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On a Revisited Moore-Penrose Inverse of a Linear Operator on Hilbert Spaces

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Abstract. For two given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and a given bounded linear operator $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ having closed range, it is well known that the Moore-Penrose inverse of A is a reflexive g-inverse $G \in \mathcal{L}(\mathcal{K},\mathcal{H})$ of A which is both minimum norm and least squares. In this paper, weaker equivalent conditions for an operator G to be the Moore-Penrose inverse of A are investigated in terms of normal, *EP*, bi-normal, bi-*EP*, ℓ-quasi-normal and r-quasi-normal and ℓ-quasi-*EP* and r-quasi-*EP* operators.

1. Introduction

The symbol $\mathcal{L}(\mathcal{H},\mathcal{K})$ stands for the algebra of bounded linear operator from the Hilbert space $\mathcal{H}$ to the Hilbert space $\mathcal{K}$, both over the field $\mathbb{C}$ of complex numbers. When $\mathcal{K} = \mathcal{H}$, it will be written $\mathcal{L}(\mathcal{H})$. The symbol A^* denotes the adjoint of an operator $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$. As usual, I and O denote the identity and the zero operators, respectively. For $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$, let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ be the range and the null space of A, respectively.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. For a given operator $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ having closed range, it is well known that the equations $AGA = A$, $GAG = G$, $(AG)^* = AG$ and $(GA)^* = GA$ have a unique common solution for $G \in \mathcal{L}(\mathcal{K},\mathcal{H})$, denoted by $G = A^\dagger$ and called the Moore-Penrose inverse of A. Moreover, an operator G satisfying $AGA = A$ and $(AG)^* = AG$ is called a least squares g-inverse of A and if it satisfies $AGA = A$ and $(GA)^* = GA$ it is called a minimum norm g-inverse of A. Also, G is called a reflexive g-inverse of A if both $AGA = A$ and $GAG = G$ hold. Thus, G is the Moore-Penrose inverse of A if G is a reflexive g-inverse of A which is both a minimum norm as well as a least squares inverse. These four conditions for defining the Moore-Penrose inverse, established in 1955, are known in the literature as the Penrose conditions. It is well known that the Moore-Penrose inverse is a very useful tool in Matrix Theory, Hilbert spaces, Ring Theory and so on. Only for a few references we refer the reader to [3], [4], [7]-[13], and for the theory on Hilbert spaces to [6].

We also recall that $A \in \mathcal{L}(\mathcal{H})$ is said to be a (a) normal operator if $AA^* = A^*A$, (b) *EP* operator if $AA^\dagger = A^\dagger A$, (c) bi-normal operator if $(AA^*)(A^*A) = (A^*A)(AA^*)$, (d) bi-*EP* operator if $(AA^\dagger)(A^\dagger A) = (A^\dagger A)(AA^\dagger)$, (e) ℓ-quasi-normal operator if $A(A^*A) = (A^*A)A$, (f) r-quasi-normal operator if $A(AA^*) = (AA^*)A$, (g) ℓ-quasi-*EP* operator if $A(A^\dagger A) = (A^\dagger A)A$ and (h) r-quasi-*EP* operator if $A(AA^\dagger) = (AA^\dagger)A$ [1, 6–8].

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The main aim of this note is to study equivalent conditions to those given in Penrose equations for an operator G to be the Moore-Penrose inverse of A by using concepts of normal, *EP*, bi-normal, bi-*EP*, ℓ- and r-quasi-normal, ℓ- and r-quasi-*EP* operators. The pursuit of the main result is due to the fact that mentioned conditions which are weaker than the one of being self-adjoint, can be adopted to define the Moore-Penrose inverse of A.

2. Main Results

Let $\mathcal{H}$ and $\mathcal{K}$ be two complex Hilbert spaces. Assume that an operator $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ having closed range is written in a matrix form with respect to mutually orthogonal subspaces decompositions $\mathcal{H} = \mathcal{R}(A^*) \oplus^\perp \mathcal{N}(A)$ and $\mathcal{K} = \mathcal{R}(A) \oplus^\perp \mathcal{N}(A^*)$ given by

$$A = \begin{bmatrix} A_1 & O \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \tag{1}$$

where $A_1 \in \mathcal{L}(\mathcal{R}(A^*), \mathcal{R}(A))$ is nonsingular. In this case, the Moore-Penrose generalized inverse of A has the following matrix decomposition

$$A^\dagger = \begin{bmatrix} A_1^{-1} & O \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}. \tag{2}$$

It is well known [6] that the general form of all g-inverses $G \in \mathcal{L}(\mathcal{K},\mathcal{H})$ of A (that is, $AGA = A$) is given by

$$G = \begin{bmatrix} A^{-1} & G_2 \\ G_3 & G_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}, \tag{3}$$

where G_i are arbitrary linear bounded operators on corresponding subspaces for $i = 2,3,4$. Clearly,

$$AG = \begin{bmatrix} I & A_1G_2 \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}. \tag{4}$$

Next technical result will be needed in the following.

