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Review

Learning Difficulties with the Concept of Function in Maths: A Literature Review

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Abstract: The concept of function is a threshold concept in mathematics since it is essential for a deep understanding of the subject and is often problematic for students. Despite the importance of this concept, numerous studies have found that students have several difficulties and misconceptions about it. This work aims to assess, classify and synthesize the existing information about the learning difficulties and misconceptions related to the concept of function. In this sense, the main achievement of the work is the synthesis and classification by topics of the information gathered from the literature.

Keywords: concept of function; mathematics teaching and learning; learning difficulties; function representation

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1. Introduction

A threshold concept can transform the understanding of a subject, and its learning is often problematic for students [1]. In teaching/learning mathematics, functions, limits, derivatives and integrals are recognized as threshold concepts [2]. Among all these concepts, functions are especially important because other concepts, such as integrals, derivatives or limits, strongly depend on a deep understanding of the concept [3].

Functions begin to be studied in secondary education and are a fundamental concept in mathematics subjects of university degrees related to basic sciences, social sciences, engineering and architecture. The lack of a deep understanding of the concept of function provokes (1) comprehension problems with subsequent concepts, such as limits, derivatives or integrals; (2) precarious application of functions in other subjects or professional future; and (3) a feeling that the concept is useless, which leads to low motivation and interest in its study [4].

Since the late 1970s, several studies have been published about the characterization of the difficulties in the teaching/learning process of the concept of function [5–9]. In 1992, Dubinsky and Harel published a book [10] about the main comprehension problems that students had with this concept. This book outlines previously documented learning difficulties and introduces new ideas, suggesting that misunderstandings can impede accurate comprehension of the concept.

However, despite the existing body of literature on this issue, we have experienced that it is challenging to become aware of the main results. This is mainly because there is a significant heterogeneity in the quality of the references. In addition, there is a lot of redundant information presented in different ways, which obscures the main findings. On the other hand, some works study issues about the concept of function that are very fragmented or too specific, and it is easy to lose sight of the problem. Finally, we missed a classification of learning difficulties based on the main characteristics of the concept. Thus, the purpose of this work is to carry out a review and analysis of the existing literature on the learning difficulties and misconceptions related to the concept of function, with the aim of (1) clarifying and unifying the results and conclusions obtained so far; (2) contributing to have a broader and more uniform knowledge about the teaching/learning process of the concept of function; (3) pointing out lines of research that remain to be explored; and (4) helping mathematics teachers, who could anticipate their students' comprehension problems once they know their main learning difficulties and misconceptions, and who could also design teaching methodologies and materials to attack them. Consequently, the research question of this work is:

What are the main results and conclusions detected in the literature with respect to the learning difficulties and misconceptions of the concept of function?

Despite all the previous studies, further research is still needed since current students continue to have learning difficulties with the concept of function. The present work may help to continue this research.

2. Materials and Methods

This work was developed as follows: First, the criteria for the literature review and the characterization of the selected works were defined, which is explained in this section. Then, the information found was classified according to the main characteristics of the concept of function, which is detailed in Section 3. Finally, results were summarized and conclusions were derived (see Section 4).

2.1. Search and Literature Analysis

An exhaustive literature search of learning difficulties related to the concept of function was carried out. The Web of Science (WOS), Scopus and ERIC were used as databases.

The search strategy included several search engines and the introduction of the following terms: «threshold concept» and «function», «learning progression» and «function concept», «mathematics education» and «function concept», «learning difficulties» and «function concept», and «misconception» and «function concept». These terms were searched within the title, abstract and keywords of WOS and Scopus, the field Topic of WOS, and the title and descriptors of ERIC. No time limit was specified. Since the term «function» is a broad term that can have many meanings, it was essential to use the term «function concept» instead of «function» to narrow the search. Articles written not only in English but also in Spanish (our mother tongue) were selected to enrich the search.

In the first step, called *selection*, all the references found were independently reviewed by two of the authors to select those works directly related to the work's aim. Most of the references to be discarded were directly detected by the title. In the case of not being sure about discarding by the title, the abstract or even the full text were read. Both authors worked independently. The criteria for the selection were:

(1) The study referred to the mathematical concept of function. We discarded works focused on functions with other meanings, such as role in economics or code in programming. Moreover, works that contained the word «function» in the title or abstract but were not related to the topic of the study were also discarded.
(2) Only studies that refer to learning difficulties and misconceptions about the concept of function were considered. It is important to remark that we did not consider in our study other types of studies related to the concept of function, such as those based on interventions to improve the understanding of the concept, or those based on how the learning of functions evolves in students.

