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Fixed point results with respect to a wt-distance in partially ordered b-metric spaces and its application to nonlinear fourth-order differential equation

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Abstract

In this paper we study the existence of the fixed points for Hardy-Rogers type mappings with respect to a wt-distance in partially ordered metric spaces. Our results provide a more general statement, since we replace a w-distance with a wt-distance and ordered metric spaces with ordered b-metric spaces. Some examples are presented to validate our obtained results and an application to nonlinear fourth-order differential equation are given to support the main results.

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1. Introduction and preliminaries

Fixed point theory is an important and useful tool for different branches of mathematical analysis and it has many applications in mathematics and sciences. In 1922, Banach proved the famous contraction mapping principle [3]. Afterward, other authors considered various definitions of contractive mappings

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and proved several fixed and common fixed point theorems (see Rhoades survey [22] and references therein). On the other hand, the symmetric spaces as metric-like spaces lacking the triangle inequality was introduced in 1931 by Wilson [24]. In the sequel, a new type of spaces which they called b-metric spaces (or metric type spaces) are defined by Bakhtin [2] and Czerwik [6]. After that, several papers have dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces (for example, see [4, 5, 15, 17]).

Definition 1.1 ([2, 6])**.** Let X be a nonempty set and $s \geq 1$ be a real number. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies

- (d_1) $d(x, y) = 0$ if and only if $x = y$;
- (d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (d_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

Then d is called a b-metric and (X, d) is called a b-metric space (or metric type space).

Obviously, for $s = 1$, a b-metric space is a metric space. Also, for notions such as convergent and Cauchy sequences, completeness, continuity, etc in b-metric spaces, we refer to [1, 5, 17].

In 1996, Kada et al. [16] introduced the concept of w-distance in metric spaces, where nonconvex minimization problems were treated. Then some fixed point results and common fixed point theorem with respect to w-distance in metric spaces were proved by Ilić and Rakočević [12] and Shioji et al. [23]. In 2014, Hussain et al. [11] introduced the concept of wt-distance on a b-metric space and proved some fixed point theorems under wt-distance in a partially ordered b-metric space. Then Demma et al. [7] considered multi-valued operators with respect to a wt-distance on b-metric spaces and proved some results on fixed point theory.

Definition 1.2 ([11])**.** Let (X, d) be a b-metric space and $s \geq 1$ be a given real number. A function $\rho : X \times X \to [0, +\infty)$ is called a wt-distance on X if the following properties are satisfied:

- (ρ_1) $\rho(x, z) \leq s[\rho(x, y) + \rho(y, z)]$ for all $x, y, z \in X$;
- (ρ_2) ρ is b-lower semi-continuous in its second variable i.e., if $x \in X$ and $y_n \to y$ in X, then $\rho(x, y) \leq s \liminf_n \rho(x, y_n)$;
- (ρ_3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(z, x) \leq \delta$ and $\rho(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Let us recall that a real-valued function f defined on a metric space X is said to be b-lower semi-continuous at a point $x \in X$ if either $\liminf\limits_{x_n \to x} f(x_n) = \infty$ or $f(x) \leq \liminf\limits_{x_n \to x} sf(x_n)$, whenever $x_n \in X$ and $x_n \to x$ for each $n \in \mathbb{N}$ [12]. Note that each b-metric d is a wt-distance, but the converse is not hold. Thus, wt-distance is a generalization of b-metric d. Obviously, for $s = 1$, every wt-distance is a w-distance. But, a w-distance is not necessary a wt-distance. Thus, each wt-distance is a generalization of w-distance.

Example 1.3 ([11])**.** Let $X = \mathbb{R}$ and define a mapping $d : X \times X \to \mathbb{R}$ by $d(x, y) = (x - y)^2$ for all $x, y \in X$. Then (X, d) is a b-metric with $s = 2$. Define a mapping $\rho : X \times X \to [0, \infty)$ by $\rho(x, y) = y^2$ or $\rho(x, y) = x^2 + y^2$ for all $x, y \in X$. Then ρ is a wt-distance.

From Example 1.3, we have two important results:

(1) for any wt-distance ρ, $\rho(x, y) = 0$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.
(2) for any wt-distance ρ, $\rho(x, y) = \rho(y, x)$ does not necessarily hold for all $x, y \in X$.

