - (A^\pi - B^\pi)^2$ is nonsingular;*

(vi) *$I - A^\pi - B^\pi$ is nonsingular.*

Proof. (i) $\Rightarrow$ (ii) : Since $R(B^s)\cap N((A^k)^*) = \{0\}$ and $R(A^k)\cap N((B^s)^*) = \{0\}$, by Lemma 2.5 (i), we have

$$\begin{aligned} \operatorname{rk}(BB^{\textcircled{\dagger}}AA^{\textcircled{\dagger}}) &= \operatorname{rk}(AA^{\textcircled{\dagger}}) - \dim(R(AA^{\textcircled{\dagger}}) \cap N(BB^{\textcircled{\dagger}})) \\ &= \operatorname{rk}(A^k) - \dim(R(A^k) \cap N((B^s)^*)) \\ &= \operatorname{rk}(A^k) \end{aligned}$$

and

$$\begin{aligned} \operatorname{rk}(AA^{\textcircled{\dagger}}BB^{\textcircled{\dagger}}) &= \operatorname{rk}(BB^{\textcircled{\dagger}}) - \dim(R(BB^{\textcircled{\dagger}}) \cap N(AA^{\textcircled{\dagger}})) \\ &= \operatorname{rk}(B^s) - \dim(R(B^s) \cap N((A^k)^*)) \\ &= \operatorname{rk}(B^s). \end{aligned}$$

From $\operatorname{rk}(AA^{\textcircled{\dagger}}BB^{\textcircled{\dagger}}) = \operatorname{rk}((AA^{\textcircled{\dagger}}BB^{\textcircled{\dagger}})^*) = \operatorname{rk}(BB^{\textcircled{\dagger}}AA^{\textcircled{\dagger}})$, we obtain that $\operatorname{rk}(B^s) = \operatorname{rk}(A^k)$. Since $L_B = B^2B^{\textcircled{\dagger}}$ and $\operatorname{ind}(B) = s$, we obtain $\operatorname{rk}(L_B) = \operatorname{rk}(B^s)$. By Lemma 2.5 (i), we have

$$\operatorname{rk}((A^k)^* L_B) = \operatorname{rk}(B^s) - \dim(R(B^s) \cap N((A^k)^*) = \operatorname{rk}(B^s)$$

and

$$\operatorname{rk}(L_B A^k) = \operatorname{rk}(A^k) - \dim(R(A^k) \cap N((B^s)^*) = \operatorname{rk}(A^k).$$

(ii) $\Rightarrow$ (iii) : We observe that $\operatorname{rk}((A^k)^* L_B A^k) \le \operatorname{rk}(A^k) = \operatorname{rk}(B^s)$. By Lemma 2.5 (ii), $\operatorname{rk}((A^k)^* L_B A^k) \ge \operatorname{rk}((A^k)^* L_B) + \operatorname{rk}(L_B A^k) - \operatorname{rk}(L_B) = \operatorname{rk}(B^s)$.

(iii) $\Rightarrow$ (iv) : From (iii) and Lemma 2.5, we have $R(A^k) \cap N((B^s)^*) = \{0\}$ and $R(B^s)\cap N((A^k)^*) = \{0\}$. Let $x \in \mathbb{C}^n$ such that $(I - A^\pi + B^\pi A^\pi)x = 0$. Then $(I - A^\pi)x = -B^\pi A^\pi x$. We obtain that

$$(I - A^\pi)x \in R(A^k) \cap R(B^\pi) = R(A^k) \cap N((B^s)^*) = \{0\}.$$

So, $x = A^\pi x$ and $B^\pi A^\pi x = 0$. Thus, $A^\pi x \in R(A^\pi) \cap N(B^\pi) = N((A^k)^*) \cap R(B^s) = \{0\}$. Hence, $x = 0$. That is, $I - A^\pi + B^\pi A^\pi$ is nonsingular.

(iv) $\Rightarrow$ (v) : Since $I - (A^\pi - B^\pi)^2 = (I - A^\pi + B^\pi A^\pi)(I - B^\pi + A^\pi B^\pi)$, it is sufficient to verify that $I - B^\pi + A^\pi B^\pi$ is nonsingular. Taking $A = U\begin{pmatrix} T_A & S_A \\ 0 & N_A \end{pmatrix}U^*$ and $B =$

$V\begin{pmatrix} T_B & S_B \\ 0 & N_B \end{pmatrix} V^*$, where U and V are unitary matrices, T_A and T_B are nonsingular. Since $\mathrm{rk}(B^s) = \mathrm{rk}(A^k)$, T_A and T_B are of the same size. Since $U^*A^\pi U = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = V^*B^\pi V$, we get $B^\pi = VU^*A^\pi UV^*$. Since $I - A^\pi + B^\pi A^\pi = I - A^\pi + VU^*A^\pi UV^*A^\pi$ is nonsingular, by Lemma 2.6, $I - A^\pi + UV^*A^\pi VU^*A^\pi = UV^*(I - B^\pi + A^\pi B^\pi)VU^*$ is nonsingular. So, $I - B^\pi + A^\pi B^\pi$ is nonsingular. Thus, $I - (A^\pi - B^\pi)^2$ is nonsingular.

(v) $\Rightarrow$ (vi) : Similar to [5, Theorem 2.1 (e) $\Rightarrow$ (f)].