Each author saved all the references selected in the same folder using this code: year of publication_first author family name. Both authors shared their findings and created a new folder with the selected references, including a reference number for each one.

In the second step, called synthesis, we built a data sheet to synthesize the information from the papers selected. Each paper was fully reviewed by two authors, who first independently filled in the table and then shared their findings. The data fields of the table were ID (number of reference), name (first author family name), year of publication, educational level (secondary or university education), type of study (experimental or without experimentation, sample size for experimental studies, qualitative or quantitative analysis, students' profile, duration of the study), objective of the study, aspects of the concept of function that authors assessed and the main ideas of the results. During this full review of the first selected references, a group of references was also discarded because we found that they were outside of the scope. Moreover, we noticed some repeated ideas in different papers written by the same author (or group of authors). In these cases, we only selected one of the papers, usually an older paper or a paper that joined more ideas, prioritizing journal over conference papers.

Citations included in the selected studies were also reviewed in order not to lose possible relevant information.

After this process, we obtained the entire sample of references for the literature review.

2.2. Characterization of the Selected Studies

In the third step, called classification, a new data sheet was compiled using Table 1 in order to record the characteristics of each paper chosen for review. Five categories were established for the topic data field in order to classify the broad aspects of the concept of function found in the synthesis step: 1: definition; 2: interpretation or meaning; 3: notation or expression; 4: graphic representation; and 5: management and characteristics. The last category (number 5) refers to types of functions, the operations made with functions and their properties.

Table 1. Information included in the data sheet used for the characterization of the selected papers.

Author	Family name of the author, two authors, or the first in a list of authors, followed by the year of publication
Sample size	Number of participants in the study. In the case of being a theoretical study without experimentation, the acronym NE is used
Educational level	Secondary education (SE) or university education (UE)
Topic	Learning issues studied related to the concept of function (1: definition; 2: interpretation or meaning; 3: notation; 4: representation; 5: management and characteristics)
Results and conclusions	Main results and conclusions classified with respect to learning difficulties and misconceptions

We named non-experimental studies (NE) those theoretical studies in which students have not participated, and we referred to experimental studies when there has been student participation. In the case of experimental studies, we divided them into two groups: those in which a small sample is used ($n < 25$), and studies with a larger sample ($n \geq 25$).

On the other hand, we thought it was necessary to distinguish between secondary education (SE) and university education (UE) since the concept is not dealt with the same complexity at both educational levels, which can provoke different learning difficulties.

Finally, the results and conclusions were classified as either learning difficulties or misconceptions based on what they referred to. A learning difficulty is a condition that affects a person's ability to learn, such as difficulty understanding or remembering new information or skills. A misconception is an incorrect or misunderstood belief, idea or opinion about something. It is typically formed without enough evidence or research and can be challenging to correct.

Repeated ideas found in different references were written with the same wording to systematize the results and conclusions field.

3. Results

3.1. Literature Review

The search produced the following number of papers according to the terms used:

- «threshold concept» and «function», 3856 papers;

- «learning progression» and «function concept», 140 papers;
- «mathematics education» and «function concept», 533 papers;
- «learning difficulties» and «function concept», 885 papers;
- «misconception» and «function concept», 329 papers.

During the selection (first step), we detected many coincidences between the references found with different queries. Additionally, most of the papers that contained the word «function» in the title or abstract were not related to the mathematical concept of function. At the end of the first step, 54 papers were selected. After the synthesis (second step), 22 papers were discarded, and one new paper was selected that came from the references analysis of the 54 papers. The total number of articles selected was 33. Table 2 shows the selected articles and the items used for the characterization.

Table 2. List of the selected papers.

