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THE DAUGAVET EQUATION FOR BOUNDED VECTOR-VALUED FUNCTIONS

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ABSTRACT. Requirements under which the Daugavet equation and the alternative Daugavet equation hold for pairs of nonlinear maps between Banach spaces are analyzed. A geometric description is given in terms of nonlinear slices. Some local versions of these properties are also introduced and studied, as well as tests for checking whether the required conditions are satisfied in relevant cases.

1. Introduction. Daugavet [**7**] proved his eponymous equation in 1963, which establishes the norm identity

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

for a compact linear operator

$$T : C[0,1] \longrightarrow C[0,1].$$

This equation was extended to more general classes of linear operators on various spaces over the years. Nowadays, investigations on this topic build upon the approach of Kadets et al. [**10**], who defined a Banach space X to have the Daugavet property if all rank 1 operators on X satisfy the Daugavet equation. This property can conveniently be characterized in terms of slices of the unit ball, and it can be shown that, on a space with the Daugavet property, all weakly compact operators and all operators not fixing a copy of ℓ_1 satisfy the Daugavet equation, see [**1, 10, 11, 15**].

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The Daugavet equation has been extended in a number of other ways as well, replacing the identity operator by a more general reference operator called a Daugavet center, **[3, 4]**, or replacing the linear operators T by nonlinear operators, **[6, 8, 12]**. Here, we attempt to combine both of these ideas. We study the equation

$$\|\Phi + \Psi\| = \|\Phi\| + \|\Psi\|,$$

where Φ and Ψ are bounded maps on the unit ball of some Banach space X having values in some (possibly different) Banach space Y, and Ψ is in some sense small with respect to Φ, the norm being the sup norm. Also, the so-called alternative Daugavet equation

$$\max_{|\omega|=1} \|\Phi + \omega\Psi\| = \|\Phi\| + \|\Psi\|$$

will be considered. We shall investigate these equations by means of suitable modifications of the notion of slice continuity introduced in **[14]**, cf., Definition 3.1 below. We also rely on some techniques from **[6, 12]**.

The paper is organized as follows. After the preliminaries in Section 2, we study the Φ-Daugavet equation in Section 3, giving complete characterizations using the notion of strong slice continuity introduced below. Likewise, we introduce weak slice continuity in order to deal with the alternative Daugavet equation in Section 4. Finally, Section 5 is devoted to some technical local versions of these Daugavet type properties, which are obtained by considering suitable subsets of those appearing in the definitions previously studied. Some tests which guarantee that the requirements in our main theorems are satisfied are also presented. In particular, examples show their usefulness, especially for the cases of $C(K)$- and $L^1(\mu)$-spaces.

Now, we introduce some fundamental definitions and notation. We will write $\mathbb{T}$ for the set of scalars of modulus 1; the field of scalars can be $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We write $\operatorname{Re}\omega$ for the real part, $\operatorname{Im}\omega$ for the imaginary part and $\overline{\omega}$ for the complex conjugate of ω. For a Banach space X, B_X is its closed unit ball, U_X its open unit ball and S_X its unit sphere, and we will denote by X^* its dual space. If L is a Banach lattice, we use the symbol L^+ to denote the positive cone and B_{L^+} for the set $B_L \cap L^+$. The space of continuous linear operators from X to Y is denoted $L(X, Y)$.

The norm of a bounded mapping $\Phi : B_X \to Y$ is defined to be the sup norm, i.e.,

$$\|\Phi\| := \sup_{x \in B_X} \|\Phi(x)\|;$$

the space of all such mappings is denoted by $\ell_\infty(B_X, Y)$. In the scalar case, an element of $\ell_\infty(B_X)$ is typically denoted by x'. The symbol $x' \otimes y$ stands for the mapping $x \mapsto x'(x)y$.

Our main characterizations are given in terms of slices. A slice $S(x^*, \varepsilon)$ of B_X defined by a norm 1 element $x^* \in X^*$ and an $\varepsilon > 0$ is defined by

$$S(x^*, \varepsilon) = \{x \in B_X : \operatorname{Re} x^*(x) \geq 1 - \varepsilon\}.$$

When a nonlinear scalar-valued function is considered, the same definition makes sense; if $p : X \to \mathbb{K}$ is a function with norm ≤ 1, we write

$$S(p, \varepsilon) = \{x \in B_X : \operatorname{Re} p(x) \geq 1 - \varepsilon\}.$$

Note that, in this case, it may occur that $S(p, \varepsilon) = \emptyset$.

2. Preliminaries. In this section, we prove fundamental characterizations of the Daugavet and the alternative Daugavet equations. The theorems in this section are adapted from results in **[6, 14]**.

Definition 2.1. Let X and Y be Banach spaces, and let $\Phi \in \ell_\infty(B_X, Y)$. We say that $\Psi \in \ell_\infty(B_X, Y)$ satisfies the Φ-*Daugavet equation* if the norm equality

$$\|\Phi + \Psi\| = \|\Phi\| + \|\Psi\| \tag{Φ-DE}$$

holds. If Φ is the restriction of the identity to B_X, we call the above equation the *Daugavet equation* (DE).

In order to connect the Daugavet equation to a set $V \subset \ell_\infty(B_X, Y)$, we establish the following terminology.

Definition 2.2. Let X and Y be Banach spaces, and let $\Phi \in \ell_\infty(B_X, Y)$.

(1) Y has the Φ-*Daugavet property* with respect to $V \subset \ell_\infty(B_X, Y)$ if (Φ-DE) is satisfied by all $\Psi \in V$.

(2) Y has the *Φ-Daugavet property* if $\|\Phi + R\| = \|\Phi\| + \|R\|$ for all $R \in L(X, Y)$ with one-dimensional range.
(3) Y has the *Daugavet property* if (2) holds for $X = Y$ and $\Phi = \mathrm{Id}$.

The following lemma, see e.g., [**1,** Lemma 11.4] or [**15**] for a proof, frequently simplifies proofs concerning the Daugavet equation, because we only need consider maps of norm 1. We will often make use of the lemma without explicitly mentioning it.

Lemma 2.3. *Two vectors u and v in a normed space satisfy $\|u+v\| = \|u\|+\|v\|$ if and only if $\|\alpha u+\beta v\| = \alpha\|u\|+\beta\|v\|$ holds for all $\alpha, \beta \geq 0$. In particular, Ψ satisfies (Φ-DE) if and only if $\alpha\Psi$ satisfies ($\beta\Phi$-DE) for all $\alpha, \beta \geq 0$.*

In order to prove the first theorem of this section, we need the following lemma.