(vi) $\Rightarrow$ (i) : Suppose that $I - A^\pi - B^\pi$ is nonsingular. Since $I - A^\pi$ and B^π are idempotent matrices, by [13, Theorem 1.2], we have $R(I - A^\pi) \cap R(B^\pi) = \{0\}$ and $N(I - A^\pi) \cap N(B^\pi) = \{0\}$. Since $R(I - A^\pi) = R(A^k)$, $R(B^\pi) = N((B^s)^*)$, $N(I - A^\pi) = N((A^k)^*)$ and $N(B^\pi) = R(B^s)$, we obtain that $R(A^k) \cap N((B^s)^*) = \{0\}$ and $R(B^s) \cap N((A^k)^*) = \{0\}$. $\square$

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then the following conditions on $B \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(B) = s$ are equivalent:*

(i) *B satisfies condition $(C_{s,*})$;*

(ii) *$I + (L_B - A)A^{\textcircled{\dagger}}$ is nonsingular, $A^\pi(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}L_B = 0$;*

(ii') *$I + (L_B - A)A^{\textcircled{\dagger}}$ is nonsingular, $A^\pi X L_B = 0$, where $X = I - (I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}A^\pi - A^\pi(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}$;*

(iii) *If A is written as in Lemma 2.1, then L_B has the following matrix form:*

$$L_B = U\begin{pmatrix} B_1 & B_1P \\ QB_1 & QB_1P \end{pmatrix} U^*,$$

for any matrices $B_1 \in \mathbb{C}^{r\times r}$, P and Q such that B_1 and $I + PQ$ are nonsingular;

(iv) *$\mathrm{rk}(B^s) = \mathrm{rk}(A^k)$, $I + (L_B - A)A^{\textcircled{\dagger}}$ is nonsingular.*

Proof. (i) $\Rightarrow$ (ii) : By Lemma 3.1, we know that $I + (L_B - A)A^{\textcircled{\dagger}}$ is nonsingular. By Lemma 2.2 and Theorem 3.2, we have

$$0 = B^\pi B^{s+1}(B^{\textcircled{\dagger}})^s = B^\pi B^2 B^{\textcircled{\dagger}} = B^\pi L_B = -X^{-1}A^\pi(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}L_B,$$

where $X = I - (I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}A^\pi - A^\pi(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}$.

(ii) $\Leftrightarrow$ (ii$'$) : From the proof of Theorem 3.2, we know that $A^\pi X = -A^\pi W$, where $W = (I + (L_B - A)A^{\textcircled{\dagger}})^{-1}$. By expression of B^π, it is clear.

(ii) $\Rightarrow$ (iii) : Suppose that $L_B = U\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} U^*$. Then we have

$$I + (L_B - A)A^{\textcircled{\dagger}} = U\begin{pmatrix} B_{11}T^{-1} & 0 \\ B_{21}T^{-1} & I \end{pmatrix} U^*.$$

Since $I + (L_B - A)A^{\textcircled{\dagger}}$ is nonsingular, we obtain that B_{11} is nonsingular. From

$$0 = A^\pi(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}L_B = U\begin{pmatrix} 0 & 0 \\ 0 & B_{22} - B_{21}B_{11}^{-1}B_{12} \end{pmatrix} U^*,$$

we get $B_{22} = B_{21}B_{11}^{-1}B_{12}$. By Lemma 2.3 (i), we know that $\mathrm{rk}(L_B) = \mathrm{rk}(B_{11})$. Since $L_B^2 = B^3B^{\textcircled{\dagger}}$, we have

$$\mathrm{rk}(L_B) = \mathrm{rk}(B^2B^{\textcircled{\dagger}}) = \mathrm{rk}(B^3B^{\textcircled{\dagger}}B^{\textcircled{\dagger}}) \leq \mathrm{rk}(B^3B^{\textcircled{\dagger}}) = \mathrm{rk}(L_B^2) \leq \mathrm{rk}(L_B).$$

So, $\mathrm{rk}(L_B) = \mathrm{rk}(L_B^2)$. That is, $\mathrm{ind}(L_B) = 1$. By Lemma 2.3 (ii), we have $I+B_{11}^{-1}B_{12}B_{21}B_{11}^{-1}$ is nonsingular. Taking $B_{11} := B_1$, $P := B_1^{-1}B_{12}$ and $Q := B_{21}B_1^{-1}$. Then $B_{22} = QB_1P$. From the nonsingularity of $I + B_{11}^{-1}B_{12}B_{21}B_{11}^{-1}$, we obtain that $I + PQ$ is nonsingular.

(iii) $\Rightarrow$ (i) : By Lemma 2.3, we have $\mathrm{rk}(L_B) = \mathrm{rk}(B_1) = \mathrm{rk}(A^k)$ and $\mathrm{ind}(L_B) = 1$. Since

$$\mathrm{rk}(B^s) = \mathrm{rk}(B^DB^2B^{\textcircled{\dagger}}B^s) \leq \mathrm{rk}(L_B) \leq \mathrm{rk}(B^s),$$

we have $\mathrm{rk}(B^s) = \mathrm{rk}(L_B) = \mathrm{rk}(A^k)$. By a direct computation, we have

$$\begin{aligned}\mathrm{rk}((A^k)^*L_BA^k) &= \mathrm{rk}(U\begin{pmatrix} I & 0 \\ (T^{-k}M)^* & I\end{pmatrix}\begin{pmatrix} (T^k)^*B_1T^k & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} I & T^{-k}M \\ 0 & I\end{pmatrix}U^*) \\ &= \mathrm{rk}((T^k)^*B_1T^k) = \mathrm{rk}(A^k),\end{aligned}$$

where $M = \sum_{i=0}^{k-1} T^iSN^{k-1-i}$. So, $\mathrm{rk}(B^s) = \mathrm{rk}(A^k) = \mathrm{rk}((A^k)^*L_BA^k)$. By Theorem 4.1 (iii), we know that B satisfies condition $(C_{s,*})$.

(iii) $\Rightarrow$ (iv) : According to the proof of (iii) $\Rightarrow$ (i), we have $\mathrm{rk}(B^s) = \mathrm{rk}(A^k)$. The rest is clear by a direct computation.

