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An improvement of the Kurchatov method by means of a parametric modification

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In this work, a uniparametric generalization of the iterative method due to Kurchatov is presented. The iterative model presented is derivative-free and approximates the solution of nonlinear equations when the operator is non-differenciable. As the accessibility of the Kurchatov method is usually a problem in the application of the method, since the set of initial guesses that guarantee the convergence of the method is small, the main objective of this work is to improve the Kurchatov iterative method in its accessibility while maintaining and even increasing its speed of convergence. For this purpose, we introduce a variable parameter in the iterative function of the Kurchatov method that allows us to get a better approximation of the derivative by using a symmetric uniparameteric first-order divided difference operator. We perform a complex dynamic study that corroborate the improvements in the accessibility region. Moreover, a complete analysis of the local and semilocal convergence is established for the new uniparametric iterative method. Finally, we apply the theoretical results to solve a nonlinear integral equation showing the usefulness of the work.

KEYWORDS
divided differences, dynamics, Kurchatov's iterative method, local convergence study, semilocal convergence study

MSC CLASSIFICATION
65H10; 65F10; 45G10

1 | INTRODUCTION

Solving systems of nonlinear equations is probably one of the most used problems in numerical calculations, especially in a wide range of engineering applications. Iterative processes are used to approximate solutions of nonlinear systems of equations when the exact solutions cannot be determined by algebraic methods. These processes build successive approximations to a solution of the system of equations considered. The convergence and performance characteristics of the iterative process can be very sensitive to the starting point considered for the iterative process. When we study the applicability of an iterative method for solving an equation, an important aspect to consider is the set of starting points that can be chosen, so as to ensure that the iterative method converges to a solution of the equation from any point of the set. We refer to this idea as the accessibility of an iterative method. So, an important aspect of the study of iterative processes is their accessibility, that is, the size of the set of starting points that guarantee the convergence of the iterative process. This will be a key piece in the study that we are going to carry out.

We will begin by posing the problem. Throughout this work, we will consider $G : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$ is a nonlinear operator $G(z) \equiv (G_1(z), G_2(z), \dots, G_m(z)), z \in \Omega$, with $G_i : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}$, $i = 1, 2, \dots, m$, and Ω a nonempty open convex domain in $\mathbb{R}^m$.

Remember that one of the most used iterative processes to approximate a solution z^* of the equation

$$G(z) = 0 \tag{1}$$

is the Newton's method, given by the following algorithm:

$$\begin{cases} z_0 \text{ given in } \Omega, \\ z_{n+1} = z_n - [G'(z_n)]^{-1} G(z_n), \;\; n \geq 0. \end{cases} \tag{2}$$

This method has quadratic convergence, low operational cost, and a good accessibility.[1] So, Newton's method is considered an efficient iterative process. But this method needs to evaluate the derivative $G'(x)$ at each iteration. This fact makes it inapplicable in situations when evaluation of the derivative is too costly or when we have to face equations with non-differentiable operators. In these cases, it is common to approximate the derivative G' by means of a divided difference to obtain an iterative process free of derivatives. Thus, one of the derivative-free iterative processes most used for its efficiency is the well-known Kurchatov method[2–4]:

$$\begin{cases} z_{-1}, z_0 \text{ given in } \Omega, \\ z_{n+1} = z_n - [z_{n-1}, 2z_n - z_{n-1}; G]^{-1} G(z_n), \;\; n \geq 0. \end{cases} \tag{3}$$

This iterative process has at least quadratic convergence and its operational cost is similar to the Newton's method one, therefore is an efficient iterative process. Notice that this iterative process use a first-order divided difference.[5,6] Remember that, if we denote by $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ the space of bounded linear operators from $\mathbb{R}^m$ to $\mathbb{R}^m$, an operator $[x, y; D] \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ is called a first-order divided difference for the operator $D : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ on the points x and y $(x \neq y)$ if

$$[x, y; D]\,(x - y) = D(x) - D(y). \tag{4}$$

Remember that, if Ω is an open convex domain of $\mathbb{R}^m$, there exists a divided difference of first order of G (see Balzs and Goldner[7]).

So, the Kurchatov's method use the approximation

$$G'(z_n) \sim [z_n - (z_n - z_{n-1}), z_n + (z_n - z_{n-1}); G]\,,$$

obviously, the Kurchatov's method use a symmetric first-order divided difference and then gives a good approximation. In fact, as we have indicated, this method preserves the quadratic convergence as the Newton, unlike other iterative processes that do not use them, such as the Secant method,[5] which loses this quadratic convergence. However, this method is still very sensitive about the initial guess. As we will see in the next section, the accessibility of the Kurchatov method is low.

Based on the above, we use a technique that consists of introducing a parameter in the first-order divided difference that uses the iterative process. This will allow us to obtain a better approximation of the derivative of the operator. Then, we will obtain an iterative process closer to Newton's method, and thus, we will increase its accessibility, approaching it to that of Newton's method. Thus, we will consider the following uniparametric family of iterative processes:

$$\begin{cases} z_0, z_{-1} \text{ given in } \Omega, \; \mu \in [0,1], \\ x_n = (1-\mu) z_n + \mu z_{n-1}, \\ y_n = (1+\mu) z_n - \mu z_{n-1}, \\ z_{n+1} = z_n - [x_n, y_n; G]^{-1} G(z_n), \;\; n \geq 0. \end{cases} \tag{5}$$

Note that this uniparametric family of iterative processes can be considered as a combination of the Kurchatov method $(\mu = 1)$ and, for differentiable case, Newton's method $(\mu = 0)$.

Our main goals in this work are the following. Firstly, the first-order divided difference considered to define this uniparametric family (5) will allow us, by varying the value of μ, to improve the approximation to $G'(x)$ provided by

Kurchatov's method. Secondly, the uniparametric family (5) maintains the quadratic convergence as Kurchatov method. And, thirdly, we improve the accessibility of the Kurchatov method considering values close to $\mu = 0$, which corresponds to the case of Newton's method. This study will be carried out experimentally, analyzing the dynamic behavior of the modification of the Kurchatov method introduced in (5), checking that its dynamics improves the dynamic behavior of the Kurchatov method (3).

Furthermore, another main objective of this work is to carry out an exhaustive study of the convergence of the uniparametric family of iterative processes (5), local and semilocal, for both differentiable and non-differentiable operators.

The paper is organized as follows. First, we introduce and motivate the main objective of this work, in Section 1. Then, in Section 2, we give some preliminaries that are the key of next parts. Section 3 is due to the dynamical study of the new iterative method. To continue, we perform the local and semilocal convergence study in Section 4. Finally, Section 5 is devoted to some numerical experiments. To conclude, we have the final remarks in Section 6.

2 | PRELIMINARIES

We begin analyzing the usefulness of the μ parameter, introduced in Kurchatov's method. Our first objective is to be able to obtain a better approximation of the derivative of G by varying this parameter. Thus, we are going to study the approximation $[x_n, y_n; G] \sim G'(z_n)$ in a simple case, such as $G(z) = arctg(z)$. As can be seen in Table 1, the approximation provided by the Kurchatov method ($\mu = 1$) is notably improved by reducing the value of the parameter μ. That is, as expected, as we approach Newton's method ($\mu = 0$), we obtain a better approximation of the derivative of G.

To continue, we study the local order of convergence for the uniparametric family of iterative processes (5). To do this, a very commonly measure of speed of convergence in $\mathbb{R}^m$ is the R-order of convergence, which is defined as follows:

Let $\{z_n\}$ a sequence of points of $\mathbb{R}^m$ converging to a point $z^* \in \mathbb{R}^m$ and let $\rho \geq 1$ and

$$e_n(\rho) = \begin{cases} n & \text{if } \rho = 1, \\ \rho^n & \text{if } \rho > 1, \end{cases} \quad n \geq 0.$$

- We say that ρ is an R-order of convergence of the sequence $\{z_n\}$ if there are two constants $b \in (0, 1)$ and $B \in (0, +\infty)$ such that
$$\|x_n - x^*\| \leq B b^{e_n(\rho)}.$$
- We say that ρ is the exact R-order of convergence of the sequence $\{x_n\}$ if there are four constants $a, b \in (0, 1)$ and $A, B \in (0, +\infty)$ such that
$$A a^{e_n(\rho)} \leq \|x_n - x^*\| \leq B b^{e_n(\rho)}, \quad n \geq 0.$$

In general, check double inequalities of (b) is complicated, so that normally only seek upper inequalities as (a). Therefore, if we find an R-order of convergence ρ of sequence $\{x_n\}$, we then say that sequence $\{x_n\}$ has order of convergence at least ρ.