Reference		Sample Size	Educational Level	Topic
[11]	Abdullah, 2010	4	SE	1, 2, 3
[12]	Akkoç y Tall, 2003	9	SE	1, 2
[13]	Bardini et al., 2014	383	UE	1, 2, 5
[14]	Borke, 2021	NE	NE	2
[15]	Cansiz et al., 2011	61	SE	2, 4, 5
[16]	Carlson y Oehrtman, 2005	NE	NE	1, 2, 3, 4, 5
[17]	Clement, 2001	60	SE	2, 4
[18]	Dogan, 2007	2	SE	1, 2
[19]	Dreyfus y Eisenberg, 1992	443	SE	1, 2, 4
[20]	Dubinsky y Wilson, 2013	15	SE	1, 3, 4, 5
[21]	Eames et al., 2021	680	SE	1, 2, 4
[22]	Evangelidou et al., 2014	164	UE	4
[8]	Even, 1990	NE	NE	2
[23]	Font et al., 2003	NE	NE	2
[24]	Hatisaru y Erbas, 2017	56	SE	5
[25]	Hitt, 1998	30	SE	4
[26]	Leinhardt et al., 1990	NE	NE	1, 2, 3, 4
[27]	Markovits et al., 1986	NE	NE	4
[28]	Martinez Planell et al., 2009	9	SE	1, 2
[29]	O'Shea et al., 2016	117	UE	1, 2, 3, 4, 5
[30]	Panaoura et al., 2015	756	SE	1, 4, 5
[31]	Parhizgar et al., 2021	74	SE	1, 4, 5
[32]	Petterson et al., 2013	4	UE	1, 2, 5
[33]	Petterson, 2012	4	UE	1, 2, 5
[34]	Sajka, 2003	1	SE	1, 2, 3
[35]	Sierpinska, 1992	NE	NE	1, 2, 3, 4, 5
[36]	Tall y Bakar, 1992	28	SE	2, 5
[37]	Thomas, 1975	50	SE	1, 2
[38]	Thompson, 1994	NE	NE	1, 2, 3, 4, 5
[39]	Vinner y Dreyfus, 1989	271	SE	2, 5

Table 2. *Cont.*

Reference		Sample Size	Educational Level	Topic
[40]	Walde, 2017	352	SE	1, 2, 3, 4, 5
[41]	Widada et al., 2020	1	UE	1, 2, 3, 4, 5
[42]	Yusof et al., 2014	52	UE	2

Figure 1 shows the distribution of the papers by educational level and topic. The number included in each sector refers to the number of papers. As shown in the last column of Table 2, some studies deal with various topics, so they may be counted in multiple sectors within the same graph. A total of 18 out of 33 papers describe interventions with SE students, 7 with UE students and 8 have no experimentation. The topic with a higher incidence is interpretation or meaning (topic 2) at all educational levels.

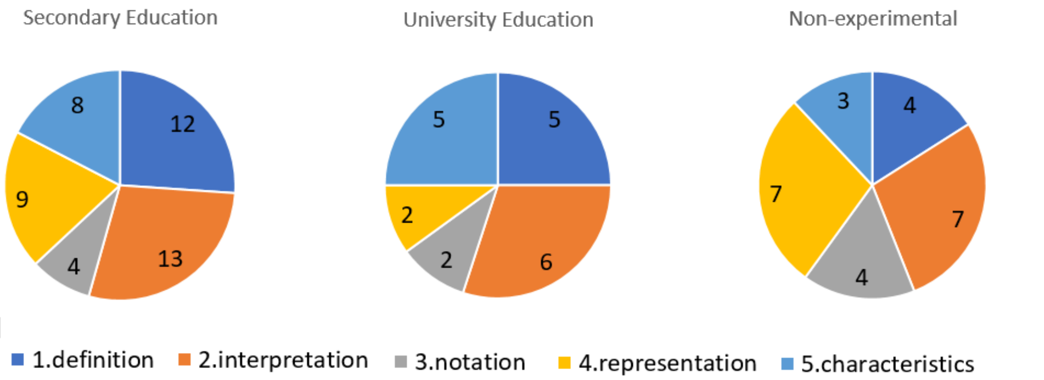

Figure 1. Number of studies, distributed by topic and educational level.

3.2. Learning Difficulties and Misconceptions in the Concept of Function

After analyzing the results and conclusions of the references selected for the study, misconceptions (M) and learning difficulties (D) related to the concept of function were identified (Tables 3 and 4, respectively). In both tables, each difficulty and misconception was coded with its corresponding letter, followed by two digits. The first digit refers to the topic (1: definition; 2: interpretation or meaning; 3: notation and expression; 4: graphic representation; 5: management and characteristics), and the second digit orders the list within the topic.