(iv) $\Rightarrow$ (iii) : Taking $L_B = U\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix}U^*$, where B_{11} and T have the same size. Since $I + (L_B - A)A^{\textcircled{\dagger}}$ is nonsingular, by the proof of (ii) $\Rightarrow$ (iii), we know that B_{11} is nonsingular. Since $\mathrm{rk}(L_B) = \mathrm{rk}(B^s) = \mathrm{rk}(A^k)$, we get that $\mathrm{rk}(L_B) = \mathrm{rk}(B_{11})$. By Lemma 2.3 (i), we know that $B_{22} = B_{21}B_{11}^{-1}B_{12}$. From $\mathrm{ind}(L_B) = 1$, by Lemma 2.3 (ii), we have $I + B_{11}^{-1}B_{12}B_{21}B_{11}^{-1}$ is nonsingular. Denoting $B_1 := B_{11}$. Then there exist matrices P and Q such that $B_{12} = B_1P$, $B_{21} = QB_1$ and $B_{22} = QB_1P$. Thus $I + PQ$ is nonsingular. $\square$

Analogously, we have the following result.

Corollary 4.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then the following conditions on $B \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(B) = s$ are equivalent:*

(i) *B satisfies condition $(C_{s,*})$;*

(ii) *$I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*)$ is nonsingular, $L_B(I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}A^\pi = 0$;*

(ii') *$I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*)$ is nonsingular, $L_BXA^\pi = 0$, where $X = I - (I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*))^{-1}A^\pi - A^\pi(I + (L_B - A)A^{\textcircled{\dagger}})^{-1}$;*

(iii) *If A is written as in Lemma 2.1, then L_B has the following matrix form:*

$$L_B = U\begin{pmatrix} B_1 & B_1P \\ QB_1 & QB_1P\end{pmatrix}U^*,$$

for any matrices B_1, P and Q such that B_1 and $I + PQ$ are nonsingular;

(iv) *$\mathrm{rk}(B^s) = \mathrm{rk}(A^k)$, $I + (A^{\textcircled{\dagger}})^*(L_B^* - A^*)$ is nonsingular.*

5 Perturbation bounds

In this section, we give the perturbation bounds for the core-EP inverse under the condition $(C_{s,*})$. First, we obtain an explicit expression for $B^{\textcircled{\dagger}}$.

Theorem 5.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and let $B \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(B) = s$ satisfying the condition $(C_{s,*})$. Denote $E = L_B - A$ and $F = L_B - A^*$, then*

$$\begin{aligned} B^{\textcircled{\dagger}} &= (A^{\textcircled{\dagger}} + (I-W)Y\Psi^{-1}A^{\textcircled{\dagger}} + (I+Y) \\ &\times (A^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}EA^{\textcircled{\dagger}}\Phi^{-1} - \Psi^{-1}(\Psi - I)A^{\textcircled{\dagger}}\Phi^{-1})(I+Y^*Y)^{-1}Y^*)\Phi^{-1}, \end{aligned} \tag{5.1}$$

where $\Phi = I + EA^{\textcircled{\dagger}}$, $\widetilde{\Phi} = I + (A^{\textcircled{\dagger}})^*F$, $W = (\widetilde{\Phi})^{-1}(A^{\textcircled{\dagger}})^*FA^{\pi}$, $Y = A^{\pi}EA^{\textcircled{\dagger}}\Phi^{-1}$ and $\Psi = I + WY$.

Proof. Let $A \in \mathbb{C}^{n\times n}$ be as in Lemma 2.1. By Theorem 4.2, we have

$$L_B = U\begin{pmatrix} B_1 & B_1P \\ QB_1 & QB_1P \end{pmatrix}U^*,$$

where B_1, P, and Q are any matrices such that B_1 and $I + PQ$ are nonsingular. It is easy to check that $\operatorname{ind}(L_B) = 1$. From [2, Lemma 2.2] and [8, Theorem 2.9], we can deduce that $B^{\textcircled{\dagger}} = (B^2B^{\textcircled{\dagger}})^{\textcircled{\dagger}} = (B^2B^{\textcircled{\dagger}})^{\textcircled{\#}} = L_B^{\textcircled{\#}}$. Now, by Lemma 2.4, we get

$$\begin{aligned} B^{\textcircled{\dagger}} &= L_B^{\textcircled{\#}} \\ &= U\begin{pmatrix} ((I+Q^*Q)B_1(I+PQ))^{-1} & ((I+Q^*Q)B_1(I+PQ))^{-1}Q^* \\ Q((I+Q^*Q)B_1(I+PQ))^{-1} & Q((I+Q^*Q)B_1(I+PQ))^{-1}Q^* \end{pmatrix}U^*. \end{aligned}$$

By denoting $E = L_B - A$, $F = L_B - A^*$, $\Phi = I + EA^{\textcircled{\dagger}}$ and $\widetilde{\Phi} = I + (A^{\textcircled{\dagger}})^*F$, we have

$$E = U\begin{pmatrix} B_1 - T & B_1P - S \\ QB_1 & QB_1P - N \end{pmatrix}U^*, \quad F = U\begin{pmatrix} B_1 - T^* & B_1P \\ QB_1 - S^* & QB_1P - N^* \end{pmatrix}U^*,$$

$$\Phi = U\begin{pmatrix} B_1T^{-1} & 0 \\ QB_1T^{-1} & I \end{pmatrix}U^* \text{ and } \widetilde{\Phi} = U\begin{pmatrix} (T^{-1})^*B_1 & (T^{-1})^*B_1P \\ 0 & I \end{pmatrix}U^*.$$

Since Φ and $\widetilde{\Phi}$ are nonsingular, we get that

$$\Phi^{-1} = U\begin{pmatrix} TB_1^{-1} & 0 \\ -Q & I \end{pmatrix}U^* \text{ and } \widetilde{\Phi}^{-1} = U\begin{pmatrix} B_1^{-1}T^* & -P \\ 0 & I \end{pmatrix}U^*.$$