Lemma 1.4 ([11])**.** *Let (X, d) be a b-metric space with parameter $s \geq 1$ and ρ be a wt-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X, and $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequences in $[0, +\infty)$ converging to zero and $x, y, z \in X$. Then the following conditions hold:*

(i) if $\rho(x_n, y) \leq \alpha_n$ and $\rho(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $\rho(x, y) = 0$ and $\rho(x, z) = 0$, then $y = z$;
(ii) if $\rho(x_n, y_n) \leq \alpha_n$ and $\rho(x_n, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;
(iii) if $\rho(x_n, x_m) \leq \alpha_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence in X;
(iv) if $\rho(y, x_n) \leq \alpha_n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

Existence of fixed points in ordered metric spaces has been applied by Ran and Reurings [21]. The key feature in this fixed point theorem is that the contractivity condition on the nonlinear map is only assumed to hold on elements that are comparable in the partial order. However, the map is assumed to be monotone. They showed that under such conditions the conclusions of Banach's fixed point theorem still hold. This fixed point theorem was extended by Nieto and Rodríguez-López [18] and was applied to the periodic boundary value problem (see, e.g., [8, 9, 13, 14, 19, 20]). A relation $\sqsubseteq$ on X is called

(i) reflexive if $x \sqsubseteq x$ for all $x \in X$;
(ii) transitive if $x \sqsubseteq y$ and $y \sqsubseteq z$ imply $x \sqsubseteq z$ for all $x, y, z \in X$;
(iii) antisymmetric if $x \sqsubseteq y$ and $y \sqsubseteq x$ imply $x = y$ for all $x, y \in X$;
(iv) pre-order if it is reflexive and transitive.

A pre-order $\sqsubseteq$ is called partial order or an order relation if it is antisymmetric. Given a partially ordered set $(X, \sqsubseteq)$; that is, the set X equipped with a partial order $\sqsubseteq$, the notation $x \sqsubset y$ stands for $x \sqsubseteq y$ and $x \neq y$. Also, let $(X, \sqsubseteq)$ be a partially ordered set. A mapping $f : X \to X$ is said to be nondecreasing if $x \sqsubseteq y$ implies that $fx \sqsubseteq fy$ for all $x, y \in X$.

2. Main results

Our main result is the following theorem for mappings satisfying Hardy-Rogers type conditions [10] with respect to a given wt-distance in a complete partially ordered b-metric space.

Theorem 2.1. *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete b-metric space with given real number $s \geq 1$ and ρ be a wt-distance on X. Suppose that there exist mappings $\alpha_i : X \to [0, 1)$ such that*

$$\alpha_i(fx) \leq \alpha_i(x) \tag{2.1}$$

for all $x \in X$ and $i = 1, 2, \cdots, 5$, where $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$ satisfying the following conditions:

$$\begin{aligned} \rho(fx, fy) \leq{}& \alpha_1(x)\rho(x, y) + \alpha_2(x)\rho(x, fx) + \alpha_3(x)\rho(y, fy) \\ &+ \alpha_4(x)\rho(x, fy) + \alpha_5(x)\rho(y, fx), \end{aligned} \tag{2.2}$$

$$\begin{aligned} \rho(fy, fx) \leq{}& \alpha_1(x)\rho(y, x) + \alpha_2(x)\rho(fx, x) + \alpha_3(x)\rho(fy, y) \\ &+ \alpha_4(x)\rho(fy, x) + \alpha_5(x)\rho(fx, y) \end{aligned} \tag{2.3}$$

for all $x, y \in X$ with $y \sqsubseteq x$ such that

$$(s(\alpha_1 + \alpha_3 + 2\alpha_4) + \alpha_2 + (s^2 + s)\alpha_5)(x) < 1. \tag{2.4}$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $\rho(z, z) = 0$.

Proof. If $fx_0 = x_0$, then x_0 is a fixed point of f and the proof is finished. Now, suppose that $fx_0 \neq x_0$. Since f is nondecreasing with respect to $\sqsubseteq$ and $x_0 \sqsubseteq fx_0$, we obtain by induction that

$$x_0 \sqsubseteq fx_0 \sqsubseteq f^2x_0 \sqsubseteq \cdots \sqsubseteq f^nx_0 \sqsubseteq f^{n+1}x_0 \sqsubseteq \cdots,$$

where $x_n = fx_{n-1} = f^nx_0$. First we shall prove that $\{x_n\}$ is a Cauchy sequence. Now, setting $x = x_n$ and $y = x_{n-1}$ in (2.2) and applying (2.1) and (ρ_1), we have