Table 3. Misconceptions (M) related to the concept of function.

M2. With the interpretation or meaning:

- M21. A function is only considered like a transformative "box" or "automatic machine" (transformation instead of an association).
- M22. A function is always given by a rule defined by an algebraic expression, or it is even the rule itself.
- M23. A function is an association of a single element of the domain with another in the range, and vice versa.

M4. With the graphic representation:

- M41. All functions have a graphical representation on the cartesian axes.
- M42. All functions have a graphical representation in the form of a line or parabola.

M5. With the management and characteristics:

- M51. Constant functions are not functions.
- M52. The non-continuous functions are not functions.

Table 4. Learning difficulties (D) related to the concept of function.

D1. With the definition:

- D11. Give a clear and precise definition using mathematical vocabulary and without graphic aids.

D2. With the interpretation or meaning:

- D21. Distinguish between function and equation.
- D22. Use functions for the modeling of a phenomenon or a real situation.

D3. With the notation and expression:

- D31. Understand what the function represents.
- D32. Understand what the role of a variable in a function is.
- D33. Manipulate different algebraic representation of the same function.

D4. With the graphic representation:

- D41. Switch between the algebraic expression and the graphic representation.
- D42. Understand that the points of a function's graphic representation are equivalent to the pairs of values which constitute the function.
- D43. Realize that $f(x) = y$.
- D45. Connect the concept of function with the table of values and the graphic.

D5. With the management and characteristics:

- D51. Distinguish whether an expression is a function or not.
- D52. Use the properties of functions and operate with them.
- D53. Realize how domain constraints also affect the range.

3.2.1. Definition (D1)

There is not a single definition of the concept of function. The Dirichlet–Bourbaki definition, which is based on the set theory, says:

A function is a relationship between two sets, called domain and range (or codomain), such that each element of the domain corresponds to exactly one element of the range.

Although this definition is broadly used, it does not express the dynamic or variational aspect of this concept, which is considered in this definition [43]:

A function, covariationally, is a conception of two quantities varying simultaneously, such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other.

This last definition eases the students' comprehension of the concept of variable, domain or range. Moreover, it makes more intuitive the use of functions to model real situations and processes. In addition, it allows students to think of a function as a process that may be reversed (inverse functions).

On the other hand, mainly for the first stages of learning, teachers usually explain functions as input–output machines in which an element is transformed into another [33]. Thus, the function is defined as a transformation instead of an association.

The three interrelated aspects of the concept of function are summarized in [44] (page 1246):

"The function as an input–output assignment. The function is an input–output assignment that helps to organize and to carry out a calculation process ... The representation of an input calculation-output chain is appropriate for this view on function.

The function as a dynamic process of co-variation. This aspect concerns the notion that the independent variable, while running through the domain set, causes the dependent variable to run through a range set. The dependent variable co-varies with the independent ... Helpful representations for studying co-variation are tables and graphs, ...

The function as a mathematical object. A function is a mathematical object which can be represented in different ways, such as arrow chains, tables, graphs, formulas, and phrases, each providing a different view on the same object ... "

It is important to remark that the image of the concept that students have usually differs from the definition. This image is the result of their experience with the context, which can range between the geometric approach (expressed as a curve), the algebraic approach (expressed as a formula) and the logic idea (input–output machine), according to Kleiner (1989) [45].

Students' difficulties begin with the definition of the concept [11,16,20,30,31,33,40,41]. Most secondary school students cannot specify what a function is. Students find it easier to give an example than a definition [18]. However, some authors have detected that the examples used are often not correct either [30,32,33]. On the other hand, students do not use abstract mathematical terminology, and without the appropriate vocabulary, it is difficult to define a function [30,32–34]. In addition to the abstraction of the concept, experts point out that textbooks use different definitions of the concept of function and, although they essentially refer to the same thing, generate confusion among students [41].

Without a clear definition of the concept, students have difficulties distinguishing when an expression corresponds to a function or not [29,41].