Then

$$B^{\textcircled{\dagger}}\Phi = U\begin{pmatrix} (I+PQ)^{-1}T^{-1} & ((I+Q^*Q)B_1(I+PQ))^{-1}Q^* \\ Q(I+PQ)^{-1}T^{-1} & Q((I+Q^*Q)B_1(I+PQ))^{-1}Q^* \end{pmatrix}U^*.$$

Denote $W = \widetilde{\Phi}^{-1}(A^{\textcircled{\dagger}})^*FA^{\pi}$ and $Y = A^{\pi}EA^{\textcircled{\dagger}}\Phi^{-1}$. By a direct computation, we obtain that

$$W = U\begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}U^* \text{ and } Y = U\begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}U^*.$$

Taking $\Psi = I + WY$. Then $\Psi = U\begin{pmatrix} I+PQ & 0 \\ 0 & I \end{pmatrix} U^*$ and $\Psi^{-1} = U\begin{pmatrix} (I+PQ)^{-1} & 0 \\ 0 & I \end{pmatrix} U^*$. By denoting

$$H_1 = A^{\textcircled{\dagger}} + (I-W)Y\Psi^{-1}A^{\textcircled{\dagger}},$$
$$H_2 = A^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}EA^{\textcircled{\dagger}}\Phi^{-1} - \Psi^{-1}(\Psi - I)A^{\textcircled{\dagger}}\Phi^{-1},$$
$$H_3 = (I+Y)H_2(I+Y^*Y)^{-1}Y^*,$$

we compute that

$$H_1 = U\begin{pmatrix} (I+PQ)^{-1}T^{-1} & 0 \\ Q(I+PQ)^{-1}T^{-1} & 0 \end{pmatrix} U^*, \quad H_2 = U\begin{pmatrix} (I+PQ)^{-1}B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*$$

and

$$\begin{aligned} H_3 &= U\begin{pmatrix} I & 0 \\ Q & I \end{pmatrix}\begin{pmatrix} (I+PQ)^{-1}B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} (I+Q^*Q)^{-1} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & Q^* \\ 0 & 0 \end{pmatrix} U^* \\ &= U\begin{pmatrix} 0 & ((I+Q^*Q)B_1(I+PQ))^{-1}Q^* \\ 0 & Q((I+Q^*Q)B_1(I+PQ))^{-1}Q^* \end{pmatrix} U^*. \end{aligned}$$

We observe that $B^{\textcircled{\dagger}}\Phi = H_1 + H_3$. Then

$$B^{\textcircled{\dagger}} = (H_1 + H_3)\Phi^{-1}.$$

By substituting H_1 and H_3, we obtain the equality (5.1). $\square$

By using the same notations as in the proof of the Theorem 5.1, we have the following results.

Theorem 5.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and let $B \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(B) = s$ satisfying the condition $(C_{s,*})$. Denote $E = L_B - A$ and $F = L_B - A^*$. If $\max\{\| EA^{\textcircled{\dagger}} \|, \| (A^{\textcircled{\dagger}})^*F \|\} < 1$ and $\| A^\pi EA^{\textcircled{\dagger}} \| < 1 - \| EA^{\textcircled{\dagger}} \|$, then*

$$\begin{aligned} \frac{\| B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} \|}{\| A^{\textcircled{\dagger}} \|} &\leq \frac{\| EA^{\textcircled{\dagger}} \|}{1 - \| EA^{\textcircled{\dagger}} \|} + \frac{\| \Psi^{-1} \| \| A^\pi EA^{\textcircled{\dagger}} \|}{(1 - \| EA^{\textcircled{\dagger}} \|)^2}\left(1 + \frac{\| (A^{\textcircled{\dagger}})^*FA^\pi \|}{1 - \| (A^{\textcircled{\dagger}})^*F \|}\right) \\ &+ \frac{\| A^\pi EA^{\textcircled{\dagger}} \|}{(1 - \| EA^{\textcircled{\dagger}} \|)(1 - \| EA^{\textcircled{\dagger}} \| - \| A^\pi EA^{\textcircled{\dagger}} \|)} \\ &\times \left(1 + \frac{\| EA^{\textcircled{\dagger}} \|}{1 - \| EA^{\textcircled{\dagger}} \|} + \frac{\| \Psi^{-1} \| \| A^\pi EA^{\textcircled{\dagger}} \| \| (A^{\textcircled{\dagger}})^*FA^\pi \|}{(1 - \| EA^{\textcircled{\dagger}} \|)^2(1 - \| (A^{\textcircled{\dagger}})^*F \|)}\right). \end{aligned} \tag{5.2}$$

If $\max\{\| EA^{\textcircled{\dagger}} \|, \| (A^{\textcircled{\dagger}})^*F \|\} < \frac{1}{1+\sqrt{\| A^\pi \|}}$, then

$$\| \Psi^{-1} \| \leq \frac{(1 - \| (A^{\textcircled{\dagger}})^*F \|)(1 - \| EA^{\textcircled{\dagger}} \|)}{(1 - \| (A^{\textcircled{\dagger}})^*F \|)(1 - \| EA^{\textcircled{\dagger}} \|) - \| (A^{\textcircled{\dagger}})^*F \| \| A^\pi EA^{\textcircled{\dagger}} \|}, \tag{5.3}$$

where $\Psi = I + (I + (A^{\textcircled{\dagger}})^*F)^{-1}(A^{\textcircled{\dagger}})^*FA^\pi EA^{\textcircled{\dagger}}(I + EA^{\textcircled{\dagger}})^{-1}$.