Lemma 2.4. *Let X be a Banach space, and assume that $x' \in \ell_\infty(B_X)$ with $\|x'\| \leq 1$. Let $0 \leq \varepsilon \leq 1$ and $x \in B_X$. Then, $\operatorname{Re} x'(x) \geq 1 - \varepsilon$ implies $|1 - x'(x)| \leq \sqrt{2\varepsilon}$.*

Proof. First, note that

$$1 \geq |x'(x)|^2 = (\operatorname{Im} x'(x))^2 + (\operatorname{Re} x'(x))^2 \geq (\operatorname{Im} x'(x))^2 + (1-\varepsilon)^2.$$

Hence,

$$(\operatorname{Im} x'(x))^2 \leq 1 - (1-\varepsilon)^2 = 2\varepsilon - \varepsilon^2.$$

Since $\operatorname{Re} x'(x) \geq 1-\varepsilon$ and $|x'(x)| \leq 1$, we see that $0 \leq 1-\operatorname{Re} x'(x) \leq \varepsilon$. Thus,

$$\begin{aligned} |1 - x'(x)|^2 &= |1 - \operatorname{Re} x'(x) - i \operatorname{Im} x'(x)|^2 \\ &= (1 - \operatorname{Re} x'(x))^2 + (\operatorname{Im} x'(x))^2 \\ &\leq \varepsilon^2 + 2\varepsilon - \varepsilon^2 \\ &= 2\varepsilon, \end{aligned}$$

i.e., $|1 - x'(x)| \leq \sqrt{2\varepsilon}$. $\square$

Theorem 2.5. *Let X and Y be Banach spaces. Let $\Phi \in \ell_\infty(B_X, Y)$, and consider a norm 1 map $x' \in \ell_\infty(B_X)$ and $y \in Y \setminus \{0\}$. Then, the following are equivalent*:

(1) $\|\Phi + x' \otimes y\| = \|\Phi\| + \|y\|$.
(2) *For every $\varepsilon > 0$, there are $x \in B_X$ and $\omega \in \mathbb{T}$ such that*

$$\operatorname{Re} \omega x'(x) \geq 1 - \varepsilon \quad \textit{and} \quad \left\| \omega \Phi(x) + \frac{y}{\|y\|} \right\| \geq \|\Phi\| + 1 - \varepsilon.$$

Proof.

(1) $\Rightarrow$ (2). By Lemma 2.3, we can assume $y \in S_Y$. Hence, there is an element $x \in B_X$ such that

$$\begin{aligned} \|\Phi\| + 1 - \frac{\varepsilon}{2} &\leq \|\Phi(x) + x'(x)y\| \\ &\leq \|\Phi(x)\| + |x'(x)|\|y\| \\ &\leq \|\Phi\| + |x'(x)|. \end{aligned}$$

Thus, $|x'(x)| \geq 1 - (\varepsilon/2)$. Writing $\omega = |x'(x)|/x'(x) \in \mathbb{T}$, we have

$$\operatorname{Re} \omega x'(x) = |x'(x)| \geq 1 - \varepsilon.$$

Moreover,

$$\begin{aligned} \|\Phi\| + 1 - \frac{\varepsilon}{2} &\leq \|\Phi(x) + x'(x)y\| \\ &= \|\omega\Phi(x) + \omega x'(x)y\| \\ &\leq \|\omega\Phi(x) + y\| + \|\omega x'(x)y - y\| \\ &= \|\omega\Phi(x) + y\| + |\omega x'(x) - 1|\|y\| \\ &= \|\omega\Phi(x) + y\| + |\,|x'(x)| - 1| \\ &\leq \|\omega\Phi(x) + y\| + \frac{\varepsilon}{2}, \end{aligned}$$

and (2) follows.

(2) $\Rightarrow$ (1). Again, by Lemma 2.3, it suffices to consider the case $\|y\| = 1$. Let $\varepsilon > 0$, and take $x \in B_X$ and $\omega \in \mathbb{T}$ such that

$$\operatorname{Re} \omega x'(x) \geq 1 - \varepsilon \quad \text{and} \quad \|\omega\Phi(x) + y\| \geq \|\Phi\| + 1 - \varepsilon.$$

Thus,

$$\begin{aligned}\|\Phi\| + 1 - \varepsilon &\le \|\omega\Phi(x) + y\| \\ &= \|\Phi(x) + \overline{\omega}y\| \\ &\le \|\Phi(x) + x'(x)y\| + \|\overline{\omega}y - x'(x)y\| \\ &= \|\Phi(x) + x'(x)y\| + \|y - \omega x'(x)y\| \\ &= \|\Phi(x) + x'(x)y\| + |1 - \omega x'(x)| \\ &\le \|\Phi(x) + x'(x)y\| + \sqrt{2\varepsilon},\end{aligned}$$

where the latter inequality is due to Lemma 2.4. Since ε was arbitrary, (1) holds. $\square$

Next, we present analogous results in the setting of the alternative Daugavet equation.

Definition 2.6. Let X and Y be Banach spaces, and let $\Phi, \Psi \in \ell_\infty(B_X, Y)$. We say that Ψ satisfies the *alternative Φ-Daugavet equation* if

$$\max_{|\omega|=1} \|\Phi + \omega\Psi\| = \|\Phi\| + \|\Psi\| \tag{Φ-ADE}$$

is true. In the case where Φ is the identity, we refer to the above equation simply as the alternative Daugavet equation (ADE).

We will also use the following definitions regarding a set $V \subset \ell_\infty(B_X, Y)$.

Definition 2.7. Let X and Y be Banach spaces, and let $\Phi \in \ell_\infty(B_X, Y)$.

(a) Y has the *alternative Φ-Daugavet property* with respect to $V \subset \ell_\infty(B_X, Y)$ if (Φ-ADE) is satisfied for all $\Psi \in V$.
(b) Y has the *alternative Φ-Daugavet property* if

$$\max_{|\omega|=1} \|\Phi + \omega R\| = \|\Phi\| + \|R\|$$

for all $R \in L(X, Y)$ with one-dimensional range.
(c) Y has the *alternative Daugavet property* if it has the alternative Id-Daugavet property.

Now that the notation is fixed, we look at the interrelation between the Daugavet and the alternative Daugavet equations.

Remark 2.8. Let X and Y be Banach spaces, and let $\Phi, \Psi \in \ell_\infty(B_X, Y)$.

(1) Ψ satisfies (Φ-ADE) if and only if there exists an $\omega \in \mathbb{T}$ such that $\omega\Psi$ fulfills (Φ-DE).
(2) (Φ-DE) implies (Φ-ADE), but, in general, the converse is not true. For example, $-\mathrm{Id}$ always satisfies (ADE), but never (DE).
(3) Ψ satisfies (Φ-ADE) if and only if $\alpha\Psi$ satisfies ($\beta\Phi$-ADE) for every $\alpha, \beta \geq 0$. This is a consequence of (1) and Lemma 2.3.

Theorem 2.9. *Let X and Y be Banach spaces. Let $\Phi \in \ell_\infty(B_X, Y)$, and consider a norm 1 map $x' \in \ell_\infty(B_X)$ and $y \in Y \setminus \{0\}$. Then, the following are equivalent*:

(1) $\max_{|\omega|=1} \|\Phi + \omega x' \otimes y\| = \|\Phi\| + \|y\|$.
(2) *For every $\varepsilon > 0$, there exist $\omega_1, \omega_2 \in \mathbb{T}$ and $x \in B_X$ such that*

$$\operatorname{Re}\omega_1 x'(x) \geq 1 - \varepsilon \quad \textit{and} \quad \left\|\omega_2\Phi(x) + \frac{y}{\|y\|}\right\| \geq \|\Phi\| + 1 - \varepsilon.$$

(3) *For every $\varepsilon > 0$, there exist $\omega \in \mathbb{T}$ and $x \in B_X$ such that*

$$|x'(x)| \geq 1 - \varepsilon \quad \textit{and} \quad \left\|\omega\Phi(x) + \frac{y}{\|y\|}\right\| \geq \|\Phi\| + 1 - \varepsilon.$$

Proof.

