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Approximating Multiple Roots of Applied Mathematical Problems Using Iterative Techniques

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Abstract: In this study, we suggest a new iterative family of iterative methods for approximating the roots with multiplicity in nonlinear equations. We found a lack in the approximation of multiple roots in the case that the nonlinear operator be non-differentiable. So, we present, in this paper, iterative methods that do not use the derivative of the non-linear operator in their iterative expression. With our new iterative technique, we find better numerical results of Planck's radiation, Van Der Waals, Beam designing, and Isothermal continuous stirred tank reactor problems. Divided difference and weight function approaches are adopted for the construction of our schemes. The convergence order is studied thoroughly in the Theorems 1 and 2, for the case when multiplicity $p \geq 2$. The obtained numerical results illustrate the preferable outcomes as compared to the existing ones in terms of absolute residual errors, number of iterations, computational order of convergence (COC), and absolute error difference between two consecutive iterations.

Keywords: Steffensen's method; nonlinear equations; optimal iterative methods; multiple roots

MSC: 41A25; 41A58; 49M15; 65G99; 65H10

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1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a complex valued function that possesses the nth order derivatives. Let $kerf = \{\alpha \in \mathbb{C} : f^{(j)}(\alpha) = 0; j = 0, 1, 2, \ldots, p-1 \text{ and } f^{(p)}(\alpha) \neq 0\, ; p \in \mathbb{N}\}$ be the kernel of function f. This $kerf$ consists of the roots of function f and p is the multiplicity of the root. Therefore, to find the $kerf$, one should work to find out the roots of the equation $f(x) = 0$. In most of the cases, it is almost impossible to find the exact roots. In such cases, one needs to apply an iterative approach to find the approximate roots. There are different research papers that provide us with iterative techniques for approximating the solution α of nonlinear equation $f(x) = 0$ with multiplicity $p > 1$. The well-known modified Newton's method [1] is one of the simplest and most popular iterative methods for multiple roots, which is provided by

$$x_{t+1} = x_t - p\frac{f(x_t)}{f'(x_t)}, t = 0, 1, 2, \ldots \tag{1}$$

The convergence order of the modified Newton's method is quadratic for $p \geq 1$. Based on this method, many methods (Hansen and Patrick [2], Osada [3], Neta [4], Sharifi et al. [5], Soleymani et al. [6], Zhou et al. [7], Li et al. [8], and Chebyshev–Halley methods [9]) have been published; the theoretical treatment of iterative methods can be found in [10,11]. However, it can be seen that all these methods use the derivative of the function f in their iterative function, or higher order derivatives.

So, we have account for how sometimes the derivatives of function f do not exist or we consume a huge time in order to have them, then we think about derivative free methods for multiple roots, but these techniques are very limited in the literature. This is because it is not an easy task to retain the same convergence order (as the simple roots) and the calculation work is very hard and time consuming. However, due to the rapid development of digital computers, advanced computer languages, and software, the production of derivative free methods for obtaining the multiple roots of nonlinear equations have become the new area of interest. Most of the time derivatives are replaced by first-order divided differences in such methods.

Traub–Steffensen's method [12] is one of the derivative free methods where the derivative $f'(x_t)$ in Equation (1) is replaced with the first-order divided difference $f[\eta_t, x_t] = \frac{f(\eta_t)-f(x_t)}{\eta_t - x_t}$ and $\eta_t = x_t + \gamma f(x_t)$, $\gamma \neq 0 \in \mathbb{R}$. Therefore, Equation (1) becomes

$$x_{t+1} = x_t - p\frac{f(x_t)}{f[\eta_t, x_t]}. \tag{2}$$

Recently, some higher order derivative free methods have been proposed by different researchers on the basis of Traub–Steffensen's method [12]. The methods by Kumar et al. [13], Behl et al. [14], Sharma et al. [15,16], Dong [17,18], and Kumar et al. [19] are some examples of derivative free methods.

Motivated by derivative-free methods for multiple roots, we try to develop a new derivative-free multipoint iterative method. The advantages of our techniques are: they have smaller residual errors, consume smaller number of iterations, and have better error difference and more stable computational order of convergence (COC). In addition, the proposed scheme also adopts as small a number of function evaluations as possible to procure a high convergence order. The convergence order of the new family is four.

The rest of the paper is summarized as follows. Section 2 includes the construction as well as the convergence analysis of new family. Some special cases of the newly developed family are discussed in Section 3. In Section 4, various numerical examples are considered to confirm the theoretical results. Finally, concluding remarks are provided in Section 5.

