Local convergence of a family of higher order iterative methods for solving nonlinear equations in Banach spaces

Eulalia Martínez, Instituto de Matemática Pura y Aplicada, Universitat Politècnica de València[1]
Sukhjit Singh, Department of Mathematics,Indian Institute of Technology Kharagpur[2]
José L. Hueso, Instituto de Matemática Multidisciplinar, Universitat Politècnica de València[3]
Dharmendra K. Gupta, Department of Mathematics,Indian Institute of Technology Kharagpur[4]

Abstract

A local convergence result of a family of higher order iterative methods for solving nonlinear equations in Banach spaces is established under the assumption that the Fréchet derivative satisfies the Lipschitz continuity condition. For some values of the parameter, these iterative methods are of fifth order. The importance of our work is that it avoids the usual practice of boundedness conditions of higher order derivatives which is a drawback for solving some practical problems. The existence and uniqueness theorem that establishes the convergence balls of these methods is obtained.

We have considered some numerical examples including a nonlinear Hammerstein equation and computed the radii of the convergence balls. It is found that the radius of convergence ball obtained by our approach is much larger when compared with some other existing methods. The global convergence properties of the family are explored by analyzing the dynamics of the corresponding operator on complex quadratic polynomials.

Keywords: Nonlinear equations; Local convergence; Banach space; Hammerstein integral equation; Lipschitz condition; Dynamical Systems; Complex dynamics.
Mathematical Subject Classification 65H10, 47H99

1. Introduction

One of the most important problems of Numerical Functional Analysis is to solve nonlinear equations in Banach spaces. This has become necessitated as the mathematical modeling [7, 8, 9, 19] of a large number of problems of science and engineering involving scalar equations, system of equations, differential equations, integral equations,etc, reduced to thousands of such equations. One such example is the dynamical systems which are mathematically modeled by difference or differential equations and their solutions usually represent the states of the systems. With the advancement in computer S/W and H/W, this problem has gained an added advantage. Generally, iterative methods along with their local and semilocal convergence analysis are used for them. The local convergence analysis[13, 3] uses information around the solution whereas semilocal convergence analysis[11, 10, 5] is based on information around an initial point. Another important problem which is to be considered for these iterative methods is the convergence domains/radii of convergence balls. In general, the convergence domain of an iterative

[1]eumarti@mat.upv.es
[2]sukhjit@maths.iitkgp.ernet.in
[3]jlhueso@mat.upv.es
[4]dkg@maths.iitkgp.ernet.in

method is small and one always tries to enlarge it by considering additional hypothesis. It is worth mentioning that most of the local convergence results are obtained under general conditions that, by using Taylor's expansions allow us to find the convergence order but not the radii of the convergence balls (see, [20, 21, 22]) and references cited therein.

The aim of this paper is to describe local convergence analysis for a unique solution x^* of nonlinear operator equation

$$F(x) = 0 \tag{1.1}$$

where, F is a Fréchet-differentiable operator defined on a subset D of a Banach space X with values in a Banach space Y. Starting from one or several initial approximations of x^*, a sequence $\{x_k\}$ of approximations is constructed so that it converges to x^*. The sequence $\{x_k\}$ can be obtained in different ways depending on the iterative method that is applied. The well known quadratically convergent Newton's method [12] is the most widely used iterative methods to solve (1.1). Starting with x_0, it is given, for $k = 0, 1, 2, \dots$, by

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k) \tag{1.2}$$

Iterative methods of higher order convergence require evaluation of higher order derivatives which are very expansively in general. For example, the third order Chebyshev-Halley type methods [6] require evaluation of second Frëchet derivative which either do not exist or computationally difficult to evaluate. But higher order methods have their importance as in some applications involving stiff system of equations need faster convergence. Also, there are integral equations where the second Frëchet derivative is diagonal by blocks and inexpensive [16]. The local convergence analysis of a family of third order iterative methods for nonlinear equations in Banach spaces is established in [2] for the method described in [18]. Argyros et al. [15] considered multi-point-parametric Chebyshev-Halley-type methods of high convergence order involving Fréchet derivative and discussed their local convergence analysis in Banach spaces. The local convergence analysis of a modified Halley-Like method of high convergence order is described in [17]. Starting with initial starting point x_0, it is defined for $k = 0, 1, 2, \dots$ by

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ u_k &= y_k + (1-a)F'(x_k)^{-1}F(x_k), \\ z_k &= y_k - \gamma A_{a,k}F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= z_k - \alpha B_{a,k}F'(x_k)^{-1}F(z_k), \end{aligned} \tag{1.3}$$

where, $\alpha, \gamma, a \in (-\infty, \infty) - \{0\}$, $H_{a,k} = \frac{1}{a}F'(x_k)^{-1}\Big(F'(u_k) - F'(x_k)\Big)$, $A_{a,k} = I - \frac{1}{2}H_{a,k}\Big(I - \frac{1}{2}H_{a,k}\Big)$, $B_{a,k} = I - H_{1,k} + H_{a,k}^2$. Recently, a local convergence analysis along with the dynamics of Chebyshev-Halley-type methods free from second derivatives is described in [4]. Starting with an initial approximation x_0, it is given for $k = 0, 1, 2 \dots$ by

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ z_k &= x_k - \Big(1 + (F(x_k) - 2\alpha F(y_k))^{-1}F(y_k)\Big)F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= z_k - \Big(F'(x_k) + \bar{F}''(x_k)(z_k - x_k)\Big)^{-1}F(z_k), \quad k \geq 0, \end{aligned} \tag{1.4}$$

where, $\bar{F}''(x_k) = 2F(y_k)F'(x_k)^2F(x_k)^{-2}$ and α is a parameter.

In this paper, a local convergence analysis of a family of iterative methods for solving nonlinear equations in Banach spaces is established under the assumption that the first Fréchet derivative satisfies the Lipschitz continuity condition. For the values of the parameter $a = \pm 1$, these iterative methods are of fifth order. The importance of our work is that it avoids the usual practice of boundedness conditions of higher order derivatives which is a drawback for solving some practical problems. The existence and uniqueness theorem that establishes the convergence balls of these methods is obtained.

We have considered some numerical examples including a nonlinear Hammerstein equation and computed the radii of the convergence balls. It is found that the radius of convergence ball obtained by our approach is much larger when compared with some other existing methods.

Finally, the complex dynamics of the family is studied for some parameter values, by analyzing the attraction basins of the iterative scheme for complex quadratic polynomials.