3.2.2. Interpretation or Meaning (M2 and D2)

Difficulties with the definition of a function are closely related to misconceptions and difficulties related to its meaning or interpretation. One of the most common interpretations that students hold of functions is that they are like transformative «boxes» or «automatic machines» [14,18,23,33] (M21-Table 3). Many students maintain this interpretation of functions since teachers sometimes use the analogy of the box and the transformation to explain the concept, as we mentioned in the previous section. This is not a wrong interpretation, and it is very useful for the initial stages of learning; however, it is incomplete because it reduces the concept to some types of functions, mainly those that are defined with algebraic expressions [17]. Furthermore, if students only have the image provided by this analogy, they understand a function as a mathematical object, but not as an association or dynamic process of co-variation. In short, they understand the concept as an action, not as a process [34], and not as a set of pairs, the elements of the domain and range, which are related. The restriction to only one approach is problematic mainly for university students.

On the other hand, due to the type of functions that students usually use, they consider that there must always be a rule (an algebraic expression) that defines the function. They even think that the rule is in fact the function [33,34,41]. Students wrongly think that all functions must be defined by an algebraic expression [8,13,16,41] (M22-Table 3).

Another misconception that students have is that the relationship between domain and range elements must be bijective (M23-Table 3); that is, one element of the domain corresponds to one element of the range, and vice versa [14,15,26,36,37,39,41]. However, the definition only says that an element of the domain corresponds to exactly one element of the range. There are no restrictions about how many elements of the domain can be related to the same range element. When the relationship is bijective, it is a function, but not only in this case.

It has been found that students have difficulty distinguishing between equations and functions [11,13,40] (D21-Table 4). For some students, functions are two algebraic expressions separated by an equal, and in this sense, they do not distinguish between functions and equations [16,34]. It is true that, in some cases, it may coincide, and that they correspond to many functions used in high school or first university courses; however, it is not something general for all functions. Students must understand that, although in some cases they share an expression, an equation and a function are different concepts.

Finally, one aspect to highlight about the interpretation of functions is that students have difficulties putting functions in a practical context, that is, to understand the sense of a function to model a phenomenon [12,29,35,38,42]. Both in the works of O'Shea et al., 2016 [29] and Yusof et al., 2014 [42], students were asked to interpret the graphs of a function which models a real phenomenon and the authors verified students' difficulties with this task (D21-Table 4).

3.2.3. Notation and Expression (D3)

For functions that depend on a single variable, x is generally used to denote an element of the domain and $f(x)$ the element of the range with which x is associated. x is called an independent variable because it refers to the elements of the domain which "choose" elements of the range. As mentioned above, in some functions, the relationship between x and $f(x)$ follows a rule that can also be expressed algebraically. For example, if the domain of the function is the set of real numbers and they are related to the numbers that double their value, the function would be expressed as:

$$f : \mathbb{R} \rightarrow \mathbb{R} f(x) = 2x \quad (1)$$

Sometimes, the elements of a range are denoted as y (that is, $y = 2x$ for the previous example), which usually leads to confusion between equation and function (D21-Table 4). Students have difficulties in understanding the meaning of the mathematical notation exemplified in expression (1) [11,16,19,28,29,34,40] (D31-Table 4).

On the other hand, x is called a variable because, as the suffix –able indicates, it can cause the function (the relationship) to vary. However, students, based on the notation used, have difficulties understanding what the variable is and what it represents [11,16,20,29]. Students do not understand that instead of x, it could be any element of the domain. In a study by Carlson and Oehrtman in 2005 [16], the authors detected that if they asked students what the value of the function $f(x) = 2x$ is when x is substituted by $x + a$, they did not understand that the variable is now called $x + a$ instead of x. As the authors reported in their work, the obtained result was often wrong: $f(x + a) = 2x + a$, instead of $f(x + a) = 2(x + a)$.

Students' difficulties with the notation and expressions used to represent functions also lead to problems in working with different algebraic representations of the same function [16,29,41] (D33-Table 4). To illustrate this difficulty, Carlson and Oehrtman [16] give an example of the same function with two different expressions: $f(x) = n^2$ y $g(x) = \sum_{k=1}^{n} 2k - 1$.

3.2.4. Graphic Representation (M4 and D4)

Functions can be represented by Venn diagrams, although the usual and practical way to graphically represent functions is using a coordinate system.