(1) $\Rightarrow$ (2). By Remark 2.8 (3), we can assume that $\|y\| = 1$. According to (1), there exists an $\omega \in \mathbb{T}$ such that $\|\Phi + \omega x' \otimes y\| = \|\Phi\| + 1$. Thus, for a given $\varepsilon > 0$, Theorem 2.5 yields $x \in B_X$ and $\omega_2 \in \mathbb{T}$ such that

$$\operatorname{Re}\omega_2\omega x'(x) \geq 1 - \varepsilon \quad \text{and} \quad \|\omega_2\Phi(x) + y\| \geq \|\Phi\| + 1 - \varepsilon.$$

Defining $\omega_1 = \omega_2\omega$, (2) follows.

(2) $\Rightarrow$ (3). If $\operatorname{Re}\omega_1 x'(x) \geq 1 - \varepsilon$, then

$$1 - \varepsilon \leq \operatorname{Re}\omega_1 x'(x) \leq |x'(x)|.$$

(3) $\Rightarrow$ (1). It suffices to consider the case $\|y\| = 1$. For a given $\varepsilon > 0$, take $\omega \in \mathbb{T}$ and $x \in B_X$ such that

$$|x'(x)| \geq 1 - \varepsilon \quad \text{and} \quad \|\omega\Phi(x) + y\| \geq \|\Phi\| + 1 - \varepsilon.$$

Denote $\omega_1 = |x'(x)|/x'(x)$ and $\omega_2 = \overline{\omega}\omega_1$. Thus,

$$\begin{aligned} \|\Phi(x) + \omega_2 x'(x)y\| &= \|\Phi(x) + \overline{\omega}\omega_1 x'(x)y\| \\ &= \|\omega\Phi(x) + \omega_1 x'(x)y\| \\ &\geq \|\omega\Phi(x) + y\| - \|y - \omega_1 x'(x)y\| \\ &= \|\omega\Phi(x) + y\| - |1 - |x'(x)|| \\ &\geq \|\Phi\| + 1 - 2\varepsilon, \end{aligned}$$

and we are finished since $\varepsilon > 0$ was arbitrary. □

3. Strong slice continuity. In **[14]**, the notion of slice continuity was introduced for studying when the Daugavet equation holds for a couple of maps Φ and Ψ between Banach spaces, i.e., when

$$\|\Phi + \Psi\| = \|\Phi\| + \|\Psi\|.$$

The functions taken into account were either linear or bilinear bounded maps. In this section, we will extend some of the results from **[14]** to the case of bounded nonlinear functions.

The following definition is from **[14]**.

Definition 3.1. Let X and Y be Banach spaces, and let $\Phi \in \ell_\infty(B_X, Y)$.

(a) If $y^* \in Y^*$ with $y^*\Phi \neq 0$, we define $\Phi_{y^*} : B_X \to \mathbb{K}$ by

$$\Phi_{y^*}(x) = 1/\|y^*\Phi\|\, y^*\Phi(x).$$

(b) The *natural set of slices* defined by Φ is given by

$$\mathcal{S}_\Phi = \{S(\Phi_{y^*}, \varepsilon) : 0 < \varepsilon < 1,\, y^* \in Y^*,\, y^*\Phi \neq 0\}.$$

(c) We write $\mathcal{S}_\Psi \leq \mathcal{S}_\Phi$ if, for every $S(\Psi_{z^*}, \varepsilon) \in \mathcal{S}_\Psi$, there is an $S(\Phi_{y^*}, \mu) \in \mathcal{S}_\Phi$ with

$$S(\Phi_{y^*}, \mu) \subset S(\Psi_{z^*}, \varepsilon).$$

In this instance, we say that Ψ is *slice continuous* with respect to Φ.

Now, we are ready to introduce the concept of strong slice continuity for bounded nonlinear maps.

Definition 3.2. Let X and Y be Banach spaces, and let $\Phi, \Psi \in \ell_\infty(B_X, Y)$. We use the symbol $\mathcal{S}_\Psi < \mathcal{S}_\Phi$ if, for every slice $S(\Psi_{z^*}, \varepsilon) \in \mathcal{S}_\Psi$, there is a slice $S(\Phi_{y^*}, \mu) \in \mathcal{S}_\Phi$ such that

$$S(\omega\Phi_{y^*}, \mu) \subset S(\omega\Psi_{z^*}, \varepsilon) \quad \text{for all } \omega \in \mathbb{T}.$$

In this case, we say that Ψ is *strongly slice continuous* with respect to Φ.

Note that Definition 3.2 and similar definitions carry over to bounded functions from X to Y by considering respective restrictions to B_X.

It is clear that strong slice continuity implies slice continuity. The following remark shows that, in the case of multilinear maps, the two concepts coincide.

Remark 3.3. Let $X_1, \dots, X_n, Z$ be Banach spaces and $A, B : X_1 \times \cdots \times X_n \to Z$ bounded multilinear maps. Then $\mathcal{S}_A < \mathcal{S}_B$ if and only if $\mathcal{S}_A \le \mathcal{S}_B$.

Proof. We only need to verify that slice continuity implies strong slice continuity. To this end, let $S(A_{x^*}, \varepsilon) \in \mathcal{S}_A$ be given. Since $\mathcal{S}_A \le \mathcal{S}_B$, we can find $S(B_{y^*}, \mu) \in \mathcal{S}_B$ with $S(B_{y^*}, \mu) \subset S(A_{x^*}, \varepsilon)$. For a given $\omega \in \mathbb{T}$ and $(x_1, \dots, x_n) \in S(\omega B_{y^*}, \mu)$, we have

$$1 - \mu \le \operatorname{Re} \omega \frac{y^* B(x_1, \dots, x_n)}{\|y^* B\|} = \operatorname{Re} \frac{y^* B(\omega x_1, \dots, x_n)}{\|y^* B\|},$$

i.e., $(\omega x_1, x_2, \dots, x_n) \in S(B_{y^*}, \mu)$. This ensures $(\omega x_1, x_2, \dots, x_n) \in S(A_{x^*}, \varepsilon)$, and the multilinearity of A leads to $(x_1, x_2, \dots, x_n) \in S(\omega A_{x^*}, \varepsilon)$. $\square$

The canonical example of when the relation $\mathcal{S}_\Psi < \mathcal{S}_\Phi$ holds is given by the case where Ψ is the concatenation of a map Φ and a bounded linear operator.

Example 3.4. Let X and Y be Banach spaces. Consider $\Phi \in \ell_\infty(B_X, Y)$ and a bounded linear operator $P : Y \to Y$. Denote $\Psi = P \circ \Phi$. Then, $S_\Psi < S_\Phi$.