2. Method and its local convergence analysis

In this section, we describe the iterative method and its local convergence analysis to solve (1.1). Consider the family iterative methods defined in [1] for $k = 0, 1, 2, \dots$ by

$$\begin{aligned} y_k &= x_k - aF'(x_k)^{-1}F(x_k), \\ z_k &= y_k - F'(x_k)^{-1}F(y_k), \\ x_{k+1} &= z_k - \left(\frac{1}{a}F'(y_k)^{-1} + \left(1 - \frac{1}{a}\right)F'(x_k)^{-1}\right)F(z_k), \end{aligned} \tag{2.1}$$

where, the parameter $a \in (-\infty, \infty) - \{0\}$ and x_0 is the starting point. It is shown there that the convergence order of this method is at least four and for $a = \pm 1$, it is five. They performed a general local convergence analysis with Taylor's developments using high order Fréchet derivatives without obtaining the convergence balls. They also assumed that an starting point x_0 is sufficiently close to the solution without estimating this closeness. In our local convergence analysis we have addressed these problems. Suppose that $B(v, \rho)$ and $\overline{B}(v, \rho)$ denote the open and closed balls, respectively with center v and radius $\rho > 0$. For the local convergence analysis of (2.1), we assume the following conditions for real numbers $L_0 > 0$, $L > 0$ and for all $x, y \in D$

$$F(x^*) = 0, F^{'}(x^*)^{-1} \in BL(Y, X)$$

$$\|F^{'}(x^*)^{-1}(F^{'}(x) - F^{'}(x^*))\| \leq L_0\|x - x^*\|, \tag{2.2}$$

$$\|F^{'}(x^*)^{-1}(F^{'}(x) - F^{'}(y))\| \leq L\|x - y\|, \tag{2.3}$$

where, $BL(Y, X)$ is the set of bounded linear operators from Y to X. Usually, a third assumption that can be written as

$$\|F^{'}(x^*)^{-1}F^{'}(x)\| \leq M$$

for real $M > 0$ is also made in several papers [17, 4]. We consider the remark made in [17] and so we drop this condition and deduce the whole process for deducing the radius of the convergence ball without using constant M. This allows us to increase the radius of the convergence ball.

Lemma 2.1. *If operator F satisfies (2.2) and (2.3), then the following holds for all $x \in D$ and $t \in [0, 1]$*

$$\|F^{'}(x^*)^{-1}F^{'}(x)\| \leq 1 + L_0\|x - x^*\| \tag{2.4}$$

$$\|F^{'}(x^*)^{-1}(F^{'}(x^* + t(x - x^*))\| \leq 1 + L_0\|x - x^*\| \tag{2.5}$$

$$\|F^{'}(x^*)^{-1}F(x)\| \leq (1 + L_0\|x - x^*\|)\|x - x^*\|. \tag{2.6}$$

Proof. Using (2.2), we get

$$\begin{aligned}\|F^{'}(x^*)^{-1}F^{'}(x)\| &= \|F^{'}(x^*)^{-1}(F^{'}(x)-F^{'}(x^*))+I)\| \leq 1+\|F^{'}(x^*)^{-1}(F^{'}(x)-F^{'}(x^*))\|\\ &\leq 1+L_0\|x-x^*\|.\end{aligned}$$

and then it follows that

$$\|F^{'}(x^*)^{-1}(F^{'}(x^*+t(x-x^*)))\| \leq 1+L_0t\|x-x^*\| \leq 1+L_0\|x-x^*\|$$

By using mean value theorem, we get

$$\begin{aligned}\|F^{'}(x^*)^{-1}F(x)\| &= \|F^{'}(x^*)^{-1}(F(x)-F(x^*))\| \leq \|F^{'}(x^*)^{-1}F^{'}(x^*+t(x-x^*))(x-x^*)\|\\ &\leq (\|1+L_0\|x-x^*\|)\|x-x^*\|.\end{aligned}$$

□

The following theorem describes the local convergence analysis of the method (2.1).

Theorem 2.1. *Let $F: D \subseteq X \to Y$ be a Fréchet differentiable operator. Suppose that there exist $x^* \in D$ and $a \in]\frac{4}{5}, \frac{5}{4}[$ such that (2.2)-(2.3) are satisfied and $\overline{B}(x^*, r) \subseteq D$, where, the radius r is to be determined. The sequence $\{x_k\}$ generated by (2.1) for $x_0 \in B(x^*, r)$ is well defined for $k = 0, 1, 2, \ldots$, remains in $B(x^*, r)$ and converges to x^*. Moreover, the following holds for $k = 0, 1, 2, \ldots$*

$$\|y_k - x^*\| \leq g_1(\|x_k - x^*\|)\|x_k - x^*\| < \|x_k - x^*\| < r, \quad (2.7)$$

$$\|z_k - x^*\| \leq g_2(\|x_k - x^*\|)\|x_k - x^*\| < \|x_k - x^*\| < r, \quad (2.8)$$

$$\|x_{k+1} - x^*\| \leq g_3(\|x_k - x^*\|)\|x_k - x^*\| < \|x_k - x^*\| < r, \quad (2.9)$$

where the " g" functions are to be defined. Furthermore, if there exists $R \in [r, \frac{2}{L_0})$ such that $\overline{B}(x^*, R) \subseteq D$, then the limit point x^* is the unique solution in $\overline{B}(x^*, R)$.

Proof. Since $x_0 \in D$ and using (2.2), we have

$$\|F^{'}(x^*)^{-1}(F^{'}(x_0)-F^{'}(x^*))\| \leq L_0\|x_0-x^*\|.$$

Assuming that $\|x_0 - x^*\| < \frac{1}{L_0}$, this gives

$$\|F^{'}(x^*)^{-1}(F^{'}(x_0)-F^{'}(x^*))\| < 1$$

Therefore, by Banach Lemma on invertible operators, $F^{'}(x_0)^{-1}$ exists and

$$\|F^{'}(x_0)^{-1}F^{'}(x^*)\| \leq \frac{1}{1-L_0\|x_0-x^*\|}. \quad (2.10)$$

Therefore, y_0 is well defined and hence z_0 is well defined.
From (2.1) for $n = 0$, we get

$$\begin{aligned}y_0 - x^* &= x_0 - x^* - aF^{'}(x_0)^{-1}F(x_0)\\ &= x_0 - x^* - F^{'}(x_0)^{-1}F(x_0) + (1-a)F^{'}(x_0)^{-1}F(x_0)\\ &= -F^{'}(x_0)^{-1}\Big(F(x_0) - F^{'}(x_0)(x_0-x^*)\Big) + (1-a)F^{'}(x_0)^{-1}F(x_0)\\ &= -F^{'}(x_0)^{-1}F^{'}(x^*)\int_0^1 F^{'}(x^*)^{-1}[F^{'}(x^*+t(x_0-x^*)) - F^{'}(x_0)](x_0-x^*)dt\\ &+ (1-a)F^{'}(x_0)^{-1}F^{'}(x^*)\int_0^1 F^{'}(x^*)^{-1}F^{'}(x^*+t(x_0-x^*))(x_0-x^*)dt\end{aligned}$$