$$\begin{aligned} \rho(x_{n+1}, x_n) &= \rho(fx_n, fx_{n-1}) \\ &\leq \alpha_1(x_n)\rho(x_n, x_{n-1}) + \alpha_2(x_n)\rho(x_n, fx_n) + \alpha_3(x_n)\rho(x_{n-1}, fx_{n-1}) \\ &\quad + \alpha_4(x_n)\rho(x_n, fx_{n-1}) + \alpha_5(x_n)\rho(x_{n-1}, fx_n) \\ &= \alpha_1(fx_{n-1})\rho(x_n, x_{n-1}) + \alpha_2(fx_{n-1})\rho(x_n, x_{n+1}) \\ &\quad + \alpha_3(fx_{n-1})\rho(x_{n-1}, x_n) + \alpha_4(fx_{n-1})\rho(x_n, x_n) \\ &\quad + \alpha_5(fx_{n-1})\rho(x_{n-1}, x_{n+1}) \\ &\leq \alpha_1(x_{n-1})\rho(x_n, x_{n-1}) + (\alpha_3 + s\alpha_5)(x_{n-1})\rho(x_{n-1}, x_n) \\ &\quad + s\alpha_4(x_{n-1})\rho(x_{n+1}, x_n) + (\alpha_2 + s\alpha_4 + s\alpha_5)(x_{n-1})\rho(x_n, x_{n+1}) \\ &\quad\ \vdots \\ &\leq \alpha_1(x_0)\rho(x_n, x_{n-1}) + (\alpha_3 + s\alpha_5)(x_0)\rho(x_{n-1}, x_n) \\ &\quad + s\alpha_4(x_0)\rho(x_{n+1}, x_n) + (\alpha_2 + s\alpha_4 + s\alpha_5)(x_0)\rho(x_n, x_{n+1}). \end{aligned} \tag{2.5}$$

Similarly, setting $x = x_n$ and $y = x_{n-1}$ in (2.3) and applying (2.1) and (ρ_1), we have

$$\begin{aligned}\rho(x_n, x_{n+1}) \le\ & \alpha_1(x_0)\rho(x_{n-1}, x_n) + (\alpha_3 + s\alpha_5)(x_0)\rho(x_n, x_{n-1}) \\ & + s\alpha_4(x_0)\rho(x_n, x_{n+1}) + (\alpha_2 + s\alpha_4 + s\alpha_5)(x_0)\rho(x_{n+1}, x_n). \end{aligned} \tag{2.6}$$

Now, adding up (2.5) and (2.6), we obtain

$$\begin{aligned}\rho(x_{n+1}, x_n) + \rho(x_n, x_{n+1}) \le\ & (\alpha_1 + \alpha_3 + s\alpha_5)(x_0)[\rho(x_n, x_{n-1}) + \rho(x_{n-1}, x_n)] \\ & + (\alpha_2 + 2s\alpha_4 + s\alpha_5)(x_0)[\rho(x_{n+1}, x_n) + \rho(x_n, x_{n+1})].\end{aligned}$$

Let $a_n = \rho(x_{n+1}, x_n) + \rho(x_n, x_{n+1})$. Then we get

$$a_n \le (\alpha_1 + \alpha_3 + s\alpha_5)(x_0)a_{n-1} + (\alpha_2 + 2s\alpha_4 + s\alpha_5)(x_0)a_n.$$

Therefore, $a_n \le ka_{n-1}$ for all $n \in \mathbb{N}$, where

$$0 \le k = \frac{(\alpha_1 + \alpha_3 + s\alpha_5)(x_0)}{1 - (\alpha_2 + 2s\alpha_4 + s\alpha_5)(x_0)} < \frac{1}{s}$$

by (2.4) and since $(\alpha_1 + \alpha_3 + s\alpha_5)(x_0) \ge 0$. By repeating the procedure, we obtain $a_n \le k^n a_0$ for all $n \in \mathbb{N}$. It follows that

$$\rho(x_n, x_{n+1}) \le a_n \le k^n[\rho(x_1, x_0) + \rho(x_0, x_1)]. \tag{2.7}$$

Let $m > n$. It follows from (2.7) and $0 \le sk < 1$ that

$$\begin{aligned}\rho(x_n, x_m) &\le s[\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_m)] \\ &\le s\rho(x_n, x_{n+1}) + s[s\rho(x_{n+1}, x_{n+2}) + \rho(x_{n+2}, x_m)] \\ &\ \ \vdots \\ &\le s\rho(x_n, x_{n+1}) + s^2\rho(x_{n+1}, x_{n+2}) + \cdots + s^{m-n}\rho(x_{m-1}, x_m)] \\ &\le (sk^n + s^2k^{n+1} \cdots + s^{m-n}k^{m-1})[\rho(x_1, x_0) + \rho(x_0, x_1)] \\ &\le \frac{sk^n}{1 - sk}[\rho(x_1, x_0) + \rho(x_0, x_1)].\end{aligned}$$