Proof. Let $S(\Psi_{y^*}, \varepsilon)$ be a slice in $\mathcal{S}_\Psi$. First, note that, since $y^*\Psi \neq 0$, we also have $(y^*P)\Phi = y^*\Psi \neq 0$, and thus, $S(\Phi_{y^*P}, \varepsilon) \in \mathcal{S}_\Phi$. Take $\omega \in \mathbb{T}$ and $x \in S(\omega\Phi_{y^*P}, \varepsilon)$, i.e.,

$$\operatorname{Re} \omega \frac{(y^*P)\Phi}{\|(y^*P)\Phi\|}(x) \geq 1 - \varepsilon.$$

By construction,

$$\operatorname{Re} \omega \frac{y^*\Psi}{\|y^*\Psi\|}(x) = \operatorname{Re} \omega \frac{(y^*P)\Phi}{\|(y^*P)\Phi\|}(x) \geq 1 - \varepsilon,$$

and therefore, $x \in S(\omega\Psi_{y^*}, \varepsilon)$. □

The next example shows that there are bounded maps Φ and Ψ with $\mathcal{S}_\Psi < \mathcal{S}_\Phi$ but $\Psi \neq P \circ \Phi$ for any bounded linear operator P.

Example 3.5. Let $C[0,1]$ denote the Banach space of continuous functions from $[0,1]$ to $\mathbb{K}$. Let

$$\Phi : C[0,1] \oplus_1 \mathbb{K} \longrightarrow C[0,1], \quad \Phi(f, \alpha) = f,$$

and

$$\Psi : C[0,1] \oplus_1 \mathbb{K} \longrightarrow C[0,1], \quad \Psi(f, \alpha) = f + \alpha^2 \mathbf{1},$$

where $\mathbf{1}$ stands for the constant one function, and $\oplus_1$ denotes the direct sum with the 1-norm. Then Ψ and Φ have norm 1. The kernel of Φ is not contained in the kernel of Ψ since $\Phi(0,1) = 0$; however, $\Psi(0,1) \neq 0$. Thus, we do not have $\Psi = P \circ \Phi$ for any bounded linear operator P. However, the slice condition $\mathcal{S}_\Psi < \mathcal{S}_\Phi$ holds. First note that, for any $x^* \in C[0,1]^* \setminus \{0\}$, we have

$$\|x^*\Phi\| = \|x^*\Psi\| = \|x^*\| \neq 0.$$

Consider some $x^* \in C[0,1]^*$ with $\|x^*\| = 1$, and let $0 < \varepsilon < 1$. We claim

$$S\left(\omega x^*\Phi, \frac{\varepsilon}{2}\right) \subset S(\omega x^*\Psi, \varepsilon) \quad \text{for all } \omega \in \mathbb{T}.$$

In order to prove this, assume

$$(f,\alpha) \in S\left(\omega x^*\Phi, \frac{\varepsilon}{2}\right),$$

i.e.,

$$\operatorname{Re}\omega x^*(f) \geq 1 - \frac{\varepsilon}{2}.$$

In particular, $\|f\| \geq 1 - \varepsilon/2$, and therefore, $|\alpha| \leq \varepsilon/2$. Hence,

$$\begin{aligned}\operatorname{Re}\omega x^*\Psi(f,\alpha) &= \operatorname{Re}\omega x^*(f + \alpha^2\mathbf{1}) \\ &= \operatorname{Re}\omega x^*(f) + \operatorname{Re}\omega x^*(\alpha^2\mathbf{1}) \\ &\geq 1 - \varepsilon.\end{aligned}$$

Now consider a closed subspace Z of a normed space X. Then,

$$q : X \longrightarrow X/Z, \qquad q(x) = x + Z$$

sends the open unit ball U_Z of Z onto the open unit ball $U_{X/Z}$ of X/Z. This motivates the following definition.

Definition 3.6. Let X and Y be Banach spaces. We call $\Phi \in \ell_\infty(B_X, Y)$ a *quotient map* if Φ is continuous and $\Phi(U_X) = U_Y$.

Given $\Phi \in \ell_\infty(B_X, Y)$, we set

$$Y^*\Phi \cdot Y = \{y^*\Phi \otimes y : y^* \in Y^*,\, y \in Y\}.$$

Lemma 3.7. *Let X and Y be Banach spaces, and assume that $\Phi \in \ell_\infty(B_X, Y)$ is a quotient map. Then the following are equivalent*:

(1) *Y has the Daugavet property.*
(2) *Y has the Φ-Daugavet property* with respect to $Y^*\Phi \cdot Y$.

Proof. This is a consequence of the assumptions that Φ is continuous and $\Phi(U_X) = U_Y$. □

Proposition 3.8. *Let X and Y be Banach spaces, and assume that Y has the Daugavet property. Consider $\Psi, \Phi \in \ell_\infty(B_X, Y)$ such that Φ is* *a quotient map and* $\|\Psi\| = 1$. *Then,* $\mathcal{S}_\Psi < \mathcal{S}_\Phi$ *implies that, for every* $y \in S_Y$ *and* $y^* \in Y^*$ *with* $y^*\Psi \neq 0$,

$$\|\Phi + \Psi_{y^*} \otimes y\| = 2.$$

Proof. By Theorem 2.5, it suffices to show that, for every $\varepsilon > 0$, there are $\omega \in \mathbb{T}$ and $x \in S(\omega\Psi_{y^*}, \varepsilon)$ such that

$$\|\omega\Phi(x) + y\| \geq 2 - \varepsilon.$$

Thus, let $\varepsilon > 0$ be given. Since $\mathcal{S}_\Psi < \mathcal{S}_\Phi$, we can find a slice $S(\Phi_{z^*}, \mu) \in \mathcal{S}_\Phi$ with $\mu \leq \varepsilon$ such that $S(\lambda\Phi_{z^*}, \mu) \subset S(\lambda\Psi_{y^*}, \varepsilon)$ for all $\lambda \in \mathbb{T}$. According to Lemma 3.7, $\|\Phi + \Phi_{z^*} \otimes y\| = 2$; therefore, Theorem 2.5 gives $\omega \in \mathbb{T}$ and $x \in S(\omega\Phi_{z^*}, \mu)$, satisfying

$$\|\omega\Phi(x) + y\| \geq 2 - \mu \geq 2 - \varepsilon.$$

By construction, $S(\omega\Phi_{z^*}, \mu) \subset S(\omega\Psi_{y^*}, \varepsilon)$. The proof is complete. $\square$

Remark 3.9. Proposition 3.8 is false if the condition $\mathcal{S}_\Psi < \mathcal{S}_\Phi$ is removed. In order to see this, consider bounded linear operators

$$\Phi, \Psi : L_1[0,1] \oplus_1 L_1[1,2] \longrightarrow L_1[0,1]$$

given by $\Phi((f,g)) = f$ and

$$\Psi((f,g)) = \left(\int_1^2 g\,dx\right) \cdot \mathbf{1},$$

where $(f,g) \in L_1[0,1] \oplus_1 L_1[1,2]$; recall that $L_1[0,1]$ has the Daugavet property. Clearly, Φ is a quotient map and $\|\Psi\| = 1$. However, if $y = \mathbf{1} \in L_1[0,1]$ and $y^* = \mathbf{1} \in L_\infty[0,1]$, then $\|\Phi + y^*\Psi \otimes y\| \leq 1$.

We shall now deal with weakly compact maps. Let us begin by recalling the definition of a (nonlinear) weakly compact map.