To establish the local R-order of convergence of the sequence $\{z_n\}$ given in (5), we first obtain a technical result. For this, we introduce a development of the first-order divided difference of a function of several variables that is given in Ezquerro et al,[8] following the ideas presented in Grau-Sánchez et al[9] and Grau-Sánchez and Noguera.[10]

z_n	$\mu = 2$	Kurchatov	$\mu = 0.5$	$\mu = 0.1$	$\mu = 0.001$
0	0.0130222 …	0.00331348 …	0.000832086 …	0.0000333313 …	$3.33333 \ldots \times 10^{-7}$
0.1	0.1228541 …	0.00312121 …	0.000783491 …	0.0000313807 …	$3.13824 \ldots \times 10^{-7}$
0.2	0.0102725 …	0.00259775 …	0.000651306 …	0.0000260762 …	$2.60772 \ldots \times 10^{-7}$
0.3	0.0074838 …	0.00187711 …	0.000469631 …	0.0000187896 …	$1.87898 \ldots \times 10^{-7}$
0.4	0.0045091 …	0.00111489 …	0.000277898 …	0.0000111052 …	$1.11047 \ldots \times 10^{-7}$
0.5	0.0018271 …	0.00043438 …	0.000107152 …	$4.26744 \ldots \times 10^{-6}$	$4.26668 \ldots \times 10^{-8}$
0.6	0.0002922 …	0.00009765 …	0.000025979 …	$1.05927 \ldots \times 10^{-6}$	$1.06011 \ldots \times 10^{-8}$
0.7	0.0017777 …	0.00046627 …	0.000117942 …	$4.73533 \ldots \times 10^{-6}$	$4.73606 \ldots \times 10^{-8}$
0.8	0.0026905 …	0.00068958 …	0.000173457 …	$6.95184 \ldots \times 10^{-6}$	$6.95241 \ldots \times 10^{-8}$
0.9	0.0031521 …	0.00079991 …	0.000200718 …	$8.03818 \ldots \times 10^{-6}$	$8.03857 \ldots \times 10^{-8}$
1	0.0032927 …	0.00083082 …	0.000208177 …	$8.33308 \ldots \times 10^{-6}$	$8.33333 \ldots \times 10^{-8}$

TABLE 1 $\|G'(z_n) - [x_n, y_n; G]\|$ obtained for different μ values

In the following, we denote by $\mathcal{L}_k(\mathbb{R}^m,\mathbb{R}^m)$ the space of bounded k-linear operators in $\mathbb{R}^m$ and $\|f(u)\| = o(\|g(u)\|^s)$ if $\lim_{g(u)\to 0}\frac{\|f(u)\|}{\|g(u)\|^s} = 0$. So, we denote $\theta_k(z^*,E)$ when $\|\theta_k(z^*,E)\| = o(\|E\|^k)$ and $\theta_{j,k}(z^*,E,\hat{E})$ when $\|\theta_{j,k}(z^*,E,\tilde{E})\| = o(\|E\|^j\|\|\hat{E}\|^k)$, $j,k\in\mathbb{N}$.

Lemma 1. *If $G\in C^{(4}(\Omega)$, with $G'(z^*)$ is continuous and nonsingular, and $A_k = \frac{1}{k!}[G'(z^*)]^{-1}G^{(k}(z^*)\in\mathcal{L}_k(\mathbb{R}^m,\mathbb{R}^m)$, with z^* a solution of (1). Then,*

$$[x_n, y_n; G] = G'(z^*)\left(I + 2A_2e_n - 2\mu A_3e_ne_{n-1} + (3+\mu^2)A_3e_n^2 + \mu^3A_3e_{n-1}^2 + \theta_2(z^*, e_{n-1})\right),$$

where $e_k = z_k - z^*$.

Proof. As G is differentiable, we have that $[-,-;G] : \Omega\times\Omega\to\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)$ and

$$[x,y;G](x-y) = G(x) - G(y) = \int_y^x G'(z)\,dz = \int_0^1 G'(y+t(x-y))\,dt\,(x-y),$$

for all $x,y\in\Omega$. So, as we assume that the operator G is differentiable, we use then the divided difference given by

$$[x,y;G] = \int_0^1 G'(y+t(x-y))\,dt.$$

As G is differentiable in Ω, at least until the fourth derivative, we obtain:

$$[x,y;G] = \int_0^1 G'(y+t(x-y))\,dt = \int_0^1\left(\sum_{j=1}^4\frac{1}{(j-1)!}G^{(j)}(y)(t(x-y))^{j-1} + \theta(x,t(x-y))\right)dt$$
$$= \sum_{j=1}^4\frac{1}{j!}G^{(j)}(y)(x-y)^{j-1} + \int_0^1\theta(y,t(x-y))\,dt, \qquad (6)$$

where $G^{(j)}\in\mathcal{L}_j(\mathbb{R}^m,\mathbb{R}^m)$ and $\theta : \Omega\times\Omega\to\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)$ with $\|\theta(x,t(x-y))\| = o\left(\|t(x-y)\|^3\right)$.

Now, by Taylor's development, the successive derivatives of $G(y)$ in a neighborhood of z^* take the form:

$$G'(y) = G'(z^*) + G''(z^*)(y-z^*) + \frac{1}{2!}G'''(z^*)(y-z^*)^2 + \frac{1}{3!}G^{(iv}(z^*)(y-z^*)^3 + o\left(\|(y-z^*)\|^3\right),$$
$$G''(y) = G''(z^*) + G'''(z^*)(y-z^*) + \frac{1}{2!}G^{(iv}(z^*)(y-z^*)^2 + o\left(\|(y-z^*)\|^2\right),$$
$$G'''(y) = G'''(z^*) + G^{(iv}(z^*)(y-z^*) + o(\|(y-z^*)\|),$$
$$G^{(iv}(y) = G^{(iv}(z^*) + O(\|(y-z^*)\|),$$

where $O(\|(y-z^*)\|)$ is such that $\lim_{(y-z^*)\to 0}\frac{\|O(\|(y-z^*)\|)\|}{\|(y-z^*)\|}$ exists and is a nonnegative real number.

Substituting these derivatives in (6) and denoting $E = y-z^*$ and $\hat{E} = x-z^*$, as $G'(z^*)$ is nonsingular, it follows that

$$[x,y;G] = G'(z^*)\left(I + \sum_{k=1}^3\left[A_{k+1}\sum_{i=0}^k E^{k-i}\hat{E}^i\right] + [G'(z^*)]^{-1}\Theta(y,E,\hat{E}-E)\right), \qquad (7)$$

where I is the identity operator in $\mathbb{R}^m$. Moreover,

$$\Theta(y,E,\hat{E}-E) = \sum_{j=0}^3\theta_j(z^*,E)(\hat{E}-E)^{3-j} + \int_0^1\theta(y,t(\hat{E}-e))\,dt$$

where $\Theta : \Omega \times \Omega \times \Omega \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ with $\|[G'(z^*)]^{-1}\Theta(y, E, \hat{E} - E)\| = o(\|e\|^p \|\tilde{e}\|^q)$, for $p + q = 3$ and $p, q = 0, 1, 2, 3$. On the other hand, we denote $\theta_k(z^*, E)$ when $\|\theta_k(z^*, E)\| = o(\|E\|^k)$.

Now, taking $y = y_n$ and $x = z_n$ in (7) and taking into account that (5), we have $x_n - z^* = (1 - \mu)e_n + \mu e_{n-1}$ and $y_n - z^* = (1 + \mu)e_n - \mu e_{n-1}$. Then, substituting these values in (7), the Lemma is proved. □

Next, we obtain a result for the R-order of convergence for (5).

