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This paper must be cited as:

Cascales Salinas, B.; Guirao Sánchez, AJ.; Kadets, V.; Soloviova, M. (2018). Gamma-flatness and Bishop-Phelps-Bollobas type theorems for operators. Journal of Functional Analysis. 274(3):863-888. doi:10.1016/j.jfa.2017.10.020

The final publication is available at

https://doi.org/10.1016/j.jfa.2017.10.020

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Γ-FLATNESS AND BISHOP–PHELPS–BOLLOBÁS TYPE THEOREMS FOR OPERATORS

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ABSTRACT. The Bishop–Phelps–Bollobás property deals with simultaneous approximation of an operator T and a vector x at which T nearly attains its norm by an operator T_0 and a vector x_0, respectively, such that T_0 attains its norm at x_0. In this note we extend the already known results about the Bishop–Phelps–Bollobás property for Asplund operators to a wider class of Banach spaces and to a wider class of operators. Instead of proving a BPB-type theorem for each space separately we isolate two main notions: Γ-flat operators and Banach spaces with ACK_ρ structure. In particular, we prove a general BPB-type theorem for Γ-flat operators acting to a space with ACK_ρ structure and show that uniform algebras and spaces with the property β have ACK_ρ structure. We also study the stability of the ACK_ρ structure under some natural Banach space theory operations. As a consequence, we discover many new examples of spaces Y such that the Bishop–Phelps–Bollobás property for Asplund operators is valid for all pairs of the form (X, Y).

1. INTRODUCTION

In this paper X, Y are Banach spaces (real or complex), $\mathbb{K}$ stands for the field of scalars $\mathbb{R}$ or $\mathbb{C}$, $L(X, Y)$ is the space of all bounded linear operators $T\colon X \to Y$, $L(X) = L(X, X)$, B_X and S_X denote the closed unit ball and the unit sphere of X, respectively and $\mathrm{aco} A$ stands for the absolute convex hull of the set A.

According to [1], a pair (X, Y) has the *Bishop–Phelps–Bollobás property* (BPB property) for operators if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every operator $T \in L(X, Y)$ of norm 1, if $x_0 \in S_X$ is such that $\|T(x_0)\| > 1 - \delta(\varepsilon)$, then there exist $u_0 \in S_X$ and $S \in S_{L(X,Y)}$ satisfying $\|S(u_0)\| = 1$, $\|x_0 - u_0\| < \varepsilon$, and $\|T - S\| < \varepsilon$.

If an analogous definition is valid for operators T, S from a subspace $\mathcal{I} \subset L(X, Y)$, then we say that (X, Y) has the *Bishop–Phelps–Bollobás property for operators from* $\mathcal{I}$.

With this terminology, the original Bishop–Phelps–Bollobás theorem [8] says that for every X, the pair $(X, \mathbb{K})$ has the BPB property for operators.

Date: April 4th, 2017.

2010 *Mathematics Subject Classification.* Primary: 46B20, 46E25 Secondary: 47B07, 47B48.

Key words and phrases. Bishop–Phelps–Bollobás, Asplund operators, norm attaining, uniform algebra.

Also, thanks to Acosta, Aron, García, and Maestre [1, Theorem 2.2], if Y has the Lindenstrauss' property β (Definition 4.8), then for every Banach space X the pair (X, Y) has the Bishop–Phelps–Bollobás property for operators.

In 2011 Aron, Cascales, and Kozhushkina [4, Theorem 2.4] showed that for every X and every compact Hausdorff space K the pair $(X, C(K))$ has the BPB property for Asplund operators (Definition 2.2). In 2013 Cascales, Guirao and Kadets [9] extended this result to uniform algebras $\mathcal{A} \subset C(K)$. The exact statement of the last result is given below.

Theorem 1.1 ([9, Theorem 3.6]). *Let $\mathcal{A} \subset C(K)$ be a uniform algebra and $T\colon X \to \mathcal{A}$ be an Asplund operator with $\|T\| = 1$. Suppose that $0 < \varepsilon < \sqrt{2}$ and $x_0 \in S_X$ are such that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{2}$. Then there exist $u_0 \in S_X$ and an Asplund operator $S \in S_{L(X,\mathcal{A})}$ satisfying that:*

$$\|Su_0\| = 1, \quad \|x_0 - u_0\| \le \varepsilon \quad \text{and} \quad \|T - S\| < 2\varepsilon.$$

In the same vein, Acosta, Becerra Guerrero, García, Kim, and Maestre [2] generalized [4, Theorem 2.4] to some spaces of continuous vector-valued functions (see Theorem 4.13 below).

The aim of this paper is to extend all these results to a wider class of Banach spaces and to a wider class of operators. The main difference of our approach is that instead of proving a Bishop–Phelps–Bollobás kind theorem for each space separately (and thus repeating essential parts of the proof many times), we introduce a new Banach space property (called ACK_ρ structure) which extracts all the useful technicalities for the BPB type of approximation. We prove a general Bishop–Phelps–Bollobás type theorem for Γ-flat operators (see Definition 2.8) acting to a space with ACK_ρ structure and show that uniform algebras and spaces with the property β have ACK_ρ structure. After that, we study the stability of the ACK_ρ structure under some natural Banach space theory operations which as a consequence gives us a wide collection of examples of pairs (X, Y) possessing the BPB property for Asplund operators.

The structure of the paper is as follows. In section 2 we collect the necessary definitions (in particular that of Asplund operators and of Γ-flat operators) and prove an important Basic Lemma. In section 3 we introduce the central concept of ACK_ρ structure and prove a general BPB type theorem for this class of Banach spaces. Finally, in section 4 we perform the announced study of spaces with ACK_ρ structure which, on the one hand, gives a unified proof of several results from [1, 2, 4] and [9], and on the other hand, leads to new BPB type theorems in concrete spaces.

For the non-defined notions used through this article, we refer to [12].

2. Γ-Flat operators and the Basic Lemma

Let (B, τ) be a topological space, ρ be a metric on B (possibly, not related with τ). B is said to be *fragmented by* ρ, if for every non-empty subset

$A \subset B$ and for every $\varepsilon > 0$ there exists a τ-open U such that $U \cap A \neq \emptyset$ and $\operatorname{diam}(U \cap A) < \varepsilon$. Some important examples of fragmented topological spaces come from Banach space theory. For instance, every weakly compact subset of a Banach space is fragmented by the norm (i.e., by the metric $\rho(x, y) = \|x - y\|$), see [16].

A Banach space X is called an *Asplund space* if, whenever f is a convex continuous function defined on an open subset U of X, the set of all points of U where f is Fréchet differentiable is a dense G_δ-subset of U. This definition is due to Asplund [3] under the name *strong differentiability space*. This concept has multiple characterizations via topology or measure theory, as in the following:

Theorem 2.1 ([17, 21, 22])**.** *Let X be a Banach space. Then the following conditions are equivalent:*

(i) *X is an Asplund space;*
(ii) *every w^*-compact subset of (X, w^*) is fragmented by the norm;*
(iii) *each separable subspace of X has separable dual;*
(iv) *X^* has the Radon-Nikodým property.*

According to the above, every reflexive space and every separable space whose dual is separable is an Asplund space. Classical example of Asplund spaces are L_p and ℓ_p with $1 < p < \infty$, and also c_0; examples of spaces that are not Asplund are $C[0, 1]$, ℓ_1, ℓ_∞, $L_1[0, 1]$ and $L_\infty[0, 1]$, see for instance [11].

Definition 2.2 ([23])**.** An operator $T \in L(X, Y)$ is said to be an *Asplund operator* if it factors through an Asplund space, i.e., there exist an Asplund Banach space Z and operators $T_1 \in L(X, Z), T_2 \in L(Z, Y)$ such that $T = T_2 \circ T_1$.

Compact and weakly compact operators are Asplund operators (every weakly compact operator factorizes through a reflexive space).

Theorem 2.1 yields the following result:

Remark 2.3 ([23])**.** *If T is an Asplund operator, then its adjoint T^* sends the unit ball of Y^* into a w^*-compact subset of (X, w^*) that is norm fragmented.*

Definition 2.4. Let Y be a Banach space. Y is said to have the *Bishop–Phelps–Bollobás property for Asplund operators* (A-BPBp for short) if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that for every Banach space X and every Asplund operator $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that $\|T(x_0)\| > 1 - \delta(\varepsilon)$, then there exist $u_0 \in S_X$ and $S \in S_{L(X,Y)}$ satisfying

$$\|S(u_0)\| = 1, \|x_0 - u_0\| < \varepsilon \text{ and } \|T - S\| < \varepsilon.$$

Definition 2.5 ([13])**.** Let A and B be topological spaces. A function $f\colon A \to B$ is said to be *quasi-continuous*, if for every non-empty open subset $U \subset A$, every $z \in U$ and every neighborhood V of $f(z)$ there exists a non-empty open subset $W \subset U$ such that $f(W) \subset V$.

Let us introduce some new terminology. Note that a similar concept of fragmentability of maps was introduced in [14].

Definition 2.6. Let A be a topological space and (M,d) be a metric space. A function $f\colon A \to M$ is said to be *openly fragmented*, if for every non-empty open subset $U \subset A$ and every $\varepsilon > 0$ there exists a non-empty open subset $V \subset U$ with $d\text{-diam}(f(V)) < \varepsilon$.

Every continuous or quasi-continuous function $f\colon A \to M$ is openly fragmented. In particular, if A is a discrete topological space then every $f\colon A \to M$ is openly fragmented. For every metric space M, every left-continuous $f\colon [0,1] \to M$ and every right-continuous function $f\colon [0,1] \to M$ are openly fragmented. Every $f\colon A \to M$ with a dense set of continuity points is openly fragmented. Every separately continuous function of two variables $f\colon [0,1]\times[0,1] \to M$ is quasi-continuous [6] and, consequently, openly fragmented. Some other easy but useful examples are given in the following theorem:

Theorem 2.7. *Let A, B be topological spaces, ρ be a metric on B (possibly, not related with the original topology), and $f\colon A \to B$ be a function.*

- (i) *If B is fragmented by ρ, and f is continuous in the original topologies, then $f\colon A \to (B,\rho)$ is openly fragmented.*
- (ii) *If A is fragmented by some metric ρ_1 and $f\colon (A,\rho_1) \to (B,\rho)$ is uniformly continuous, then $f\colon A \to (B,\rho)$ is openly fragmented.*

Let, moreover, $(B, \|\cdot\|)$ be a Banach space. Then

- (iii) *If $f, g\colon A \to (B,\|\cdot\|)$ are openly fragmented then $f+g\colon A \to (B,\|\cdot\|)$ is openly fragmented.*
- (iv) *If $f\colon A \to (B,\|\cdot\|)$ and $g\colon A \to \mathbb{K}$ are openly fragmented then $gf\colon A \to (B,\|\cdot\|)$ is openly fragmented.*

Proof. (i) For a given non-empty open subset $U \subset A$ consider $f(U) \subset B$. By ρ-fragmentability of B, for every $\varepsilon > 0$ there exits an open subset W of B with $f(U) \cap W \neq \emptyset$ and $\operatorname{diam}(f(U) \cap W) < \varepsilon$. By continuity of f the set $f^{-1}(W)$ is open and $V := f^{-1}(W) \cap U$ will be the non-empty open subset $V \subset U$ we need.