2. Construction of Higher-Order Scheme

Here, we construct a fourth-order family of Steffensen-type method [12] for multiple zeros $(p \geq 2)$, which is defined by

$$\begin{aligned} y_t &= x_t - p\frac{f(x_t)}{f[\eta_t, \theta_t]} \\ x_{t+1} &= y_t - G(\kappa_t)\frac{f(x_t)}{f[\eta_t, \theta_t]}, \end{aligned} \tag{3}$$

where $\eta_t = x_t + \gamma f(x_t)$, $\theta_t = x_t - \gamma f(x_t)$, $\gamma \in \mathbb{R}, \gamma \neq 0$, and $p \geq 2$ is known as the multiplicity of the required zero. In addition, $G(\kappa_t)$ is a single variable weight function and $f[\eta_t, \theta_t]$ is a finite difference of order one and is provided by $f[\eta_t, \theta_t] = \frac{f(\eta_t)-f(\theta_t)}{\eta_t - \theta_t}$. Moreover, $\kappa_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{p}}$, is a multi-valued function. Suppose their principal analytic branches (see [20]), s_t as a principal root provided by $\kappa_t = \exp\left[\frac{1}{p}\log\left(\frac{f(y_t)}{f(x_t)}\right)\right]$, with $\log\left(\frac{f(y_t)}{f(x_t)}\right) = \log\left|\frac{f(y_t)}{f(x_t)}\right| + i\arg\left(\frac{f(y_t)}{f(x_t)}\right)$ for $-\pi < \arg\left(\frac{f(y_t)}{f(x_t)}\right) \leq \pi$. The choice of $\arg(z)$ for $z \in \mathbb{C}$ agrees with that of $\log(z)$ to be employed later in the numerical experiments of section. We have an analogous way $\kappa_t = \left|\frac{f(y_t)}{f(x_t)}\right|^{\frac{1}{p}}.\exp\left[\frac{1}{p}\arg\left(\frac{f(y_t)}{f(x_t)}\right)\right] = O(e_t)$.

In Theorem 1, we illustrate that the constructed scheme (3) attains a maximum fourth-order of convergence for all $\gamma \neq 0$, without adopting the evaluation of the derivative.

Theorem 1. *Let us consider function $f : \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ an analytic function in $\mathbb{D}$ surrounding the required zero x_*, which is a solution of multiplicity $p = 2$ of equation $f(x) = 0$. Then, the scheme (3) has fourth-order convergence, when*

$$G(0) = 0, \quad G'(0) = 2, \quad G''(0) = 8, \quad |G'''(0)| < \infty, \tag{4}$$

and satisfies the following error equation

$$e_{t+1} = \left(-\frac{\left(G'''(0) - 66\right)B_1^2}{96} - \frac{1}{4}B_1B_2 \right) e_t^4 + O(e_t^5),$$

where $e_t = x_t - x_*$ and $B_i = \frac{2!}{(2+i)!}\frac{f^{(2+i)}(x_*)}{f^{(2)}(x_*)}$, $i = 1, 2, 3, \ldots$ are the errors in the t–th iteration and the asymptotic error constant numbers, respectively.

Proof. We develop the Taylor's series expansions for the functions $f(x_t)$, $f(\eta_t)$ and $f(\theta_t)$ around $x = x_*$ with the assumption $f(x_*) = f'(x_*) = 0$ and $f^{(2)}(x_*) \neq 0$, which are given, respectively, by

$$f(x_t) = \frac{f^{(2)}(x_*)}{2!}e_t^2\left(1 + B_1e_t + B_2e_t^2 + B_3e_t^3 + B_4e_t^4 + O(e_t^5)\right) \tag{5}$$

$$\begin{aligned} f(\eta_t) =& \frac{f^{(2)}(x_*)}{2!}e_t^2\Big[1 + \left(\gamma\Delta_2 + B_1\right)e_t + \frac{1}{4}\left(10B_1\gamma\Delta_2 + 4B_2 + \gamma^2\Delta_2^2\right)e_t^2 \\ &+ \frac{1}{4}\left(5B_1\gamma^2\Delta_2^2 + 6B_1^2\gamma\Delta_2 + 4(3B_2\gamma\Delta_2 + B_3)\right)e_t^3 \\ &+ \frac{1}{8}\left(B_1(28B_2\gamma\Delta_2 + \gamma^3\Delta_2^3) + 4(4B_2\gamma^2\Delta_2^2 + 7B_3\gamma\Delta_2 + 2B_4) + 14B_1^2\gamma^2\Delta_2^2\right)e_t^4 + O(e_t^5)\Big], \end{aligned} \tag{6}$$

and

$$\begin{aligned} f(\theta_t) =& \frac{f^{(2)}(x_*)}{2!}e_t^2\Big[1 + \left(B_1 - \gamma\Delta_2\right)e_t + \frac{1}{4}\left(-10B_1\gamma\Delta_2 + 4B_2 + \gamma^2\Delta_2^2\right)e_t^2 \\ &+ \frac{1}{4}\left(5B_1\gamma^2\Delta_2^2 - 6B_1^2\gamma\Delta_2 + 4(B_3 - 3B_2\gamma\Delta_2)\right)e_t^3 \\ &+ \frac{1}{8}\left(14B_1^2\gamma^2\Delta_2^2 - B_1(28B_2\gamma\Delta_2 + \gamma^3\Delta_2^3) + 4(4B_2\gamma^2\Delta_2^2 - 7B_3\gamma\Delta_2 + 2B_4)\right)e_t^4 + O(e_t^5)\Big], \end{aligned} \tag{7}$$

where $\Delta_2 = f^{(2)}(x_*)$.