Generally, secondary school and first-year undergraduate students use functions that depend on a single variable, in which the elements of the domain are usually real numbers. The representation of these functions is made in the X-Y Cartesian coordinate system. The x values are located on the abscissa axis and the y or $f(x)$ values on the ordinate axis. Figure 2 shows the graphical representation of a function whose expression is (1).

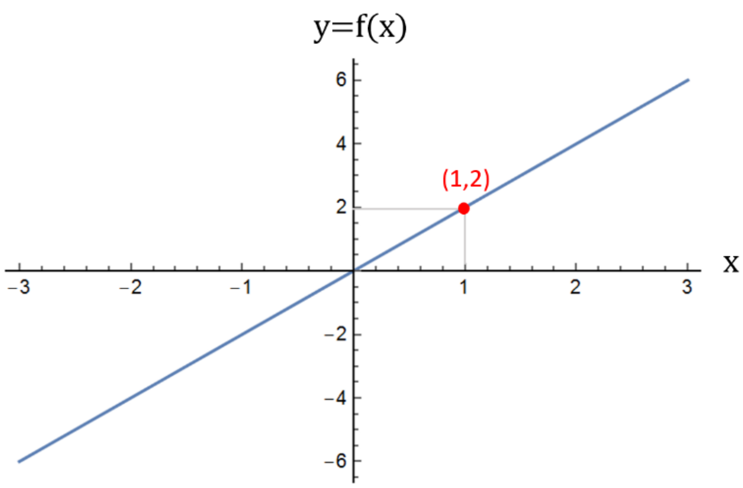

Figure 2. Graphic representation of the function defined by $f(x) = 2x$. A pair of values which belong to the function are marked in red, corresponding to $x = 1, y = f(x) = 2$.

In this regard, students find it challenging to switch from the algebraic expression of the function to its graphic representation, and vice versa [15,25,30,37,40] (D41-Table 4). Again, since they do not understand the notation used and do not deeply understand the definition of a function, they cannot interpret that they are being asked to locate in a graph the pairs of values which represent the function [21] (D42-Table 4). That is, the pairs of values are the points on the graph, in which the first coordinate of the point represents the element of the domain and the second coordinate represents the associated element of the range. In Figure 2, one of these points is represented in red color, although there would be infinite pairs of values (red points) that form the function (blue line). Students usually substitute x by defined values in Equation (1) mechanically, but without having a clear idea of what they are doing, frequently leading them to a misrepresentation. They consider that the graphic representation is like a photo of a physical situation; they do not interpret it as a relationship between some input values located on the abscissa axis and some output values located on the ordinate axis [16]. In addition, they have problems relating the ordinate axis to the range values, i.e., understanding that $f(x) = y$ [11,31] (D43-Table 4). In fact, in the work of Tall and Bakar (1992) [36], the authors found that almost two thirds of the high school students in their study thought that a circle could be the graph of a function.

Tables of values are also used to represent functions, especially those that do not follow a particular rule or cannot be expressed by an algebraic expression. In these cases, it has been detected that students have difficulties connecting the concept of function with the table of values, and this table with the graphic representation [17,20,41] (D44-Table 4). Again, as students do not consider a function as a set of pairs of values, it is difficult for them to understand the different ways of representing functions.

A misconception related to graphic representation is that only graphs that represent straight lines or parabolas correspond to a function's representation [16,27,41] (M41-Table 3). It seems surprising that university students have this idea, but the truth is that many examples of functions used in high school (such as $f(x) = 2x$) have a graphic representation in the form of a straight line or parabola, so students make a false association: «All functions have a graph in the form of a line or parabola». Moreover, students often have another false idea: «All functions have a graphic representation in a Cartesian diagram» [22] (M42-Table 3), which is true for functions that associate two sets of numbers, but not for those associating non-numeric objects.

3.2.5. Management and Characteristics (M5 and D5)

As previous studies have confirmed, students have difficulties working with the properties of functions, i.e., operating with them [29,31] (D52-Table 3).

In addition, there are various types of functions around which they have misconceptions. One of them is that constant functions ($f(x) = k$, k being a constant) are not really functions [16,36] (M51-Table 3). The graphic representation of a constant function is a straight line parallel to the abscissa axis, so students consider that these functions are not functions because they confuse them with non-functions whose graph is a straight line parallel to the ordinate axis [15]. Furthermore, they have difficulties in recognizing these functions because they expect an independent variable on the right-hand side of the algebraic expression [24,36,39].