By taking norm on both sides and using (2.3) and (2.10), we get

$$\begin{aligned}
\|y_0 - x^*\| &\leq \|F^{'}(x_0)^{-1}F^{'}(x^*)\| \left\| \int_0^1 F^{'}(x^*)^{-1}[F^{'}(x^* + t(x_0 - x^*)) - F^{'}(x_0)](x_0 - x^*)dt \right\| \\
&+ |1-a| \|F^{'}(x_0)^{-1}F^{'}(x^*)\| \left\| \int_0^1 F^{'}(x^*)^{-1}F^{'}(x^* + t(x_0 - x^*))(x_0 - x^*)dt \right\| \\
&\leq \frac{1}{1 - L_0\|x_0 - x^*\|} \left[\frac{L}{2}\|x_0 - x^*\| + |1-a|(1 + L_0\|x_0 - x^*\|) \right] \|x_0 - x^*\| \\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\|, \qquad (2.11)
\end{aligned}$$

where,

$$g_1(t) = \frac{1}{1 - L_0 t}\left(\frac{L}{2}t + |1-a|(1 + L_0 t)\right).$$

Consider the function $h_1(t) = g_1(t) - 1$. Since, $h_1(0) = |1-a| - 1 < 0$ if $a \in]0, 2[$ and $h_1(1/L_0) \to +\infty$. Therefore, by intermediate value theorem, $h_1(t)$ has at least one root in $]0, 1/L_0[$. Let r_1 be the smallest root of $h_1(t)$ in $]0, 1/L_0[$. Then, we get

$$0 < r_1 < 1/L_0, \qquad (2.12)$$

and

$$0 \leq g_1(t) < 1, \ \forall t \in [0, r_1). \qquad (2.13)$$

Therefore, by using (2.11) and (2.13), we get

$$\|y_0 - x^*\| \leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\|.$$

Again, from (2.1) for $n = 0$, and by using (2.6) and (2.10), we get

$$\begin{aligned}
\|z_0 - x^*\| &\leq \|y_0 - x^*\| + \|F^{'}(x_0)^{-1}F(y_0)\| \\
&\leq \|y_0 - x^*\| + \|F^{'}(x_0)^{-1}F^{'}(x^*)\|\|F^{'}(x^*)^{-1}F(y_0)\|, \\
&\leq \left(1 + \frac{1 + L_0\|y_0 - x^*\|}{1 - L_0\|x_0 - x^*\|}\right)\|y_0 - x^*\| \\
&\leq \left(1 + \frac{1 + L_0 g_1(\|x_0 - x^*\|)\|x_0 - x^*\|)}{1 - L_0\|x_0 - x^*\|}\right) g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= g_2(\|x_0 - x^*\|)\|x_0 - x^*\|, \qquad (2.14)
\end{aligned}$$

where,

$$g_2(t) = \left(1 + \frac{1 + L_0 t g_1(t)}{1 - L_0 t}\right) g_1(t).$$

Consider the function $h_2(t) = g_2(t) - 1$. Since $g_1(0) = |1-a|$. Then, $h_2(0) = 2|1-a| - 1 < 0$ if $a \in]\frac{1}{2}, \frac{3}{2}[$ and $h_2(r_1) = \frac{1 + L_0 r_1}{1 - L_0 r_1} > 0$. Therefore, $h_2(t)$ has at least one root in $]0, r_1[$. Let r_2 be the smallest root of $h_2(t)$ in $]0, r_1[$. Therefore,

$$0 < r_2 < r_1, \qquad (2.15)$$

and

$$0 \leq g_2(t) < 1, \ \forall t \in [0, r_2). \qquad (2.16)$$

Therefore, by using (2.14) and (2.16), we get

$$\|z_0 - x^*\| \quad \leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\|.$$

Since $y_0 \in D$, and using the assumption (2.2), we get

$$\|F^{'}(x^*)^{-1}(F^{'}(y_0) - F^{'}(x^*))\| \leq L_0\|y_0 - x^*\| \leq L_0\|x_0 - x^*\| < 1,$$

Therefore, by Banach's Lemma on invertible operators, $F^{'}(y_0)^{-1}$ exists and

$$\|F^{'}(y_0)^{-1}F^{'}(x^*)\| \leq \frac{1}{1 - L_0\|y_0 - x^*\|}. \tag{2.17}$$

Therefore x_1 is well defined and we have

$$\begin{aligned}
\|x_1 - x^*\| &\leq \|z_0 - x^*\| + \left\|\left(\frac{1}{a}F^{'}(y_0)^{-1} + (1 - \frac{1}{a})F^{'}(x_0)^{-1}\right)F(z_0)\right\| \\
&\leq \|z_0 - x^*\| + \left(\frac{1}{|a|}\|F^{'}(y_0)^{-1}F^{'}(x^*)\| + \left|1 - \frac{1}{a}\right|\|F^{'}(x_0)^{-1}F^{'}(x^*)\|\right)\|F^{'}(x^*)^{-1}F(z_0)\| \\
&\leq \left[1 + \left(\frac{1}{|a|}\frac{1}{1 - L_0\|y_0 - x^*\|} + \left|1 - \frac{1}{a}\right|\frac{1}{1 - L_0\|x_0 - x^*\|}\right)(1 + L_0\|z_0 - x^*\|)\right]\|z_0 - x^*\|, \\
&\leq \left[1 + \left(\frac{1}{|a|}\frac{1}{1 - L_0\|y_0 - x^*\|} + \left|1 - \frac{1}{a}\right|\frac{1}{1 - L_0\|x_0 - x^*\|}\right)(1 + L_0\|z_0 - x^*\|)\right]g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= g_3(\|x_0 - x^*\|)\|x_0 - x^*\|,
\end{aligned} \tag{2.18}$$

where,

$$g_3(t) = \left[1 + \left(\frac{1}{|a|}\frac{1}{1 - L_0 g_1(t)t} + \left|1 - \frac{1}{a}\right|\frac{1}{1 - L_0 t}\right)(1 + L_0 g_2(t)t)\right]g_2(t).$$

Consider the function $h_3(t) = g_3(t) - 1$. Since $g_2(0) = 2g_1(0) = 2|1 - a|$. Then,

$$h_3(0) = \left[1 + \left(\frac{1}{|a|} + \left|1 - \frac{1}{a}\right|\right)\right]g_2(0) - 1 < 0$$

if $a \in]\frac{4}{5}, \frac{5}{4}[$ and $h_3(r_2) = \left[\left(\frac{1}{|a|}\frac{1}{1 - L_0 g_1(r_2)r_2} + \left|1 - \frac{1}{a}\right|\frac{1}{1 - L_0 r_2}\right)(1 + L_0 g_2(r_2)r_2)\right] > 0$. Therefore, $h_3(t)$ has at least one root in $]0, r_2[$. Let r be the smallest root of $h_3(t)$ in $]0, r_2[$. Then, we have

$$r < r_2 < r_1 < \frac{1}{L_0}, \tag{2.19}$$

and

$$0 \leq g_3(t) < 1, \quad \forall \quad t \in [0, r). \tag{2.20}$$

Therefore, for $a \in]\frac{4}{5}, \frac{5}{4}[$, we have

$$0 < r < r_2 < r_1 < 1/L_0.$$

By using (2.18) and (2.20), we have

$$\|x_1 - x^*\| \quad \leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r.$$