By adopting expressions (5)–(7), we obtain further

$$\frac{f(x_t)}{f[\eta_t, \theta_t]} = \frac{1}{2}e_t - \frac{B_1}{4}e_t^2 + \frac{1}{8}(3B_1^2 - 4B_2)e_t^3 + \frac{1}{16}\left(B_1\left(20B_2 - \gamma^2\Delta_2^2\right) - 9B_1^3 - 12B_3\right)e_t^4 + O(e_t^5), \tag{8}$$

and, by using the expression (8) in the first step of (3), we have

$$y_t = \frac{B_1}{2}e_t^2 + \left(B_2 + \frac{3}{4}B_1^2\right)e_t^3 + \frac{1}{8}\left(B_1(\gamma^2\Delta_2^2 - 20B_2) + 9B_1^3 + 12B_3\right)e_t^4 + O(e_t^5). \tag{9}$$

Now, we use the expression (9), so we obtain

$$\begin{aligned} f(y_t) =& \frac{1}{2}\Delta_2e_t^2\Big[\frac{1}{4}B_1^2e_t^2 + B_1\left(B_2 - \frac{3B_1^2}{4}\right)e_t^3 \\ &+ \frac{1}{16}\left(2B_1^2(\gamma^2\Delta_2^2 - 32B_2) + 29B_1^4 + 24B_3B_1 + 16B_2^2\right)e_t^4 + O(e_t^5)\Big], \end{aligned} \tag{10}$$

and

$$\kappa_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{2}} = \frac{B_1}{2}e_t + (B_2 - B_1^2)e_t^2 + \frac{1}{16}\Big(2B_1(\gamma^2\Delta_2^2 - 26B_2) + 29B_1^3 + 24B_3\Big)e_t^3 + O(e_t^4). \tag{11}$$

The expression (11) demonstrates that $\kappa_t = O(e_t)$. So, we expand the weight function $G(\kappa_t)$ in the neighborhood of the origin obtaining:

$$G(\kappa_t) = G(0) + G'(0)\kappa_t + \frac{1}{2!}G''(0)\kappa_t^2 + \frac{1}{3!}G'''(0)\kappa_t^3. \tag{12}$$

Now, we have the following expression by inserting Equation (11) in the scheme (3)

$$\begin{aligned} e_{t+1} = &-\frac{G(0)}{2}e_t + \frac{1}{4}B_1\Big(G(0) - G'(0) + 2\Big)e_t^2 \\ &+ \frac{1}{16}\Big[8B_2\Big(G(0) - G'(0) + 2\Big) - B_1^2\Big(6G(0) - 10G'(0) + G''(0) + 12\Big)\Big]e_t^3 \\ &+ \Omega_1 e_t^4 + O(e_t^5), \end{aligned} \tag{13}$$

where Ω_1 is a function that depends on the parameters defined previously,

$$\Omega_1\Big(\gamma, \Delta_2, B_1, B_2, B_3, G(0), G'(0), G''(0), G''(0)\Big).$$

From expression (13), we deduce that the scheme (3) reaches at the least fourth-order convergence, if

$$\begin{aligned} G(0) &= 0, \\ G(0) - G'(0) + 2 &= 0, \\ 6G(0) - 10G'(0) + G''(0) + 12 &= 0 \end{aligned}$$

which further provide

$$G(0) = 0, \;\; G'(0) = 2, \;\; G''(0) = 8. \tag{14}$$

Next, by inserting above expression (14) in (12), we obtain

$$e_{t+1} = \left(-\frac{1}{96}B_1^3(G'''(3) - 66) - \frac{B_2B_1}{4}\right)e_t^4 + O(e_p^5), \tag{15}$$

provided $|G'''(0)| < \infty$. Hence, the scheme (3) has fourth-order convergence for $p = 2$. □

Theorem 2. *Let us consider function $f : \mathbb{D} \subset \mathbb{C} \to \mathbb{C}$ an analytic function in $\mathbb{D}$ surrounding the required zero x_*, which is a solution of multiplicity $p \geq 3$ of equation $f(x) = 0$. Then, the scheme (3) has fourth-order convergence, when*

$$G(0) = 0, \;\; G'(0) = p, \;\; G''(0) = 4p, \;\; |G'''(0)| < \infty, \tag{16}$$

and satisfies the following error equation

$$e_{t+1} = \left(-\frac{\Big(G'''(0) - 3p(p+9)\Big)M_1^2}{6p^4} - \frac{1}{p^2}M_1M_2\right)e_t^4 + O(e_t^5),$$

where $e_t = x_t - x_*$ and $M_i = \frac{p!}{(p+i)!}\frac{f^{(p+i)}(x_*)}{f^{(p)}(x_*)}$, $i = 1, 2, 3, \ldots$ are the errors in t-th iteration and asymptotic error constant numbers, respectively.