Proof. Since $B^{\textcircled{\dagger}}\Phi - A^{\textcircled{\dagger}} = B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} + (B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} + A^{\textcircled{\dagger}})EA^{\textcircled{\dagger}}$, we have

$$B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} + (B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} + A^{\textcircled{\dagger}})EA^{\textcircled{\dagger}} = H_1 - A^{\textcircled{\dagger}} + H_3,$$

where $\Phi = I + EA^{\textcircled{\dagger}}$, $Y = A^{\pi}EA^{\textcircled{\dagger}}\Phi^{-1}$, $H_1 = A^{\textcircled{\dagger}} + (I - W)Y\Psi^{-1}A^{\textcircled{\dagger}}$ and $H_3 = (I + Y)H_2(I + Y^*Y)^{-1}Y^*$. Then

$$\| B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} \| \leq \| EA^{\textcircled{\dagger}} \| \| B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}} \| + \| EA^{\textcircled{\dagger}} \| \| A^{\textcircled{\dagger}} \| + \| H_1 - A^{\textcircled{\dagger}} \| + \| H_3 \| .$$

Since $\max\{\| EA^{\textcircled{\dagger}} \|, \| (A^{\textcircled{\dagger}})^*F \|\} < 1$ and $\| A^{\pi}EA^{\textcircled{\dagger}} \| < 1- \| EA^{\textcircled{\dagger}} \|$, we compute that

$$\| \Phi^{-1} \| \leq \frac{1}{1- \| EA^{\textcircled{\dagger}} \|} \text{ and } \| \widetilde{\Phi}^{-1} \| \leq \frac{1}{1- \| (A^{\textcircled{\dagger}})^*F \|},$$

$$\| H_1 - A^{\textcircled{\dagger}} \| \leq \frac{\| A^{\textcircled{\dagger}} \| \| \Psi^{-1} \| \| A^{\pi}EA^{\textcircled{\dagger}} \|}{1- \| EA^{\textcircled{\dagger}} \|}(1 + \frac{\| (A^{\textcircled{\dagger}})^*FA^{\pi} \|}{1- \| (A^{\textcircled{\dagger}})^*F \|}),$$

$$\begin{aligned}
\| H_3 \| &\leq \frac{\| H_2 \| \| Y \| (1+ \| Y \|)}{1- \| Y \|^2} = \frac{\| Y \| \| H_2 \|}{1- \| Y \|} \\
&\leq \frac{\| A^{\textcircled{\dagger}} \| \| A^{\pi}EA^{\textcircled{\dagger}} \|}{1- \| EA^{\textcircled{\dagger}} \| - \| A^{\pi}EA^{\textcircled{\dagger}} \|} \\
&\times (1 + \frac{\| EA^{\textcircled{\dagger}} \|}{1- \| EA^{\textcircled{\dagger}} \|} + \frac{\| \Psi^{-1} \| \| A^{\pi}EA^{\textcircled{\dagger}} \| \| (A^{\textcircled{\dagger}})^*FA^{\pi} \|}{(1- \| EA^{\textcircled{\dagger}} \|)^2(1- \| (A^{\textcircled{\dagger}})^*F \|)}).
\end{aligned}$$

By substitution and simplification, we obtain inequality (5.2).

If $\max\{\| EA^{\textcircled{\dagger}} \|, \| (A^{\textcircled{\dagger}})^*F \|\} < \frac{1}{1+\sqrt{\|A^{\pi}\|}}$, then

$$\| \Psi - I \| \leq \frac{\| (A^{\textcircled{\dagger}})^*F \| \| A^{\pi}EA^{\textcircled{\dagger}} \|}{(1- \| EA^{\textcircled{\dagger}} \|)(1- \| (A^{\textcircled{\dagger}})^*F \|)} < \frac{\| A^{\pi} \| (\frac{1}{1+\sqrt{\|A^{\pi}\|}})^2}{(1 - \frac{1}{1+\sqrt{\|A^{\pi}\|}})^2} = 1.$$

Thus,

$$\| \Psi^{-1} \| \leq \frac{(1- \| (A^{\textcircled{\dagger}})^*F \|)(1- \| EA^{\textcircled{\dagger}} \|)}{(1- \| (A^{\textcircled{\dagger}})^*F \|)(1- \| EA^{\textcircled{\dagger}} \|)- \| (A^{\textcircled{\dagger}})^*F \| \| A^{\pi}EA^{\textcircled{\dagger}} \|}.$$

□

Theorem 5.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ and let $B \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(B) = s$ satisfying the condition $(C_{s,*})$. Denote $E = L_B - A$. If $\| EA^{\textcircled{\dagger}} \| < 1$ and $\| A^{\pi}EA^{\textcircled{\dagger}} \| < 1- \| EA^{\textcircled{\dagger}} \|$, then*

$$\| B^{\pi} - A^{\pi} \| \leq \frac{\| A^{\pi}EA^{\textcircled{\dagger}} \|}{1- \| EA^{\textcircled{\dagger}} \|} + \frac{\| A^{\pi}EA^{\textcircled{\dagger}} \|}{(1- \| EA^{\textcircled{\dagger}} \|)(1- \| EA^{\textcircled{\dagger}} \| - \| A^{\pi}EA^{\textcircled{\dagger}} \|)}. \quad (5.4)$$

Proof. Suppose that A is written as in Lemma 2.1. By Theorem 3.2, we have $B^{\pi} + B^{\pi}EA^{\textcircled{\dagger}} = -X^{-1}A^{\pi}$, where $X = I - A^{\pi}(I + EA^{\textcircled{\dagger}})^{-1} - ((I + EA^{\textcircled{\dagger}})^{-1})^*A^{\pi}$. By the proof of Theorem 5.1, we obtain

$$X = U \begin{pmatrix} I & Q^* \\ Q & -I \end{pmatrix} U^* \text{ and } X^{-1} = U \begin{pmatrix} (I + Q^*Q)^{-1} & (I + Q^*Q)^{-1}Q^* \\ Q(I + Q^*Q)^{-1} & Q(I + Q^*Q)^{-1}Q^* - I \end{pmatrix} U^*.$$