Theorem 1. *Under conditions of Lemma 1, and the initial approximations z_{-1}, z_0 are chosen sufficiently close to z^*, then the sequence $\{z_n\}$, given by (5), converges to z^* with at least R-order of convergence 2. Moreover, the error equation is*

$$e_{n+1} = A_2 e_n^2 + \mu^2 A_3 e_{n-1}^2 e_n + \theta_{2,1}(z^*, e_{n-1}, e_n). \tag{8}$$

Proof. From the Lemma 1, taking into account that $[x_n, y_n; G]^{-1}[x_n, y_n; G] = I$, by expanding in formal power series of e_{n-1} and e_n, we then obtain

$$[x_n, y_n; G]^{-1} = \left(I - 2A_2 e_n - \mu^2 A_3 e_{n-1}^2) + \theta_2(z^*, e_{n-1})\right) [G'(\alpha)]^{-1}.$$

Now, from (5), by subtracting the root z^* to both sides of the equality that defines z_{n+1}, we deduce

$$\begin{aligned} e_{n+1} &= e_n - \left(I - 2A_2 e_n - \mu^2 A_3 e_{n-1}^2 + \theta_2(\alpha, e_{n-1})\right) [G'(z^*)]^{-1} G'(z^*) \\ &\quad \times \left(e_n + A_2 e_n^2 + w_2(z^*, e_n)\right), \end{aligned}$$

and, therefore, we obtain (8). Moreover, we get

$$\|e_{n+1}\| \leq \|A_2\| \|e_n\|^2 + \mu^2 \|A_3\| \|e_{n-1}\|^2 \|e_n\|.$$

Next, we suppose that the sequence $\{z_n\}$, given in (5), has R-order of convergence lower 2, therefore exists C and $N > 0$, that for all $n \geqslant N$, the following inequality holds $\|e_n\| \geqslant C\|e_{n-1}\|^2$. Since

$$\|e_{n+1}\| \leq \|A_2\| \|e_n\|^2 + \frac{\mu^2}{C} \|A_3\| \|e_n\|^2 = \left(\|A_2\| + \frac{\mu^2}{C} \|A_3\|\right) \|e_n\|^2. \tag{9}$$

But inequality (9) means that the order of convergence is not lower 2. Thus, the R-order of convergence of sequence $\{z_n\}$ is at least 2. □

Obviously, from the error Equation (8), we see that for lower values of the parameter μ, this error is reduced.

3 | DYNAMICAL STUDY

Another objective raised with the introduction of the μ parameter is to improve the accessibility of the Kurchatov method. We will see that the uniparametric family of iterative processes (5) has a dynamical behavior similar to that of Newton method in the differentiable case. Besides, we compare the dynamics of the Kurchatov method with the dynamics of the uniparametric family of iterative processes (5), obtaining that, both in the case of differentiable and non-differentiable operators, the dynamics of the Kurchatov's method (3) is improved by reducing the value of the parameter μ in (5).

To analyze the dynamics of an iterative process we use complex dynamics.[11,12] For this purpose, we will consider z_0 as a complex number which will be chosen on the axis where $\Re e(z_0)$ will be consider in the horizontal axis and $\Im m(z_0)$ in the vertical axis, so each point of the plane will be a pair of values $(\Re e(z_0), \Im m(z_0))$. On the other hand, it is known[2] that to ensure the convergence of an iterative process with memory, it is favorable to take the initial points, z_{-1} and z_0, close. Therefore, we will consider with $z_{-1} = z_0 - 0.1$ and will see how is the dynamical behavior.

The painting algorithm in the complex dynamics is as follows: Each fixed point of the plane is considered as the starting point as $z_0 = x_0 + y_0 i$ and $z_{-1} = z_0 - 0.1$ (except in Newton's method in which we only fix z_0) and the iterative method starting in the fixed point; if the sequence generated after 10 iterations does not converge to any of the roots with a tolerance of $tol = 10^{-15}$, the point is painted in black, but if the sequence converges to any of the roots, it is painted in any of the colors associated to the roots.

Along this section, we will consider two different kind of equations, one differentiable and one non-differentiable.

3.1 | The differentiable case

In this section, we consider the complex differentiable function

$$H_1(z) = z^3 - 2z^2 - z,$$

whose roots are $z^* = 1 - \sqrt{2}, z^{**} = 0$ and $z^{***} = 1 + \sqrt{2}$. So, the convergence to z^* is colored in blue, and the convergence to z^{**} is colored in red and the convergence to z^{***} in yellow.

We begin by showing the complex dynamics of Newton's and Kurchatov's method when it is applied to H_1, as it appears in Figure 1. Moreover, we present the complex dynamics behavior of the uniparametric family (5) where it can be seen in Figure 2.

Although graphically there do not appear to be excessive differences, we want to see its behavior in a numerical way, and, for that purpose, we compute the percentage of points which converges to any of the roots. We get this information in Table 2, in which it can be seen that the accessibility of the family of iterative processes (5) is similar as the Newton's method and better than the one of Kurchatov's method. In any case, we see that by reducing the parameter, we improve the accessibility of the family of iterative processes (5), and we even improve the accessibility of Newton's method.

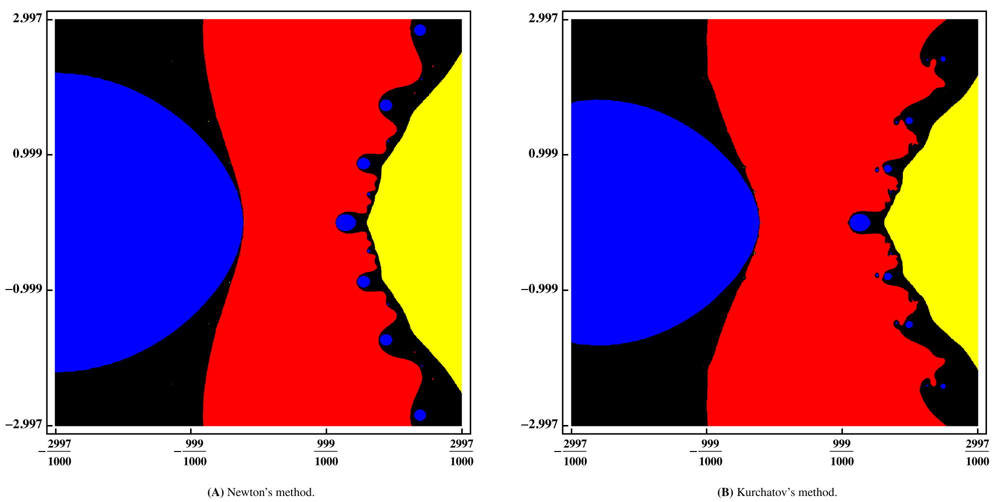


FIGURE 1 Basins of attraction to polynomial $H_1(z) = z^3 - 2z^2 - z$. (A) Newton's method; (B) Kurchatov's method [Colour figure can be viewed at wileyonlinelibrary.com]

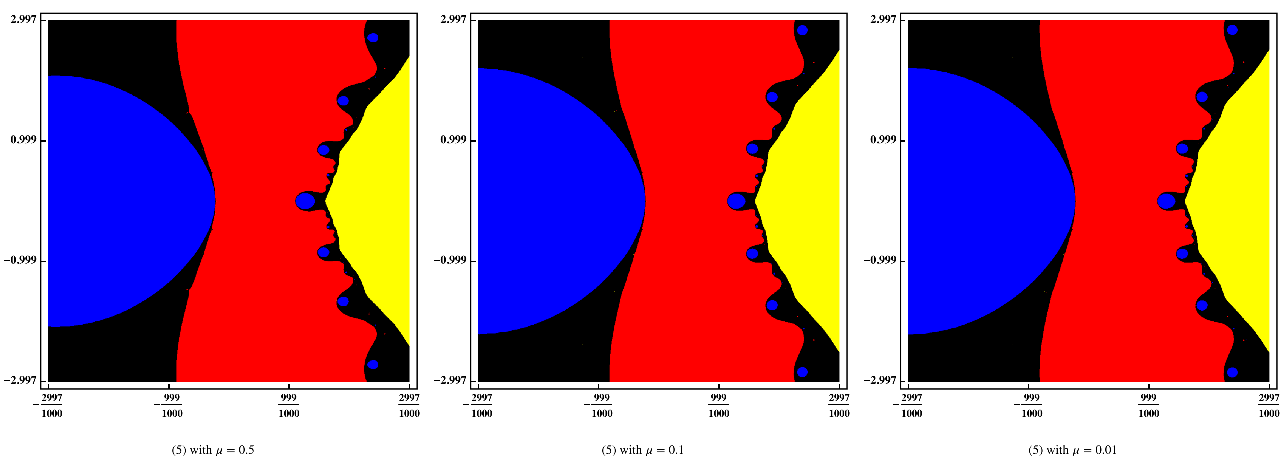


FIGURE 2 Basins of attraction to polynomial $H_1(z) = z^3 - 2z^2 - z$ of iterative processes (5) [Colour figure can be viewed at wileyonlinelibrary.com]

3.2 | Non-differentiable case

In this case, in the numerical study, we compare the accessibilities of the iterative methods of the uniparametric family (5) and the Kurchatov method when they are applied to solve the equation $H_2(z) = z^3 - z|z| - 2z = 0$, whose roots are the real numbers $z^* = -1$, $z^{**} = 0$, and $z^{***} = 1$. So, the convergence to $z^* = -1$ is colored in blue, the convergence to $z^{**} = 0$ in red, and the convergence to $z^{***} = 1$ in yellow. We will prove that by varying the parameter μ, with family (5), we improve the accessibility of Kurchatov method when it is applied to a non differentiable case.