The statements (ii), (iii) and (iv) are routine. □

Definition 2.8. Let X, Y be Banach spaces and $\Gamma \subset Y^*$. An operator $T \in L(X,Y)$ is said to be Γ*-flat*, if $T^*|_\Gamma\colon (\Gamma, w^*) \to (X^*, \|\cdot\|_{X^*})$ is openly fragmented. In other words, for every w^*-open subset $U \subset Y^*$ with $U \cap \Gamma \neq \emptyset$ and every $\varepsilon > 0$ there exists a w^*-open subset $V \subset U$ with $V \cap \Gamma \neq \emptyset$ such that $\operatorname{diam}(T^*(V \cap \Gamma)) < \varepsilon$. The set of all Γ-flat operators in $L(X,Y)$ will be denoted by $\mathrm{Fl}_\Gamma(X,Y)$.

Statements (iii) and (iv) of the previous theorem imply that $\mathrm{Fl}_\Gamma(X,Y)$ is a linear subspace of $L(X,Y)$. Let us list some examples of Γ-flat operators.

Example A. *Every Asplund operator $T \in L(X,Y)$ is Γ-flat for every $\Gamma \subset B_{Y^*}$. This follows from Remark 2.3 and Theorem 2.7, (i).*

Example B. *If* $(\Gamma, w^*) \subset Y^*$ *is norm fragmented, then every bounded operator in* $L(X,Y)$ *is* Γ*-flat (Theorem 2.7, (ii)). In particular, we have the next concrete example.*

Example C. *If* $(\Gamma, w^*) \subset Y^*$ *is discrete, then every operator* $T \in L(X,Y)$ *is* Γ*-flat.*

The notion of Γ-flatness generalizes the property of Asplund operators that allowed to prove [4, Lemma 2.3]. The immediate generalization of that lemma is the following result:

Lemma 2.9 (Basic Lemma)**.** *Let* X, Y *be Banach spaces,* $\Gamma \subset B_{Y^*}$ *be a* 1*-norming set,* $T \in \mathrm{Fl}_\Gamma(X,Y)$ *be a* Γ*-flat operator with* $\|T\| = 1$ *,* $0 < \varepsilon < 2/3$*, and* $x_0 \in S_X$ *be such that* $\|Tx_0\| > 1-\varepsilon$*. Then for every* $r > 0$ *and for every* $k \in [\frac{\varepsilon}{2(1-\varepsilon)}, 1)$ *there exist:*

(i) *a* w^**-open set* $U_r \subset Y^*$ *with* $U_r \cap \Gamma \neq \emptyset$*, and*
(ii) *points* $x_r^* \in S_{X^*}$ *and* $u_r \in S_X$ *with* $|x_r^*(u_r)| = 1$ *such that*

$$\|x_0 - u_r\| \leq \frac{\varepsilon}{k} \quad \text{and} \quad \|T^*z^* - x_r^*\| \leq r + 2k \text{ for every } z^* \in U_r \cap \Gamma. \tag{2.1}$$

The proof of this fact is a modification of that of [4, Lemma 2.3]. First, we use the following fact:

Proposition 2.10 ([19, Corollary 2.2])**.** *Let* X *be a real Banach space,* $z^* \in S_{X^*}$*,* $z \in S_X$*,* $\eta > 0$ *and* $z^*(z) \geq 1-\eta$*. Then for every* $k \in (0,1)$ *there exist* $y^* \in S_{X^*}$ *and* $u \in S_X$ *such that*

$$y^*(u) = 1, \qquad \|z - u\| \leq \frac{\eta}{k}, \qquad \|z^* - y^*\| \leq 2k.$$

In the next proposition, we relax the condition $z^* \in S_X$ allowing $\|z^*\|$ to be smaller than 1. Note that x^* plays the role of z^*.

Proposition 2.11. *Let* X *be a Banach space,* $\varepsilon \in (0, 2/3)$*,* $x \in S_X, x^* \in B_{X^*}$ *and* $|x^*(x)| \geq 1-\varepsilon$*. Then, for every* $k \in [\frac{\varepsilon}{2(1-\varepsilon)}, 1)$ *there exist* $y^* \in S_{X^*}$ *and* $u \in S_X$ *such that*

$$|y^*(u)| = 1, \qquad \|x - u\| \leq \frac{\varepsilon}{k}, \qquad \|x^* - y^*\| \leq 2k.$$

Proof. Without loss of generality we can assume that $x^*(x) \geq 1-\varepsilon$. Then $\|x^*\| \geq 1-\varepsilon$. Set $z^* := x^*/\|x^*\|, z := x$. Then $z^*(z) \geq 1-\eta$ for $\eta = 1 - (1-\varepsilon)\|x^*\|^{-1} \in [0, \varepsilon]$. If $\eta = 0$, then $z^*(z) = 1$, so we can take $y^* = z^*$ and $u = x$, which satisfy the inequalities we want. So we may assume that $0 < \eta \leq \varepsilon$. Set $k_0 := \frac{k\eta}{\varepsilon} \in (0,1)$. So, according to Proposition 2.10, there exist $y^* \in S_{X^*}$ and $u \in S_X$ such that

$$y^*(u) = 1, \qquad \|z - u\| \leq \frac{\eta}{k_0}, \qquad \|z^* - y^*\| \leq 2k_0.$$

Therefore, $\|x-u\| \le \eta/k_0 = \varepsilon/k$. Also, we have

$$\|x^*-y^*\| \le \|x^*-z^*\| + \|z^*-y^*\| \le \left\| x^* - \frac{x^*}{\|x^*\|} \right\| + 2k_0$$
$$= 1 - \|x^*\| + 2k_0 = 1 - \|x^*\| + \frac{2k}{\varepsilon}\left(1 - \frac{1-\varepsilon}{\|x^*\|}\right).$$

Observe that the function $\psi(t) = 1 - t + \frac{2k}{\varepsilon}(1 - \frac{1-\varepsilon}{t})$ is increasing when $t \in \left(0, \sqrt{\frac{2k(1-\varepsilon)}{\varepsilon}}\right)$. So, if $k \ge \frac{\varepsilon}{2(1-\varepsilon)}$, we have $\psi(\|x^*\|) \le \psi(1) = 2k$. In this case, we get our conclusion. □

Proof of Lemma 2.9. Use that $\Gamma \subset B_{Y^*}$ is 1-norming and pick $y_0^* \in \Gamma$ such that

$$|T^*(y_0^*)(x_0)| = |y_0^*(Tx_0)| > 1 - \varepsilon.$$

Set $U := \{y^* \in Y^* : |T^*y^*(x_0)| > 1-\varepsilon\}$. We have that $y_0^* \in U \cap \Gamma \subset B_{Y^*}$. Since U is w^*-open in Y^* and $U \cap \Gamma \ne \emptyset$, according to Definition 2.8, for every $r > 0$ there exists a w^*-open subset $U_r \subset U$ with $U_r \cap \Gamma \ne \emptyset$ such that $\operatorname{diam}(T^*(U_r \cap \Gamma)) < r$.

Fix some $y_1^* \in U_r \cap \Gamma$ and set $x_1^* = T^*y_1^*$. Then, $1 \ge \|x_1^*\| \ge |x_1^*(x_0)| > 1 - \varepsilon$ which, by applying Proposition 2.11 to any $\frac{\varepsilon}{2(1-\varepsilon)} \le k < 1$, gives $x_r^* \in S_{X^*}$ and $u_r \in S_X$ with $|x_r^*(u_r)| = 1$ and such that

$$\|x_0 - u_r\| \le \frac{\varepsilon}{k} \quad \text{and} \quad \|x_1^* - x_r^*\| \le 2k.$$

Finally, let $z^* \in U_r \cap \Gamma$ be arbitrary. Then,

$$\|T^*z^* - x_r^*\| \le \|T^*z^* - x_1^*\| + \|x_1^* - x_r^*\| \le r + 2k,$$

which finishes the proof. □

3. The ACK structure

In the definition below we extract the structural properties of $C(K)$ and its uniform subalgebras that were essential in the proof of [9, Th. 3.6]. The name "ACK structure" comes from the words "Asplund" and "$C(K)$".

Definition 3.1. Let X be a Banach space and $\mathcal{O}$ be a non-emtpy subset of $L(X)$. We will say that X has *$\mathcal{O}$-ACK structure with parameter ρ*, for some $\rho \in [0,1)$ ($X \in \mathcal{O}$-ACK$_\rho$, for short) whenever there exists a 1-norming set $\Gamma \subset B_{X^*}$ such that for every $\varepsilon > 0$ and every non-empty relatively w^*-open subset $U \subset \Gamma$ there exist a non-empty subset $V \subset U$, vectors $x_1^* \in V$, $e \in S_X$ and an operator $F \in \mathcal{O}$ with the following properties:

(I) $\|Fe\| = \|F\| = 1$;
(II) $x_1^*(Fe) = 1$;
(III) $F^*x_1^* = x_1^*$;
(IV) denoting $V_1 = \{x^* \in \Gamma : \|F^*x^*\| + (1-\varepsilon)\,\|(I_{X^*} - F^*)(x^*)\| \le 1\}$, then $|x^*(Fe)| \le \rho$ for every $x^* \in \Gamma \setminus V_1$;

(V) $\operatorname{dist}(F^*x^*, \operatorname{aco}\{0, V\}) < \varepsilon$ for every $x^* \in \Gamma$ (recall, that aco abbreviates the absolute convex hull); and

(VI) $|v^*(e) - 1| \leq \varepsilon$ for every $v^* \in V$.

The Banach space X is said to have *simple $\mathcal{O}$-ACK structure* ($X \in \mathcal{O}$-ACK) if $V_1 = \Gamma$. In other words, for $X \in \mathcal{O}$-ACK the above definition holds true in a stronger form: the property (IV) is substituted by

(IV)' $\|F^*x^*\| + (1-\varepsilon)\|(I_{X^*} - F^*)(x^*)\| \leq 1$ for every $x^* \in \Gamma$,

which makes the original (IV) unnecessary, because now it would speak about the empty set $\Gamma \setminus V_1$. In case of $\mathcal{O} = L(X)$, we will simply say ACK_ρ (and simple ACK) structure.

Remark 3.2. *If X belongs to the class ACK_ρ, then X also belongs to ACK_σ for every $\sigma \in [\rho, 1)$. Moreover, $\mathrm{ACK} \subset \mathrm{ACK}_\rho$ for every $\rho \in [0, 1)$.*

Definition 3.3. A linear subspace $\mathcal{I} \subset L(X, Y)$ is said to be a Γ-*flat ideal*, if all elements of $\mathcal{I}$ are Γ-flat operators, $\mathcal{I}$ contains all operators of finite rank, and for every $T \in \mathcal{I}$ and every $F \in L(Y)$ their composition $F \circ T$ belongs to $\mathcal{I}$.

Observe that the subspace of Asplund operators in $L(X, Y)$ is an example of Γ-flat ideal. The theorem below motivates the above definition.