Now, Lemma 1.4 (iii) implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x' \in X$ such that $x_n \to x'$ as $n \to \infty$. The continuity of f implies that $x_{n+1} = fx_n \to fx'$ as $n \to \infty$, and since the limit of a sequence is unique, we get that $fx' = x'$. Thus, x' is a fixed point of f. Further, suppose that $fz = z$. Then, by using (2.2), we have

$$\begin{aligned}\rho(z, z) &= \rho(fz, fz) \\ &\le \alpha_1(z)\rho(z, z) + \alpha_2(z)\rho(z, fz) + \alpha_3(z)\rho(z, fz) \\ &\quad + \alpha_4(z)\rho(z, fz) + \alpha_5(z)\rho(z, fz) \\ &\le (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)(z)\rho(z, z).\end{aligned}$$

Since $\sum_{i=1}^{5} \alpha_i(z) < s(\alpha_1 + \alpha_3 + 2\alpha_4)(z) + \alpha_2 + (s^2 + s)\alpha_5)(z) < 1$, we obtain that $\rho(z, z) = 0$ by using Lemma 1.4 (i). This completes the proof. $\square$

Example 2.2. Let $X = [0,1]$ and define a mapping $d : X \times X \to \mathbb{R}$ by $d(x,y) = (x-y)^2$ for all $x, y \in X$. Then (X,d) is a complete b-metric space with $s = 2$. Define a function $\rho : X \times X \to [0,\infty)$ by $\rho(x,y) = d(x,y)$ for all $x, y \in X$. Then ρ is a wt-distance. Let an order relation $\sqsubseteq$ be defined by $x \sqsubseteq y$ if and only if $x \leq y$ Also, let a mapping $f : X \to X$ be defined by $fx = \frac{x^2}{5}$ for all $x \in X$. Then f is a continuous and nondecreasing mapping with respect to $\sqsubseteq$ and there exists a $0 \in X$ such that $0 \sqsubseteq f0$. Define the mappings $\alpha_1(x) = \frac{(x+1)^2}{25}$ and $\alpha_i(x) = 0$ for all $x \in X$ and $i = 2,3,4,5$. Observe that

$$s(\alpha_1 + \alpha_3 + 2\alpha_4)(x) + \alpha_2 + (s^2 + s)\alpha_5)(x) = 2\frac{(x+1)^2}{25} < 1.$$

Also,

$$\alpha_1(fx) = \frac{1}{25}\big(\frac{x^2}{5} + 1\big)^2 \leq \frac{1}{25}\big(x^2 + 1\big)^2 \leq \frac{(x+1)^2}{25} = \alpha_1(x)$$

for all $x \in X$ and $\alpha_i(fx) = 0 = \alpha_i(x)$ for all $x \in X$ and $i = 2,3,4,5$. Moreover, for all $x, y \in X$ with $y \sqsubseteq x$, we get

$$\begin{aligned}
\rho(fx, fy) &= \left(\frac{x^2}{5} - \frac{y^2}{5}\right)^2 \\
&= \frac{(x+y)^2(x-y)^2}{25} \\
&\leq \frac{(x+1)^2}{25}(x-y)^2 \\
&\leq \alpha_1(x)\rho(x,y) + \alpha_2(x)\rho(x,fx) + \alpha_3(x)\rho(y,fy) \\
&\quad + \alpha_4(x)\rho(x,fy) + \alpha_5(x)\rho(y,fx).
\end{aligned}$$

Similarly, for all $x, y \in X$ with $y \sqsubseteq x$, we get

$$\begin{aligned}
\rho(fy, fx) &\leq \alpha_1(x)\rho(y,x) + \alpha_2(x)\rho(fx,x) + \alpha_3(x)\rho(fy,y) \\
&\quad + \alpha_4(x)\rho(fy,x) + \alpha_5(x)\rho(fx,y).
\end{aligned}$$

Therefore, all the conditions of Theorem 2.1 are satisfied. Hence, f has a fixed point $x = 0$ with $\rho(0,0) = 0$.

Several consequences of Theorem 2.1 follow now for particular choices of the contractions.