We begin by showing the dynamics of Kurchatov's method as it appears in Figure 3. The dynamics behavior families of iterative processes (5) are shown in Figure 4. We get the information of the percentage of convergent points in Table 3, in which it can be seen that the accessibility of the family of iterative processes (5) is better than the Kurchatov one.

Method	Percentage of convergence points
Newton	76.23%
Kurchatov	70.23%
(5) with $\mu = 0.5$	78.24%
(5) with $\mu = 0.1$	79.01%
(5) with $\mu = 0.01$	79.04%

TABLE 2 Percentage of convergence points for $H_1(z) = z^3 - 2z^2 - z$

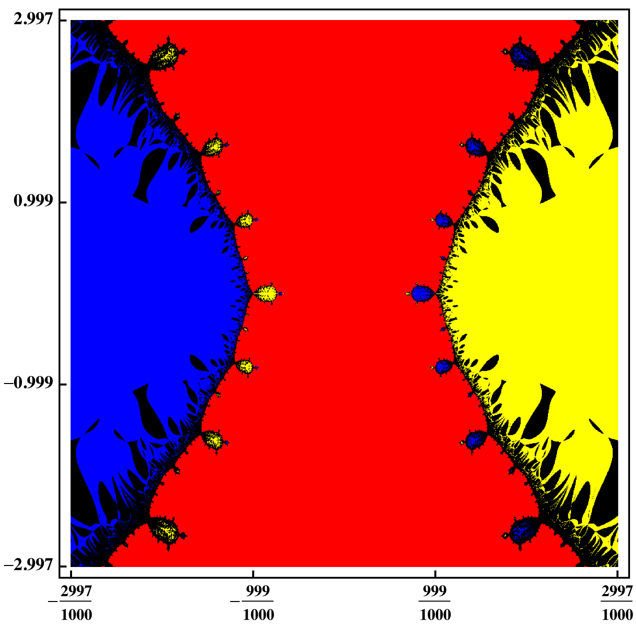
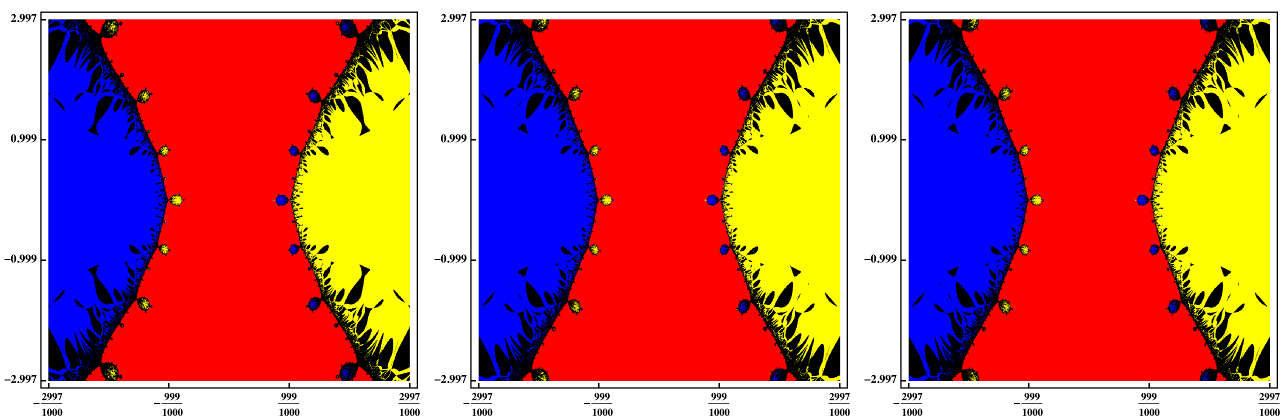


FIGURE 3 Basins of attraction to $H_2(z) = z^3 - z|z| - 5z$ for the Kurchatov's method [Colour figure can be viewed at wileyonlinelibrary.com]



(5) with $\mu = 0.5$ (5) with $\mu = 0.1$ (5) with $\mu = 0.01$

FIGURE 4 Basins of attraction to $H_2(z) = z^3 - z|z| - 5z$ for the iterative processes (5) [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 3 Percentage of convergence points for $H_2(z) = z^3 - z|z| - 5z$

Method	Percentage of convergence points
Kurchatov	86.77%
(5) with $\mu = 0.5$	87.43%
(5) with $\mu = 0.1$	88.03%
(5) with $\mu = 0.01$	88.05%

4 LOCAL CONVERGENCE

In this section, we analyze the local convergence of the uniparametric family of iterative processes (5). For this, two types of conditions are imposed to guarantee the local convergence: conditions on a solution z^* of the Equation (1) and conditions on the operator G. In this case, as the algorithm (5) uses first-order divided differences $[x, y; G]$, then it will be necessary to give conditions on them.

Regarding the condition on the solution z^*, there are many known results of local convergence (see Rena and Argyros and Shakhno[13,14]) which usually include the condition of the existence of the operator $[z^*, z^*; G]^{-1} = [G'(z^*)]^{-1}$, forcing the operator G to be differentiable. However, in this paper, we obtain a result for the local convergence from requiring a weaker type of assumptions to obtain a local convergence result when operator G is non-differentiable. So, as this condition is necessary to prove the existence of operators $[x, y; G]^{-1}$ at each step of the algorithm, we consider an auxiliary point $\tilde{z}$ in Ω, and the existence of operator $[z^*, \tilde{z}; G]^{-1}$ is the condition considered.

Regarding the condition on the first-order divided differences, remember that there exists a divided difference of first-order of G. As it appears in Argyros,[5] a possible required condition is that $[x, y; G]$ is Lipschitz-continuous. In some situations, this condition is generalized to Hölder-continuous condition,[15] but in the above cases, it is known[5] that the Fréchet derivative of G exists in Ω and satisfies $[x, x; G] = G'(x)$. Therefore, these conditions cannot be verified if the operator G is non-differentiable. So, when the operator G is non-differentiable, the following condition[16] is usually used

$$\|[x, y; G] - [u, v; G]\| \leq \omega(\|x - u\|, \|y - v\|), \tag{10}$$

for all $x, y, u, v \in \Omega$, with $x \neq y$ and $u \neq v$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function in both components. Obviously, this condition (10) generalizes the Lipschitz and Hölder cases, but it is shown in Hernández and Rubio[17] that if $\omega(0, 0) = 0$, then G is differentiable, so in non-differentiable situations, we will obtain that $\omega(0, 0) > 0$.

From the above comments, we first assume the following conditions:

($L1$) There exist $z^* \in \Omega$ a solution of (1), two positive real numbers d, b and $\tilde{z} \in \Omega$, with $\|\tilde{z} - z^*\| = d$, such that there exists $[z^*, \tilde{z}; G]^{-1} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ with $\|[z^*, \tilde{z}; G]^{-1}\| \leq b$.

($L2$) The first-order difference operator $[, ; G]$ satisfies (10).

To study the local convergence of family of iterative processes (5), we will use the technique related to the construction of recurrence relations for the sequence $\{z_n\}$ (see Hernández and Rubio[18]). For this purpose, it is necessary to study two or three steps of the iterative process and intuit the recurrence relations that the sequence $\{z_n\}$ satisfies, so that we can prove its convergence in the space $\mathbb{R}^m$.