Therefore, the theorem holds for $k = 0$. Replacing x_0, y_0, z_0 and x_1 by x_k, y_k, z_k, x_{k+1} in the preceding way, we get the inequalities (2.7)-(2.9). Using the estimate $\|x_{k+1} - x^*\| \leq \|x_k - x^*\| < r$, we get $x_{k+1} \in B(x^*, r)$. Obviously the function g_3 is increasing in its domain, so we have

$$\begin{aligned}
\|x_{k+1} - x^*\| &\leq g_3(t)\|x_k - x^*\| \leq g_3(t)g_3(\|x_{k-1} - x^*\|)\|x_{k-1} - x^*\| \\
&\leq g_3(t)^2 g_3(\|x_{k-2} - x^*\|)\|x_{k-2} - x^*\| \leq \ldots \leq g_3(t)^{k+1}\|x_0 - x^*\|.
\end{aligned}$$

Then, by taking limits in the last expression and using that $\lim_{k\to\infty} g_3(t)^{k+1} = 0$, we get $\lim_{k\to\infty} x_k = x^*$, and so, the method converges to the solution.

In order to prove the uniqueness part, let $y^* \in B(x^*, r)$, $y^* \neq x^*$ with $F(y^*) = 0$.

Let $T = \int_0^1 F^{'}(y^* + t(x^* - y^*))dt$. Then by using (2.2), we have

$$\|F^{'}(x^*)^{-1}(T - F^{'}(x^*))\| \leq \int_0^1 L_0\|y^* + t(x^* - y^*) - x^*\|dt \leq \frac{L_0}{2}\|x^* - y^*\| = \frac{L_0}{2}R < 1,$$

therefore, T^{-1} exists. Then, from the identity

$$0 = F(x^*) - F(y^*) = T(x^* - y^*),$$

we obtain $x^* = y^*$. □

3. Numerical examples

In this section, a number of numerical examples are worked out to demonstrate the efficiency of our local convergence analysis. All the numerical examples are worked out by using high level language MATLAB R2012b on an Intel(R) core (TM) i5-3470 CPU 3.20GHz with 4GB of RAM running on the windows 7 Professional version 2009 Service Pack 1.

Example 3.1. *Consider the function f defined on $D = [-\frac{1}{2}, \frac{5}{2}]$ by*

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

The unique solution is $x^* = 1$. The successive derivatives of f are

$$\begin{aligned} f^{'}(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \\ f^{''}(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ f^{'''}(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

It can be easily observed that $f^{'''}$ is unbounded on D. However, all the conditions of the iterative method (2.1) are satisfied and hence applying Theorem (2.1) with $x^* = 1$ we obtained $L_0 = L = 96.6628$. Taking $a = 1$, we get

$$r = 0.002818 < r_2 = 0.004206 < r_1 = 0.006897.$$

Next, we consider a nonlinear integral equation of Hammerstein type. These equations have many applications in Chemistry and appear in problems of electro-magnetic fluid dynamics, in the kinetic theory of gases, and in reformulation of boundary value problems etc.[14]. This equation is of the form

$$x(s) = u(s) + \int_l^m G(s,t)H(x(t))dt, \quad l \leq s \leq m,$$

for $x(s), u(s) \in C[l,m]$ with $-\infty < l < m < \infty$, G is the Green function and H, is a polynomial function. The usual technique to solve these kind of equations consist in expressed it in a Banach space as a nonlinear operator, that is

$$F(x) = 0,$$

where $F : D \subseteq C[l, m] \to C[l, m]$ with D a non-empty open convex subset,

$$[F(x)](s) = x(s) - u(s) - \int_l^m G(s,t)H(x(t))dt$$

considering the uniform norm $\|\nu\| = \max_{s\in[l,m]} |\nu(s)|$. It is more convenient using the local convergence results obtained in our study in order to give the radius of a convergence ball. We apply our theoretical study presented in Theorem (2.1) to a particular Hammerstein equation given by

Example 3.2.

$$F(x(s)) = x(s) - 5\int_0^1 s\,t\,x(t)^3 dt, \tag{3.1}$$

with $x(s)$ in $\mathcal{C}[0,1]$.

The derivative can be given by

$$F'(x(s)) = v(s) - 15\int_0^1 s\,t\,x(t)^2 v(s) dt, \tag{3.2}$$

So, we obtain $L_0 = 7.5$ and $L = 15$. Now using the iterative method (2.1) for $a = 1$, we get

$$r = 0.023064 < r_2 = 0.037592 < r_1 = 0.066667.$$

Example 3.3. *Let $X = Y = \mathbb{R}$. Define F on $D = [1,3]$ by*

$$F(x) = \frac{2}{3}x^{\frac{3}{2}} - x$$

Then, $x^* = \frac{9}{4}$, $F^{'}(x^*)^{-1} = 2$, $L_0 = L = 1$.
Taking

$$a = 0.9870,$$

we get

$$r = 0.250885 < r_2 = 0.390015 < r_1 = 0.652346.$$

Example 3.4. *Let $X = Y = \mathbb{R}^3$, $D = \overline{B}(0,1)$. Define F on D for $v = (x,y,z)$ by*

$$F(v) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right).$$

The Fréchet derivative is given by

$$F^{'}(v) = \begin{pmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For $x^* = (0,0,0)$, $F^{'}(x^*) = F^{'}(x^*)^{-1} = diag\{1,1,1\}$, $L_0 = e-1,$, $L = e$ and taking

$$a = 1.0125,$$

we have

$$r = 0.111057 < r_2 = 0.181094 < r_1 = 0.318661.$$

Table 1: Values of parameters

Examples	a	γ	α
3.1	1	0.005	0.008
3.2	1	0.575	0.03
3.3	0.987	0.6	0.001
3.4	1.0125	0.3	0.03

Table 2: Comparison of radius of convergence ball

Examples	Method (2.1)	Method (1.3)	Method (1.4)
3.1	0.002818	2.5×10^{-9}	4.4×10^{-9}
3.2	0.023064	0.004999	0.007218
3.3	0.250885	0.059554	0.105844
3.4	0.111057	0.27226	0.001728