Corollary 2.3. *Let $(X, \sqsubseteq)$ be a partially ordered set, (X,d) be a complete b-metric space with given real number $s \geq 1$ and ρ be a wt-distance on X. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0,1)$ such that*

$$\alpha(fx) \leq \alpha(x), \quad \beta(fx) \leq \beta(x), \quad \gamma(fx) \leq \gamma(x)$$

for all $x \in X$, where $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$ satisfying the following conditions:

$$\begin{aligned}
\rho(fx, fy) &\leq \alpha(x)\rho(x,y) + \beta(x)\rho(x,fy) + \gamma(x)\rho(y,fx), \\
\rho(fy, fx) &\leq \alpha(x)\rho(y,x) + \beta(x)\rho(fy,x) + \gamma(x)\rho(fx,y)
\end{aligned}$$

for all $x, y \in X$ with $y \sqsubseteq x$ such that

$$(s(\alpha + 2\beta) + (s^2 + s)\gamma)(x) < 1.$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $\rho(z, z) = 0$.

Proof. We obtain this result by applying Theorem 2.1 with $\alpha_1(x) = \alpha(x)$, $\alpha_2(x) = \alpha_3(x) = 0$, $\alpha_4(x) = \beta(x)$ and $\alpha_5(x) = \gamma(x)$. □

In the process of proving Theorem 2.1, consider $x = x_{n-1}$ and $y = x_n$ with $x \sqsubseteq y$ (instead of $x = x_n$ and $y = x_{n-1}$ with $y \sqsubseteq x$). Then, we only need one condition for some following types of the contractions.

Corollary 2.4. *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete b-metric space with given real number $s \geq 1$ and ρ be a wt-distance on X. Suppose that there exist mappings $\alpha_i : X \to [0, 1)$ such that $\alpha_i(fx) \leq \alpha_i(x)$ for all $x \in X$ and $i = 1, 2, 3, 4$, where $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$ satisfying the following condition:*

$$\rho(fx, fy) \leq \alpha_1(x)\rho(x, y) + \alpha_2(x)\rho(x, fx) + \alpha_3(x)\rho(y, fy) + \alpha_4(x)\rho(x, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$ such that

$$(s(\alpha_1 + \alpha_2) + \alpha_3 + (s^2 + s)\alpha_4)(x) < 1.$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $\rho(z, z) = 0$.

Corollary 2.5. *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete b-metric space with given real number $s \geq 1$ and ρ be a wt-distance on X. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that*

$$\alpha(fx) \leq \alpha(x), \quad \beta(fx) \leq \beta(x), \quad \gamma(fx) \leq \gamma(x)$$

for all $x \in X$, where $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$ satisfying the following condition:

$$\rho(fx, fy) \leq \alpha(x)\rho(x, y) + \beta(x)\rho(x, fx) + \gamma(x)\rho(y, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$ such that

$$(s(\alpha + \beta) + \gamma)(x) < 1.$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $\rho(z, z) = 0$.

Theorem 2.6. *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete b-metric space with given real number $s \geq 1$ and ρ be a wt-distance on X. Suppose that there exists a continuous and nondecreasing mapping $f : X \to X$ with respect to $\sqsubseteq$ such that the following conditions hold:*

$$\rho(fx, fy) \leq \alpha_1\rho(x, y) + \alpha_2\rho(x, fx) + \alpha_3\rho(y, fy) + \alpha_4\rho(x, fy) + \alpha_5\rho(y, fx),$$
$$\rho(fy, fx) \leq \alpha_1\rho(y, x) + \alpha_2\rho(fx, x) + \alpha_3\rho(fy, y) + \alpha_4\rho(fy, x) + \alpha_5\rho(fx, y)$$

for all $x, y \in X$ *with* $y \sqsubseteq x$*, where* α_i *are nonnegative coefficients for* $i = 1, 2, \cdots, 5$ *with*

$$s(\alpha_1 + \alpha_3 + 2\alpha_4) + \alpha_2 + (s^2 + s)\alpha_5 < 1.$$

If there exists $x_0 \in X$ *such that* $x_0 \sqsubseteq fx_0$*, then* f *has a fixed point. Moreover, if* $fz = z$*, then* $\rho(z, z) = 0$*.*

Proof. We can prove this result by applying Theorem 2.1 with $\alpha_i(x) = \alpha_i$ for $i = 1, 2, \cdots, 5$. □

Several consequences of Theorem 2.6 follow now for particular choices of the contractions.