Proof. We obtain Taylor's series expansions for the functions $f(x_t)$, $f(\eta_t)$, and $f(\theta_t)$ around $x = x_*$ with the assumption $f(x_*) = f'(x_*) = f''(x_*) = \cdots = f^{(p-1)}(x_*) = 0$ and $f^{(p)}(x_*) \neq 0$, which are provided by, respectively,

$$f(x_t) = \frac{f^{(p)}(x_*)}{p!} e_t^p \Big(1 + M_1 e_t + M_2 e_t^2 + M_3 e_t^3 + M_4 e_t^4 + O(e_p^5)\Big) \tag{17}$$

$$f(\eta_t) = \frac{f^p(x_*)}{p!} e_t^p \Big[1 + M_1 e_t + \Gamma_1 e_t^2 + \Gamma_2 e_t^3 + \Gamma_3 e_t^4 + O(e_t^5)\Big], \tag{18}$$

and

$$f(\theta_t) = \frac{f^p(x_*)}{p!} e_t^p \Big[1 + M_1 e_t + \bar{\Gamma}_1 e_t^2 + \bar{\Gamma}_2 e_t^3 + \bar{\Gamma}_3 e_t^4 + O(e_t^5)\Big], \tag{19}$$

where $\Delta_p = f^{(p)}(x_*), p = 3,4,5,\ldots$, $\Gamma_i = \Gamma_i\Big(\gamma, \Delta_p, M_1, M_2, M_3, M_4\Big)$, and $\bar{\Gamma}_i = \bar{\Gamma}_i\Big(\gamma, \Delta_p, M_1, M_2, M_3, M_4\Big)$; some of them are provided below:

$$\Gamma_1 = \begin{cases} \frac{1}{2}(\gamma\Delta_3 + 2M_2), & p = 3 \\ M_2 & p \geq 4 \end{cases},$$

and

$$\bar{\Gamma}_1 = \begin{cases} \frac{1}{2}(-\gamma\Delta_3 + 2M_2), & p = 3 \\ M_2 & p \geq 4 \end{cases}.$$

By adopting expressions (17)–(19), we obtain further

$$\begin{aligned} \frac{f(x_t)}{f[\eta_t, \theta_t]} = {} & \frac{1}{p} e_t - \frac{M_1}{p^2} e_t^2 + \frac{1}{p^3}\Big((p+1)M_1^2 - 2pM_2\Big)e_t^3 \\ & - \frac{1}{p^4}\Big[(p+1)^2 M_1^3 - p(3p+4)M_1M_2 + 3p^2 M_3\Big]e_t^4 + O(e_p^5). \end{aligned} \tag{20}$$

For the expression (20) used in the first substep of (3), we have

$$\begin{aligned} y_t = e_{y_t} = {} & \frac{M_1}{p} e_t^2 - \frac{1}{p^2}\Big((p+1)M_1^2 - 2pM_2\Big)e_t^3 \\ & + \frac{1}{p^3}\Big[(p+1)^2 M_1^3 - p(3p+4)M_1M_2 + 3p^2 M_3\Big]e_t^4 + O(e_p^5). \end{aligned} \tag{21}$$

Now, use the expression (21), we obtain

$$f(y_t) = \frac{f^{(p)}(x_*)}{p!} e_{y_t}^p \Big(1 + M_1 e_{y_t} + M_2 e_{y_t}^2 + M_3 e_{y_t}^3 + M_4 e_{y_t}^4 + O(e_p^5)\Big) \tag{22}$$

and

$$\begin{aligned} \kappa_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{p}} = {} & \frac{M_1}{2} e_t + (M_2 - M_1^2)e_t^2 + \frac{1}{16}\Big(2M_1(\gamma^2\Delta_2^2 - 26M_2) + 29M_1^3 + 24M_3\Big)e_t^3 \\ & + O(e_t^4). \end{aligned} \tag{23}$$

The expression (23) demonstrates that $\kappa_t = O(e_t)$. So, we expand the weight function $G(\kappa_t)$ in the neighborhood of the origin in the following way

$$G(\kappa_t) = G(0) + G'(0)\kappa_t + \frac{1}{2!}G''(0)\kappa_t^2 + \frac{1}{3!}G'''(0)\kappa_t^3. \tag{24}$$

We have the following expression by inserting equation (23) in the scheme (3)

$$e_{t+1} = -\frac{G(0)}{p}e_t + \frac{1}{p^2}M_1\Big(G(0) - G'(0) + p\Big)e_t^2 + \frac{1}{2p^3}\Big[4pM_2\Big(G(0) - G'(0) + p\Big) - M_1^2\Big(G''(0) + 2(p+1)G(0) - 2(p+3)G'(0) + 2p(p+1)\Big)\Big]e_t^3 + Ce_t^4 + O(e_t^5), \tag{25}$$

where C depends on parameters defined before, $C\Big(\gamma, \Delta_p, M_1, M_2, M_3, G(0), G'(0), G''(0), G''(0)\Big)$.