So, $-X^{-1}A^{\pi} = A^{\pi} + U\begin{pmatrix} 0 & -(I+Q^*Q)^{-1}Q^* \\ 0 & -Q(I+Q^*Q)^{-1}Q^* \end{pmatrix}U^*$. Then $-X^{-1}A^{\pi} = A^{\pi} - (I+Y)(I+Y^*Y)^{-1}Y^*$, where $Y = A^{\pi}EA^{\textcircled{\dagger}}\Phi^{-1}$. Thus, we have

$$\begin{aligned} B^{\pi} - A^{\pi} &= -X^{-1}A^{\pi} - A^{\pi} - (B^{\pi} - A^{\pi} + A^{\pi})EA^{\textcircled{\dagger}} \\ &= -(B^{\pi} - A^{\pi} + A^{\pi})EA^{\textcircled{\dagger}} - (I + A^{\pi}EA^{\textcircled{\dagger}}\Phi^{-1}) \\ &\times\ (I + (A^{\pi}EA^{\textcircled{\dagger}}\Phi^{-1})^*A^{\pi}EA^{\textcircled{\dagger}}\Phi^{-1})^{-1}(A^{\pi}EA^{\textcircled{\dagger}}\Phi^{-1})^*. \end{aligned}$$

By substitution and simplification, we have inequality (5.4). $\square$

Remark 5.4. *The upper bound of $\| B^{\pi} - A^{\pi} \|$ is the upper bound of $\| BB^{\textcircled{\dagger}} - AA^{\textcircled{\dagger}} \|$. In fact, $\| B^{\pi} - A^{\pi} \| = \| -(BB^{\textcircled{\dagger}} - AA^{\textcircled{\dagger}}) \| = \| BB^{\textcircled{\dagger}} - AA^{\textcircled{\dagger}} \|$.*

Finally, the following examples illustrate the above theorems.

Example 5.5. *Let* $A = \begin{pmatrix} 1 & 2 & \frac{1}{10} & \frac{1}{10} \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *with* $\mathrm{ind}(A) = 2$*. Then* $A^{\textcircled{\dagger}} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
Set $B \in \mathbb{C}^{4\times 4}$ with $\mathrm{ind}(B) = s$, where $0 < s < 4$. By the equivalence of condition (i) *and condition* (iii) *in Theorem 4.2, we set* $B_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $P = \begin{pmatrix} \frac{1}{10} & \frac{1}{10} \\ 0 & 0 \end{pmatrix}$ *and* $Q = \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{1}{10} & 0 \end{pmatrix}$*. By using MATLAB, we obtain*

$$L_B = \begin{pmatrix} 1 & 2 & \frac{1}{10} & \frac{1}{10} \\ 2 & 1 & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{50} & \frac{1}{50} \\ \frac{1}{10} & \frac{1}{5} & \frac{1}{100} & \frac{1}{100} \end{pmatrix},\ B^{\textcircled{\dagger}} = L_B^{\#} = \begin{pmatrix} -\frac{409}{1327} & \frac{200}{309} & -\frac{400}{6489} & -\frac{200}{6489} \\ \frac{40}{63} & -\frac{1}{3} & \frac{8}{63} & \frac{4}{63} \\ -\frac{400}{6489} & \frac{40}{309} & -\frac{80}{6489} & -\frac{40}{6489} \\ -\frac{200}{6489} & \frac{20}{309} & -\frac{40}{6489} & -\frac{20}{6489} \end{pmatrix},$$

$$A^{\pi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ B^{\pi} = \begin{pmatrix} \frac{1}{21} & 0 & -\frac{4}{21} & -\frac{2}{21} \\ 0 & 0 & 0 & 0 \\ -\frac{4}{21} & 0 & \frac{101}{105} & -\frac{2}{105} \\ -\frac{2}{21} & 0 & -\frac{2}{105} & \frac{104}{105} \end{pmatrix},$$

$$E = L_B - A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{50} & -\frac{49}{50} \\ \frac{1}{10} & \frac{1}{5} & \frac{1}{100} & \frac{1}{100} \end{pmatrix},\ F = L_B - A^* = \begin{pmatrix} 0 & 0 & \frac{1}{10} & \frac{1}{10} \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{10} & \frac{2}{5} & \frac{1}{50} & \frac{1}{50} \\ 0 & \frac{1}{5} & -\frac{99}{100} & \frac{1}{100} \end{pmatrix}.$$

In this case, we have $\| EA^{\textcircled{\dagger}} \| = \frac{646}{2889} < 1$, $\| (A^{\textcircled{\dagger}})^*F \| = \frac{197}{1393} < 1$ and $\| EA^{\textcircled{\dagger}} \| + \| A^{\pi}EA^{\textcircled{\dagger}} \| = \frac{1292}{2889} < 1$. Denoting by v, v_1 and v_2 the value of the right side of inequalities (5.2), (5.3) and (5.4), respectively. By Theorem 5.2 and Theorem 5.3, we obtain $v = \frac{1450}{1019}$ and $v_2 = \frac{1292}{1597}$. By a direct computation, we have $\frac{\|B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\|}{\|A^{\textcircled{\dagger}}\|} = \frac{1973}{11791} < v$ and $\| B^{\pi} - A^{\pi} \| = \frac{769}{3524} < v_2$. Since $\frac{1}{1+\sqrt{\|A^{\pi}\|}} = \frac{985}{1393}$, we know that $\max\{\| EA^{\textcircled{\dagger}} \|, \| (A^{\textcircled{\dagger}})^*F \|\} < \frac{1}{1+\sqrt{\|A^{\pi}\|}}$. By using the same notation as in the proof of Theorem 5.1, we obtain $\| \Psi^{-1} \| = 1$. By a direct computation, we get $v_1 = \frac{1855}{1769}$. So, $\| \Psi^{-1} \| < v_1$.