Firstly, the local study of the convergence is based on providing the so-called ball of convergence of iterative process that shows the accessibility to z^* from the initial approximations $z_{-1}, z_0 \in B(z^*, R)$, with $z_{-1} \neq z_0$. Obviously, we suppose that $z_{-1} \neq z^*$ and $z_0 \neq z^*$; in other case, the solution is obtained. We denote the ball of convergence as $B(z^*, R)$ and then we have to locate $R > 0$ such that $z_n \in B(z^*, R)$, for $n \geqslant 1$, and $B(z^*, R) \subseteq \Omega$.

Secondly, for the first step of iterative processes (5), we must prove the existence of the operator $[x_0, y_0; G]^{-1}$. Since we will consider G any operator, differentiable or non-differentiable, we will eliminate from our study the case $\mu = 0$, corresponding to Newton's method. Thus, from now on, we will consider $\mu \in (0, 1]$. So, as $z_{-1} \neq z_0$, then $x_0 \neq y_0$ and $[x_0, y_0; G]$ is well-defined.

Notice that, from the condition (10), it is verified that

$$\|[x, y; G] - [z^*, \tilde{z}; G])\| \leq \omega_0(\|x - z^*\|, \|y - \tilde{z}\|) \tag{11}$$

holds for each pair of different points $(x, y) \in \Omega \times \Omega$, where $\omega_0 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function in its two variables.

Moreover,

$$\|x_0 - z^*\| \le (1-\mu)\|z_0 - z^*\| + \mu\|z_{-1} - z^*\| < R,$$

so that $x_0 \in B(z^*, R)$. Now, if we denote $\|z_0 - z^*\| = a$, we have

$$\|y_0 - z^*\| \le (1+\mu)\|z_0 - z^*\| + \mu\|z_{-1} - z^*\| \le (1+\mu)a + \mu\|z_{-1} - z^*\|.$$

Then, if we consider z_{-1} such that $\|z_{-1} - z^*\| \le \frac{R-(1+\mu)a}{\mu}$, under condition $R > (1+\mu)a$, we obtain that $y_0 \in B(z^*, R)$. Therefore, we get

$$\|I - [z^*, \tilde{z}; G]^{-1}[x_0, y_0; G]\| \le \|[z^*, \tilde{z}; G]^{-1}\|\|[z^*, \tilde{z}; G] - [x_0, y_0; G]\| \tag{12}$$

$$\le b\omega_0(\|z^* - x_0\|, \|\tilde{z} - y_0)\| \le b\omega_0(R, \|\tilde{z} - z^*\| + \|z^* - y_0\|) < b\omega_0(R, d+R)$$

$$< b\omega_0(R, d+R),$$

and, if we assume that $b\omega_0(R, d+R) < 1$, by the Banach Lemma for invertible operators,[3] we obtain that $[x_0, y_0; G]^{-1}$ exists with

$$\|[x_0, y_0; G]^{-1}\| \le \frac{b}{1 - b\omega_0(R, d+R)}.$$

On the other hand, from (5), it follows

$$\begin{aligned} z_1 - z^* &= z_0 - [x_0, y_0; G]^{-1}G(z_0) - z^* \\ &= [x_0, y_0; G]^{-1}([x_0, y_0; G]\,(z_0 - z^*) - G(z_0) + G(z^*)) \\ &= [x_0, y_0; G]^{-1}([x_0, y_0; G] - \left[z_0, z^*; G\right])(z_0 - z^*). \end{aligned}$$

Then, by applying conditions (10) and (11), we obtain

$$\|z_1 - z^*\| \le \|[x_0, y_0; G]^{-1}\|\omega(\|x_0 - z_0\|, \|y_0 - z^*\|)\|z_0 - w^*\| \tag{13}$$

$$\le \frac{b\omega(\mu\|z_0 - z_{-1}\|, (1+\mu)\|z_0 - z^*\| + \mu\|z_{-1} - z^*\|)\|z_0 - z^*\|}{1 - b\omega_0(R, d+R)} < M(R)\|z_0 - z^*\|,$$

where $M(R) = \frac{b\omega(2\mu R, (1+2\mu)R)}{1 - b\omega_0(R, d+R)}$. So, if $b(\omega(2\mu R, (1+2\mu)R) + \omega_0(R, d+R)) < 1$ then $M(R) < 1$ and therefore $\|z_1 - z^*\| < \|z_0 - z^*\|$.

Moreover, we consider $z_1 \neq z_0$, in other case $z_0 = z_1 = z^*$ and then the iterative processes (5) converge.

Thirdly, for the second step of iterative processes (5), we get

$$\|x_1 - z^*\| \le (1-\mu)\|z_1 - z^*\| + \mu\|z_0 - z^*\| < R,$$

$$\begin{aligned} \|y_1 - z^*\| &\le (1+\mu)\|z_1 - z^*\| + \mu\|z_0 - z^*\| \\ &< ((1+\mu)M(R) + \mu)\|z_0 - z^*\| \\ &= ((1+\mu)M(R) + \mu)a = R. \end{aligned}$$

Then, we consider the scalar equation

$$((1+\mu)M(t) + \mu)a = t. \tag{14}$$

So, if this scalar equation has at least a positive real root and denote by R the smallest positive real root of this equation, then $x_1, y_1 \in B(z^*, R)$. Next, it follows that $x_1 \neq y_1$ because $z_1 \neq z_0$, then $[x_1, y_1; G]$ is well-defined. Now, proceeding as in (12), by the Banach Lemma for invertible operators, we obtain that $[x_1, y_1; G]^{-1}$ exists, and it is verified

$$\|[x_1, y_1; G]^{-1}\| \leq \frac{b}{1 - b\omega_0(R, d + R)}.$$

Now, as in (13), it follows that $\|z_2 - z^*\| < M(R)\|z_1 - z^*\| < M(R)^2\|z_0 - z^*\| < \|z_0 - z^*\|$.

In view of the previous reasoning, to ensure the convergence of the iterative processes given in (5), it is necessary to include the following hypotheses:

($L3$) The scalar equation

$$ab(1 + \mu)\omega(2\mu t, (1 + 2\mu))t) + (\mu a - t)(1 - b\omega_0(t, d + t)) = 0, \tag{15}$$

has at least one positive root and we denote by R the smallest positive root.

($L4$) $B(z^*, R) \subseteq \Omega$, with $R > (1 + \mu)a$ and $b(\omega(2\mu R, (1 + 2\mu)R) + \omega_0(R, d + R)) < 1$.

To continue, from the previous development, applying a mathematical induction procedure, it is easy to prove the following recurrence relations for the sequence $\{z_n\}$ given by (5).

Theorem 2. *Suppose that the conditions (L1)–(L4) hold. If* $z_{-1}, z_0 \in B(z^*, R) \subseteq \Omega$, *with* $z_{-1} \neq z_0$, *and*

$$\|z_{-1} - z^*\| < \frac{R - (1 + \mu)a}{\mu}, \tag{16}$$

then, for $n \geq 1$, *it follows:*

(i) $x_n, y_n \in B(z^*, R) \subseteq \Omega$ *with* $x_n \neq y_n$,

(ii) There exists $[x_n, y_n; G]^{-1}$, *and it is verified:* $\|[x_n, y_n; G]^{-1}\| \leq \frac{b}{1 - b\omega_0(R, d+R)}$,

(iii) $\|z_n - z^*\| < M(R)\|z_{n-1} - z^*\| < M(R)^n\|z_0 - z^*\| < \|z_0 - z^*\|$, *and* $z_n \in B(z^*, R)$.

Next, from the recurrence relations established in the previous result, we are now in a position to obtain a local convergence result for the sequence $\{z_n\}$ given in (5).

Theorem 3. *Under conditions of the previous Theorem, then the sequence* $\{z_n\}$ *given in (5), starting at* $z_{-1}, z_0 \in B(z^*, R)$, *remains in* $B(z^*, R)$ *and converges to* z^*, *a solution of Equation (1).*

Moreover, if we suppose that there exists $\tilde{R} \geq R$ *such that* $b\omega_0(0, d + \tilde{R}) < 1$. *Then,* z^* *is the unique solution of the Equation (1) in* $B(z^*, \tilde{R}) \cap \Omega$.