Theorem 3.4. *Let X be a Banach space, $Y \in \mathrm{ACK}_\rho$, $\Gamma \subset Y^*$ be the corresponding 1-norming set from Defintion 3.1 and $T \in L(X, Y)$ be a Γ-flat operator with $\|T\| = 1$. Let $0 < \varepsilon \leq 1/2$ and let $x_0 \in S_X$ be such that $\|Tx_0\| > 1 - \varepsilon$. Then there exist $u_0 \in S_X$ and an operator $S \in S_{L(X,Y)}$ with $\|Su_0\| = 1$ such that*

$$\max\{\|x_0 - u_0\|, \|T - S\|\} < \sqrt{2\varepsilon}\left(1 + \frac{2}{1 - \rho + \sqrt{2\varepsilon}}\right).$$

Moreover, if $Y \in \mathrm{ACK}$ then the estimate can be improved to

$$\max\{\|x_0 - u_0\|, \|T - S\|\} < \sqrt{2\varepsilon}.$$

Additionally, S can be chosen from $\mathcal{I}$ whenever T belongs to a Γ-flat ideal $\mathcal{I}$. In particular, every $Y \in \mathrm{ACK}_\rho$ (ACK) has the A-BPBp.

Before proving the theorem, we need a preliminary result.

Lemma 3.5. *Under the conditions of Theorem 3.4 above, for every $k \in (\varepsilon/(2(1-\varepsilon)), 1)$ and for every*

$$\nu > 2k\left(1 + \frac{2}{1 - \rho + 2k}\right),$$

there exist $u_0 \in S_X$ and $S \in S_{L(X,Y)}$ satisfying $\|Su_0\| = 1$, $\|x_0 - u_0\| \leq \frac{\varepsilon}{k}$ and $\|T - S\| < \nu$. In the case of $Y \in \mathrm{ACK}$ the same is true for every $\nu > 2k$.

If, moreover, T belongs to a Γ-flat ideal $\mathcal{I}$, then S can be chosen from $\mathcal{I}$ as well.

Proof. First, consider the more involved case of $Y \in \mathrm{ACK}_\rho$. Fix $r > 0$ and $0 < \varepsilon' < 2/3$. Now, we can apply Lemma 2.9 with Y, Γ, r and $\varepsilon > 0$. We produce a w^*-open set $U_r \subset Y^*$ with $U_r \cap \Gamma \neq \emptyset$, and points $x_r^* \in S_{X^*}$ and $u_r \in S_X$ with $|x_r^*(u_r)| = 1$ such that (2.1) holds true.

Since $U_r \cap \Gamma \neq \emptyset$, we can apply Definition 3.1 to $U = U_r \cap \Gamma$ and ε' and obtain a non-empty $V \subset U$, $y_1^* \in V$, $e \in S_Y$, $F \in L(Y)$ and $V_1 \subset \Gamma$ which satisfy properties (I) – (VI). In particular, for every $z^* \in V \subset U_r \cap \Gamma$ according to (2.1) we have

$$\|T^* z^* - x_r^*\| \leq r + 2k. \tag{3.1}$$

Define now the linear operator $S \colon X \to Y$ by the formula

$$S(x) := x_r^*(x) F e + (1 - \widetilde{\varepsilon})(I_Y - F) T x, \tag{3.2}$$

where the value of $\widetilde{\varepsilon} \in [\varepsilon', 1)$ will be specified below in such a way that $\|S\| \leq 1$. In order to do this, bearing in mind the fact that Γ is 1-norming, we can write

$$\|S\| = \|S^*\| = \sup \left\{ \|S^* y^*\| : y^* \in \Gamma \right\}.$$

So our first goal is to estimate

$$\|S^* y^*\| = \|y^*(Fe) x_r^* + (1 - \widetilde{\varepsilon}) T^* (I_{Y^*} - F^*)(y^*)\| \tag{3.3}$$

from above for all $y^* \in \Gamma$. For $y^* \in V_1$, the sought estimate $\|S^* y^*\| \leq 1$ follows immediately from the definition of V_1 (see property (IV)). So, it remains to consider the case $y^* \in \Gamma \setminus V_1$. Thanks to (V), for every $y^* \in \Gamma$, there exists an element $v^* = \sum_{k=1}^n \lambda_k v_k^*$ with

$$\|F^* y^* - v^*\| < \varepsilon' \tag{3.4}$$

such that $\{v_k^*\}_{k=1}^n \subset V$, and $\sum_{k=1}^n |\lambda_k| \leq 1$. According to (3.1) we have $\|T^* v_k^* - x_r^*\| \leq r + 2k$, consequently

$$\begin{aligned} \|v^*(e) x_r^* - T^* v^*\| &\leq \sum_{k=1}^n |\lambda_k| \, \|v_k^*(e) x_r^* - T^* v_k^*\| \\ &\overset{\text{(VI)}}{\leq} \varepsilon' + \sum_{k=1}^n |\lambda_k| \, \|x_r^* - T^* v_k^*\| \leq \varepsilon' + r + 2k. \end{aligned} \tag{3.5}$$

Now, for every $y^* \in \Gamma \setminus V_1$

$$\begin{aligned} \|S^* y^*\| &\leq \widetilde{\varepsilon} |y^*(Fe)| + (1 - \widetilde{\varepsilon}) \|y^*(Fe) x_r^* + T^* y^* - T^* F^* y^*\| \\ &\overset{\text{(IV)}}{\leq} \widetilde{\varepsilon} \rho + (1 - \widetilde{\varepsilon}) \|T^* y^*\| + (1 - \widetilde{\varepsilon}) \|(F^* y^*)(e) x_r^* - T^* F^* y^*\| \\ &\overset{(3.4)}{\leq} \widetilde{\varepsilon} \rho + (1 - \widetilde{\varepsilon}) + 2\varepsilon'(1 - \widetilde{\varepsilon}) + (1 - \widetilde{\varepsilon}) \|v^*(e) x_r^* - T^* v^*\| \\ &\overset{(3.5)}{\leq} \widetilde{\varepsilon} \rho + (1 - \widetilde{\varepsilon}) + 2\varepsilon'(1 - \widetilde{\varepsilon}) + (1 - \widetilde{\varepsilon})(\varepsilon' + r + 2k) \\ &\leq \widetilde{\varepsilon} \rho + (1 - \widetilde{\varepsilon})(1 + 3\varepsilon' + r + 2k). \end{aligned}$$

This means, that if we choose $\widetilde{\varepsilon} = (3\varepsilon' + r + 2k)/(1 - \rho + 3\varepsilon' + r + 2k)$, then we have $\|S\| \leq 1$. In this case,

$$1 = |x_r^*(u_r)| \overset{\text{(II)}}{=} |y_1^*(x_r^*(u_r)Fe)|,$$

and by using de definition of S and keeping in mind that (III) implies $y_1^*(FTu_r) = F^*y_1^*(Tu_r) = y_1^*(Tu_r)$, we deduce that

$$|y_1^*(x_r^*(u_r)Fe)| = |y_1^*(S(u_r))| \leq \|S(u_r)\| \leq 1.$$

Therefore, $\|S\| = 1$ and S attains the norm at the point $u_0 := u_r \in S_X$ for which by (2.1) we already had that $\|u_0 - x_0\| \leq \frac{\varepsilon}{k}$.

Now, let us estimate

$$\begin{aligned} \|S - T\| = \|S^* - T^*\| &= \sup_{y^* \in \Gamma} \|S^*y^* - T^*y^*\| \\ &\leq \sup_{y^* \in \Gamma} \|y^*(Fe)x_r^* - T^*F^*y^*\| + 2\widetilde{\varepsilon}. \end{aligned} \tag{3.6}$$

For every $y^* \in \Gamma$ we can proceed the same way as before. Namely,

$$\begin{aligned} \|(F^*y^*)(e)x_r^* - T^*F^*y^*\| &\overset{(3.4)}{\leq} 2\varepsilon' + \|v^*(e)x_r^* - T^*v^*\| \\ &\overset{(3.5)}{\leq} 3\varepsilon' + r + 2k. \end{aligned}$$

Combining this with the inequalities (3.6) and the value of $\widetilde{\varepsilon}$ we conclude that

$$\|T - S\| \leq 3\varepsilon' + r + 2k + 2\frac{3\varepsilon' + r + 2k}{1 - \rho + 3\varepsilon' + r + 2k}. \tag{3.7}$$

Since $r > 0$ and $0 < \varepsilon' < 2/3$ are arbitrary, for suitable values we will have the desired estimate $\|T - S\| < \nu$.

To finish the proof in the case of $Y \in \mathrm{ACK}_\rho$ we observe that if T belongs to a Γ-flat ideal $\mathcal{I}$ then $S \in \mathcal{I}$.

Now the simpler case of $Y \in \mathrm{ACK}$. In this case $\|S^*y^*\| \leq 1$ for all $y^* \in \Gamma$ thanks to (IV)'. So, $\|S\| \leq 1$ for all values of $\widetilde{\varepsilon} \in [\varepsilon', 1)$ and we can simply take $\widetilde{\varepsilon} = \varepsilon'$. With such a choice of $\widetilde{\varepsilon}$ the estimate (3.7) changes to $\|T - S\| \leq 5\varepsilon' + r + 2k$, which again for small values of r and ε' gives us $\|T - S\| < \nu$ for the ν which corresponds to this case. □

Proof of Theorem 3.4. First, select $\varepsilon_0 \in (0, \varepsilon)$ in such a way that the inequality $\|Tx_0\| > 1 - \varepsilon_0$ is still valid. Now we apply Lemma 3.5 with ε_0 instead of ε and substitute $k = \sqrt{\varepsilon_0/2}$. In the case of $Y \in \mathrm{ACK}_\rho$ we take $\nu \in \left(\sqrt{2\varepsilon_0}\left(1 + \frac{2}{1-\rho+\sqrt{2\varepsilon_0}}\right), \sqrt{2\varepsilon}\left(1 + \frac{2}{1-\rho+\sqrt{2\varepsilon}}\right)\right)$, and in the case of $Y \in \mathrm{ACK}$ we take $\nu \in (\sqrt{2\varepsilon_0}, \sqrt{2\varepsilon})$. □

Remark 3.6. *The statements of Lemma 3.5 and Theorem 3.4 remain correct if in the definition of* ACK_ρ *and* ACK *the property (IV) is substituted by the following weaker one, in which* V_1 *is larger than in the original definition:*

Denote $V_1 = \{y^* \in \Gamma : |y^*(Fe)| + (1 - \varepsilon') \|(I_{Y^*} - F^*)(y^*)\| \leq 1\}$. *Then* $|v^*(Fe)| \leq \rho$ *for every* $v^* \in \Gamma \setminus V_1$.

Also, a look at the proof of Lemma 2.9 shows that the condition of T being Γ-flat can be weaken in the following way: for every $y \in B_Y$ and every $\delta > 0$ if the w^*-slice $S(\Gamma, x, \delta) := \{y^* \in \Gamma : \operatorname{Re} y^*(y) > 1 - \delta\}$ is not empty, then for every $\varepsilon > 0$ there exists a non-empty relatively w^*-open subset $V \subset S(\Gamma, x, \delta)$ such that $\operatorname{diam}(T^*(V)) < \varepsilon$.

There are two reasons why we have selected the more restrictive variants. Firstly, with the restrictive definition of (IV) we are able to prove a nice stability result (Theorem 4.12 below), and secondly, all the examples with "relaxed" versions of (IV) and of Γ-flatness that we have in hand, satisfy the restrictive variant of (IV) and of Γ-flatness.

4. Banach spaces with ACK structure

The aim of this section is presenting those *natural* examples of Banach spaces having ACK structure as well as showing the stability of the ACK structure under some operations, such us ℓ_∞-sums or injective tensor products.