The values of the different parameters used by the iterative methods (2.1), (1.3) and (1.4) are listed in Table 1. Next, the radii of the convergence balls enclosing unique solution of each numerical example worked out by these method are obtained and compared in Table 2. From the Table 2, it can be easily observed that the iterative method (2.1) gives larger radius of convergence ball as compared to existing methods. The comparison of radii of convergence balls of our iterative method (2.1) for different values of parameter a for all the examples considered is displayed in Table 3. From the Table 3, it can be easily

Table 3: Radii of convergence balls for different values of a

Examples	$a = 0.85$	$a = 0.95$	$a = 1$	$a = 1.24$
3.1	6.07×10^{-4}	1.999×10^{-3}	2.818×10^{-3}	8.2×10^{-5}
3.2	5.307×10^{-3}	1.6789×10^{-2}	2.3064×10^{-2}	7.59×10^{-4}
3.3	5.8680×10^{-2}	0.193260	0.272422	7.966×10^{-3}
3.4	2.6752×10^{-2}	8.5657×10^{-2}	0.118532	3.762×10^{-3}

observed that we get bigger radius of convergence ball as a is closer to 1.

4. Dynamics

Here we study the dynamics of the family of iterative methods (2.1) for complex polynomials of second degree proving scaling and conjugacy results. Similar studies have been performed in [24, 25, 26] for other families of iterative methods. The dynamics of the relaxed Newton's method has been studied in [23]. The motivation for studying the dynamics of a family of methods is to choose the values of the parameters that ensure a better behaviour of the method for different initial conditions.

Let us establish some notation. The iterates obtained starting from $z_0 \in \mathbb{C}$ can be denoted by $\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\}$, where R is a rational function defined on the Riemann sphere $\hat{\mathbb{C}}$. This set is called the *orbit* of z_0.

Let $z \in \hat{\mathbb{C}}$ be a *fixed point* of the rational function R, that is to say $R(z) = z$. The *basin of attraction* of z consists of the points whose orbit tends to z. The behaviour of the orbits near a fixed point z depends on the derivative $R'(z)$. If $|R'(z)| < 1$, the fixed point z is *attracting* and if $|R'(z)| > 1$, it is *repelling*. If $R'(z) = 0$, the fixed point is *superattracting*.

The set of points $z_0 \in \hat{\mathbb{C}}$ such that their families $R_n(z_0),\ n \in \mathbb{N}$ are normal in some neighbourhood $U(z_0)$ is the *Fatou set*, $\mathcal{F}(R)$ and its complement in $\hat{\mathbb{C}}$ is the *Julia set* $\mathcal{J}(R)$. Roughly speaking, the orbits of the points in $\mathcal{F}(R)$ present a stable behaviour whereas the orbits of the points in $\mathcal{J}(R)$ have chaotic behaviour. In particular, the Fatou set contains the attraction basins of the attracting fixed points whereas the Julia set contains the boundaries of the attraction basins.

Given an analytic function $f(z)$, consider the function associated to a step of the iterative method (2.1) $M_f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, such that $M_f(x_k) = x_{k+1}$. The following scaling result holds for M_f:

Theorem 4.1. *Let f be an analytic function on $\hat{\mathbb{C}}$, and $A(z) = \alpha z + \beta$, with $\alpha \neq 0$, an affine map. If $g(z) = \lambda(f \circ A)(z)$, $\lambda \neq 0$, then M_f is analytically conjugated to M_g by A, that is, $A \circ M_g \circ A^{-1} = M_f$.*

Any polynomial of second degree is conjugated by an affine transformation to a polynomial of the form $f(z) = z^2 + c,\ c \in \mathbb{C}$, so that in order to study the dynamics of M_f on quadratic polynomials, it suffices to consider only polynomials of this form.

Then, if $f(z) = z^2 + c$, with $c \neq 0$, M_f has the form

$$\begin{aligned} M_f(z) = {} & \frac{1}{32\left(acz + (-2+a)z^3\right)^3}\Big(16a^3z^2\left(-c+z^2\right)\left(c+z^2\right)^3 - a^5\left(c+z^2\right)^5 \\ & + a^4\left(c+z^2\right)^4\left(c+3z^2\right) + 8a^2z^2\left(c+z^2\right)^2\left(c^2+14cz^2-11z^4\right) \\ & - 48az^4\left(c+z^2\right)\left(c^2+6cz^2-3z^4\right) + 16z^4\left(c^3+5c^2z^2+15cz^4-5z^6\right)\Big). \end{aligned}$$

The equation $M_f(z) = z$ can be written as

$$\begin{aligned} & \frac{c+z^2}{32\left(acz+(-2+a)z^3\right)^3}\Big(-16a^3z^2\left(c+z^2\right)^3 - a^5\left(c+z^2\right)^4 \\ & + a^4\left(c+z^2\right)^3\left(c+3z^2\right) - 48az^4\left(c+z^2\right)\left(c+5z^2\right) \\ & + 8a^2z^2\left(c+z^2\right)^2\left(c+13z^2\right) + 16z^4\left(c^2+4cz^2+11z^4\right)\Big) = 0, \end{aligned}$$

so that, M_f has ten fixed points. Eight of them depend on a and the two remaining are the roots of $f(z),\ \ \pm\sqrt{-c}$, which do not depend on a. The two last fixed point are superattracting, because

$$\begin{aligned} M_f'(z) = {} & \frac{\left(c+z^2\right)^3}{32\left(acz+(-2+a)z^3\right)^4}\left(3(-1+a)a^5c^3 + (-2+a)a^3(4+5(-3+a)a)c^2z^2\right. \\ & \left. + (-2+a)^3a(-2+(-11+a)a)cz^4 - (-2+a)^5(5+a)z^6\right) \end{aligned}$$

and then, $M_f'(\pm\sqrt{-c}) = 0$. The character of the remaining fixed points depends on a.

The dynamical study can be simplified further by using the idea of analytical conjugation. If B(z) is a Möbius map

$$B(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \neq 0, \tag{4.1}$$

the rational maps M and N are analytically conjugated via B if $N = BMB^{-1}$. Then, $\mathcal{F}(N) = B(\mathcal{F}(M))$ and $\mathcal{J}(N) = B(\mathcal{J}(M))$.