Example 5.6. *Let* $A = \begin{pmatrix} \frac{1}{10} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{10} & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ *with* $\operatorname{ind}(A) = 3$. *It is easy to obtain that* $A^{\textcircled{\dagger}} = \begin{pmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. *Setting* $L_B = \begin{pmatrix} \frac{1}{10} & 0 & \frac{1}{100} & 0 & \frac{1}{400} \\ 0 & \frac{1}{10} & 0 & \frac{1}{200} & 0 \\ \frac{1}{300} & \frac{1}{300} & \frac{1}{3000} & \frac{1}{6000} & \frac{1}{12000} \\ \frac{1}{500} & 0 & \frac{1}{5000} & 0 & \frac{1}{20000} \\ 0 & \frac{1}{600} & 0 & \frac{1}{12000} & 0 \end{pmatrix}$ *with* $\operatorname{ind}(B) = s$, *where* $0 < s < 5$. *Since* $\operatorname{rk}(B^s) = \operatorname{rk}(L_B) = 2 = \operatorname{rk}(A^k) = \operatorname{rk}((A^k)^* L_B A^k)$, *by Theorem 4.1, B satisfies consition* $(C_{s,*})$. *By using MATLAB, it is easy to obtain*

$$A^{\pi} = \operatorname{diag}(0,0,1,1,1), \ B^{\textcircled{\dagger}} = \begin{pmatrix} \frac{5165}{519} & -\frac{37}{765} & \frac{1744}{5283} & \frac{1033}{5190} & -\frac{24}{29773} \\ -\frac{71}{3376} & \frac{5063}{507} & \frac{286}{861} & -\frac{15}{35662} & \frac{241}{1448} \\ \frac{287}{867} & \frac{373}{1126} & \frac{121}{5481} & \frac{221}{33381} & \frac{111}{20105} \\ \frac{1033}{5190} & -\frac{14}{14473} & \frac{119}{18024} & \frac{119}{29894} & -\frac{12}{744325} \\ -\frac{24}{68471} & \frac{241}{1448} & \frac{116}{20953} & -\frac{1}{142648} & \frac{229}{82554} \end{pmatrix},$$

$$B^{\pi} = \begin{pmatrix} \frac{59}{39135} & \frac{18}{16247} & -\frac{89}{2677} & -\frac{53}{2654} & \frac{3}{162470} \\ \frac{18}{16247} & \frac{25}{18041} & -\frac{40}{1203} & \frac{2}{90261} & -\frac{1129}{67834} \\ -\frac{89}{2677} & -\frac{40}{1203} & \frac{450}{451} & -\frac{15}{22559} & -\frac{2}{3609} \\ -\frac{53}{2654} & \frac{2}{90261} & -\frac{15}{22559} & \frac{2503}{2504} & 0 \\ \frac{3}{162470} & -\frac{1129}{67834} & -\frac{2}{3609} & 0 & \frac{3604}{3605} \end{pmatrix},$$

$$E = L_B - A = \begin{pmatrix} 0 & 0 & -\frac{49}{100} & 0 & \frac{1}{400} \\ 0 & 0 & 0 & \frac{1}{200} & -\frac{1}{5} \\ \frac{1}{300} & \frac{1}{300} & \frac{1}{3000} & -\frac{5999}{6000} & \frac{1}{12000} \\ \frac{1}{500} & 0 & \frac{1}{5000} & 0 & -\frac{19999}{20000} \\ 0 & \frac{1}{600} & 0 & \frac{1}{12000} & 0 \end{pmatrix},$$

$$F = L_B - A^* = \begin{pmatrix} 0 & 0 & \frac{1}{100} & 0 & \frac{1}{400} \\ 0 & 0 & 0 & \frac{1}{200} & 0 \\ -\frac{149}{300} & \frac{1}{300} & \frac{1}{3000} & \frac{1}{6000} & \frac{1}{12000} \\ \frac{1}{500} & 0 & -\frac{4999}{5000} & 0 & \frac{1}{20000} \\ 0 & -\frac{119}{600} & 0 & -\frac{11999}{12000} & 0 \end{pmatrix}.$$

In this case, we have $\| EA^{\textcircled{\dagger}} \| = \frac{288}{5689} < 1$, $\| (A^{\textcircled{\dagger}})^* F \| = \frac{211}{2047} < 1$ *and* $\| EA^{\textcircled{\dagger}} \| + \| A^{\pi} E A^{\textcircled{\dagger}} \| = \frac{576}{5689} < 1$. *Denoting by v, v_1 and v_2 the value of the right side of inequalities (5.2), (5.3) and (5.4), respectively. By Theorem 5.2 and Theorem 5.3, we obtain* $v = \frac{680}{3801}$ *and* $v_2 = \frac{576}{5113}$. *By a direct computation, we have* $\frac{\|B^{\textcircled{\dagger}} - A^{\textcircled{\dagger}}\|}{\|A^{\textcircled{\dagger}}\|} = \frac{505}{9601} < v$ *and* $\| B^{\pi} - A^{\pi} \| = \frac{113}{2235} < v_2$. *Since* $\frac{1}{1+\sqrt{\|A^{\pi}\|}} = \frac{1}{2}$, *we know that* $\max\{\| EA^{\textcircled{\dagger}} \|, \| (A^{\textcircled{\dagger}})^* F \|\} < \frac{1}{1+\sqrt{\|A^{\pi}\|}}$. *By using the same notation as in the proof of Theorem 5.1, we obtain* $\| \Psi^{-1} \| = \frac{810}{809}$. *By a direct computation, we get* $v_1 = \frac{816}{811}$. *So,* $\| \Psi^{-1} \| < v_1$.

These two examples highlight the powerful of our theorems because they show that we can compute the core-EP inverse of B without needing the explicit computation of matrix B.

Acknowledgments

The authors thank the Editor and Reviewers sincerely for their constructive comments and suggestions that have improved the quality of the paper. This research is supported by the National Natural Science Foundation of China (No. 11771076, 11871145), the Postgraduate Research and Practice Innovation Program of Jiangsu Province (No. KYCX18_0053), the China Scholarship Council (File No. 201906090122). The third author is partially supported by Ministerio de Economía y Competitividad of Spain (grant Red de Excelencia MTM2017-90682-REDT) and partially supported by Universidad de Buenos Aires. EXP-UBA: 13.019/2017, 20020170100350BA.