First of all, let us introduce the first natural class of Banach spaces with ACK structure. As commented above, Definition 3.1, comes from an analysis of the proofs in [9]. We shall show next that, indeed, every uniform algebra $\mathcal{A}$ has simple ACK structure. The key tool is Lemma 4.2, that was proved in [9, Lemma 2.5 and Lemma 2.7], and is about the existence of peak functions $f \in S_{\mathcal{A}}$ whose range is contained in the Stolz's region

$$\mathrm{St}_\varepsilon = \{z \in \mathbb{C} : |z| + (1-\varepsilon)|1-z| \le 1\}.$$

For a topological space (T, τ), we denote by $C_b(T)$ the space of bounded continuous functions $f : T \to \mathbb{K}$ equipped with the sup-norm.

Definition 4.1. Let (T, τ) be a topological space. A subalgebra $\mathcal{A} \subset C_b(T)$ is said to be an ACK-*subalgebra*, if for every non-empty open set $W \subset T$ and $0 < \varepsilon < 1$, there exist $f \in \mathcal{A}$ and $t_0 \in W$ such that $f(t_0) = \|f\|_\infty = 1$, $|f(t)| < \varepsilon$ for every $t \in T \setminus W$ and $f(T) \subset \mathrm{St}_\varepsilon$.

Lemma 4.2. *Let $\mathcal{A} \subset C(K)$ be a uniform algebra. Then there exists a topological space $\Gamma_{\mathcal{A}}$ such that $\mathcal{A}$ is isometric to an* ACK*-subalgebra of $C_b(\Gamma_{\mathcal{A}})$. In the case of K being the space of multiplicative functionals on $\mathcal{A}$ the corresponding $\Gamma_{\mathcal{A}}$ can be selected as a topological subspace of K.*

We will use the following elementary property of St_ε.

Lemma 4.3. *If z belongs to the Stolz region St_ε, then $z^n \in \mathrm{St}_\varepsilon$.*

Proof. For every $z \in \mathrm{St}_\varepsilon$ it holds

$$\begin{aligned}
|z^n| + (1-\varepsilon)|1-z^n| &= |z^n| + (1-\varepsilon)|1-z||1+z+\ldots+z^{n-1}| \\
&\le |z|^n + (1-|z|)|1+z+\ldots+z^{n-1}| \\
&\le |z|^n + (1-|z|)(1+|z|+\ldots+|z|^{n-1}) \\
&= |z|^n + (1-|z|^n) = 1,
\end{aligned}$$

which finishes the proof. □

The following simple lemma gives an essential property that turns uniform algebras into Banach spaces with simple ACK structure.

Lemma 4.4. *Let $\mathcal{A} \subset C_b(\Gamma_{\mathcal{A}})$ be an* ACK*-subalgebra. Then, for every non-empty open set $W \subset \Gamma_{\mathcal{A}}$ and $0 < \varepsilon < 1$, there exist a non-emtpy subset $W_0 \subset W$, functions f, $e \in \mathcal{A}$, and $t_0 \in W_0$ such that $f(t_0) = \|f\| = 1$, $e(t_0) = \|e\| = 1$, $|f(t)| < \varepsilon$ for every $t \in \Gamma_{\mathcal{A}} \setminus W_0$, $|1 - e(t)| < \varepsilon$ for every $t \in W_0$ and $f(\Gamma_{\mathcal{A}}) \subset \mathrm{St}_\varepsilon$.*

Proof. By using Definition 4.1 for the open set $W \subset \Gamma_{\mathcal{A}}$ and ε, we get a function $e \in \mathcal{A}$ and $t_0 \in W$ such that $e(t_0) = \|e\| = 1$, $|e(t)| < \varepsilon$ for every $t \in \Gamma_{\mathcal{A}} \setminus W$ and $e(\Gamma_{\mathcal{A}}) \subset \mathrm{St}_\varepsilon$. Let $W_0 := \{t \in W : |1 - e(t)| < \varepsilon\}$. Define the function $f_n \colon \Gamma_{\mathcal{A}} \to \mathbb{K}$ by $f_n(t) := (e(t))^n$ whose range, by Lemma 4.3, is contained in St_ε. From the very definition of W_0 and the fact that $e(\Gamma_{\mathcal{A}}) \subset \mathrm{St}_\varepsilon$, we deduce that $|e(t)| \leq 1 - \varepsilon(1 - \varepsilon) < 1$ for every $t \in \Gamma_{\mathcal{A}} \setminus W_0$. Thus, taking a suitable $n_0 \in \mathbb{N}$, we can assume that $|f_{n_0}(t)| = |e(t)|^{n_0} < \varepsilon$ on $\Gamma_{\mathcal{A}} \setminus W_0$. Therefore, $f := f_{n_0} \in \mathcal{A}$ gives the conclusions of the lemma. □

Theorem 4.5. *Let $\mathcal{A} \subset C_b(\Gamma_{\mathcal{A}})$ be an* ACK*-subalgebra, and let X be a subspace $\mathcal{A} \subset X \subset C_b(\Gamma_{\mathcal{A}})$ that has the following property: $fx \in X$ for every $x \in X$ and $f \in \mathcal{A}$. Then $X \in$* ACK *with the corresponding 1-norming subset of B_{X^*} being $\Gamma = \{\delta_t : t \in \Gamma_{\mathcal{A}}\}$.*

Proof. Fix $\varepsilon > 0$ and a non-emtpy relatively w^*-open subset $U = \{\delta_t : t \in W \subset \Gamma_{\mathcal{A}}\} \subset \Gamma$. Observe that $W \subset \Gamma_{\mathcal{A}}$ is open. Now, by applying Lemma 4.4 to W with ε we obtain the corresponding $W_0 \subset \Gamma_{\mathcal{A}}$, $t_0 \in W_0$, f, $e_{\mathcal{A}} \in \mathcal{A}$. Let us define $V \subset U$, $x_1^* \in V$, $e \in S_X$ and $F \in L(X)$ as follows:

$$V := \{\delta_t : t \in W_0\}, \quad x_1^* := \delta_{t_0}, \quad e := e_{\mathcal{A}}, \quad Fx := fx, \text{ for } x \in X.$$

Then, $F^*x^* = f(t)x^*$ for every $x^* = \delta_t \in \Gamma$. We shall show that properties (I) – (VI) are satisfied. First, $\|F\| \leq 1$ and $\|Fe\| = e(t_0)f(t_0) = 1$, which proves (I). Property (II) is straightforward from $x_1^*(Fe) = x_1^*(fe) = e(t_0)f(t_0) = 1$. From $(F^*x_1^*)(x) = x(t_0)f(t_0) = x(t_0) = x_1^*(x)$ we deduce that $F^*x_1^* = x_1^*$, which is (III). To show (IV)', take $x^* = \delta_t \in \Gamma$ and estimate

$$\begin{aligned} &\|F^*x^*\| + (1 - \varepsilon) \|(I_{X^*} - F^*)(x^*)\| \\ &\qquad \leq |f(t)| + (1 - \varepsilon)|1 - f(t)| \leq 1. \end{aligned}$$

Let us show now (V). Take $x^* = \delta_t \in \Gamma$. In case t belongs to $\Gamma_{\mathcal{A}} \setminus W_0$, then $\|F^*x^*\| = |f(t)| < \varepsilon$. Otherwise, $t \in W_0$ (that is, $x^* \in V$), using that $F^*x^* = f(t)x^*$ and that $f \in S_X$, we deduce that $f(t)x^* \in \mathrm{aco}\{0, V\}$. Hence, in both cases

$$\mathrm{dist}(F^*x^*, \mathrm{aco}\{0, V\}) < \varepsilon.$$

Finally, for every $v^* \in V$ we have that $v^*(e) = e(t)$ for some $t \in W_0$. So,

$$|v^*(e) - 1| = |e(t) - 1| \leq \varepsilon,$$

which shows (VI) and finishes the proof. □

From Lemma 4.2 and Theorem 4.5 taking $X = \mathcal{A}$ we obtain the promised example.

Corollary 4.6. *Every uniform algebra $\mathcal{A}$ has simple* ACK *structure.*

Theorem 4.5 gives more examples of spaces with simple ACK structure. For instance, let $\mathbb{T}$ be the unit disk in $\mathbb{C}$, $A(\mathbb{T}) \subset C(\mathbb{T})$ be the disc-algebra, i.e., $A(\mathbb{T})$ is the closure in $C(\mathbb{T})$ of the set $\{\sum_{k=0}^{m} a_k z^k : a_k \in \mathbb{C}, m \in \mathbb{N}\}$ of all polynomials. For a given $n \in \mathbb{N}$ denote $A_n(\mathbb{T})$ the closure in $C(\mathbb{T})$ of the set $\{\sum_{k=-n}^{m} a_k z^k : a_k \in \mathbb{C}, m \in \mathbb{N}\}$. Then $A(\mathbb{T})$ and $X = A_n(\mathbb{T})$ satisfy all the conditions of Theorem 4.5, so $A_n(\mathbb{T}) \in \mathrm{ACK}$, but $A_n(\mathbb{T})$ is not an algebra. Another example: let $c_0 \subset X \subset \ell_\infty$. Then $X \in \mathrm{ACK}$.

The first example is of illustrative character, because the space $A_n(\mathbb{T})$ is isometric to the algebra $A(\mathbb{T})$. In contrast, the second example gives a big variety of mutually non-isomorphic spaces with ACK structure. Observe that the simple ACK structure of those X such that $c_0 \subset X \subset \ell_\infty$ can be also deduced from Theorem 4.9 below.

Remark 4.7. *In general, it is not clear whether for a given $T \in \mathrm{Fl}_\Gamma(X, Y)$ the formula (3.2) gives a Γ-flat operator S. But, under the conditions of Theorem 4.5, we have an additional property $F^*x^* = f(t)x^*$. Combining this property with (iv) of Theorem 2.7, we get $S \in \mathrm{Fl}_\Gamma(X, Y)$. In particular, in the case of uniform algebras the Bishop–Phelps–Bollobás type approximation of Γ-flat operators can be made by operators that are Γ-flat as well.*

Now we show that Banach spaces with Lindenstrauss' property β (see for instance [18]) have ACK structure.

Definition 4.8. A Banach space X is said to have the property β if there exist two sets $\{x_\alpha : \alpha \in \Lambda\} \subset S_X$, $\{x_\alpha^* : \alpha \in \Lambda\} \subset S_{X^*}$ and $\rho \in [0, 1)$ such that the following conditions hold:

(i) $x_\alpha^*(x_\alpha) = 1$;
(ii) $|x_\alpha^*(x_\gamma)| \leq \rho < 1$ if $\alpha \neq \gamma$; and
(iii) $\|x\| = \sup\{|x_\alpha^*(x)| : \alpha \in \Lambda\}$, for all $x \in X$.