Theorem 4.2. *Let $f(z)$ be a quadratic polynomial with simple roots. The fixed point operator $M_f(z)$ associated to the family of iterative methods* **??** *verifies:*

a. *$M_f(z)$ is analytically conjugated with*

$$N_f(z) = -\frac{p(z)}{q(z)} \tag{4.2}$$

where

$$p(z) = z^4(1-a+z)^2\left(2a^3z + a^2\left(-1+3z^2\right) + (1+z)^2(5+z(4+z)) \right.$$
$$\left. -2a(1+z)(1+z(5+2z))\right)$$

and

$$q(z) = (1+z-az)^2\left(-1+z\left(-6+4a+(-14+(14-3a)a)z \right.\right.$$
$$\left.\left. -2\left(7-6a+a^3\right)z^2 + (-5+a(2+a))z^3\right)\right).$$

b. *The Julia set of this operator contains the unit circle.*

c. *The Fatou set consists of the attraction basins of 0 and infinity. Both are superattracting fixed points.*

Proof:

a. Due to the scaling theorem 4.1, we suppose $f(z) = z^2 + c$. Then, the Möbius transform

$$B(z) = \frac{z-\sqrt{-c}}{z+\sqrt{-c}}$$

has the following properties:

$$B(\infty) = 1, \quad B(\sqrt{-c}) = 0, \text{ and } B(-\sqrt{-c}) = \infty.$$

By conjugating M_f with B one gets (4.2), which does not depend on c.

b. It is easy to check that the unit circle $z : |z| = 1$ is invariant under N_f. Figure 2 shows that the Julia set contains the unit circle.

c. Expression (4.2) shows that 0 and ∞ are fixed points. The derivative N_f' is 0 at these points, so that, they are superattracting.

For values of a between -2 and about 2, the only attracting fixed points of the method are the roots of f and the behaviour of the method is similar to Newton's method. Figures 1 and 2 show the attraction basins of M_f for $f(z) = z^2 + i$ and the Julia set of the corresponding conjugate operator N_f. The roots, marked in red, are in its attraction basin and the other repelling fixed pints, marked in green, are in the boundary of the basins, in the Julia set.

For $a = 2$ the attraction basins are slightly more intricated. Figure 3 shows a detail of the attraction basins near the origin. The points for which the algorithm does not converge in 200 iterations are marked in dark grey. The corresponding Julia set is more complex for this case, Figure 4.

The method has different dynamical behaviour for the two parameter values that give order 5. Whereas for $a = 1$ the basins are similar to that of Figure 1, for $a = -1$ the attraction basins present isolated regions as shown in Figure 5. Figure 6 shows the Julia set for this case.

5. Conclusions

A local convergence of a family of higher order iterative methods for solving nonlinear equations in Banach spaces is established under the assumption that the Fréchet derivative satisfies the Lipschitz continuity condition. The method is of fifth order for $a = \pm 1$. The existence and uniqueness theorem that establishes the convergence balls of these methods is obtained.

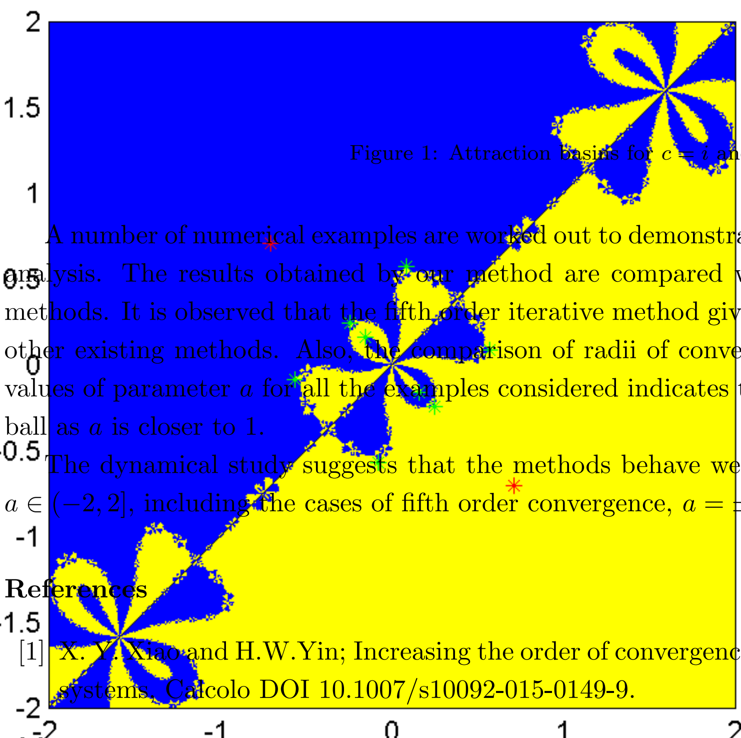

Figure 1: Attraction basins for $c = i$ and $a = 0.75$.

A number of numerical examples are worked out to demonstrate the efficiency of our local convergence analysis. The results obtained by our method are compared with the results of some of the existing methods. It is observed that the fifth order iterative method gives larger radius of convergence ball than other existing methods. Also, the comparison of radii of convergence balls of the method for different values of parameter a for all the examples considered indicates that we get bigger radius of convergence ball as a is closer to 1.

The dynamical study suggests that the methods behave well for a wide range of parameter values, $a \in (-2, 2]$, including the cases of fifth order convergence, $a = \pm 1$.

References

[1] X. Y. Xiao and H.W.Yin; Increasing the order of convergence for iterative methods to solve nonlinear systems, Calcolo DOI 10.1007/s10092-015-0149-9.

[2] I.K Argyros and S.K.Khattri; Local convergence for a family of third order methods in Banach spaces, Journal of Mathematics, Vol-46 (2014) pp. 53-62.

[3] A. Cordero, J.A. Ezquerro, M.A. Hernàndez-Verön, and J.R. Torregrosa; On the local convergence of a fifth-order iterative method in Banach spaces, Applied Mathematics and Computation, Vol-251 (2015) pp. 396-403.

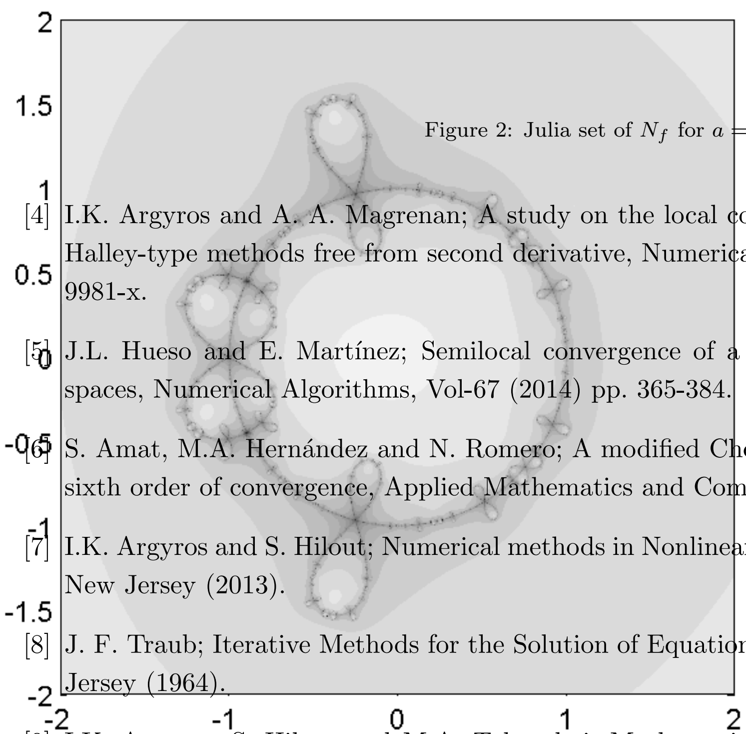

Figure 2: Julia set of N_f for $a = 0.75$

[4] I.K. Argyros and A. A. Magrenan; A study on the local convergence and dynamics of Chebyshev-Halley-type methods free from second derivative, Numerical Algorithms, DOI 10.1007/s11075-015-9981-x.