References

[1] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications. 2nd ed. New York: Springer-Verlag, 2003.

[2] O.M. Baksalary, G. Trenkler, Core inverse of matrices, Linear Multilinear Algebra 58(6) (2010) 681-697.

[3] S.L. Campbell, C.D. Meyer, Generalized Inverses of Linear Transformations, Pitman, London, 1979; SIAM, Philadelphia, 2009.

[4] N. Castro-González, J.J. Koliha, Y.M. Wei, Perturbation of the Drazin inverse for matrices with equal eigenprojection at zero, Linear Algebra Appl. 312 (2000) 181-189.

[5] N. Castro-González, J. Robles, J.Y. Vélez-Cerrada, Characterizations of a class of matrices and perturbation of the Drazin inverse, SIAM J. Matrix Anal. Appl. 30 (2008) 882-897.

[6] D.S. Cvetković-Ilić, D.S. Djordjević, Y.M. Wei, Additive results for the generalized Drazin inverse in a Banach algebra, Linear Algebra Appl. 418 (2006) 53-61.

[7] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506-514.

[8] D.E. Ferreyra, F.E. Levis, N. Thome, Revisiting the core-EP inverse and its extension to rectangular matrices, Quaest. Math. 41 (2018) 265-281.

[9] Y.F. Gao, J.L. Chen, Pseudo core inverses in rings with involution, Comm. Algebra 46(1) (2018) 38-50.

[10] Y.F. Gao, J.L. Chen, P. Patrício, Continuity of the core-EP inverse and its applications, Linear Multilinear Algebra (2019) doi:10.1080/03081087.2019.1608899.

[11] J. Ji, Y.M. Wei, The core-EP, weighted core-EP inverse of matrices and constrained systems of linear equations, Commun. Math. Res. doi:10.4208/cmr.2020-0028.

[12] J.J. Koliha, Error bound for a general perturbation of the Drazin inverse, Appl. Math. Comput. 126 (2002) 181-185.

[13] J.J. Koliha, V. Rakočević, I. Straškraba, The difference and sum of projectors, Linear Algebra Appl. 388 (2004) 279-288.

[14] Ivan I. Kyrchei, Determinantal representations of the weighted core-EP, DMP, MPD, and CMP inverses, J. Math. (2020) doi:10.1155/2020/9816038.

[15] X.Z. Li, Y.M. Wei, An expression of the Drazin inverse of a perturbed matrix, Appl. Math. Comput. 153 (2004) 187-198.

[16] H.F. Ma, Optimal perturbation bounds for the core inverse, Appl. Math. Comput. 336 (2018) 176-181.

[17] H.F. Ma, P.S. Stanimirović, Characterizations, approximation and perturbations of the core-EP inverse, Appl. Math. Comput. 359 (2019) 404-417.

[18] K. Manjunatha Prasad, M. David Raj, M. Vinay, Iterative methods to find core-EP inverse, Bull. Kerala Math. Assoc. 16(1) (2018) 139-152.

[19] K. Manjunatha Prasad, K.S. Mohana, Core-EP inverse, Linear Multilinear Algebra 62 (2014) 792-802.

[20] K. Manjunatha Prasad, M.D. Raj, Bordering method to compute core-EP inverse, Spec. Matr. 6 (2018) 193-200.

[21] D. Mosić, Core-EP pre-order of Hilbert space operators, Quaest. Math. 41 (2018) 585-600.

[22] D. Mosić, Weighted core-EP inverse of an operator between Hilbert spaces, Linear Multilinear Algebra 67 (2019) 278-298.

[23] D. Mosić, Perturbation of the weighted core-EP inverse, Ann. Funct. Anal. 11 (2020) 75-86.

[24] D. Mosić, P.S. Stanimirović, V.N. Katsikis, Solvability of some constrained matrix approximation problems using core-EP inverses, Comput. Appl. Math. 39(4) (2020) doi:10.1007/s40314-020-01360-y.

[25] R. Penrose. A generalzied inverse for matrices, Proc. Cambridge Philos. Soc. 51 (1955) 406-413.

[26] G.H. Rong, The error bound of the perturbation of the Drazin inverse, Linear Algebra Appl. 47 (1982) 159-168.

[27] J.K. Sahoo, R. Behera, P.S. Stanimirović, V.N. Katsikis, H.F. Ma, Core and core-EP inverses of tensors, Comput. Appl. Math. 39 (2020) 168-187.

[28] J.K. Sahoo, G. Maharana, R. Behera, Further results on weighted core-EP inverse of matrices, arXiv:2005.02974v1.

[29] N. Thome, Inequalities and equalities for $\ell = 2$ (Sylvester), $\ell = 3$ (Frobenius), and $\ell > 3$ matrices, Aequationes Math. 90(5) (2016) 951-960.

[30] H.X. Wang, Core-EP decomposition and its applications, Linear Algebra Appl. 508 (2016) 289-300.

[31] H.X. Wang, H. Wu, X.J. Liu, The m-core-EP inverse in Minkowski space, arXiv:2011.04851v2.

[32] H.X. Wang, X.Y. Zhang, The core inverse and constrained matrix approximation problem, 18 (2020) 653-661.

[33] Y.M. Wei, X.Z. Li, An improvement on perturbation bounds for the Drazin inverse, Linear Algebra Appl. 10 (2003) 563-575.

[34] S.Z. Xu, J.L. Chen, X.X. Zhang, New characterizations for core inverses in rings with involution, Front. Math. China 12(1) (2017) 231-246.

[35] F.Z. Zhang, Matrix Theory: Basic Results and Techniques, Springer-Verlag, New York, 1999.