Corollary 2.7. *Let* $(X, \sqsubseteq)$ *be a partially ordered set,* (X, d) *be a complete b-metric space with given real number* $s \geq 1$ *and* ρ *be a wt-distance on* X*. Suppose that there exists a continuous and nondecreasing mapping* $f : X \to X$ *with respect to* $\sqsubseteq$ *such that the following conditions hold:*

$$\rho(fx, fy) \leq \alpha\rho(x, y) + \beta\rho(x, fy) + \gamma\rho(y, fx),$$
$$\rho(fy, fx) \leq \alpha\rho(y, x) + \beta\rho(fy, x) + \gamma\rho(fx, y)$$

for all $x, y \in X$ *with* $y \sqsubseteq x$*, where* α, β, γ *are nonnegative coefficients with*

$$s(\alpha + 2\beta) + (s^2 + s)\gamma < 1.$$

If there exists $x_0 \in X$ *such that* $x_0 \sqsubseteq fx_0$*, then* f *has a fixed point. Moreover, if* $fz = z$*, then* $\rho(z, z) = 0$*.*

Proof. We obtain this result by applying Theorem 2.6 with $\alpha_1 = \alpha$, $\alpha_2 = \alpha_3 = 0$, $\alpha_4 = \beta$ and $\alpha_5 = \gamma$. □

In the process of proving Theorem 2.6, consider $x = x_{n-1}$ and $y = x_n$ with $x \sqsubseteq y$ (instead of $x = x_n$ and $y = x_{n-1}$ with $y \sqsubseteq x$). Then, we only need one condition for some types of the contractions.

Corollary 2.8. *Let* $(X, \sqsubseteq)$ *be a partially ordered set,* (X, d) *be a complete b-metric space with given real number* $s \geq 1$ *and* ρ *be a wt-distance on* X*. Suppose that there exists a continuous and nondecreasing mapping* $f : X \to X$ *with respect to* $\sqsubseteq$ *such that the following condition hold:*

$$\rho(fx, fy) \leq \alpha_1\rho(x, y) + \alpha_2\rho(x, fx) + \alpha_3\rho(y, fy) + \alpha_4\rho(x, fy)$$

for all $x, y \in X$ *with* $x \sqsubseteq y$*, where* α_i *for* $i = 1, 2, 3, 4$ *are nonnegative coefficients with*

$$s(\alpha_1 + \alpha_2) + \alpha_3 + (s^2 + s)\alpha_4 < 1.$$

If there exists $x_0 \in X$ *such that* $x_0 \sqsubseteq fx_0$*, then* f *has a fixed point. Moreover, if* $fz = z$*, then* $\rho(z, z) = 0$*.*

Corollary 2.9. *Let* $(X, \sqsubseteq)$ *be a partially ordered set,* (X, d) *be a complete b-metric space with given real number* $s \geq 1$ *and* ρ *be a wt-distance on* X*.*

Suppose that there exists a continuous and nondecreasing mapping $f : X \to X$ with respect to $\sqsubseteq$ such that the following condition hold:

$$\rho(fx, fy) \leq \alpha\rho(x, y) + \beta\rho(x, fx) + \gamma\rho(y, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$, where α, β, γ are nonnegative coefficients with

$$s(\alpha + \beta) + \gamma < 1.$$

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $\rho(z, z) = 0$.

Example 2.10. Consider X, d, s and order relation $\sqsubseteq$ as in Example 2.2. Define a function $\rho : X \times X \to [0, \infty)$ by $\rho(x, y) = y^2$ for all $x, y \in X$. Then ρ is a wt-distance. Also, let a mapping $f : X \to X$ be defined by $fx = \frac{x^2}{3}$ for all $x \in X$. Then f is a continuous and nondecreasing mapping with respect to $\sqsubseteq$ and there exists a $0 \in X$ such that $0 \sqsubseteq f0$. Take $\alpha = \frac{1}{9}$, $\beta = \frac{1}{8}$ and $\gamma = \frac{1}{4}$. Then we obtain

$$\begin{aligned}\rho(fx, fy) = (fy)^2 = (\frac{y^2}{3})^2 = \frac{y^4}{9}& \\ \leq \frac{1}{9}y^2& \\ = \frac{1}{9}\rho(x, y) \leq \alpha\rho(x, y) + \beta\rho(x, fx) + \gamma\rho(y, fy).&\end{aligned}$$

Also, we have

$$s(\alpha + \beta) + \gamma = 2(\frac{1}{9} + \frac{1}{8}) + \frac{1}{4} = \frac{26}{36} < 1.$$

Hence, all the conditions of Corollary 2.9 are satisfied. Therefore, f has a fixed point $x = 0$. Moreover, $\rho(0, 0) = 0$.