Definition 3.10. Let X and Y be Banach spaces. A function $\Psi \in \ell_\infty(B_X, Y)$ is called *weakly compact* if the weak closure of $\Psi(B_X)$ is a weakly compact set.

We now prove the main result of this section, namely, Theorem 3.11.

Theorem 3.11. *Let X and Y be Banach spaces, and let $\Phi, \Upsilon, \Psi \in \ell_\infty(B_X, Y)$ with $\|\Phi\| = \|\Upsilon\| = \|\Psi\| = 1$. Assume that Y has the Φ-Daugavet property with respect to $Y^*\Upsilon \cdot Y$. Then, if $\mathcal{S}_\Psi < \mathcal{S}_\Upsilon$ and Ψ is weakly compact,*

$$\|\Phi + \Psi\| = 2.$$

Proof. Since the set $K = \overline{\text{co}}(\mathbb{T}\Psi(B_X))$ is weakly compact by Krein's theorem, we can conclude that K coincides with the closed convex hull of its strongly exposed points ([**2,** Corollary 5.18], [**5**]). Therefore, given $\varepsilon > 0$, we may take a strongly exposed point $y_0 \in K$ with $\|y_0\| > 1 - \varepsilon$. Since y_0 is a strongly exposed point, there are $z^* \in Y^*$ and $\eta > 0$ such that the set

$$\{y \in K : \operatorname{Re} z^*(y) \geq \operatorname{Re} z^*(y_0) - \eta\}$$

has diameter less than ε and $\operatorname{Re} z^*(y_0) > \operatorname{Re} z^*(y)$ for all $y \in K \setminus \{y_0\}$. After defining $y_0^* = z^*/\operatorname{Re} z^*(y_0)$ and $\delta = \min\{\varepsilon/2, \eta/\operatorname{Re} z^*(y_0)\}$, we have found a slice

$$S = \{y \in K : \operatorname{Re} y_0^*(y) \geq 1 - \delta\}$$

containing y_0 and having diameter less than ε. In particular,

$$y \in K,\ \operatorname{Re} y_0^*(y) \geq 1 - \delta \Longrightarrow \|y - y_0\| < \varepsilon.$$

Also note that, since K is balanced,

$$\sup_{y \in K} \operatorname{Re} y_0^*(y) = \sup_{y \in K} |y_0^*(y)| = 1.$$

Denote $\psi := y_0^* \circ \Psi$. We have

$$\|\psi\| = \sup_{x \in B_X} |y_0^*(\Psi(x))| = \sup_{y \in K} |y_0^*(y)| = 1;$$

hence, $S(\psi, \delta) \in \mathcal{S}_\Psi$. Due to $\mathcal{S}_\Psi < \mathcal{S}_\Upsilon$, there are $\mu \leq \delta$ and $S(\Upsilon_{z^*}, \mu) \in \mathcal{S}_\Upsilon$ such that

$$S(\lambda\Upsilon_{z^*}, \mu) \subset S(\lambda\psi, \delta) \quad \text{for all } \lambda \in \mathbb{T}.$$

Since, by assumption, $\|\Phi + \Upsilon_{z^*} \otimes y_0\| = 1 + \|y_0\|$, Theorem 2.5 yields $\omega \in \mathbb{T}$ and $x \in S(\omega\Upsilon_{z^*}, \mu)$ such that

$$\|\omega\Phi(x) + \frac{y_0}{\|y_0\|}\| \geq 2 - \mu \geq 2 - \varepsilon.$$

By construction, $x \in S(\omega\Upsilon_{z^*}, \mu) \subset S(\omega\psi, \delta)$, and therefore,

$$\operatorname{Re} y_0^*(\omega\Psi(x)) = \operatorname{Re}\omega\psi(x) \geq 1 - \delta;$$

thus, the fact that $\omega\Psi(x) \in K$ gives $\|\omega\Psi(x) - y_0\| < \varepsilon$.

We calculate

$$\begin{aligned} \|y_0 + \omega\Phi(x)\| &\geq \left\|\omega\Phi(x) + \frac{y_0}{\|y_0\|}\right\| - \left\|y_0 - \frac{y_0}{\|y_0\|}\right\| \\ &= \left\|\omega\Phi(x) + \frac{y_0}{\|y_0\|}\right\| - |\|y_0\| - 1| \\ &\geq 2 - 2\varepsilon. \end{aligned}$$

Finally,

$$\begin{aligned} \|\Phi + \Psi\| &\geq \|\Phi(x) + \Psi(x)\| \\ &= \|\omega\Phi(x) + \omega\Psi(x)\| \\ &\geq \|\omega\Phi(x) + y_0\| - \|\omega\Psi(x) - y_0\| \\ &\geq 2 - 3\varepsilon. \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we conclude that Ψ satisfies (Φ-DE). □

Remark 3.12. The requirement on the weak compactness of the function Ψ can be substituted in the above result by the more general notion of the Radon-Nikodým function, which fits exactly with what is needed; see the definition and how to use it in this setting, for example, in [**4**]. One method of defining the Radon-Nikodým property for a closed convex set A is that every closed convex subset $B \subset A$ is the closed convex hull of its strongly exposed points. (See [**2,** Theorems 5.8, 5.17].) Thus, a function is said to be a *Radon-Nikodým function* if the closure of $T(B_X)$ has the Radon-Nikodým property.

Corollary 3.13. *Let X and Y be Banach spaces, and assume that Y has the Daugavet property. Consider $\Phi, \Psi \in \ell_\infty(B_X, Y)$ such that Φ is a quotient map and $\|\Psi\| = 1$. If $\mathcal{S}_\Psi < \mathcal{S}_\Phi$ and Ψ is weakly compact, then*

$$\|\Phi + \Psi\| = 2.$$

Proof. In Lemma 3.7 we observed that, if Y has the Daugavet property and Φ is a quotient map, then Y has the Φ-Daugavet property with respect to $Y^*\Phi \cdot Y$. Thus, Theorem 3.11 yields $\|\Phi + \Psi\| = 2$. $\square$

Remark 3.14. Corollary 3.13 is invalid if the condition $\mathcal{S}_\Psi < \mathcal{S}_\Phi$ is removed. For instance, consider

$$\Phi, \Psi : L_1[0,1] \oplus_1 L_2[1,2] \longrightarrow L_1[0,1]$$

given by

$$\Phi((f,g)) = f \quad \text{and} \quad \Psi((f,g))(x) = g(x+1),$$

where $(f,g) \in L_1[0,1] \oplus_1 L_2[1,2]$. Then, Ψ is weakly compact and $\|\Phi\| = \|\Psi\| = 1$; however, $\|\Phi + \Psi\| \leq 1$.