From the expression (25), we deduce that the scheme (3) reaches at least fourth-order convergence, if

$$G(0) = 0,$$
$$G(0) - G'(0) + p = 0,$$
$$G''(0) + 2(p+1)G(0) - 2(p+3)G'(0) + 2p(p+1) = 0$$

which further provide

$$G(0) = 0, \quad G'(0) = p, \quad G''(0) = 4p. \tag{26}$$

Next, by inserting above expression (26) in (24), we obtain

$$e_{t+1} = \left(-\frac{\Big(G'''(3) - 3m(m+9)\Big)}{6p^4}M_1^3 - \frac{M_2M_1}{p^2}\right)e_t^4 + O(e_p^5), \tag{27}$$

provided $|G'''(0)| < \infty$. Hence, the scheme (3) has fourth-order convergence for $p \geq 3$. □

3. Special Cases

In this section, we want to show that we can develop as many new derivative-free methods for multiple roots as we can build weight functions. However, all the weight functions should satisfy the conditions of Theorem 1. Some of the important special cases are mentioned in Table 1.

Table 1. Some special cases of our scheme (3).

Cases (Naming)	Weight Functions	Corresponding Iterative Method
Case-1 (RM1)	$G(\kappa_t) = \frac{p\kappa_t}{1-2\kappa_t}$	$y_t = x_t - p\frac{f(x_t)}{f[\eta_t,\theta_t]}$ $x_{t+1} = y_t - \frac{p\kappa_t}{1-2\kappa_t}\frac{f(x_t)}{f[\eta_t.\theta_t]}$.
Case-2 (RM2)	$G(\kappa_t) = \frac{p\kappa_t}{1-2\kappa_t-\kappa_t^2}$	$y_t = x_t - p\frac{f(x_t)}{f[\eta_t,\theta_t]}$ $x_{t+1} = y_t - \frac{p\kappa_t}{1-2\kappa_t-\kappa_t^2}\frac{f(x_t)}{f[\eta_t,\theta_t]}$.
Case-3 (RM3)	$G(\kappa_t) = \frac{p\kappa_t+\kappa_t^2}{1-2\kappa_t} - \kappa_t^2$	$y_t = x_t - p\frac{f(x_t)}{f[\eta_t,\theta_t]}$ $x_{t+1} = y_t - (\frac{p\kappa_t+\kappa_t^2}{1-2\kappa_t} - \kappa_t^2)\frac{f(x_t)}{f[\eta_t,\theta_t]}$.
Case-4 (RM4)	$G(\kappa_t) = \frac{p\kappa_t}{1-2\kappa_t+\kappa_t^2}$	$y_t = x_t - p\frac{f(x_t)}{f[\eta_t,\theta_t]}$ $x_{t+1} = y_t - \frac{p\kappa_t}{1-2\kappa_t+\kappa_t^2}\frac{f(x_t)}{f[\eta_t,\theta_t]}$.
Case-5 (RM5)	$G(\kappa_t) = \frac{p}{1-\kappa_t} - p + p\kappa_t^2$	$y_t = x_t - p\frac{f(x_t)}{f[\eta_t,\theta_t]}$ $x_{t+1} = y_t - (\frac{p}{1-\kappa_t} - p + p\kappa_t^2)f[\eta_t.\theta_t]$.
Case-6 (RM6)	$G(\kappa_t) = p\kappa_t + 2p\kappa_t^2$	$y_t = x_t - p\frac{f(x_t)}{f[\eta_t,\theta_t]}$ $x_{t+1} = y_t - (p\kappa_t + 2p\kappa_t^2)\frac{f(x_t)}{f[\eta_t,\theta_t]}$
Case-7 (RM7)	$G(\kappa_t) = p\exp 2\kappa_t - \sin\kappa_t - 1$	$y_t = x_t - p\frac{f(x_t)}{f[\eta_t,\theta_t]}$ $x_{t+1} = y_t - (p\exp 2\kappa_t - \sin\kappa_t - 1)\frac{f(x_t)}{f[\eta_t,\theta_t]}$

4. Numerical Results

In this section, the efficiency and convergence of the newly generated methods are checked on some nonlinear problems. For this purpose, we consider RM1, RM2, RM3, and RM4 methods. We compared them with other existing derivative free and fourth-order convergence methods.