[5] J.L. Hueso and E. Martínez; Semilocal convergence of a family of iterative methods in Banach spaces, Numerical Algorithms, Vol-67 (2014) pp. 365-384.

[6] S. Amat, M.A. Hernández and N. Romero; A modified Chebyshev's iterative method with at least sixth order of convergence, Applied Mathematics and Computation, Vol-206 (2008) pp.164-174.

[7] I.K. Argyros and S. Hilout; Numerical methods in Nonlinear Analysis, World Scientific Publ. Comp. New Jersey (2013).

[8] J. F. Traub; Iterative Methods for the Solution of Equations, Prentice-Hall, Englewood Cliffs, New Jersey (1964).

[9] I.K. Argyros, S. Hilout and M.A. Tabatabai; Mathematical Modelling with Applications in Biosciences and Engineering, Nova Publishers, New York, (2011).

[10] X. Wang, J. Kou, and C. Gu; Semilocal convergence of a sixth-order Jarratt method in Banach spaces, Numerical Algorithms, Vol-57, (2011) pp. 441-456.

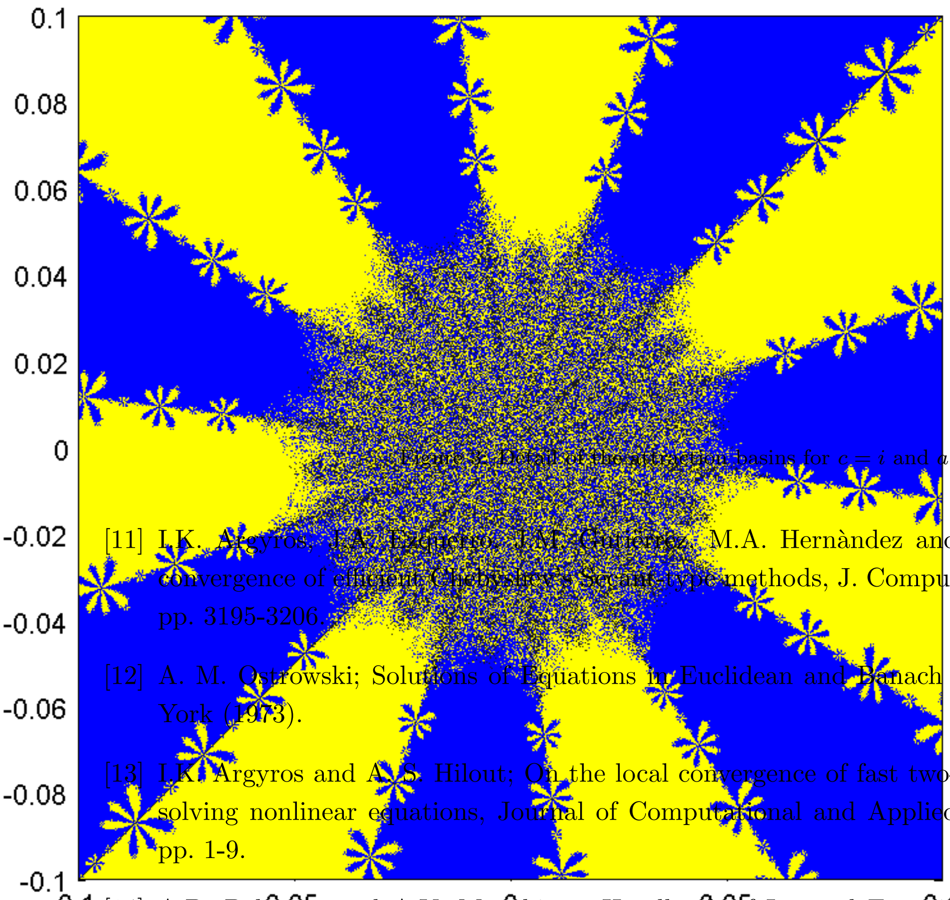

Figure 3: Detail of the attraction basins for $c = i$ and $a = 2$.

[11] I.K. Argyros, J.A. Ezquerro, J.M. Gutièrrez, M.A. Hernàndez and S. Hilout, On the semilocal convergence of efficient Chebyshev's Secant-type methods, J. Comput. Appl. Math., Vol-235 (2011) pp. 3195-3206.

[12] A. M. Ostrowski; Solutions of Equations in Euclidean and Banach Spaces, Academic Press, New York (1973).

[13] I.K. Argyros and A. S. Hilout; On the local convergence of fast two-step Newton-like methods for solving nonlinear equations, Journal of Computational and Applied Mathematics Vol-245 (2013) pp. 1-9.

[14] A.D. Polyanin and A.V. Manzhirov; Handbook of Integral Equations, CRC Press, Boca Raton, (1998).

[15] I.K. Argyros, Santhosh George and A. A. Magrenan; Local convergence for multi-point-parametric

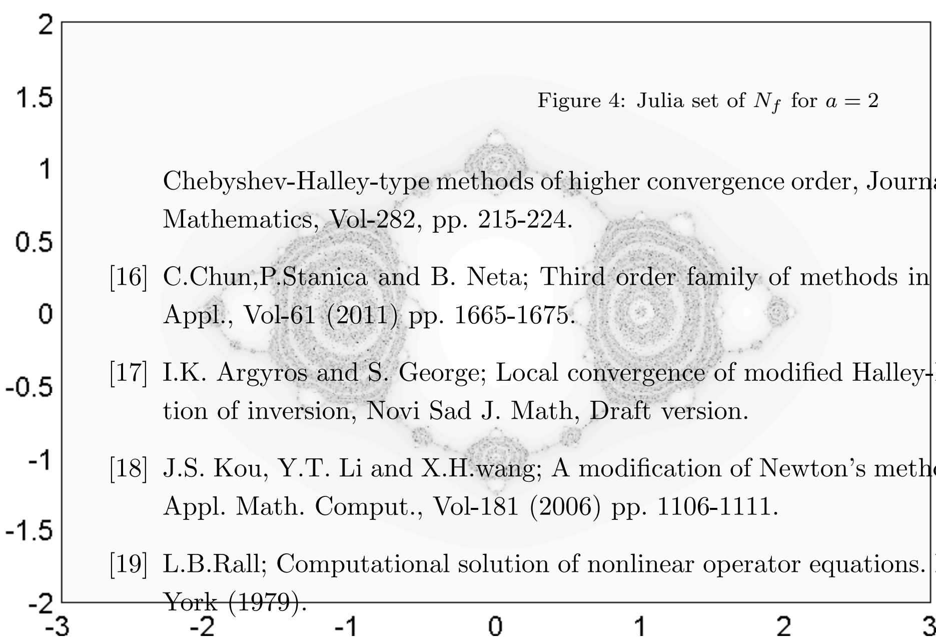

Figure 4: Julia set of N_f for $a = 2$

Chebyshev-Halley-type methods of higher convergence order, Journal of Computational and Applied Mathematics, Vol-282, pp. 215-224.