Corollary 2.11. *Let $(X, \sqsubseteq)$ be a partially ordered set, (X, d) be a complete b-metric space with given real number $s \geq 1$ and ρ be a wt-distance on X. Suppose that there exists a continuous and nondecreasing mapping $f : X \to X$ with respect to $\sqsubseteq$ such that*

$$\rho(fx, fy) \leq \lambda\rho(x, y) \tag{2.8}$$

for all $x, y \in X$ with $x \sqsubseteq y$, where $\lambda \in [0, \frac{1}{s})$. If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $\rho(z, z) = 0$.

Example 2.12. Let $X = [0, 2]$ and consider d, s and order relation $\sqsubseteq$ as in Example 2.2. Also, let a mapping $f : X \to X$ be defined by $fx = \frac{x^2}{2}$ for all $x \in X$. Since $d(f0, f2) = d(0, 2)$, there is not $0 \leq \alpha < \frac{1}{s}$ such that $d(fx, fy) \leq \alpha d(x, y)$ for all $x, y \in X$. Hence, Banach-type result on b-metric space cannot be applied for this example. Now, let $X = [0, 1]$ and define a function $\rho : X \times X \to [0, \infty)$ by $\rho(x, y) = y^2 + x^2$ for all $x, y \in X$. Then ρ is a

wt-distance.

$$\begin{aligned}\rho(fx,fy) = (fy)^2 + (fx)^2 &= (\frac{y^2}{2})^2 + (\frac{x^2}{2})^2 \\ &= \frac{y^4+x^4}{4} \leq \frac{y^2+x^2}{4} = \frac{1}{4}\rho(x,y).\end{aligned}$$

Thus, (2.8) is hold with $\lambda = \frac{1}{4} \in [0, \frac{1}{2})$. Hence, all conditions of Banach-type fixed point results (or same Corollary 2.11) with respect to the wt-distance on b-metric spaces are satisfied. Note that f has a (trivial) fixed point $0 \in [0,1] \subseteq [0,2]$ and $\rho(0,0) = 0$.

Corollary 2.13. *Let $(X, \sqsubseteq)$ be a partially ordered set, (X,d) be a complete b-metric space with given real number $s \geq 1$ and ρ be a wt-distance on X. Suppose that there exists a continuous and nondecreasing mapping $f : X \to X$ with respect to $\sqsubseteq$ such that*

$$\rho(fx,fy) \leq \delta(\rho(x,fx) + \rho(y,fy))$$

for all $x, y \in X$ with $x \sqsubseteq y$, where $\delta \in [0, \frac{1}{s+1})$. If there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $\rho(z,z) = 0$.

3. An application

It is well-known that fourth-order differential equations are important and useful tools for modeling the elastic beam deformation. Precisely, we refer to beams in equilibrium state, whose two ends are simply supported. Consequently, this study has many applications in engineering and physical science. Now, we establish the existence of solutions of fourth-order boundary value problems as a consequence of Theorem 2.1. In particular, the focus is on the equivalent integral formulation of the boundary value problem below and the use of Green's functions. At the first, we introduce the mathematical background as follows (also, see [14]).

Let $X = C([0,1],\mathbb{R})$ be the set of all non-negative real-valued continuous functions on the interval $[0,1]$. Also, let X be endowed with the supremum norm $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$ and define a mapping $d : X \times X \to \mathbb{R}$ by $d(x,y) = \sup_{t\in[0,1]}(x(t) - y(t))^2$ for all $x, y \in X$. Also, consider the partial order

$$(x,y) \in X \times X,\ x \sqsubseteq y \Longleftrightarrow x(t) \leq y(t) \text{ for all } t \in [0,1].$$

Clearly, $(X, \sqsubseteq)$ is a partially ordered set and (X,d) is a complete b-metric space with $s = 2$. Finally, consider the wt-distance $\rho : X \times X \to \mathbb{R}$ given by $\rho(x,y) = d(x,y)$ for all $x, y \in X$. Thus, we study the following fourth-order two-point boundary value problem

$$\left\{\begin{array}{l} x^{iv}(t) = k(t,x(t)), \quad 0 < t < 1, \\ \\ x(0) = x'(0) = x''(1) = x'''(1) = 0, \end{array}\right. \tag{3.1}$$

with $k \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

It is well-known that the problem (3.1) may be equivalently expressed in integral form: find $x^* \in X$ solution of

$$x(t) = \int_0^1 G(t,\tau)k(\tau, x(\tau))\, d\tau, \quad t \in [0,1], \tag{3.2}$$

where the Green function $G(t,\tau)$ is given by

$$G(t,\tau) = \frac{1}{6}\begin{cases} \tau^2(3t-\tau), & 0 \leq \tau \leq t \leq 1, \\ t^2(3\tau - t), & 0 \leq t \leq \tau \leq 1. \end{cases}$$