First of all, we compare them with the following fourth-order method proposed by Kumar et al. [19]:

$$\begin{aligned} y_t &= x_t - p\frac{f(x_t)}{f[\eta_t, x_t]} \\ x_{t+1} &= y_t - \left(\frac{(p+2)\kappa_t}{1-2\kappa_t}\right)\frac{f(x_t)}{f[\eta_t, x_t] + 2f[\eta_t, y_t]}, \end{aligned} \tag{28}$$

where $\eta_t = x_t + \gamma f(x_t),\ \gamma \neq 0 \in \mathbb{R}$, $p \geq 2$ is the known multiplicity of the required zero, $\kappa_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{p}}$ is multi-valued functions, and $f[\eta_t, x_t] = \frac{f(\eta_t)-f(x_t)}{\eta_t - x_t}$ is finite difference of order one. We called the expression (28) using $(SM2)$.

We also contrast them with a fourth-order scheme provided by Zafar et al. [21], which is defined as follows:

$$\begin{aligned} y_t &= x_t - p\frac{f(x_t)}{f'(x_t) + \frac{p}{2}f(x_t)} \\ x_{t+1} &= y_t - p\kappa_t(\frac{11}{2}(\kappa_t)^2 + 2\kappa_t + 1)\frac{f(x_t)}{f'(x_t) + 2pf(x_t)}, \end{aligned} \tag{29}$$

where $\kappa_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{p}}$ and $p \geq 2$ is the known multiplicity of the required zero. We symbolized the scheme (29) by (ZM).

We chose a fourth-order method presented by Behl et al. [22], which is provided below:

$$\begin{aligned} y_t &= x_t - p\frac{f(x_t)}{f[\eta_t, x_t]} \\ x_{t+1} &= x_t + p\left(1 + \frac{\kappa_t}{1-2\alpha\kappa_t}\right)\left(\frac{1}{2}\kappa_t - (1 + \frac{s_t}{2} + 2(1-\alpha)s_t^2)\right)\frac{f(x_t)}{f[\eta_t, x_t]}, \end{aligned} \tag{30}$$

where $\eta_t = x_t + \gamma f(x_t),\ \gamma \neq 0 \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $p \geq 2$ is the known multiplicity of the required zero, and $f[\eta_t, x_t] = \frac{f(\eta_t)-f(x_t)}{\eta_t - x_t}$ is finite difference of order one. Further, $\kappa_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{p}}$ and $s_t = \left(\frac{f(y_t)}{f(\eta_t)}\right)^{\frac{1}{p}}$ are multi-valued functions. We denoted the scheme (30) using (BM).

We consider the following method provided by Kansal et al. [23]:

$$\begin{aligned} y_t &= x_t - p\frac{f(x_t)}{f'(x_t)} \\ x_{t+1} &= y_t - p\kappa_t\left(1 + 2s_t + \frac{13}{2}s_t^2\right)\frac{f(x_t)}{f'(x_t)}, \end{aligned} \tag{31}$$

where $\kappa_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{p}}$ and $s_t = \frac{\kappa_t}{1+\kappa_t}$ are multi-valued functions. We called the scheme (30) using (MKM).

The computational work is performed on Mathematica programming software [24] by selecting the value of the parameter $\gamma = 0.01$. The numerical results are depicted in Tables 2–5. The tables include the number of iterations required to obtain the root with stopping criterion $|x_{t+1} - x_t| + |f(x_t)| < 10^{-200}$, estimated errors $|x_{t+1} - x_t|$, and residual errors of the considered function $|f(x_t)|$. In addition, the computational order of convergence (COC) by using the proceeding formula:

$$COC = \frac{log|\frac{x_{t+2}-\alpha}{x_{t+1}-\alpha}|}{log|\frac{x_{t+1}-\alpha}{x_t-\alpha}|}, \text{where } t = 1, 2, \ldots \quad (32)$$

In order to illustrate the applicability of our scheme, we chose the following four real life problems. The considered problems are mentioned in Examples 1–4, which are defined as follows:

Example 1. *Van Der Waals equation of state:*

In 1873, Van Der Waals modified the Ideal Gas Law ($PV = nRT$) when he realised no gas in the universe is ideal. He just adjusted the Pressure and Volume. This equation of state is presented as

$$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT,$$

where P is the pressure, V is the volume, R is the universal gas constant, and T is the absolute temperature. The constants a and b represent the magnitude of intermolecular attraction and excluded, respectively. These constants are specific to a particular gas. The above equation can also be written as

$$PV^3 - (nbP + nRT)V^2 + \alpha n^2 V - \alpha\beta n^2 = 0.$$

For a particular gas, the problem reduces to the following polynomial in x of degree three

$$f(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675. \quad (33)$$

According to the Fundamental Theorem of arithmetic, the above polynomial has three roots and, among them, $x = 1.75$ is a multiple root of multiplicity $p = 2$ and $x = 1.72$ is a simple zero. The numerical results for this problem are mentioned in Table 2.