Theorem 3.15. *Let X and Y be Banach spaces, and let $\mathcal{Z}$ be a linear subspace of $\ell_\infty(B_X)$. Assume that $\Phi \in \ell_\infty(B_X, Y)$ with $\|\Phi\| = 1$. Then, the following are equivalent*:

(1) *for every $x' \in \mathcal{Z}$ and $y \in Y$, $x' \otimes y$ satisfies* (Φ-DE).
(2) *For every $x' \in S_{\mathcal{Z}}$, $y \in S_Y$ and $\varepsilon > 0$, there exist $\omega \in \mathbb{T}$ and $x \in B_X$ such that*

$$\operatorname{Re} \omega x'(x) \geq 1 - \varepsilon \quad \textit{and} \quad \|\omega\Phi(x) + y\| \geq 2 - \varepsilon.$$

(3) *Every weakly compact $\Psi \in \ell_\infty(B_X, Y)$ such that $y^* \circ \Psi \in \mathcal{Z}$ for all $y^* \in Y^*$ satisfies* (Φ-DE).

Proof.

(1) $\Leftrightarrow$ (2). This equivalence follows from Theorem 2.5.

(1) $\Rightarrow$ (3). Let Ψ be as in (3). Due to (1), Y has the Φ-Daugavet property with respect to $Y^*\Psi \cdot Y$. Since, trivially $\mathcal{S}_\Psi < \mathcal{S}_\Psi$, Theorem 3.11 yields (3).

(3) $\Rightarrow$ (1). Given $x' \in \mathcal{Z}$ and $y \in Y$, $x' \otimes y$ has finite-dimensional range and, consequently, is a weakly compact map. $\square$

For completeness, we note the n-linear version of [**14,** Corollary 3.10].

Corollary 3.16. *Let $X_1, \dots, X_n$ and Y be Banach spaces, and consider a continuous multilinear map*

$$B_0 : X_1 \times \cdots \times X_n \longrightarrow Y$$

satisfying $B_0(U_{X_1 \times \cdots \times X_n}) = U_Y$. Consider the subsets R, C and WC of $L(Y, Y)$ of rank 1, compact and weakly compact linear operators. Denote

$$R \circ B_0 = \{T \circ B_0 : T \in R\},$$
$$C \circ B_0 = \{T \circ B_0 : T \in C\}$$

and

$$WC \circ B_0 = \{T \circ B_0 : T \in WC\}.$$

Then, the following are equivalent:

(1) *Y has the Daugavet property.*
(2) *Y has the B_0-Daugavet property with respect to $R \circ B_0$.*
(3) *Y has the B_0-Daugavet property with respect to $C \circ B_0$.*
(4) *Y has the B_0-Daugavet property with respect to $WC \circ B_0$.*

Proof. The equivalence of (1) and (2) follows from Lemma 3.7. Items (2) and (4) are equivalent by using $\mathcal{Z} = \{y^* \circ B_0 : y^* \in Y^*\}$ in Theorem 3.15. The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are due to the inclusions $R \subset C \subset WC$. $\square$

4. Weak slice continuity. In the previous section, we defined the notion of strong slice continuity and related it to the Daugavet equation. This section is the analogue of Section 3 for the alternative Daugavet equation. We introduce the concept of weak slice continuity to further investigate when two maps Ψ and Φ satisfy the alternative Daugavet equation, i.e., when

$$\max_{|\omega|=1} \|\Phi + \omega\Psi\| = \|\Phi\| + \|\Psi\|.$$

Definition 4.1. Let X be a Banach space, $x' \in \ell_\infty(B_X)$ with $\|x'\| = 1$ and $\varepsilon > 0$. We write

$$S'(x', \varepsilon) = \{x \in B_X : |x'(x)| \geq 1 - \varepsilon\}$$

for the *weak slice* of B_X determined by x' and ε.

Secondly, we extend the above definition to Banach space-valued functions.

Definition 4.2. Let X and Y be Banach spaces and $\Phi \in \ell_\infty(B_X, Y)$. The *natural set of weak slices* defined by Φ is given by

$$\mathcal{S}'_\Phi = \{S'(\Phi_{y^*}, \varepsilon) : 0 < \varepsilon < 1,\, y^* \in Y^*,\, y^*\Phi \neq 0\}.$$

Now, we are in a position to define weak slice continuity in analogy to strong slice continuity, cf., Definition 3.2.

Definition 4.3. Let X and Y be Banach spaces and $\Phi, \Psi \in \ell_\infty(B_X, Y)$. We write $\mathcal{S}'_\Psi < \mathcal{S}'_\Phi$ if, for every weak slice $S'(\Psi_{z^*}, \varepsilon) \in \mathcal{S}'_\Psi$, there is a weak slice $S'(\Phi_{y^*}, \mu) \in \mathcal{S}'_\Phi$ such that

$$S'(\Phi_{y^*}, \mu) \subset S'(\Psi_{z^*}, \varepsilon).$$

In this case, we say that Ψ is *weakly slice continuous* with respect to Φ.

If Φ and Ψ are two maps such that Ψ is strongly slice continuous with respect to Φ, then Ψ is also slice continuous with respect to Φ. We check that a similar implication holds for strong and weak slice continuity.

Remark 4.4. Let X and Y be Banach spaces and $\Phi, \Psi \in \ell_\infty(B_X, Y)$. Then, $\mathcal{S}_\Psi < \mathcal{S}_\Phi$ implies $\mathcal{S}'_\Psi < \mathcal{S}'_\Phi$.

Proof. Assume $S'(\Psi_{z^*}, \varepsilon) \in \mathcal{S}'_\Psi$. Since $\mathcal{S}_\Psi < \mathcal{S}_\Phi$, there is an $S(\Phi_{y^*}, \mu) \in \mathcal{S}_\Phi$ satisfying

$$S(\lambda\Phi_{y^*}, \mu) \subset S(\lambda\Psi_{z^*}, \varepsilon) \quad \text{for all } \lambda \in \mathbb{T}.$$

We claim

$$S'(\Phi_{y^*}, \mu) \subset S'(\Psi_{z^*}, \varepsilon).$$

In order to prove this, let $x \in B_X$ with $|\Phi_{y^*}(x)| \geq 1 - \mu$ and denote $\omega = |\Phi_{y^*}(x)|/\Phi_{y^*}(x)$. Then, $\operatorname{Re} \omega\Phi_{y^*}(x) = |\Phi_{y^*}(x)| \geq 1 - \mu$, and therefore, $\operatorname{Re} \omega\Psi_{z^*}(x) \geq 1 - \varepsilon$. In particular, $|\Psi_{z^*}(x)| \geq 1 - \varepsilon$, i.e., $x \in S'(\Psi_{z^*}, \varepsilon)$. $\square$

The next example shows that the reverse implication in Remark 4.4 does not hold.

Example 4.5. Let $\Psi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\Psi(x) = \begin{cases} 1 & \text{if } x = 0, \\ -|x| & \text{if } x \neq 0. \end{cases}$$

Then Ψ is weakly slice continuous with respect to the identity; however, Ψ is not strongly slice continuous with respect to the identity.

Proof. Consider the slice $S(\Psi, 1/2) \in \mathcal{S}_\Psi$. Then, $S(c\,\mathrm{Id}, \varepsilon) \not\subset S(\Psi, 1/2)$ for any $c \in \{-1, 1\}$ and $0 < \varepsilon < 1$. Thus, Ψ is not strongly slice continuous with respect to Id. However, if $c \in \{-1, 1\}$ and $0 < \varepsilon < 1$ are given, then $S'(\mathrm{Id}, \varepsilon) \subset S'(c\Psi, \varepsilon)$. Therefore, Ψ is weakly slice continuous with respect to Id. $\square$

Example 4.6. Let X and Y be Banach spaces. Consider $\Phi \in \ell_\infty(B_X, Y)$ and a bounded linear operator

$$P : Y \longrightarrow Y.$$

Denote $\Psi = P \circ \Phi$. Then, $\mathcal{S}'_\Psi < \mathcal{S}'_\Phi$.