Theorem 4.9. *Let X have the property β. Then $X \in \mathrm{ACK}_\rho$ with the same value of ρ as in Definition 4.8 and with $\Gamma = \{x_\alpha^* : \alpha \in \Lambda\}$ from that definition. Moreover, if X has property β with $\rho = 0$, then $X \in \mathrm{ACK}$.*

Proof. Since X has property β, the set $\Gamma = \{x_\alpha^* : \alpha \in \Lambda\}$ is a 1-norming subset of B_{X^*}. Observe that property β implies that (Γ, w^*) is a discrete topological space. Fix $\varepsilon > 0$ and a non-empty relatively w^*-open subset $U \subset \Gamma$. Take $x_{\alpha_0}^* \in U$. Let us define the corresponding V, $x_1^* \in V$, $e \in S_X$, and $F \in L(X)$ as follows:

$$V := \{x_{\alpha_0}^*\} \subset U, \quad x_1^* := x_{\alpha_0}^*, \quad e := x_{\alpha_0}, \quad F(x) := x_{\alpha_0}^*(x) x_{\alpha_0}.$$

It is clear that $F^*x^* = x^*(x_{\alpha_0})x^*_{\alpha_0}$ for every $x^* \in X^*$. We shall show that properties (I) – (VI) of Definition 3.1 hold true. Properties (I) – (III) are routine. To show (IV) observe first that

$$\left\|F^*x^*_{\alpha_0}\right\| + (1-\varepsilon)\left\|(I_{X^*} - F^*)(x^*_{\alpha_0})\right\| = \left\|x^*_{\alpha_0}(x_{\alpha_0})x^*_{\alpha_0}\right\| = 1,$$

that is, $x^*_{\alpha_0} \in V_1$. Consequently, whenever $v^* = x^*_\alpha \in \Gamma \setminus V_1$, then $\alpha \neq \alpha_0$ and thus $|v^*(Fe)| = |x^*_\alpha(x_{\alpha_0})| \leq \rho$.

In case that $\rho = 0$, we have that $F^*x^*_\alpha = 0$ for every $\alpha \neq \alpha_0$, so

$$\|F^*x^*_\alpha\| + (1-\varepsilon)\|(I_{X^*} - F^*)x^*_\alpha\| = (1-\varepsilon)\|x^*_\alpha\| < 1,$$

i.e., $V_1 = \Gamma$.

Property (V) is a consequence of the fact that $F^*x^* \in \operatorname{aco}\{0, V\}$ for every $x^* = x^*_\alpha \in \Gamma$, because $F^*x^* = x^*_\alpha(x_{\alpha_0})x^*_{\alpha_0}$. Finally, property (VI) and in turn our conclusions are consequence of the fact that the unique $v^* \in V$ is $v^* = x^*_{\alpha_0}$, so $|v^*(e) - 1| = 0 \leq \varepsilon$. □

Corollary 4.10 ([1, Theorem 2.2])**.** *Let Y have property β. Then, for every Banach space X, the pair (X,Y) has the Bishop–Phelps–Bollobás property for operators.*

Proof. In the proof of Theorem 4.9, (Γ, w^*) is a discrete topological space. Therefore every operator $T \in L(X,Y)$ is Γ-flat (Example C after Definition 2.8). Now the application of Theorem 3.4 completes the proof. □

Now we show the stability of the ACK structure with respect to the operations of ℓ_∞-sum and injective tensor product of two spaces (Theorem 4.11 and Theorem 4.12)

Theorem 4.11. *Let X, Y be Banach spaces having* ACK *structure with parameters ρ_X and ρ_Y respectively. Then $Z := X \bigoplus_\infty Y \in \mathrm{ACK}_\rho$ with $\rho = \max\{\rho_X, \rho_Y\}$. Moreover, $Z \in \mathrm{ACK}$ whenever $X, Y \in \mathrm{ACK}$.*

Proof. Observe that both X and Y have ACK structure with parameter ρ. Let $\Gamma_X \subset B_{X^*}$ and $\Gamma_Y \subset B_{Y^*}$ be the corresponding 1-norming subsets in Definition 3.1. Then, the set

$$\Gamma := \{(x^*, 0) : x^* \in \Gamma_X\} \cup \{(0, y^*) : y^* \in \Gamma_Y\}$$

is a 1-norming subset of B_{Z^*}. Take a non-empty relatively w^*-open subset $U \subset \Gamma$. Then, there exist relatively w^*-open subsets $U_X \subset \Gamma_X$ and $U_Y \subset \Gamma_Y$ that are not both empty and such that $(U_X \times \{0\}) \cup (\{0\} \times U_Y) \subset U$. Without loss of generality we may assume that $U_X \neq \emptyset$.

Fix $\varepsilon > 0$. By using Definition 3.1 for X, ε, and U_X we obtain a non-empty subset $V_X \subset U_X$, $x^*_1 \in V_X$, $e_X \in S_X$, $F_X \in L(X)$ with the properties (I) – (VI). Thus, we can define the corresponding $V \subset U$, $z^*_1 \in V$, $e \in S_Z$ and $F \in L(Z)$ as follows:

$$V := \{(x^*, 0) : x^* \in V_X\} \subset U, \quad z^*_1 := (x^*_1, 0), \quad e := (e_X, 0),$$

and for $(x,y) \in Z$,

$$F(x,y) := (F_X(x), 0).$$

Let us check the required properties. It is clear that $\|F\| = 1$ and that $\|Fe\| = \|F_X(e_X)\| = 1$, which shows (I). (II) follows easily; $z^*(Fe) = x_1^*(F_X e_X) = 1$. Due to the fact that $(F_X x_1^*, 0) = (x_1^*, 0)$, we deduce that $F^* z_1^* = z_1^*$, showing that (III) holds. Now, for every $z^* = (x^*, 0) \in V$ with $x^* \in V_{X,1}$ we have

$$\begin{aligned}\|F^* z^*\| &+ (1-\varepsilon)\,\|(I_{Z^*} - F^*)(z^*)\| \\ &= \|F_X^* x^*\| + (1-\varepsilon)\,\|(I_{X^*} - F_X^*)(x^*)\| \\ &\leq 1,\end{aligned}$$

which can be easily deduced from $F^* z^* = (F_X^* x^*, 0)$. Consequently, for every $x^* \in V_{X,1}$ we have $z^* = (x^*, 0) \in V_1$. (Observe that in the case of *simple ACK structure* we have already proved (IV)'). Let $v^* \in \Gamma \setminus V_1$. Then, either $v^* = (0, y^*)$, or $v^* = (x^*, 0)$ with $x^* \in \Gamma_X \setminus V_{X,1}$. On the one hand, when $v^* = (0, y^*)$, we have $|v^*(Fe)| = 0 \leq \rho$. On the other hand, whenever $v^* = (x^*, 0)$ with $x^* \in \Gamma_X \setminus V_{X,1}$, then $|v^*(Fe)| = |x^*(F_X e_X)| \leq \rho$, which proves (IV). Now, let $z^* \in \Gamma$. Whenever $z^* = (0, y^*)$ we have $F^* z^* = 0$. Otherwise, $z^* = (x^*, 0)$ and we have $\operatorname{dist}(F_X^* x^*, \operatorname{aco}\{0, V_X\}) < \varepsilon$. Thus, in both cases

$$\operatorname{dist}(F^* z^*, \operatorname{aco}\{0, V\}) < \varepsilon.$$

Finally, for every $v^* = (x^*, 0) \in V$ we have $|v^*(e) - 1| = |x^*(e_X) - 1| \leq \varepsilon$, which proves (VI) and concludes our proof. $\square$

Recall that given two normed spaces X and Y, one can define their injective tensor product $X \hat{\otimes}_\varepsilon Y$, as the completion of $(X \otimes Y, \|\cdot\|_\varepsilon)$, where

$$\|z\|_\varepsilon := \sup\{|\langle x^* \otimes y^*, z\rangle| \colon x^* \in B_{X^*},\ y^* \in B_{Y^*}\},$$

for every $z \in X \otimes Y$ and $\langle x^* \otimes y^*, x \otimes y\rangle := x^*(x)\, y^*(y)$, for every $x \otimes y \in X \otimes Y$ and for every $x^* \in X^*$ and $y^* \in Y^*$.

An important example of such a product is the Banach space $C(K) \hat{\otimes}_\varepsilon Y$, which can be naturally identified with $C(K, Y)$, that is, the Banach space of continuous $(Y, \|\cdot\|)$-valued functions defined on K, endowed with the supremum norm $\|f\| = \sup\{\|f(t)\| : t \in K\}$.

Note that it follows from the definition of the injective norm that if $X_0 \subset B_{X^*}$ and $Y_0 \subset B_{Y^*}$ are 1-norming, then for every $z \in X \hat{\otimes}_\varepsilon Y$ the following equality holds:

$$\|z\|_\varepsilon = \sup\{|\langle x^* \otimes y^*, z\rangle| : x^* \in X_0,\ y^* \in Y_0\}.$$

Recall also that $\|x^* \otimes y^*\|_{(X \hat{\otimes}_\varepsilon Y)^*} = \|x^*\| \cdot \|y^*\|$ for every $x^* \in X^*$ and $y^* \in Y^*$.

This is all the information about tensor products that will be used in Theorem 4.12 below. We refer to Ryan's book [20] for tensor products theory in general and the above definitions and statements in particular.

Theorem 4.12. *Let X and Y be Banach spaces both of which have* ACK *(resp.* ACK_ρ*) structure. Then, $X \hat{\otimes}_\varepsilon Y$ has* ACK *(resp.* ACK_ρ*) structure.*

Proof. Since X and Y have ACK (resp. ACK_ρ) structure, there exist 1-norming sets $\Gamma_X \subset S_{X^*}$ and $\Gamma_Y \subset S_{Y^*}$ satisfying Definition 3.1. Define the map $\phi\colon (B_{X^*}, w^*) \times (B_{Y^*}, w^*) \to (B_{(X\hat{\otimes}_\varepsilon Y)^*}, w^*)$ by $\phi(x^*, y^*) = x^* \otimes y^*$, for every $x^* \in B_{X^*}$ and for every $y^* \in B_{Y^*}$.

First, we shall show that the map ϕ is continuous. Let $\{(x^*_\alpha, y^*_\alpha)\}_{\alpha\in\Lambda}$ be a convergent net to $(x^*, y^*) \in B_{X^*} \times B_{Y^*}$. Then, for every $x \otimes y \in X \otimes Y$, we can estimate

$$\begin{aligned}
|\langle \phi(x^*_\alpha, y^*_\alpha) - \phi(x^*, y^*), x \otimes y\rangle| &= |x^*_\alpha(x)y^*_\alpha(y) - x^*(x)y^*(y)| \\
&\le |(x^*_\alpha(x) - x^*(x))y^*_\alpha(y)| + |x^*(x)(y^*_\alpha(y) - y^*(y))| \\
&\le |x^*_\alpha(x) - x^*(x)|\, \|y^*_\alpha\|\, \|y\| + \|x^*(x)\|\, |y^*_\alpha(y) - y^*(y)| \\
&\le |x^*_\alpha(x) - x^*(x)|\, \|y\| + \|x\|\, |y^*_\alpha(y) - y^*(y)|,
\end{aligned}$$

which tends to zero. This argument extends easily to every element in $X\otimes Y$ and, in turn, to every $z \in X \hat{\otimes}_\varepsilon Y$ (due to the boundedness of the range of the map ϕ).