Example 2. *Planck's radiation problem:*

Consider the Planck's radiation equation that determines the spectral density E_λ of electromagnetic radiations emitted by a black body in the thermal equilibrium at a definite temperature [25] as

$$E_\lambda = \frac{8\pi ch}{\lambda^5} \times \frac{1}{e^{\frac{ch}{kT\lambda}} - 1},$$

where $T, \lambda, k, h,$ and c are, respectively, the absolute temperature of the black body, the wavelength of radiation, the Boltzmann constant, the Plank's constant, and the speed of light in the medium (vacuum).

To evaluate the wavelength λ, for which the energy density E_λ is maximum, the necessary condition is $E'_\lambda = 0$, provides us with the following equation:

$$\frac{(\frac{ch}{\lambda kT})e^{\frac{ch}{\lambda kT}}}{e^{\frac{ch}{\lambda kT}} - 1} = 5.$$

Using $x = \frac{ch}{\lambda kT}$, the corresponding non-linear function is as follows:

$$f(x) = \left(e^{-x} - 1 + \frac{x}{5}\right)^p. \quad (34)$$

The approximated zero of $f(x)$ is $x_* = 4.96511423174427630369875 9$ with multiplicity $p = 4$ and, by using this solution, one can easily determine the wave length λ from the relation $x = \frac{ch}{\lambda kT}$. The computational results are provided in Table 3.

Example 3. *Beam Designing Model*

Consider a beam designing problem where a beam of length r unit is leaning against the edge of a cubical box with sides of length 1 unit each, such that one end of the beam touches the wall and the other end touches the floor. What should the distance along the floor from the base of the wall to the bottom of the beam be? Suppose y is the distance along the beam from the floor to the edge of the box, and let x be the distance from the bottom of the box to the bottom of the beam. Then for a given value of r, we have

$$x^4 + 4x^3 - 24x^2 + 16x + 16 = 0.$$

The solution $x = 2$ of the above equation is a double root. We consider the initial guess $x_* = 1.8$ to find the root. Table 4 shows the numerical results of various methods corresponding to the problem.

Example 4. *Isothermal Continuous Stirred Tank Reactor Problem [26]*

The test equation corresponding to this problem is provided as follows:

$$x^4 + 11.50x^3 + 47.49x^2 + 83.06325x + 51.23266875 = 0.$$

The solution of the equation is $x = -2.85$ with multiplicity 2. We consider the initial guess as $x_* = 2.7$ to find the root. The numerical results of various methods for this problem are shown in Table 5.

Table 2. Comparison of different iterative methods based on Example (1).

Methods	t	$\|e_{t-2}\|$	$\|e_{t-1}\|$	$\|e_t\|$	$\|f(x_{t+1})\|$	COC
RM1	7	6.11×10^{-26}	9.67×10^{-98}	6.08×10^{-385}	2.71×10^{-3068}	4.0
RM2	7	3.24×10^{-32}	2.57×10^{-123}	1.00×10^{-487}	1.66×10^{-3891}	4.0
RM3	7	2.55×10^{-30}	9.77×10^{-116}	2.11×10^{-457}	6.39×10^{-3649}	4.0
RM4	7	3.44×10^{-22}	1.63×10^{-82}	8.13×10^{-324}	7.65×10^{-2579}	4.0
ZM	8	1.61×10^{-48}	3.95×10^{-189}	5.36×10^{-376}	7.09×10^{-2999}	6.0
BM ($\alpha = 0$)	7	1.21×10^{-19}	3.52×10^{-72}	2.48×10^{-282}	1.14×10^{-2246}	4.0
($\alpha = 0$)	7	6.04×10^{-26}	9.22×10^{-98}	5.01×10^{-385}	5.78×10^{-3069}	4.0
MKM	7	3.47×10^{-22}	6.70×10^{-83}	9.33×10^{-326}	3.68×10^{-2595}	4.0
SKM	7	7.75×10^{-28}	1.67×10^{-105}	3.59×10^{-416}	1.77×10^{-3318}	4.0

Table 3. Comparison of different iterative methods based on Example (2).

Methods	t	$\|e_{t-2}\|$	$\|e_{t-1}\|$	$\|e_t\|$	$\|f(x_{t+1})\|$	COC
RM1	5	6.78×10^{-26}	2.43×10^{-105}	3.99×10^{-423}	9.96×10^{-6778}	4.0
RM2	5	5.63×10^{-26}	1.09×10^{-105}	1.56×10^{-424}	2.29×10^{-6800}	4.0
RM3	5	6.19×10^{-26}	1.64×10^{-105}	8.12×10^{-424}	7.79×10^{-6789}	4.0
RM4	5	8.11×10^{-26}	5.23×10^{-105}	9.04×10^{-422}	5.82×10^{-6756}	4.0
ZM	7	1.55×10^{-33}	2.73×10^{-132}	7.74×10^{-264}	1.11×10^{-4214}	6.0
BM ($\alpha = 0$)	5	9.61×10^{-26}	1.08×10^{-104}	1.72×10^{-420}	2.01×10^{-6735}	4.0
($\alpha = 1$)	5	6.75×10^{-26}	2.38×10^{-105}	3.71×10^{-423}	3.07×10^{-6778}	4.0
MKM	5	6.30×10^{-26}	1.76×10^{-105}	1.07×10^{-423}	6.52×10^{-6787}	4.0
SKM	5	6.77×10^{-26}	2.42×10^{-105}	3.90×10^{-423}	7.03×10^{-6778}	4.0