Proof. According to Example 3.4, the assumptions imply $\mathcal{S}_\Psi < \mathcal{S}_\Phi$. Hence, $\mathcal{S}'_\Psi < \mathcal{S}'_\Phi$ by Remark 4.4. $\square$

Note that we have shown in Example 4.5 that there are bounded maps Φ and Ψ with $\mathcal{S}'_\Psi < \mathcal{S}'_\Phi$; however, $\Psi \neq P \circ \Phi$ for any bounded linear operator P.

Recall from Definition 3.6 that a quotient map is a continuous function mapping the open unit ball of its domain onto the open unit ball of its range space. These properties allow for the next lemma.

Lemma 4.7. *Let X and Y be Banach spaces, and assume that $\Phi \in \ell_\infty(B_X, Y)$ is a quotient map. Then, the following are equivalent*:

(1) *Y has the alternative Daugavet property.*
(2) *Y has the alternative Φ-Daugavet property with respect to*

$$Y^*\Phi \cdot Y.$$

Proposition 4.8. *Let X and Y be Banach spaces, and assume that Y has the alternative Daugavet property. Consider $\Psi, \Phi \in \ell_\infty(B_X, Y)$ such that Φ is a quotient map and $\|\Psi\| = 1$. Then, $\mathcal{S}'_\Psi < \mathcal{S}'_\Phi$ implies that, for every $y \in S_Y$ and $y^* \in Y^*$ with $y^*\Psi \neq 0$,*

$$\max_{|\omega|=1} \|\Phi + \omega\Psi_{y^*} \otimes y\| = 2.$$

Proof. We will use Theorem 2.9, i.e., we need to show that, for every $\varepsilon > 0$, there exist $\omega \in \mathbb{T}$ and $x \in S'(\Psi_{y^*}, \varepsilon)$ such that

$$\|\omega\Phi(x) + y\| \geq 2 - \varepsilon.$$

Since $\mathcal{S}'_\Psi < \mathcal{S}'_\Phi$, there is a slice $S'(\Phi_{z^*}, \mu) \in \mathcal{S}'_\Phi$ such that

$$S'(\Phi_{z^*}, \mu) \subset S'(\Psi_{y^*}, \varepsilon)$$

and $\mu \leq \varepsilon$. The alternative Daugavet property of Y in conjunction with Lemma 4.7 yields the norm equality

$$\max_{|\omega|=1} \|\Phi + \omega\Phi_{y^*} \otimes y\| = 2.$$

Hence, another application of Theorem 2.9 yields $\omega \in \mathbb{T}$ and $x \in S'$ (Φ_{y^*}, μ) such that

$$\|\omega\Phi(x) + y\| \geq 2 - \mu \geq 2 - \varepsilon.$$

Due to

$$S'(\Phi_{z^*}, \mu) \subset S'(\Psi_{y^*}, \varepsilon),$$

we also have $x \in S'(\Psi_{y^*}, \varepsilon)$, which completes the proof. $\square$

Remark 4.9. In Proposition 4.8, the assumption $\mathcal{S}'_\Psi < \mathcal{S}'_\Phi$ cannot be removed. This may be shown by using the functions from Remark 3.9.

Theorem 4.10. *Let X and Y be Banach spaces, and let $\Phi, \Upsilon, \Psi \in \ell_\infty(B_X, Y)$ with $\|\Phi\| = \|\Upsilon\| = \|\Psi\| = 1$. Assume that Y has the alternative Φ-Daugavet property with respect to $Y^*\Upsilon \cdot Y$. Then, if $\mathcal{S}'_\Psi < \mathcal{S}'_\Upsilon$ and Ψ is weakly compact,*

$$\max_{|\omega|=1} \|\Phi + \omega\Psi\| = 2.$$

Proof. Denote $K = \overline{\mathrm{co}}(\mathbb{T}\Psi(B_X))$, and let $\varepsilon > 0$ be given. In the same manner as in the proof of Theorem 3.11, we may find $y_0 \in K$ with $\|y_0\| > 1-\varepsilon$, $\delta \in (0, \varepsilon/2)$ and $y_0^* \in Y^*$ such that

$$y \in K,\ \operatorname{Re} y_0^*(y) \geq 1-\delta \Longrightarrow \|y - y_0\| < \varepsilon$$

and

$$\sup_{y\in K} |y_0^*(y)| = 1.$$

Setting $\psi := y_0^* \circ \Psi$, we obtain

$$\|\psi\| = \sup_{x\in B_X} |y_0^*(\Psi(x))| = \sup_{y\in K} |y_0^*(y)| = 1,$$

i.e., $S'(\psi,\delta) \in \mathcal{S}'_\Psi$. From $\mathcal{S}'_\Psi < \mathcal{S}'_\Upsilon$, we deduce the existence of $\mu \leq \delta$ as well as $S'(\Upsilon_{z^*},\mu) \in \mathcal{S}'_\Upsilon$, satisfying

$$S'(\Upsilon_{z^*},\mu) \subset S'(\psi,\delta).$$

Since Y has the alternative Φ-Daugavet property with respect to $Y^*\Upsilon \cdot Y$, we can use Theorem 2.9 to obtain $\omega_1 \in \mathbb{T}$ and $x \in S'(\Upsilon_{z^*},\mu)$ such that

$$\left\|\omega_1\Phi(x) + \frac{y_0}{\|y_0\|}\right\| \geq 2-\mu \geq 2-\varepsilon,$$

in particular, $x \in S'(\Upsilon_{z^*},\mu) \subset S'(\psi,\delta)$. Writing $\omega_2 = |\psi(x)|/\psi(x)$, we observe

$$\operatorname{Re} y_0^*(\omega_2\Psi(x)) = \operatorname{Re}\omega_2\psi(x) = |\psi(x)| \geq 1-\delta;$$

thus, the fact that $\omega_2\Psi(x) \in K$ gives

$$\|\omega_2\Psi(x) - y_0\| < \varepsilon.$$

On the other hand,

$$\begin{aligned}\|y_0 + \omega_1\Phi(x)\| &\geq \left\|\omega_1\Phi(x) + \frac{y_0}{\|y_0\|}\right\| - \left\|y_0 - \frac{y_0}{\|y_0\|}\right\| \\ &= \left\|\omega_1\Phi(x) + \frac{y_0}{\|y_0\|}\right\| - |\|y_0\| - 1| \\ &\geq 2-2\varepsilon.\end{aligned}$$