Proof. From the previous theorem, obviously the sequence $\{z_n\}$, given in (5), is well-defined and remains in $B(z^*, R)$. Finally, it follows from the item (iii) of Theorem 2 that $\{z_n\}$ converges to z^*.

To prove uniqueness, suppose that $w^* \in B(z^*, \tilde{R}) \cap \Omega$ be such that $G(w^*) = 0$. Then, from (11), we obtain that $\|I - [z^*, \tilde{z}; G]^{-1}[z^*, w^*; G]\| \leq \left\|[z^*, \tilde{z}; G]^{-1}\right\| \|[z^*, \tilde{z}; G] - [z^*, w^*; G]\| \leq b\omega_0(0, d + \tilde{R}) < 1$. Therefore, there exists $[z^*, w^*; G]^{-1}$, and, as $0 = G(z^*) - G(w^*) = [z^*, w^*; G](z^* - w^*)$, we deduce that $z^* = w^*$. □

5 | SEMILOCAL CONVERGENCE

In this section, we consider G a continuous operator in Ω. As we have indicated previously, there exists a first-order divided difference for G in Ω. Then, we establish the semilocal convergence of the sequence $\{z_n\}$, given in (5), for $\mu \in (0, 1]$, under the following assumptions:

(SL1) Let $z_0, z_{-1} \in \Omega$, with $\|z_0 - z_{-1}\| = \alpha > 0$, such that $x_0, y_0 \in \Omega$.

(SL2) $B_0^{-1} = [x_0, y_0; G]^{-1}$ exists such that $\|B_0^{-1}\| \leq \beta$, $\|G(z_0)\| \leq \delta$ and $\|z_1 - z_0\| \leq \eta$.

(SL3) $\|[x, y; G] - [u, v; G]\| \leq \omega(\|x - u\|, \|y - v\|)$, $x, y, u, v \in \Omega$, where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function in two arguments.

(SL4) Consider the scalar equation:

$$t(1 - T(t)) - \eta = 0, \tag{17}$$

where

$$T(t) = \frac{\tilde{T}}{1 - \beta\omega\left(t + \mu\alpha, (1+2\mu)t + \mu\alpha\right)},$$

being $\tilde{T} = \beta \max\{\omega\left(\eta + \mu\alpha, \mu\alpha\right), \omega\left((1+\mu)\eta, \mu\eta\right)\}$. Suppose that the scalar Equation (17) has at least a positive real root, and we denote by R the smallest positive real root of this equation.

(SL5) $B(z_0, R + \mu\eta) \subseteq \Omega$ and $\beta\omega\left(R + \mu\alpha, (1+2\mu)R + \mu\alpha\right) < 1$.

Note that, as $\alpha > 0$, we have $z_{-1} \neq z_0$, and therefore, $x_0 \neq y_0$ with $x_0, y_0 \in \Omega$. So, $[x_0, y_0; G]$ is well-defined.
We will start with a technical lemma that we will use later.

Lemma 2. *Let $\{z_n\}$ be the sequence generated by (5). If $z_{n-2}, z_{n-1}, z_n \in \Omega$, with $z_n \neq z_{n-1}$, and $z_{n-1} \neq z_{n-2}$, then*

$$G(z_n) = \left([z_n, z_{n-1}; G] - [x_{n-1}, y_{n-1}; G]\right)(z_n - z_{n-1}).$$

Proof. Notice that, if $z_{n-1} = z_{n-2}$ or $z_n = z_{n-1}$, then $z_{n-1} = z_{n-2} = z^*$ or $z_n = z_{n-1} = z^*$, with which the sequence $\{z_n\}$ would already be convergent.

Now, taking into account that

$$G(z_n) = G(z_{n-1}) + G(z_n) - G(z_{n-1}) = \left(-[x_{n-1}, y_{n-1}; G](z_n - z_{n-1}) + [z_n, z_{n-1}; G]\right)(z_n - z_{n-1}),$$

the result is obtained. □

Next, we establish the recurrence relations that the sequence $\{z_n\}$ verifies, which will later allow us to verify its convergence. We will assume that $z_n \neq z_{n-1}$, for all $n \geqslant 1$; otherwise, we have that the sequence $\{z_n\}$ is convergent since if $z_k = z_{k+1}$ then $z_k = z_{k-1} = z^*$.

Lemma 3. *Suppose that the conditions (SL1)–(SL5) hold. Then, for $n \geq 1$, it follows:*

(i_n) $x_n, y_n \in B(z_0, R + \mu\eta)$, *with* $x_n \neq y_n$, *and there exists* B_n^{-1} *such that*

$$\|B_n^{-1}\| \leq \frac{\beta}{1 - \beta\omega\left(R + \mu\alpha, (1+2\mu)R + \mu\alpha\right)},$$

(ii_n) $\|G(z_n)\| \leq \tilde{T}\|z_n - z_{n-1}\|$

(iii_n) $\|z_{n+1} - z_n\| \leq T(R)\|z_n - z_{n-1}\| < T(R)^n\|z_1 - z_0\| < \eta$,

(iv_n) $\|z_{n+1} - z_0\| < \frac{1}{1 - T(R)}\eta = R$.

Proof. Using the condition (II), z_1 is well-defined and

$$\|z_1 - z_0\| \leq \|B_0^{-1}\|\|G(z_0)\| \leq \beta\delta = \eta < R.$$

As $z_1 \in B(z_0, R) \subseteq B(z_0, R + \mu\eta) \subseteq \Omega$, we have

$$\|x_1 - z_0\| = \|(1-\mu)z_1 + \mu z_0 - z_0\| \leq \|(1-\mu)(z_1 - z_0)\| \leq (1-\mu)R < R + \mu\eta,$$

and, from condition (SL4), it follows

$$\|y_1 - z_0\| = \|(1+\mu)z_1 - \mu z_0 - z_0\| \leq \|(z_1 - z_0) + \mu(z_1 - z_0)\| \leq \eta + \mu\eta < R + \mu\eta.$$

Then, $x_1, y_1 \in B(z_0, R + \mu\eta) \subseteq \Omega$ and $x_1 \neq y_1$ since $z_1 \neq z_0$. Therefore, $B_1 = [x_1, y_1; G]$ is well-defined. To prove the existence of B_1^{-1}, we consider

$$\begin{aligned}
\|I - B_0^{-1}B_1\| &\leq \|B_0^{-1}\|\|B_1 - B_0\| \\
&\leq \beta\, \|[x_1, y_1; G] - [x_0, y_0; G]\| \\
&\leq \beta\omega\left((1-\mu)\|z_1 - z_0\| + \mu\|z_0 - z_{-1}\|, (1+\mu)\|z_1 - z_0\| + \mu\|z_0 - z_{-1}\|\right) \\
&\leq \beta\omega\left((1-\mu)\eta + \mu\alpha, (1+\mu)\eta + \mu\alpha\right) \\
&\leq \beta\omega\left((1-\mu)R + \mu\alpha, (1+\mu)R + \mu\alpha\right) \\
&\leq \beta\omega\left(R + \mu\alpha, (1+2\mu)R + \mu\alpha\right).
\end{aligned} \tag{18}$$

As $\beta\omega(R + \mu\alpha, (1+2\mu)R + \mu\alpha) < 1$, from condition (SL5), then B_1^{-1} exists by the Banach Lemma for invertible operators, and

$$\|B_1^{-1}\| \leq \frac{\beta}{1 - \beta\omega(R + \mu\alpha, (1+2\mu)R + \mu\alpha)}, \tag{19}$$

what it proves (i_1). Next, using Lemma 2, we prove (ii_1) since that

$$\begin{aligned}
\|G(z_1)\| &\leq \|[z_1, z_0; G] - [x_0, y_0; G]\|\, \|z_1 - z_0\| \\
&\leq \omega\left(\|z_1 - z_0\| + \mu\|z_0 - z_{-1}\|, \mu\|z_0 - z_{-1}\|\right) \|z_1 - z_0\| \\
&\leq \omega(\eta + \mu\alpha, \mu\alpha)\, \|z_1 - z_0\| \\
&\leq \tilde{T}\|z_1 - z_0\|.
\end{aligned} \tag{20}$$

Hence, the iterate z_2 is well-defined, and using (19) and (5), we get

$$\begin{aligned}
\|z_2 - z_1\| \leq \|B_1^{-1}\|\|G(z_1)\| &\leq \frac{\tilde{T}}{1 - \beta\omega(R + \mu\alpha, (1+2\mu)R + \mu\alpha)}\|z_1 - z_0\| \\
&= T(R)\|z_1 - z_0\| < \eta,
\end{aligned} \tag{21}$$

given that $T(R) < 1$, since that $T(R) > 0$, by the condition (SL5), and also $R > 0$ and $R(1 - T(R)) = \eta > 0$. So, (iii_1) is proved.