Discontinuous functions are also especially problematic for students [16,24,33,36,39–41]. The fact that their graph is not a continuous line makes them assume that what is represented is not a function (M52-Table 3).

The misconceptions found in the two types of functions mentioned are closely related to difficulty D51 in Table 4, which states that students have difficulty distinguishing when an expression represents a function or not, either from its algebraic expression or its graphic representation [20,29,30].

Lastly, students find it difficult to consider that domain restrictions have an impact on range [13,31,41] (D53-Table 4). Again, since they do not consider a function as an association, they cannot extend the domain restriction to the range.

4. Discussion

The general objective of this review was to analyze the existing results with respect to the difficulties and misconceptions that students have regarding the concept of function in mathematics.

It is overwhelming that students have difficulties and/or misconceptions in all main aspects related to the concept of function and that most of these difficulties were detected more than 30 years ago and remain today. Sierpisnka [35] already published a paper on a study of the concept of function, and in her conclusions, she proposed some ideas about how to improve understanding of this key concept in mathematics. The first idea refers to the use of functions in real contexts, in problem modeling. The author highlights the importance of the fact that students are protagonists in the meaning of the functions. However, it should be noted that, years later, as we have seen in the results (D22), both in [29] and in [42], the authors have verified that students continue to have great difficulties with this aspect. This is also a problem for other aspects of mathematics, such as abstract algebra [46]

The difficulties and misconceptions detected are found both in secondary and higher education students. This leads us to think that the difficulties drag from secondary to higher courses, although perhaps with less intensity, as indicated in the works of [30,32,33].

It is true that there a considerable number of references used with a small sample size (n < 25), or that they are theoretical studies without experimentation with students (classified as NE). However, the ideas found in these works can also be found in other experimental studies with much larger sample sizes, therefore validating them. On the other hand, studies with a small sample size usually use qualitative data collection and analysis techniques such as interviews, which is why they provide another methodology for detecting learning difficulties necessary for the validity of results.

Our motivation for this work has focused on extracting from the literature only references to difficulties and misconceptions detected with the concept of function. However, since function is a key concept in any mathematics course, much research has been carried out focused on different issues related to the learning of the concept. As an example, in [47], the author studied the learning process of the concept of function by developing and assessing a teaching unit. Obviously, trying to understand the learning process of students is related to knowledge of students' difficulties with the concept. For this reason, the author carried out an intervention to detect his students' main difficulties with the concept of function. Therefore, this is a good reference in terms of knowing how to detect students' difficulties and misconceptions related to the concept of function. In [48], we have another example of intervention, but in this case related to the detection of learning difficulties with abstract algebra.

References relating to the opinions of experts are very limited. We have not found much information about this; the work of Bardini et al. [13] is the one that has contributed the most. In this sense, it would be interesting to continue the research to find out why the experts consider that the concept of function is difficult for students.

Most studies agree that students tend to memorize the concept of function, as they do with many other concepts, but without reasoning [41]. The deep understanding of a threshold concept is a process; if it starts with memorization and no reasoning, this process cannot evolve. This same idea is found in the results of Veith et al. [46] in the case of abstract algebra.

Regarding the literature review, the use of the query «threshold concept» and «function» produced a great number of papers that were discarded and was not productive. The use of queries with «function concept» instead of «function» narrowed the search and focused on conceptual understanding works.

5. Conclusions

The concept of function is a key concept in mathematics. In this work, we conducted an extensive review of the literature to gather clear, precise and schematic information about the main difficulties and misconceptions that students have with the concept of function. The review identified the main difficulties and misconceptions detected up to now, both in secondary and in higher education. In addition, we also classified the results according to the aspect of the function concept to which they referred. We summarized them in two tables, which is the main contribution of this work, as these extensive categorized lists cannot be found in the literature.

The review and synthesis undertaken in this work are especially interesting for teachers of mathematics or related disciplines, such as physics, economics or engineering. The information from this study allows teachers to anticipate the comprehension problems that may arise in their students. Moreover, with this information, it is possible for teachers to design teaching materials or implement methodologies to help their students to have a better understanding of the concept of function.

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