The 1-norming set Γ that we need for our theorem can be introduced as follows:

$$\Gamma := \{x^* \otimes y^* \colon x^* \in \Gamma_X,\ y^* \in \Gamma_Y\} = \phi(\Gamma_X \times \Gamma_Y).$$

Let $\varepsilon > 0$ and U be a non-empty relatively w^*-open subset of Γ. Let $x^*_0 \in \Gamma_X$ and $y^*_0 \in \Gamma_Y$ be such that $\phi(x^*_0, y^*_0) \in U$. The continuity of ϕ ensures that there exist non-empty relatively w^*-open subsets $W_X \subset \Gamma_X$, $W_Y \subset \Gamma_Y$ such that $x^*_0 \in W_X$, $y^*_0 \in W_Y$ and $\phi(W_X \times W_Y) \subset U$.

We can apply Definition 3.1 to X and Y, to the former with $\varepsilon/2$ and W_X and to the latter with $\varepsilon/2$ and W_Y, to find two non-empty sets $V_X \subset W_X$ and $V_Y \subset W_Y$, two functionals $x^*_1 \in V_X$ and $y^*_1 \in V_Y$, two points $e_X \in S_X$ and $e_Y \in S_Y$ and finally, two operators $F_X \in L(X)$ and $F_Y \in L(Y)$, satisfying respectively the properties (I) – (VI), or with their corresponding modifications for the the simple ACK structure. Denote also by $V_{X,1}$ and $V_{Y,1}$ the corresponding variants for X and Y of the set V_1 from property (IV) of Definition 3.1.

Now, define the non-emtpy set $V \subset U$ and corresponding $z^*_1 \in V$, $e \in S_{X\otimes_\varepsilon Y}$, $F \in L(X \hat{\otimes}_\varepsilon Y)$ as follows: $V := \phi(V_X \times V_Y) \subset U$, $z^*_1 := \phi(x^*_1, y^*_1) = x^*_1 \otimes y^*_1$, $e := e_X \otimes e_Y$, and $F(x \otimes y) := F_X(x) \otimes F_Y(y)$ for every $x \otimes y \in X \otimes Y$. It remains to check the properties (I) – (VI). First, observe that $F^*(x^* \otimes y^*) = F^*_X x^* \otimes F^*_Y y^*$ for every $x^* \in X^*$ and $y^* \in Y^*$.

(I) Let z belong to $B_{X\hat{\otimes}_\varepsilon Y}$, then

$$\begin{aligned}
\|Fz\|_\varepsilon &= \sup_{x^*\in\Gamma_X} \sup_{y^*\in\Gamma_Y} |\langle x^* \otimes y^*, Fz\rangle| = \sup_{x^*\in\Gamma_X} \sup_{y^*\in\Gamma_Y} |\langle F^*(x^* \otimes y^*), z\rangle| \\
&= \sup_{x^*\in\Gamma_X} \sup_{y^*\in\Gamma_Y} |\langle F^*_X x^* \otimes F^*_Y y^*, z\rangle| \le \sup_{x^*\in\Gamma_X} \sup_{y^*\in\Gamma_Y} \|F^*_X x^*\|\, \|F^*_Y y^*\| \\
&\le \|F^*_X\|\, \|F^*_Y\| \le 1,
\end{aligned}$$

which implies that $\|F\| = 1$, since

$$\|Fe\| = \|F_X e_X \otimes F_Y e_Y\| = \|F_X e_X\|\, \|F_Y e_Y\| = 1.$$

(II) $z_1^*(Fe) = (x_1^* \otimes y_1^*)(F_X e_X \otimes F_Y e_Y) = x_1^*(F_X e_X) y_1^*(F_Y e_Y) = 1$.

(III) $F^* z_1^* = z_1^*$, since for every $x \otimes y \in X \otimes Y$ we have

$$(F^* z_1^*)(x \otimes y) = (x_1^* \otimes y_1^*)(F_X x \otimes F_Y y) = (F_X^* x_1^*)(x)(F_Y^* y_1^*)(y),$$

which, in turn, implies that $(F^* z_1^*)(x \otimes y) = x_1^*(x) y_1^*(y) = z_1^*(x \otimes y)$.

(IV) For $(x^*, y^*) \in \Gamma_X \times \Gamma_Y$, denote $z^* = x^* \otimes y^*$. Firstly, let us show that for every $x^* \in V_{X,1}$ and $y^* \in V_{Y,1}$ the functional z^* belongs to V_1, i.e., that

$$\|F^* z^*\| + (1-\varepsilon) \left\| (I_{(X \hat{\otimes}_\varepsilon Y)^*} - F^*)(z^*) \right\| \leq 1.$$

First of all, observe that

$$\begin{aligned} &\|x^* \otimes y^* - F_X^* x^* \otimes F_Y^* y^*\| = \\ &\quad = \|x^* \otimes (y^* - F_Y^* y^*) - (x^* - F_X^* x^*) \otimes F_Y^* y^*\| \\ &\quad \leq \|y^* - F_Y^* y^*\| + \|F_Y^* y^*\| \, \|(x^* - F_X^* x^*)\| . \end{aligned}$$

Therefore,

$$\begin{aligned} &\|F_X^* x^*\| \, \|F_Y^* y^*\| + (1-\varepsilon) \|x^* \otimes y^* - F_X^* x^* \otimes F_Y^* y^*\| \\ &= \|F_Y^* y^*\| \left(\|F_X^* x^*\| + (1-\varepsilon) \|(x^* - F_X^* x^*)\| \right) + (1-\varepsilon) \|y^* - F_Y^* y^*\| \\ &\leq \|F_Y^* y^*\| + (1-\varepsilon) \|y^* - F_Y^* y^*\| \leq 1. \end{aligned}$$

This implies that for every $z^* = x^* \otimes y^* \in \Gamma \setminus V_1$ we have two possibilities: either $x^* \notin V_{X,1}$ or $y^* \notin V_{Y,1}$. By symmetry, it is sufficient to consider $x^* \notin V_{X,1}$. In this case $|x^*(F_X e_X)| \leq \rho$, so

$$|z^*(Fe)| = |x^*(F_X e_X)| \, |y^*(F_Y e_Y)| \leq |x^*(F_X e_X)| \leq \rho.$$

(V) We shall show that $\operatorname{dist}(F^* z^*, \operatorname{aco}\{0, V\}) < \varepsilon$ for every $z^* = x^* \otimes y^* \in \Gamma$. Due to the facts that $\operatorname{dist}(F_X^* x^*, \operatorname{aco}\{0, V_X\}) < \varepsilon/2$ and that $\operatorname{dist}(F_Y^* y^*, \operatorname{aco}\{0, V_Y\}) < \varepsilon/2$, there exist $v_X^* \in \operatorname{aco}\{0, V_X\}$ and $v_Y^* \in \operatorname{aco}\{0, V_Y\}$ such that $\|F_X^* x^* - v_X^*\| < \varepsilon/2$ and $\|F_Y^* y^* - v_Y^*\| < \varepsilon/2$. Then $v^* := v_X^* \otimes v_Y^*$ belongs to $\operatorname{aco}\{0, V\}$ and

$$\begin{aligned} \|F^* z^* - v^*\| &\leq \|(F_X^* x^* - v_X^*) \otimes F_Y^* y^*\| + \|v_X^* \otimes (F_Y^* y^* - v_Y^*)\| \\ &\leq \|F_X^* x^* - v_X^*\| \, \|F_Y^* y^*\| + \|v_X^*\| \, \|F_Y^* y^* - v_Y^*\| \leq \varepsilon. \end{aligned}$$

(VI) For every $v^* = x^* \otimes y^* \in V$ we get

$$\begin{aligned} |v^*(e) - 1| = |x^*(e_X) y^*(e_Y) - 1| &\leq |x^*(e_X) y^*(e_Y) - y^*(e_Y)| \\ &+ |y^*(e_Y) - 1| \leq \frac{\varepsilon}{2} |y^*(e_Y)| + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

This finishes the proof. □

4.1. **Sup-normed spaces of vector-valued functions.** As we mentioned in the introduction, Acosta, Becerra Guerrero, García, Kim, and Maestre considered the A-BPB property in spaces of continuous vector-valued functions. Let us recall their result explicitly. Here, as usual, $\sigma(Z, \Delta)$ denotes the weakest topology on Z in which all elements of $\Delta \subset Z^*$ are continuous.

Theorem 4.13 ([2, Theorem 3.1])**.** *Let X, Z be Banach spaces, K be a compact Hausdorff topological space. Let Z satisfy property β for the subset of functionals $\Delta = \{z^*_\alpha : \alpha \in \Delta\}$. Let $\tau \supseteq \sigma(Z, \Delta)$ be a linear topology on Z dominated by the norm topology. Then for every closed operator ideal $\mathcal{I}$ contained in the ideal of Asplund operators, we have that $(X, C(K, (Z, \tau)))$ has the Bishop–Phelps–Bollobás property for operators from $\mathcal{I}$.*

The next proposition together with Theorem 3.4 generalize Theorem 4.13 for the case of Z endowed with its strong topology.

Proposition 4.14. *Let K be a compact Hausdorff topological space. Then,*

$$(Y \in \mathrm{ACK}_\rho) \Rightarrow (C(K, Y) \in \mathrm{ACK}_\rho);$$

$$(Y \in \mathrm{ACK}) \Rightarrow (C(K, Y) \in \mathrm{ACK}).$$

Proof. Bearing in mind Corollary 4.6 and Theorem 4.12, the fact that the space $C(K) \hat{\otimes}_\varepsilon Y$ is isometric to $C(K, Y)$ concludes the proof. □

Our aim now is showing a generalization of Theorem 4.13 in the spirit of the ACK structure, that covers all topologies τ from that theorem. In order to do this we need some terminology.

For a topological space T and a Banach space Z denote by $C_{\mathrm{bof}}(T, Z)$ the space of all bounded openly fragmented (see Definiton 2.6) functions $f \colon T \to Z$ equipped with the $\sup$-norm. For a topology τ on Z denote by $C_b(T, (Z, \tau))$ the space of bounded τ-continuous functions $f \colon T \to Z$ equipped with the sup-norm.

Definition 4.15. Let $Z \in \mathrm{ACK}_\rho$ and let $\Gamma \subset B_{Z^*}$ be the corresponding 1-norming set. A linear topology τ on Z is said to be Γ-*acceptable*, if it is dominated by the norm topology and dominates $\sigma(Z, \Gamma)$.

The following result simultaneously generalizes our Theorem 4.5 and Theorem 4.13. We state the result in the most general settings, which makes the statement bulky. Some "elegant" partial cases will be given as corollaries.

Theorem 4.16. *Let $\mathcal{A} \subset C_b(\Gamma_{\mathcal{A}})$ be an ACK-subalgebra. Let Z be a Banach space and $\mathcal{O} \subset L(Z)$ such that $Z \in \mathcal{O}$-ACK_ρ ($Z \in \mathcal{O}$-ACK) with $\Gamma_Z \subset B_{Z^*}$ being the corresponding 1-norming set. Finally, let τ be a Γ_Z-acceptable topology on Z. Let $X \subset C_b(\Gamma_{\mathcal{A}}, (Z, \tau))$ be a Banach space satisfying the following properties:*

(i) *For every $x \in X$ and $f \in \mathcal{A}$ the function fx belongs to X.*
(ii) *X contains all functions of the form $f \otimes z$, $f \in \mathcal{A}$, $z \in Z$.*
(iii) *$F \circ x \in X$ for every $x \in X$ and $F \in \mathcal{O}$.*
(iv) *For every finite collection $\{x_k\}_{k=1}^n \subset X$ the corresponding function of two variables $\varphi \colon \Gamma_{\mathcal{A}} \times (\Gamma_Z, w^*) \to \mathbb{K}^n$, defined by $\varphi(t, z^*) = (z^*(x_k(t)))_{k=1}^n$, is quasi-continuous.*

Then $X \in \mathrm{ACK}_\rho$ ($X \in \mathrm{ACK}$, respectively) with the corresponding 1-norming subset of B_{X^*} being $\Gamma = \{\delta_t \otimes z^* : t \in \Gamma_{\mathcal{A}}, z^* \in \Gamma_Z\}$, where the functional $\delta_t \otimes z^* \in X^*$ acts as follows: $(\delta_t \otimes z^*)(x) = z^*(x(t))$.