Finally, we prove (iv_1) taking into account that

$$\|z_2 - z_0\| \leq \|z_2 - z_1\| + \|z_1 - z_0\| \leq (1 + T(R))\|z_1 - z_0\| < \frac{1}{1 - T(R)}\eta = R. \tag{22}$$

Hence, $z_2 \in B(z_0, R) \subseteq B(z_0, R + \mu\eta) \subseteq \Omega$.

In a similar manner, using the principle of mathematical induction, we can establish the previous recurrence relations for $n \geq 1$. Thus, as an induction hypothesis, we consider that $(i_k), (ii_k), (iii_k)$, and (iv_k) are verified, for $k = 1, 2, \ldots, n-1$, and we will prove that $(i_n), (ii_n), (iii_n)$, and (iv_n) are verified.

From (iv_{n-1}), $z_{n-1} \in B(z_0, R) \subseteq B(z_0, R + \mu\eta) \subseteq \Omega$, and then we have

$$\|x_n - z_0\| = \|(1-\mu)z_n + \mu z_{n-1} - z_0\| \leq (1-\mu)\|z_n - z_0\| + \mu\|z_{n-1} - z_0\| \leq R < R + \mu\eta,$$

and, from (iii_{n-1}), we get

$$\|y_n - z_0\| = \|(1+\mu)z_n - \mu z_{n-1} - z_0\| \leq \|z_n - z_0\| + \mu\|z_n - z_{n-1}\| < \|z_n - z_0\| + \mu T(R)^{n-1}\|z_1 - z_0\| < R + \mu\eta.$$

Then, $x_n, y_n \in B(z_0, R + \mu\eta) \subseteq \Omega$ and $x_n \neq y_n$ since $z_n \neq z_{n-1}$. Therefore, $B_n = [x_n, y_n; G]$ is well-defined. To prove (i_n), proceeding as in (18), we obtain

$$\begin{aligned}
\|I - B_0^{-1}B_n\| &\leq \|B_0^{-1}\|\|B_n - B_0\| \\
&\leq \beta \, \|[x_n, y_n; G] - [x_0, y_0; G]\| \\
&\leq \beta\omega\left((1-\mu)\|z_n - z_0\| + \mu\|z_n - z_{-1}\|, (1+\mu)\|z_n - z_0\| + \mu\|z_n - z_{-1}\|\right) \\
&\leq \beta\omega\left((1-\mu)R + \mu(R+\alpha), (1+\mu)R + \mu(R+\alpha)\right) \\
&\leq \beta\omega\left(R + \mu\alpha, (1+2\mu)R + \mu\alpha\right) < 1.
\end{aligned}$$

Then, by the Banach Lemma for inverse operators, (i_n) is proved.

Next, to prove (ii_n), we consider, as in (5):

$$\begin{aligned}
\|G(z_n)\| &\leq \|[z_n, z_{n-1}; G] - [x_{n-1}, y_{n-1}; G]\| \, \|z_n - z_{n-1}\| \\
&\leq \omega\left(\|z_n - z_{n-1}\| + \mu\|z_{n-1} - z_{n-2}\|, \mu\|z_{n-1} - z_{n-2}\|\right) \|z_n - z_{n-1}\| \\
&\leq \omega\left((1+\mu)\eta, \mu\eta\right) \|z_n - z_{n-1}\| \\
&\leq \tilde{T}\|z_n - z_{n-1}\|.
\end{aligned}$$

Now, as in (21), we can easily get (iii_n).

Finally, from the expression of the sum of a geometric progression with a ratio less than 1, we have

$$\|z_{n+1} - z_0\| \leq \sum_{i=0}^{n} \|z_{i+1} - z_i\| \leq \sum_{i=0}^{n} T(R)^i \|z_1 - z_0\| < \frac{1}{1 - T(R)}\eta = R,$$

what it proves (iv_n). □

Theorem 4. *Suppose that the conditions (SL1)–(SL5) hold. Then, the sequence $\{z_n\}$ given in (5), starting at $z_0, z_{-1} \in \Omega$, remains in $B(z_0, R)$ and converges to z^* a solution of Equation (1).*

Moreover, z^* is the unique solution of Equation (1) in $\overline{B(z_0, R)}$.

Proof. As we have already indicated above, suppose that $z_n \neq z_{n-1}$, for all $n \geqslant 1$, otherwise we have that the sequence $\{z_n\}$ is convergent since if $z_k = z_{k+1}$ then $z_k = z_{k-1} = z^*$, and the result is proved.

To prove the convergence of the sequence generated by (5), it is sufficient to prove that the sequence $\{z_n\}$ is a Cauchy sequence. Using $T(R) < 1$, for $n \geqslant 1$, we have

$$\|z_{n+k} - z_n\| \leq \sum_{i=1}^{k} \|z_{n+i} - z_{n+i-1}\| \leq \sum_{i=1}^{k} T(R)^{i-1} \|z_{n+1} - z_n\| \leq \frac{1 - T(R)^k}{1 - T(R)} T(R)^n \|z_1 - z_0\|.$$

Hence, $\{z_n\}$ is a Cauchy sequence which converges to z^*. Since

$$\|G(z_n)\| \leq \tilde{T}\|z_n - z_{n-1}\|,$$

with $\tilde{T}$ a finite real number and $\|z_n - z_{n-1}\| \to 0$ as $n \to \infty$, then $G(z^*) = 0$ by using the continuity of G.

In order to prove the uniqueness, let y^* be another solution of $G(z) = 0$ in $\overline{B(z_0, R)}$. If we denote $A = [y^*, z^*; G]$, we get

$$\begin{aligned}
\|I - B_0^{-1}A\| &\leq \|B_0^{-1}\|\|A - B_0\| \\
&\leq \beta \, \|[y^*, z^*; G] - [(1-\mu)z_0 + \mu z_{-1}, (1+\mu)z_0 - \mu z_{-1}; G]\| \\
&\leq \beta\omega\left(\|y^* - z_0\| + \mu\|z_0 - z_{-1}\|, \|z^* - z_0\| + \mu\|z_0 - z_{-1}\|\right) \\
&\leq \beta\omega\left(R + \mu\alpha, R + \mu\alpha\right) \\
&< \beta\omega\left(R + \mu\alpha, (1+2\mu)R + \mu\alpha\right) < 1
\end{aligned}$$

Hence, by the Banach Lemma, A^{-1} exists. Now, as $0 = G(y^*) - G(z^*) = A(y^* - z^*)$ and there exists A^{-1}, the uniqueness of solution is proved. □

5.1 Numerical experiments

In this section, we show the applicability of the theoretical results obtained in previous ones. We focus on nonlinear integral equations, that appear in different applied physics and chemistry problems like radioactive transfer, and the oscillation of a string, membrane, or axle. Moreover, different initial and boundary value problems can be converted to integral equations. The potential theory, scattering in quantum mechanics, conformal mapping, and water waves also contributed more than any field to give rise to integral equations.[19,20]

We consider the nonlinear integral equation of Hammerstein type given by

$$[\mathcal{H}(x)](s) = x(s) - f(s) - \int_a^b G(s,t)(\lambda x(t)^2 + \sigma|x(t)|),\, dt,\ s \in [a,b], \tag{23}$$

where $-\infty < a < b < +\infty$, G is the Green's function, f is known function, $\lambda, \sigma \in \mathbb{R}$, and x is the solution to be obtained.