Theorem 2.1. *A necessary and sufficient condition for a closed range operator $M \in \mathcal{L}(\mathcal{K})$ in the form*

$$M = \begin{bmatrix} I & Y \\ O & O \end{bmatrix} : \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M^*) \end{bmatrix},$$

to be (a) normal, (b) EP, (c) bi-normal (d) bi-EP, (e) ℓ-quasi-normal, (f) r-quasi-normal, (g) ℓ-quasi-EP or (h) r-quasi-EP is that $Y = O$.

Proof. First note that the bounded operator $I + YY^*$ is self-adjoint positive definite. Hence, it has a bounded inverse [3, pp. 334]. Now, we have

$$M^* = \begin{bmatrix} I & O \\ Y^* & O \end{bmatrix} \quad \text{and} \quad M^\dagger = \begin{bmatrix} (I+YY^*)^{-1} & O \\ Y^*(I+YY^*)^{-1} & O \end{bmatrix}$$

by Lemma 3.3.1 in [4]. Thus, simple computations give

$$MM^* = \begin{bmatrix} I+YY^* & O \\ O & O \end{bmatrix}, \quad MM^\dagger = \begin{bmatrix} I & O \\ O & O \end{bmatrix}, \quad M^*M = \begin{bmatrix} I & Y \\ Y^* & Y^*Y \end{bmatrix}$$

and

$$M^\dagger M = \begin{bmatrix} (I+YY^*)^{-1} & (I+YY^*)^{-1}Y \\ Y^*(I+YY^*)^{-1} & Y^*(I+YY^*)^{-1}Y \end{bmatrix}.$$

We now consider each of the cases.

(a) If M is normal then $MM^* = M^*M$ and directly yields $Y = O$.

(b) Assume that M is EP. So, from $MM^\dagger = M^\dagger M$ and their matrix forms we get $(I + YY^*)^{-1}Y = O$. Hence, $Y = O$.

(c) If M is bi-normal then $(MM^*)(M^*M) = (M^*M)(MM^*)$. Using that

$$(MM^*)(M^*M) = \begin{bmatrix} I + YY^* & (I + YY^*)Y \\ O & O \end{bmatrix}$$

and

$$(M^*M)(MM^*) = \begin{bmatrix} I + YY^* & O \\ Y^*(I + YY^*) & O \end{bmatrix}$$

we get $(I + YY^*)Y = O$. Since $I + YY^*$ is nonsingular, we thus arrive at $Y = O$.

(d) If M is bi-EP, the equality $(MM^\dagger)(M^\dagger M) = (M^\dagger M)(MM^\dagger)$ leads to

$$\begin{bmatrix} (I + YY^*)^{-1} & (I + YY^*)^{-1}Y \\ O & O \end{bmatrix} = \begin{bmatrix} (I + YY^*)^{-1} & O \\ Y^*(I + YY^*)^{-1} & O \end{bmatrix}$$

which implies $(I + YY^*)^{-1}Y = O$ and again $Y = O$.

(e) If M is ℓ-quasi-normal then $M(M^*M) = (M^*M)M$. So, from

$$\begin{bmatrix} I + YY^* & Y(I + Y^*Y) \\ O & O \end{bmatrix} = \begin{bmatrix} I & Y \\ Y^* & Y^*Y \end{bmatrix}$$

we get $Y = O$.

(f) The proof in case of M is r-quasi-normal is similar to that of (e).

(g) If M is ℓ-quasi-EP then $M(M^\dagger M) = (M^\dagger M)M$. Thus,

$$\begin{bmatrix} I & Y \\ O & O \end{bmatrix} = \begin{bmatrix} (I + YY^*)^{-1} & (I + YY^*)^{-1}Y \\ Y^*(I + YY^*)^{-1} & Y^*(I + YY^*)^{-1}Y \end{bmatrix}$$

gives $Y^*(I + YY^*)^{-1} = O$ and then $Y = O$.

(h) The proof in case M is r-quasi-EP is similar to that of (g).

□

Theorem 2.2. *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be a closed range operator. If $G \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ is a g-inverse of A such that AG is (a) normal, (b) EP, (c) bi-normal, (d) bi-EP, (e) ℓ-quasi-normal, (f) r-quasi-normal, (g) ℓ-quasi-EP or (h) r-quasi-EP then G is a least squares g-inverse of A.*

Proof. Assume that $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is written in the matrix form (1) and the general form for its g-inverses G is expressed as in (3). So, AG has the expression

$$AG = \begin{bmatrix} I_r & A_1G_2 \\ O & O \end{bmatrix} \quad (5)$$

as it was given in (4).