Table 4. Comparison of different iterative methods based on Example (3).

Methods	t	$\lvert e_{t-2} \rvert$	$\lvert e_{t-1} \rvert$	$\lvert e_t \rvert$	$\lvert f(x_{t+1}) \rvert$	COC
RM1	**6**	1.71×10^{-18}	1.56×10^{-73}	1.09×10^{-293}	1.57×10^{-2346}	4.0
RM2	**6**	3.97×10^{-22}	6.44×10^{-89}	4.48×10^{-356}	2.65×10^{-2847}	4.0
RM3	**6**	2.09×10^{-21}	4.93×10^{-86}	1.54×10^{-344}	5.02×10^{-2755}	4.0
RM4	**6**	1.71×10^{-17}	2.87×10^{-69}	2.29×10^{-276}	2.08×10^{-2207}	4.0
ZM	10	1.06×10^{-35}	1.31×10^{-140}	2.56×10^{-280}	4.92×10^{-2236}	6.0
BM ($\alpha = 0$)	7	4.22×10^{-45}	6.32×10^{-179}	3.18×10^{-714}	9.96×10^{-5709}	4.0
($\alpha = 1$)	7	9.11×10^{-26}	6.14×10^{-51}	4.75×10^{-203}	6.70×10^{-809}	1.3
MKM	6	1.20×10^{-18}	2.20×10^{-74}	2.42×10^{-297}	3.07×10^{-2376}	4.0
SKM	8	7.55×10^{-50}	1.37×10^{-198}	1.40×10^{-396}	6.22×10^{-3169}	6.0

Table 5. Comparison of different iterative methods based on Example (4).

Methods	t	$\lvert e_{t-2} \rvert$	$\lvert e_{t-1} \rvert$	$\lvert e_t \rvert$	$\lvert f(x_{t+1}) \rvert$	COC
RM1	6	1.37×10^{-49}	2.02×10^{-198}	9.39×10^{-794}	4.12×10^{-6349}	4.0
RM2	6	1.36×10^{-49}	1.94×10^{-198}	8.12×10^{-794}	1.27×10^{-6349}	4.0
RM3	6	1.36×10^{-49}	1.94×10^{-198}	8.11×10^{-794}	1.27×10^{-6349}	4.0
RM4	6	1.38×10^{-49}	2.09×10^{-198}	1.09×10^{-793}	1.33×10^{-6348}	4.0
ZM	6	3.15×10^{-35}	9.46×10^{-70}	2.24×10^{-277}	4.84×10^{-1107}	1.3
BM ($\alpha = 0$)	6	4.90×10^{-48}	4.80×10^{-192}	4.41×10^{-768}	2.10×10^{-6143}	4.0
($\alpha = 1$)	6	4.44×10^{-48}	3.18×10^{-192}	8.31×10^{-769}	3.17×10^{-6149}	4.0
MKM	6	1.37×10^{-49}	1.97×10^{-198}	8.62×10^{-794}	2.07×10^{-6349}	4.0
SKM	6	4.43×10^{-48}	3.14×10^{-192}	7.93×10^{-769}	2.18×10^{-6149}	4.0

5. Concluding Remarks

- We presented new derivative-free and multi-point iterative techniques that can handle multiple zeros $(p \geq 2)$ of nonlinear models.
- Divided difference and weight function approaches are the main pillar where the construction of our scheme lies.
- Our expression (3) consuming is an optimal scheme in the regard of Kung–Traub conjecture. Because, it adopts only three values of f at different points.
- Many new weight functions are depicted in Table 1 that satisfy the hypotheses of the Theorems 1 and 2. These new weight functions also correspond to new iterative techniques.
- Our techniques provide better numerical solutions in terms of the residual errors, stable COC, absolute error between two iterations, and number iterations as compared to the existing ones (see Tables 2–5). We have emphasized, in the Tables, the better result in all problems and coincides in Tables 2 and 3 with the new method RM2, while, in Table 5, we remark that new methods perform less iterations for reaching the same tolerance than the known ones.
- Finally, we wind up with this statement that "our schemes is a good alternative to the existing methods". Our scheme is not valid for the solutions of nonlinear system. In the future, we will try to work on this direction.

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