Proof. Fix $\varepsilon > 0$ and a non-empty relatively w^*-open subset $U \subset \Gamma$. Let $t_0 \in \Gamma_{\mathcal{A}}$ and $z_0^* \in \Gamma_Z$ be such that $\delta_{t_0} \otimes z_0^* \in U$. Since U is relatively w^*-open, there exist $\{x_k\}_{k=1}^n \subset X$ such that $\delta_t \otimes z^* \in \Gamma$ belongs to U whenever

$$\max_{1\le k\le n} |\langle(\delta_{t_0} \otimes z_0^*) - (\delta_t \otimes z^*), x_k\rangle| < 1.$$

Consider the non-emtpty open set

$$B := \{t \in \Gamma_{\mathcal{A}} \colon |z_0^*(x_k(t)) - z_0^*(x_k(t_0))| < 1 \ \text{ for } 1 \le k \le n\},$$

and define the following non-empty relatively w^*-open subset of Γ_Z:

$$D := \{z^* \in \Gamma_Z : |z^*(x_k(t_0)) - z_0^*(x_k(t_0))| < 1 \ \text{ for } 1 \le k \le n\}.$$

Using property (iv) for $\{x_k\}_{k=1}^n \subset X$ we can find a non-empty open subset $B_1 \subset B$ and a non-empty relatively w^*-open subset $D_1 \subset D$ such that for every $t \in B_1$ and every $z^* \in D_1$ it holds

$$\max_{1\le k\le n} |z^*(x_k(t)) - z_0^*(x_k(t_0))| < 1.$$

Define the non-empty subset $W := \{\delta_t \otimes z^* : t \in B_1, z^* \in D_1\} \subset \Gamma$. It is clear that $W \subset U$.

By applying Definition 3.1 to Z, Γ_Z, D_1 and $(\varepsilon/2)$, we get $V_Z \subset D_1$, $z_1^* \in V_Z$, $e_Z \in S_Z$ and $F_Z \in \mathcal{O}$ satisfying (I) – (VI). Denote also $V_{Z,1} \subset \Gamma_Z$, the subset that appears in property (IV) (in the case of $Z \in \mathrm{ACK}$ we have $V_{Z,1} = \Gamma_Z$). By applying Lemma 4.4 to $\mathcal{A}$, $\Gamma_{\mathcal{A}}$, the non-empty open set B_1 and $(\varepsilon/2)$, we find a non-empty subset $B_2 \subset B_1$, functions f_0, $e_{\mathcal{A}}$ (both belonging to $\mathcal{A}$) and $s_0 \in B_2$, satisfying its conclusions.

Finally, let us define the requested non-empty subset $V \subset U$ and corresponding $x_1^* \in V$, $e \in S_X$, $F \in L(X)$ as follows:

$$V := \{\delta_t \otimes z^* \colon t \in B_2, z^* \in V_Z\} \subset W \subset U,$$
$$x_1^* := \delta_{s_0} \otimes z_1^*, \quad e(t) := e_{\mathcal{A}}(t)e_Z, \text{ for every } t \in \Gamma_{\mathcal{A}}$$

(condition (ii) implies $e \in X$), and

$$(Fx)(t) := f_0(t)F_Z(x(t)),$$

for every $x \in X$ and for every $t \in \Gamma_{\mathcal{A}}$. Conditions (i) and (iii) ensure that $F(x) \in X$. Observe that for every $x^* = \delta_t \otimes z^* \in \Gamma$

$$F^*x^* = f_0(t)\left(\delta_t \otimes F_Z^* z^*\right).$$

It remains to check the properties (I) – (VI).

(I) It is clear that $\|F\| = \|F_Z\| = 1$ and $\|Fe\| = \|f_0 e_{\mathcal{A}}\| \, \|F_Z(e_Z)\| = 1$.

(II) $x_1^*(Fe) = z_1^*(f_0(s_0)e_{\mathcal{A}}(s_0)F_Z(e_Z)) = 1$.

(III) $F^*x_1^* = x_1^*$, since for every $x \in X$ we have

$$(F^*x_1^*)(x) = z_1^*\left(f_0(s_0)F_Z x(s_0)\right) = (F_Z^* z_1^*)(x(s_0)) = z_1^*(x(s_0)) = x_1^*(x).$$

(IV) For every $x^* \in \Gamma$, we have $x^* = \delta_t \otimes z^*$, $t \in \Gamma_{\mathcal{A}}$ and $z^* \in \Gamma_Z$. First, consider the case $z^* \in V_{Z,1}$ and observe that

$$\begin{aligned}\|(I_{X^*} - F^*)(x^*)\| &= \|z^* - f_0(t)F_Z^* z^*\| \\ &\leq |1 - f_0(t)|\,\|z^*\| + |f_0(t)| \cdot \|(I_{Z^*} - F_Z^*)(z^*)\| \\ &= |f_0(t)| \cdot \|(I_{Z^*} - F_Z^*)(z^*)\| + |1 - f_0(t)|.\end{aligned}$$

Therefore, in this case

$$\begin{aligned}&\|F^* x^*\| + (1-\varepsilon)\,\|(I_{X^*} - F^*)(x^*)\| \\ &= |f_0(t)| \cdot \|F_Z^* z^*\| + (1-\varepsilon)\,\|z^* - f_0(t)F_Z^* z^*\| \\ &\leq |f_0(t)|\big(\|F_Z^* z^*\| + (1-\varepsilon)\,\|(I_{Z^*} - F_Z^*)(z^*)\|\big) + (1-\varepsilon)|1 - f_0(t)| \\ &\leq |f_0(t)| + (1-\varepsilon)|1 - f_0(t)| \leq 1.\end{aligned}$$

Whenever $Z \in \mathrm{ACK}$, then $V_{Z,1} = \Gamma_Z$, so the above inequality holds for every $z^* \in \Gamma_Z$. Thus, we have proved (IV)'. If $Z \in \mathrm{ACK}_\rho$ we still must consider those x^* belonging to $\Gamma \setminus V_1$. The above inequality implies that $z^* \notin V_{Z,1}$ and, consequently, $|z^*(F_Z e_Z)| \leq \rho$ which, in turn, implies that

$$|x^*(Fe)| = |f_0(t)e_{\mathcal{A}}(t)z^*(F_Z e_Z)| \leq \rho.$$

(V) Let $x^* = \delta_t \otimes z^* \in \Gamma$. Recall that $F^* x^* = f_0(t)\delta_t \otimes F_Z^* z^*$. Set $V_{\mathcal{A}} := \{\delta_t \colon t \in B_2\}$. In the proof of Theorem 4.5 it was proved that for every $t \in \Gamma_{\mathcal{A}}$ it holds

$$\operatorname{dist}(f(t)\delta_t, \operatorname{aco}\{0, V_{\mathcal{A}}\}) < \frac{\varepsilon}{2}.$$

On the other hand, by our construction, we deduce that

$$\operatorname{dist}\left(F_Z^* z^*, \operatorname{aco}\{0, V_Z\}\right) < \frac{\varepsilon}{2}.$$

Thus, there exist $a^* \in \operatorname{aco}\{0, V_{\mathcal{A}}\}$ and $b^* \in \operatorname{aco}\{0, V_Z\}$ such that

$$\|f(t)\delta_t - a^*\| < \frac{\varepsilon}{2} \text{ and } \|F_Z^* z^* - b^*\| < \frac{\varepsilon}{2}.$$

In particular, since $a^* \otimes b^*$ belongs to $\operatorname{aco}\{0, V\}$, we can deduce that

$$\begin{aligned}\operatorname{dist}(F^* x^*, \operatorname{aco}\{0, V\}) &\leq \|f_0(t)\delta_t \otimes F_Z^* z^* - a^* \otimes b^*\| \\ &\leq \|f_0(t)\delta_t \otimes F_Z^* z^* - f_0(t)\delta_t \otimes b^*\| + \\ &\quad + \|f_0(t)\delta_t \otimes b^* - a^* \otimes b^*\| \\ &\leq \|F_Z^* z^* - b^*\| + \|f_0(t)\delta_t - a^*\| < \varepsilon.\end{aligned}$$

(VI) For every $x^* = \delta_t \otimes z^* \in V$ we have $t \in B_2$ and $z^* \in V_Z$. Consequently, $|e_{\mathcal{A}}(t) - 1| \leq \frac{\varepsilon}{2}$ and $|z^*(e_Z) - 1| \leq \frac{\varepsilon}{2}$. From this we get

$$|x^*(e) - 1| = |e_{\mathcal{A}}(t)z^*(e_Z) - 1| = |e_{\mathcal{A}}(t)(z^*(e_Z) - 1) + (e_{\mathcal{A}}(t) - 1)| \leq \varepsilon,$$

which completes the proof. □

Remark 4.17. *Under the hypothesis of the previous theorem, given $F \in L(Z)$ and $f \in \mathcal{A}$ we can consider the operators $C_F\colon X \to X$ and $P_f\colon X \to X$ defined, respectively, by $C_F(x) = F \circ x$ and $P_f(x) = fx$, for every $x \in X$. Then, if we set $\mathcal{O}' := \{C_F \circ P_f : F \in \mathcal{O},\ f \in \mathcal{A}\}$, then X has $\mathcal{O}'$-ACK_ρ (resp. $\mathcal{O}'$-ACK) structure.*

Conditions (i) – (iii) in Theorem 4.16 are easily verified in concrete examples. In contrast, condition (iv) looks technical. So, in order to make Theorem 4.16 more applicable, we shall present easy-to-verify sufficient conditions for (iv).

Before passing to these sufficient conditions, observe that the function of two variables $\varphi\colon \Gamma_A \times (\Gamma_Z, w^*) \to \mathbb{K}^n$ from condition (iv) is separately continuous. Therefore, the role of sufficient condition for (iv) can be played by any theorem about quasi-continuity of a separately continuous function $f\colon U \times V \to W$. There is a number of such theorems (see Encyclopedia of Mathematics article "Separate and joint continuity" or the introduction to [7]). For example, according to Namioka's theorem [15] this (and a much stronger result) occurs for U being a regular, strongly countably complete topological space, V being a locally compact σ-compact space and W being a pseudo-metric space. The results of the kind "separate continuity implies quasi-continuity" that we list and apply below do not pretend to be new.

Proposition 4.18. *Let U, V, W be topological spaces, V be discrete and $f\colon U \times V \to W$ be separately continuous. Then, f is continuous (and consequently quasi-continuous).*

If Z has property β, the corresponding (Γ_Z, w^*) is a discrete topological space. Thus, the above proposition guaranties the validity of (iv) of Theorem 4.16 in this case.