If we set $Y = A_1G_2$, and assume any of the assumptions (a)-(h) for AG, an application of Lemma 2.1 yields $Y = O$, that is $G_2 = O$ because A_1 is nonsingular. Hence, from (5) we have $(AG)^* = AG$. □

Corollary 2.3. *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be a closed range operator and $G \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ be a g-inverse of A. Then the following conditions are equivalent:*

(i) *AG is self-adjoint,*

(ii) *AG is normal,*
(iii) *AG is EP,*
(iv) *AG is bi-normal,*
(v) *AG is bi-EP,*
(vi) *AG is ℓ-quasi-normal,*
(vii) *AG is r-quasi-normal,*
(viii) *AG is ℓ-quasi-EP,*
(ix) *AG is r-quasi-EP.*

Proof. We know that a self-adjoint operator is normal, *EP*, bi-normal, bi-*EP*, ℓ- and r-quasi-normal and ℓ- and r-quasi-*EP*. So, item (i) implies items (ii)-(ix). If we assume that any of the conditions (ii)-(ix) holds, then (i) is satisfied by Theorem 2.2. Hence, the corollary follows. □

Next theorem provides a property related to minimum norm taking advantage of the one corresponding to least squares and remarking that G is a minimum norm g-inverse of A if and only if G^* is a least squares g-inverse of A^*.

Theorem 2.4. *Let $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ be a closed range operator. If $G \in \mathcal{L}(\mathcal{K},\mathcal{H})$ is a g-inverse of A such that GA is (a) normal, (b) EP, (c) bi-normal, (d) bi-EP, (e) ℓ-quasi-normal, (f) r-quasi-normal, (g) ℓ-quasi-EP or (h) r-quasi-EP then G is a minimum norm g-inverse of A.*

Proof. We first show that if an operator $B \in \mathcal{L}(\mathcal{H})$ is (a) normal, (b) *EP*, (c) bi-normal, (d) bi-*EP* then so is B^*. In fact, it is straightforward to check the normal, bi-normal and bi-*EP* cases by definition and using that $(B^*)^\dagger = (B^\dagger)^*$. Now, B is *EP* if and only if B and B^* have the same range [4, 6]. Evidently, this last condition and the fact that B^* and $(B^*)^*$ have the same range are equivalent, which means that B^* is *EP*. Now, it is easy to see that if B is ℓ-(or r-)quasi-normal then B^* is r-(or ℓ-)quasi-normal by taking adjoint operator. Similarly, it can be shown that if B is ℓ-(or r-)quasi-*EP* then B^* is r-(or ℓ-)quasi-*EP* by using $(B^*)^\dagger = (B^\dagger)^*$.

Let assume now that G is a g-inverse of A such that GA is (a) normal, (b) *EP*, (c) bi-normal, (d) bi-*EP*, (e) ℓ-quasi-normal, (f) r-quasi-normal, (g) ℓ-quasi-*EP* or (h) r-quasi-*EP*. Then, G^* is a g-inverse of A^* such that A^*G^* satisfies any of the conditions (a), (b), (c), (d), (f), (e), (h) or (g), respectively. Applying Theorem 2.2 we obtain that G^* is a least squares g-inverse of A^*. Hence, G is a minimum norm g-inverse of A. □

Now, we are ready to give the main result, which provides a new characterization of the Moore-Penrose inverse operator in terms of weaker conditions than those by Penrose.

Theorem 2.5. *Let $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ be a closed range operator and $G \in \mathcal{L}(\mathcal{K},\mathcal{H})$ be a reflexive g-inverse of A. If both AG and GA satisfy any of the following statements:*

(a) *normal,*
(b) *EP,*
(c) *bi-normal,*
(d) *bi-EP,*
(e) *ℓ-quasi-normal,*
(f) *r-quasi-normal,*
(g) *ℓ-quasi-EP,*
(h) *r-quasi-EP,*

then G is the Moore-Penrose inverse of A.

Proof. It follows from Theorem 2.2 and Theorem 2.4 and from the uniqueness of the Moore-Penrose inverse operator. □

Finally, if we denote the following subclasses of $\mathcal{L}(\mathcal{H})$: hermitian, normal, bi-normal, *EP*, bi-*EP*, quasi-normal and quasi-*EP* by the symbols $\mathbf{H}$, $\mathbf{N}$, $bi-\mathbf{N}$, $\mathbf{EP}$, $bi-\mathbf{EP}$, $\ell-q-\mathbf{N}$, $r-q-\mathbf{N}$, $\ell-q-\mathbf{EP}$ and $r-q-\mathbf{EP}$, respectively, it is remarkable that

$$\mathbf{H} \subsetneq \mathbf{N} \subsetneq bi-\mathbf{N} \cap \mathbf{EP} \subsetneq bi-\mathbf{EP}.$$

These inclusions can be seen using [2, pp. 2799] and the following finite-dimensional examples. The matrix

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is normal but not hermitian. The matrix

$$B_2 = \begin{bmatrix} 0 & 1 \\ 2i & 0 \end{bmatrix}$$

is bi-normal and also *EP* but is not normal. The matrix

$$B_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is bi-*EP* and bi-normal but it is not *EP*. Moreover, it is well known that ℓ- and r-quasi-normal and ℓ- and r-quasi-*EP* classes are different from each other as it can be seen in [5, 6], even different from the normal class.

The previous (strict) inclusions clarify the fact that conditions used in Theorem 2.5, which are weaker than the one of being self-adjoint, can be now adopted to define the Moore-Penrose inverse of A.

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