[16] C.Chun,P.Stanica and B. Neta; Third order family of methods in Banach spaces, Comput. Math. Appl., Vol-61 (2011) pp. 1665-1675.

[17] I.K. Argyros and S. George; Local convergence of modified Halley-like methods with less computation of inversion, Novi Sad J. Math, Draft version.

[18] J.S. Kou, Y.T. Li and X.H.wang; A modification of Newton's method with third order convergence, Appl. Math. Comput., Vol-181 (2006) pp. 1106-1111.

[19] L.B.Rall; Computational solution of nonlinear operator equations. In: Robert E. Krieger (ed.) New York (1979).

[20] S. Weerakoon and T.G.I. Fernando; A variant of Newton's method with accelerated third-order convergence, Applied Mathematics Letters, Vol-13 (2000) pp. 87-93.

[21] X. Feng and Y. He; High order oterative methods without derivatives for solving nonlinear equations, Applied Mathematics and Computation, Vol-186 (2007) pp. 1617-1623.

[22] X. Wang, J. Kou and Y. Li; Modified Jarratt method with sixth-order convergence, Applied Mathematics Letters, Vol-22 (2009) pp. 1798-1802.

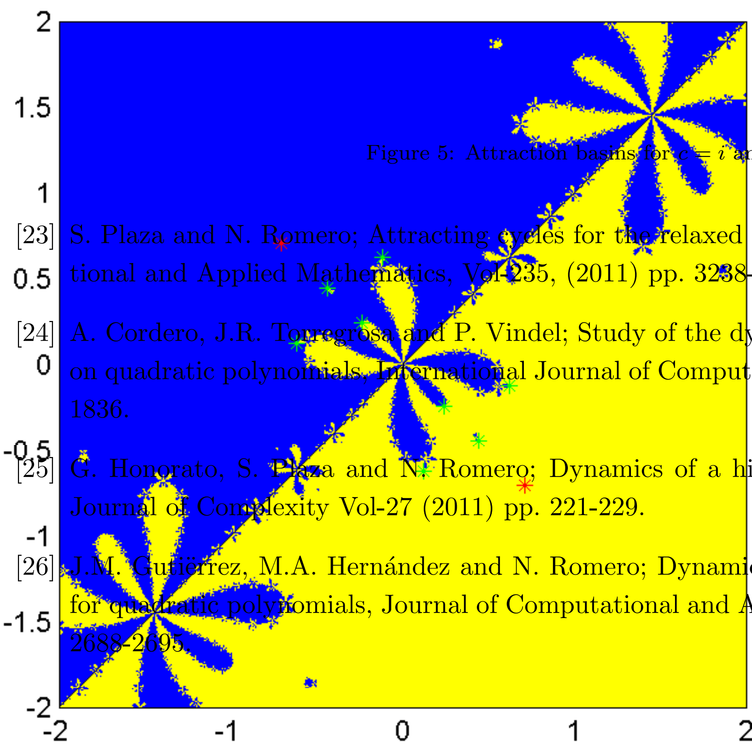

Figure 5: Attraction basins for $c = i$ and $a = -1$.

[23] S. Plaza and N. Romero; Attracting cycles for the relaxed Newton's method, Journal of Computational and Applied Mathematics, Vol-235, (2011) pp. 3238-3244.

[24] A. Cordero, J.R. Torregrosa and P. Vindel; Study of the dynamics of third-order iterative methods on quadratic polynomials, International Journal of Computer Mathematics, Vol-89 (2012) pp. 1826-1836.

[25] G. Honorato, S. Plaza and N. Romero; Dynamics of a higher-order family of iterative methods, Journal of Complexity Vol-27 (2011) pp. 221-229.

[26] J.M. Gutiérrez, M.A. Hernández and N. Romero; Dynamics of a new family of iterative processes for quadratic polynomials, Journal of Computational and Applied Mathematics, Vol-233 (2010) pp. 2688-2695.

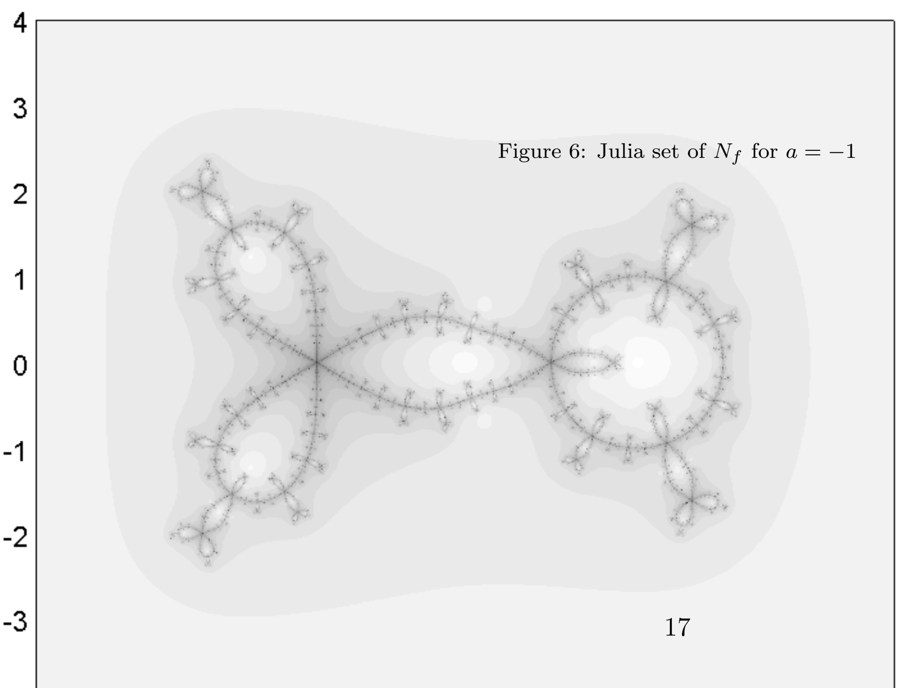

Figure 6: Julia set of N_f for $a = -1$