Corollary 4.19. *Under the conditions of Theorem 4.13, $C(K,(Z,\tau)) \in \mathrm{ACK}_\rho$, where ρ is the parameter from the property β of Z. If $\beta = 0$, then $C(K,(Z,\tau)) \in \mathrm{ACK}$. In particular, this implies the conclusion of Theorem 4.13.*

Proposition 4.18 also guaranties (iv) of Theorem 4.16 in the case of $\Gamma_{\mathcal{A}} = \mathbb{N}$ (just change the roles of U and V in Proposition 4.18). If we apply Theorem 4.16 with $\mathcal{A} = c_0 \subset C_b(\mathbb{N}) = \ell_\infty$, this leads to the following result:

Corollary 4.20. *Let $Z \in \mathrm{ACK}_\rho$ ($Z \in \mathrm{ACK}$), $c_0(Z) \subset X \subset \ell_\infty(Z)$, and X has the following property: $(Fz_1, Fz_2, \dots) \in X$ for every $x = (z_1, z_2, \dots) \in X$ and $F \in L(Z)$. Then $X \in \mathrm{ACK}_\rho$ ($X \in \mathrm{ACK}$ respectively).*

This corollary is applicable to $c_0(Z)$ and $\ell_\infty(Z)$ themselves and also for some intermediate spaces like $c_0(Z,w)$ of weakly null sequences in Z.

Proposition 4.21. *Let Z be a Banach space, $(\Gamma_{\mathcal{A}}, \tau)$ be a topological space, $\Gamma_Z \subset (B_{Z^*}, w^*)$, and $x_k\colon \Gamma_{\mathcal{A}} \to Z$ for $k \in \{1, 2, \ldots, n\}$ be τ-$\sigma(Z, \Gamma_Z)$-continuous and τ-$\|\cdot\|$-openly fragmented functions. Then, the function $\varphi\colon (\Gamma_{\mathcal{A}}, \tau) \times (\Gamma_Z, w^*) \to \mathbb{K}^n$ given by $\varphi(t, z^*) = (z^*(x_k(t)))_{k=1}^n$ is quasi-continuous.*

Proof. Fix $(t_0, z_0^*) \in \Gamma_{\mathcal{A}} \times \Gamma_Z$. Let $U_{\mathcal{A}} \subset \Gamma_{\mathcal{A}}$, $U_Z \subset \Gamma_Z$ be open and w^*-open neighborhoods of t_0 and z_0^* respectively. Set $U := U_{\mathcal{A}} \times U_Z$. We have to show that, for a given $\varepsilon > 0$, there exist a non-empty open subset $W_{\mathcal{A}} \subset U_{\mathcal{A}}$ and a non-empty relatively w^*-open subset $W_Z \subset U_Z$ such that for every $t \in W_{\mathcal{A}}$ and every $z^* \in W_Z$

$$\max_{1\le k\le n} |z^*(x_k(t)) - z_0^*(x_k(t_0))| < \varepsilon. \tag{4.1}$$

Fix $\delta < \varepsilon/4$ and define

$$V_{\mathcal{A}} := \left\{ t \in U_{\mathcal{A}} \colon \max_{1\le k\le n} |z_0^*(x_k(t)) - z_0^*(x_k(t_0))| < \delta \right\}.$$

The set $V_{\mathcal{A}} \subset U_{\mathcal{A}}$ is a non-emtpy open neighborhood of t_0 because of the τ-$\sigma(Z, \Gamma_Z)$ continuity of x_k (the map $z_0^* \circ x_k$ is a $\mathbb{K}$-valued τ-continuous function). Applying inductively the definition of openly fragmented function, we define a non-empty open set $W_{\mathcal{A}} \subset (V_{\mathcal{A}}, \tau)$ in such a way that for all $k = 1, \ldots, n$ it holds

$$\operatorname{diam}(x_k(W_{\mathcal{A}})) < \delta.$$

Fix a $t_1 \in W_{\mathcal{A}}$ and define the non-empty relatively w^*-open subset $W_Z \subset U_Z$ as follows:

$$W_Z := \left\{ z^* \in U_Z \colon \max_{1\le k\le n} |z^*(x_k(t_1)) - z_0^*(x_k(t_1))| < \delta \right\}.$$

Let us show, for every $t \in W_{\mathcal{A}}$ and every $z^* \in W_Z$, the validity of inequality (4.1):

$$\begin{aligned} |z_0^*(x_k(t_0)) - z^*(x_k(t))| \le{}& |z_0^*(x_k(t_0)) - z_0^*(x_k(t))| \\ &+ |z_0^*(x_k(t)) - z_0^*(x_k(t_1))| \\ &+ |z_0^*(x_k(t_1)) - z^*(x_k(t_1))| \\ &+ |z^*(x_k(t_1)) - z^*(x_k(t))|. \end{aligned}$$

The first summand in the right-hand side of the previous inequality does not exceed δ since $t \in V_{\mathcal{A}}$. Accordingly, the second and fourth summands are both bounded by δ since $z_0^*, z^* \in B_{Z^*}$ and $\|x_k(t) - x_k(t_1)\| < \delta$ since $t, t_1 \in W_{\mathcal{A}}$ and $\operatorname{diam}(x_k(W_{\mathcal{A}})) < \delta$. Finally, the corresponding third summand is bounded by δ since $z^* \in W_Z$. Therefore,

$$|z_0^*(x_k(t_0)) - z^*(x_k(t))| \le 4\delta < \varepsilon,$$

which completes the proof of (4.1) and that of the proposition. □

As an application of the previous proposition we get the following corollaries which contain as a particular case the space $C_w(K,Z)$ of Z-valued weakly continuous functions for $Z \in \mathrm{ACK}_\rho$ (or $Z \in \mathrm{ACK}$).

Corollary 4.22. *Let $Z \in \mathcal{O}\text{-}\mathrm{ACK}_\rho$ (or $Z \in \mathcal{O}\text{-}\mathrm{ACK}$) and $\mathcal{A} \subset C(K)$ be a uniform algebra with K being the space of multiplicative functionals on $\mathcal{A}$. Fix $\Gamma_Z \subset H \subset Z^*$, where Γ_Z is the 1-norming set given by the ACK structure of Z. Denote by $\mathcal{A}_{\sigma(Z,H)}(K,Z)$ the following subspace of $C(K,(Z,\sigma(Z,H)))$:*

$$\mathcal{A}_{\sigma(Z,H)}(K,Z) = \left\{ f \in Z^K \colon z^* \circ f \in \mathcal{A} \text{ for all } z^* \in H \right\}.$$

Let us assume that

(i) *$F^*H \subset H$ for every $F \in \mathcal{O}$.*
(ii) *$(f(K),\sigma(Z,H))$ is fragmented by the norm for every f belonging to $\mathcal{A}_{\sigma(Z,H)}(K,Z)$.*

Then, $\mathcal{A}_{\sigma(Z,H)}(K,Z) \in \mathrm{ACK}_\rho$ (resp. $\mathcal{A}_{\sigma(Z,H)}(K,Z) \in \mathrm{ACK}$).

Sketch of the proof: It relays on the use of Theorem 4.16. Let $\Gamma_{\mathcal{A}} \subset K$ be the corresponding subset from Lemma 4.2. Then, restrictions of elements of $\mathcal{A}$ to $\Gamma_{\mathcal{A}}$ form an ACK-subalgebra $C_b(\Gamma_{\mathcal{A}})$ isometric to $\mathcal{A}$ (that we identify with $\mathcal{A}$) and restrictions of elements of $\mathcal{A}_{\sigma(Z,H)}(K,Z)$ to $\Gamma_{\mathcal{A}}$ form a subspace $X \subset C_b(\Gamma_{\mathcal{A}},(Z,\sigma(Z,H)))$ isometric to $\mathcal{A}_{\sigma(Z,H)}(K,Z)$. The conditions (i) and (ii) of Theorem 4.16 follow from the definition of $\mathcal{A}_{\sigma(Z,H)}(K,Z)$. The condition (iii) of Theorem 4.16 is reduced to the present condition (i). And, finally, the condition (iv) of Theorem 4.16 is reduced to the present (ii) by using Proposition 4.21. □

The condition (i) above could be quite demanding, for instance, when $\mathcal{O} = L(Z)$ in which case H is forced to be Z^*. However, in all concrete examples that we know of ACK structure, the family $\mathcal{O}$ can be taken really small. Thus, for concrete examples of Z, the condition (i) could be easily satisfied for every election of H.

By using the results from [5] it can be shown that condition (ii) above is satisfied for every H whenever (Z,w) is Lindelöf. Indeed, given f belonging to $\mathcal{A}_{\sigma(Z,H)}(K,Z)$, $f(K) \subset Z$ is $\sigma(Z,H)$-compact, thus, it is also Lindelöf. A straightforward application of [5, Corollary E] ensures that $(f(K),\sigma(Z,H))$ is norm-fragmented. Hence, in this case, Corollary 4.22 can be simplified as follows:

Corollary 4.23. *Let $Z \in \mathcal{O}\text{-}\mathrm{ACK}_\rho$ (or $Z \in \mathcal{O}\text{-}\mathrm{ACK}$) such that (Z,w) is Lindelöf and $\mathcal{A} \subset C(K)$ be a uniform algebra with K being the space of multiplicative functionals on $\mathcal{A}$. Fix $\Gamma_Z \subset H \subset Z^*$ such that $F^*H \subset H$ for every $F \in \mathcal{O}$, where Γ_Z is the 1-norming set given by the ACK structure of Z. Then, $\mathcal{A}_{\sigma(Z,H)}(K,Z) \in \mathrm{ACK}_\rho$ (resp. $\mathcal{A}_{\sigma(Z,H)}(K,Z) \in \mathrm{ACK}$).*

Observe that when Z has property β, the set $\mathcal{O}$ coincides with the set $\{x_\alpha^*(\cdot)\, x_\alpha : \alpha \in \Lambda\}$. Therefore, in this case, $F^*H \subset H$ for every H and for every $F \in \mathcal{O}$. Thus, we have proved the following corollary.

Corollary 4.24. *Let Z be a Banach space with property β such that (Z, w) is Lindelöf and $\mathcal{A} \subset C(K)$ be a uniform algebra with K being the space of multiplicative functionals on $\mathcal{A}$. Fix $\Gamma_Z \subset H \subset Z^*$, where $\Gamma_Z = \{x_\alpha^* : \alpha \in \Lambda\}$. Then, $\mathcal{A}_{\sigma(Z,H)}(K, Z) \in \mathrm{ACK}_\rho$.*

However, this technique can not fully generalize Theorem 4.13 by Acosta et al. to the case of vector-valued uniform algebras, since here the Lindelöf property is essential and property β does not imply in general weak Lindelöf. Observe that nevertheless the original statement of Theorem 4.13 is covered completely by our Corollary 4.19.

Acknowledgements

The research of first, second and third authors was partially supported by MINECO grant MTM2014-57838-C2-1-P and Fundación Séneca, Región de Murcia grant 19368/PI/14. The research of the third author is done in frames of Ukranian Ministry of Science and Education Research Program $0115U000481$. The research of fourth author has been partially performed during her stay in University of Murcia in frames of Erasmus+ program. We thank the referee for his/her suggestions that helped to improve the original manuscript.

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