In order to solve the equation $\mathcal{H}(x) = 0$, where $\mathcal{H} : \Omega \subset C[a,b] \to C[a,b]$, we transform the problem into a nonlinear system in $\mathbb{R}^n$, and we consider the max-norm. So, we approximate the integral by Gauss–Legendre quadrature formula with the corresponding weighs $x_j = x(t_j)$ and $f_j = f(t_j)$, with this discretization of the problem we have the following nonlinear system:

$$x_j = f_j + \sum_{i=1}^{n} p_{ji}(\lambda x_i^2 + \sigma|x_i|)^T \ j = 1, 2, \ldots, n, \tag{24}$$

where

$$p_{ij} = q_i G(t_j, t_i) = \begin{cases} q_i \frac{(b-t_j)(t_i-a)}{b-a}, & i \le j, \\ q_i \frac{(b-t_i)(t_j-a)}{b-a}, & i > j, \end{cases} \tag{25}$$

with $x_j = x(t_j)$ and $f_j = f(t_j)$, with $j = 1, \ldots, n$.

Now, the system (24) can be written as

$$H(\mathbf{x}) \equiv \mathbf{x} - \mathbf{f} - P\bar{\mathbf{x}} = 0, \quad H : \mathbb{R}^n \to \mathbb{R}^n, \quad H = (H_1, H_2, \ldots, H_n), \tag{26}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T$, the matrix $P = (p_{ij})_{i,j=1}^n$, and $\bar{\mathbf{x}} = \left(\lambda x_1^2 + \sigma|x_1|, \lambda x_2^2 + \sigma|x_2|, \ldots, \lambda x_n^2 + \sigma|w_n|\right)^T$, so that H is nonlinear and non-differentiable.

Next, we consider $n = 8$ and $f(s) = 0$ for all $s \in [0,1]$; moreover, $\lambda = \frac{1}{4}, \sigma = \frac{3}{2}$ for the local convergence study, so the solution is $z^* = \mathbf{0}$. We work in the domain $\Omega = B(\mathbf{0}, 1)$, and by taking $\tilde{\mathbf{z}} = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)^T$ and $\mathbf{z_0} = \left(\frac{1}{5}, \ldots, \frac{1}{5}\right)^T$, we have parameters appearing in $(L1)$ $b = 1.2549$ and $d = 0.5$. The w–function verifying condition $(L2)$ given by (10) has the following form: $w(s,t) = \frac{1}{8}(2\sigma + \lambda(s+t))$, and in this case, $w_0(s,t) = w(s,t)$, for all $s, t \in [0,1]$.

Then, for this problem, the previous study of local convergence gives us the results of Table 4, where we can see for different values of parameter μ the radius of the convergence local ball. We observe that, if we take values of μ smaller, we improve the accessibility region.

Now, we solve the nonlinear system (24) by applying Kurchatov method and different versions of its generalization given in (5).

We program the iterative schemes in Matlab20 by using variable precision arithmetic with 50 digits, using as stopping criteria $||\mathbf{x_{n+1}} - \mathbf{x_n}|| < 10^{-25}$ and with the starting point $\mathbf{z_0} = (0.2, 0.2, \ldots, 0.2)^T$ and $\mathbf{z_{-1}} = (0.5, 0.5, \ldots, 0.5)^T$. Then, for different values of μ, we obtain the approximated solution to the problem. We can check in Table 5 that the behavior of the introduced method is always better that the initial Kurchatov's iterative method.

Finally, we apply the theoretical results obtained in Theorem 7 and previous Lemmas in order to set the semilocal convergence ball. In this case, we consider the nonlinear equation given by (26) with $n = 8$ and $f(s) = \frac{1}{4}$ for all $s \in [0,1]$;

TABLE 4 Radii of the local convergence balls for different values of μ

μ	R
0.15	0.2587
0.1	0.2366
0.05	0.2148

moreover, $\lambda = \sigma = \frac{1}{4}$. In this case, we do not know the exact solution of the nonlinear system, even more we do not know if there exists solution of the problem. But, if we work in the domain $\Omega = B(\mathbf{0}, 2)$ and by taking $\mathbf{z_0} = \left(\frac{1}{5}, \dots, \frac{1}{5}\right)^T$ and $\mathbf{z_{-1}} = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)^T$, we have parameters appearing in ($SL1$) and $SL2$ $\alpha = 0.3$ and $\beta = 1.0458$ and $\delta = 0.4575$. The w-function has the same form that the one in the local convergence study. So, by solving equation in ($SL4$), we get the radius of semilocal convergence given in Table 6. We point out that with these results, we confirm the existence and uniqueness of a solution in the domain $\overline{B(\mathbf{z_0}, R)} \subseteq \Omega$.

Now, we obtain the approximated solution in this domain by using the proposed schemes and by working with the same conditions that in local convergence and with the starting guesses proposed in the theoretical semilocal study, so we have in Table 7 the results. One can check that the uniparametric family (5) improves the convergence results obtained for the Kurchatov method. The approximated solution with six digits is

$$\mathbf{x}^* = (0.295568, 0.292161, 0.287938, 0.285097, 0.285097, 0.287938, 0.292161, 0.295568)^T.$$

To conclude this case of semilocal convergence, we consider a well-known family of secant-type iterative processes, and we make a comparative study with the uniparametric family (5). In previous work,[17] the authors construct the following uniparametric family of secant-type iterative methods for solving $H(\mathbf{x}) = 0$:

$$\begin{cases} w_{-1}, w_0 \text{ given in } \Omega, \ \lambda \in [0,1], \\ u_n = \lambda w_n + (1-\lambda) w_{n-1}, \ n \geq 0, \\ w_{n+1} = w_n - [u_n, w_n; H]^{-1} H(w_n), \end{cases} \tag{27}$$

which depends on the parameter $\lambda \in [0, 1]$. The family (27) is reduced to the secant method if $\lambda = 0$ and to Newton's method if $\lambda = 1$ and H is differentiable. In Table 8, we can see the results by using these processes for different values of λ and working with the same conditions that we have used for obtaining Table 7. As one can check, with these secant-type methods (27), we need one more iteration for reaching the same tolerance that we have obtained with the new methods, and even so, the residual error and the value of the operator at the approximated solution do not reach the same accuracy that in the results obtained in Table 7, where we have used the variants of Kurchatov's method introduced in this work

TABLE 5 Numerical results with different values of parameter μ

Method	**Kurchatov** $\mu = 1$	**(5)** $\mu = 0.1$	**(5)** $\mu = 0.05$
k	12	8	9
$\|\|\mathbf{x_{n+1}} - \mathbf{x_n}\|\|$	8.5087e-26	3.1372e-26	1.0182e-46
$\|\|H(\mathbf{x_{n+1}})\|\|$	1.2237e-26	4.5119e-27	1.4644e-47

TABLE 6 Radii of the semilocal convergence balls for different values of μ

μ	R
0.15	1.1754
0.1	1.1651
0.05	1.1544

TABLE 7 Numerical results with different values of parameter μ

Method	**Kurchatov** $\mu = 1$	**(5)** $\mu = 0.1$	**(5)** $\mu = 0.05$
k	6	5	5
$\|\|\mathbf{x_{n+1}} - \mathbf{x_n}\|\|$	1.9789e-51	1.1557e-30	1.1557e-30
$\|\|H(\mathbf{x_{n+1}})\|\|$	6.2180e-59	6.6046e-59	6.6046e-59

TABLE 8 Numerical results with different values of parameter λ for (27)

Method	**Secant like (27)** $\lambda = 0$	**Secant like (27)** $\lambda = 0.5$	**Secant like (27)** $\lambda = 0.8$
k	7	6	6
$\|\|\mathbf{x_{n+1}} - \mathbf{x_n}\|\|$	2.5456e-37	1.1028e-27	4.5699e-32
$\|\|H(\mathbf{x_{n+1}})\|\|$	5.2687e-59	2.1741e-45	1.0203e-52

6 | CONCLUSIONS

In this work, we have introduced a variable parameter in the function iteration of Kurchatov's iterative method that allows us to improve the accessibility region for this iterative method. The behavior of the new uniparametric family of iterative methods is very close to the one of Newton's method, but the main advantage is that the new model can be used in the non-differentiable case. We perform a complete theoretical study of local and semilocal convergence for the new models. After that, we check the results by applying the theory to some real nonlinear integral equation. As a conclusion, it can be stated that we have obtained a uniparametric family of iterative processes whose behavior is similar to that of Newton's method but which is also applicable to situations in which the operator is non-differentiable and improves the behavior of the Kurchatov method. This is corroborated by solving an applied problem formulated as a nonlinear non-differentiable integral equation of Hammerstein type.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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