Also, it is immediate to show that

$$0 \leq G(t,\tau) \leq \frac{1}{2}t^2\tau \text{ for all } t,\tau \in [0,1]. \tag{3.3}$$

Next, we consider the following hypotheses:

(I) There exists $\alpha_1 : X \to [0, \frac{1}{2})$ such that

$$0 \leq k(t, y(t)) - k(t, x(t)) \leq 4\sqrt{\alpha_1(x)\rho(x,y)} \tag{3.4}$$

for all $x, y \in X$ with $y \sqsubseteq x$ and for all $t \in [0,1]$ and

$$\alpha_1\left(\int_0^1 G(t,\tau)k(\tau, x(\tau))\, d\tau\right) \leq \alpha_1(x) \text{ for all } x \in X. \tag{3.5}$$

(II) There exists $x_0 \in X$ such that

$$x_0(t) \leq \int_0^1 G(t,\tau)k(\tau, x_0(\tau))\, d\tau, \quad t \in [0,1];$$

that is, the integral equation (3.2) admits a lower solution in X.

Now, we prove the existence of at least a solution of (3.1) in X.

Theorem 3.1. *The existence of at least a solution of problem* (3.1) *in X is established, provided that the function $k \in C([0,1]\times\mathbb{R}, \mathbb{R})$ satisfies the hypotheses (I) and (II).*

Proof. The problem in study is equivalent to the fixed point problem obtained by introducing the continuous integral operator $f : X \to X$ given as

$$(fx)(t) = \int_0^1 G(t,\tau)k(\tau, x(\tau))\, d\tau, \quad t \in [0,1] \text{ and } x \in X.$$

Now, we show that the operator f satisfies all the conditions in Theorem 2.1 to conclude that there exists a fixed point of f in X. By using the inequality (3.4) in hypothesis (I), we deduce that f is a nondecreasing mapping with respect to $\sqsubseteq$. Also, by using (3.4), for all $t \in [0,1]$ and for all $x, y \in X$ with $y \sqsubseteq x$, we get

$$\begin{aligned}|(fy)(t)-(fx)(t)| &= \int_0^1 G(t,\tau)[k(\tau,y(\tau))-k(\tau,x(\tau))]\,d\tau \\ &\leq \int_0^1 G(t,\tau)4\sqrt{\alpha_1(x)\rho(x,y)}\,d\tau \\ &\leq \left(\int_0^1 G(t,\tau)\,d\tau\right)4\sqrt{\alpha_1(x)\rho(x,y)} \\ &\leq \sqrt{\alpha_1(x)\rho(x,y)} \quad \text{(from (3.3))}.\end{aligned}$$

Since $\rho(x,y)=d(x,y)$ for all $x,y\in X$, by passing to square and taking the supremum with respect to t, we get

$$\begin{aligned}\rho(fx,fy) &= d(fx,fy) \\ &= \sup_{t\in[0,1]}((fy)(t)-(fx)(t))^2 \\ &\leq \alpha_1(x)\rho(x,y)\end{aligned}$$

for all $x,y\in X$ with $y\sqsubseteq x$. It follows that the conditions (2.2) and (2.3) of Theorem 2.1 hold $\alpha_i(x)=0$ for all $x\in X$ and $i=2,3,4,5$. By hypothesis (II), we get that there exists $x_0\in X$ such that $x_0\sqsubseteq fx_0$. Also, from (3.5) and the fact that the function α_1 assumes values in the interval $[0,\frac{1}{2})$, we have

$$\alpha_1(fx)\leq\alpha_1(x)<\frac{1}{2} \quad \text{for all } x\in X;$$

that is, the condition (2.1) of Theorem 2.1 hold with $\alpha_i(x)=0$ for all $x\in X$ and $i=2,3,4,5$. We conclude that all the conditions of Theorem 2.1 hold and so we deduce the existence of a fixed point of f; that is, there exists a solution of problem (3.1) in X. □

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