From the above, we conclude

$$\begin{aligned}\max_{|\omega|=1}\|\Phi+\omega\Psi\| &\geq \|\Phi+\overline{\omega_1}\omega_2\Psi\|\\ &\geq \|\Phi(x)+\overline{\omega_1}\omega_2\Psi(x)\|\\ &= \|\omega_1\Phi(x)+\omega_2\Psi(x)\|\\ &\geq \|\omega_1\Phi(x)+y_0\|-\|\omega_2\Psi(x)-y_0\|\\ &\geq 2-3\varepsilon,\end{aligned}$$

which proves the assertion since $\varepsilon>0$ was arbitrarily chosen. □

Corollary 4.11. *Let X and Y be Banach spaces, and assume that Y has the alternative Daugavet property. Consider $\Phi,\Psi\in\ell_\infty(B_X,Y)$ such that Φ is a quotient map and $\|\Psi\|=1$. If $\mathcal{S}'_\Psi<\mathcal{S}'_\Phi$ and Ψ is weakly compact, then*

$$\max_{|\omega|=1}\|\Phi+\omega\Psi\|=2.$$

Proof. The space Y has the alternative Φ-Daugavet property with respect to $Y^*\Phi\cdot Y$, by Lemma 4.7. Therefore, Ψ satisfies the alternative Φ-Daugavet equation according to Theorem 4.10. □

Remark 4.12. In Corollary 4.11, assumption $\mathcal{S}'_\Psi<\mathcal{S}'_\Phi$ cannot be dropped. For instance, this follows with the aid of the functions constructed in Remark 3.14.

Theorem 4.13. *Let X and Y be Banach spaces, and let $\mathcal{Z}$ be a linear subspace of $\ell_\infty(B_X)$. Assume that $\Phi\in\ell_\infty(B_X,Y)$ with $\|\Phi\|=1$. Then, the following are equivalent*:

(1) *for every $x'\in\mathcal{Z}$ and $y\in Y$, $x'\otimes y$ satisfies* (Φ-ADE).

(2) *For every $x'\in S_{\mathcal{Z}}$, $y\in S_Y$ and $\varepsilon>0$, there exist $\omega_1,\omega_2\in\mathbb{T}$ and $y\in B_X$ such that*

$$\operatorname{Re}\omega_1x'(x)\geq 1-\varepsilon \quad \textit{and} \quad \|\omega_2\Phi(x)+y\|\geq 2-\varepsilon.$$

(3) *For every $x'\in S_{\mathcal{Z}}$, $y\in S_Y$ and $\varepsilon>0$, there exist $\omega\in\mathbb{T}$ and $x\in B_X$ such that*

$$|x'(x)|\geq 1-\varepsilon \quad \textit{and} \quad \|\omega\Phi(x)+y\|\geq 2-\varepsilon.$$

(4) *Every weakly compact* $\Psi \in \ell_\infty(B_X, Y)$ *such that* $y^* \circ \Psi \in \mathcal{Z}$ *for all* $y^* \in Y^*$ *satisfies* (Φ-ADE).

Proof. The equivalence of (1), (2) and (3) is a consequence of Theorem 2.9. The implication (1) $\Rightarrow$ (4) follows from Theorem 4.10 since, trivially, $\mathcal{S}'_\Psi < \mathcal{S}'_\Psi$ for any Ψ as in (4). The direction (4) $\Rightarrow$ (1) holds since finite-rank maps are weakly compact. $\square$

The next corollary is analogous to Corollary 3.16.

Corollary 4.14. *Let* $X_1, \dots, X_n$ *and* Y *be Banach spaces, and consider a continuous multilinear map*

$$B_0 : X_1 \times \cdots \times X_n \longrightarrow Y$$

satisfying

$$B_0(U_{X_1 \times \cdots \times X_n}) = U_Y.$$

Consider the subsets R, C *and* WC *of* $L(Y, Y)$ *of rank* 1, *compact and weakly compact operators. Denote*

$$R \circ B_0 = \{T \circ B_0 : T \in R\},$$
$$C \circ B_0 = \{T \circ B_0 : T \in C\}$$

and

$$WC \circ B_0 = \{T \circ B_0 : T \in WC\}.$$

Then the following are equivalent:

(1) Y *has the alternative Daugavet property.*
(2) Y *has the alternative* B_0*-Daugavet property with respect to* $R \circ B_0$.
(3) Y *has the alternative* B_0*-Daugavet property with respect to* $C \circ B_0$.
(4) Y *has the alternative* B_0*-Daugavet property with respect to* $WC \circ B_0$.

Proof. The equivalence of (1) and (2) is due to Lemma 4.7. Items (2) and (4) are equivalent by letting $\mathcal{Z} = \{y^* \circ B_0 : y^* \in Y^*\}$ in Theorem 4.13. The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) follow from the inclusions $R \subset C \subset WC$. $\square$

5. Local Φ-Daugavet type properties and applications. From our previous main results, we are ready to present more technical versions of the tools obtained there, which can be proved using the same arguments and are useful for applications. Essentially, we introduce the notion of a norm determining set $\Gamma \subset B_X$ for a class of functions and some new elements that allow the definition of the notion of the Φ-Daugavet property with respect to particular sets of scalar functions and vectors in Y, using a norm that can be defined as the supremum of the evaluation of the functions involved merely for some subset of vectors in S_X.

In this section, all Banach spaces are supposed to be $\mathbb{R}$-vector spaces for simplicity of notation.

Let X and Y be Banach spaces, and let $V \subset \ell_\infty(B_X, Y)$. We say that a subset $\Gamma \subset B_X$ is *norm determining* for V, if

$$\|\Psi\| = \|\Psi\|_\Gamma := \sup_{x \in \Gamma} \|\Psi(x)\|$$

for all $\Psi \in V$.

We begin by formulating a version of Theorem 2.5 which considers norm determining subsets for the functions involved. Its proof follows the same lines as that of Theorem 2.5; thus, we omit it.

Proposition 5.1. *Let X and Y be Banach spaces, and let $\Gamma \subset B_X$. Let $\Phi \in \ell_\infty(B_X, Y)$ be a norm 1 map, and consider a norm 1 function $x' \in \ell_\infty(B_X)$. Let $y \in S_Y$. The following assertions are equivalent.*

(1) $\|\Phi + x' \otimes y\|_\Gamma = 2$.
(2) *For every $\varepsilon > 0$, there is an $\omega \in \mathbb{T}$ and an element*

$$x \in S(\omega x', \varepsilon) \cap \Gamma$$

such that

$$\|\omega\Phi(x) + y\| \geq 2 - 2\varepsilon.$$

Remark 5.2. Note that the condition in the above result implies that, for a norm 1 scalar function $x' \in \ell_\infty(B_X, Y)$,

$$2 \leq \|x' \otimes y + \Phi\|_\Gamma \leq \|x'\|_\Gamma \|y\| + \|\Phi\| \leq \|x'\|_\Gamma + 1,$$

and thus, $\|x'\|_\Gamma = \|x'\| = 1$. Therefore, Γ is norm determining for x'; the same argument gives that it is so for Φ.

We now define a type of "local version" of the notion of Φ-Daugavet property.

Definition 5.3. Let X and Y be Banach spaces, and let

$$\Phi : B_X \longrightarrow Y$$

be a norm 1 function. Let $\Gamma \subset B_X$ be a norm determining set for Φ, and consider subsets $\mathcal{W} \subset \ell_\infty(B_X)$ and $\Delta \subset S_Y$. Based on this, Y has the Φ-Daugavet property with respect to $(\Gamma, \mathcal{W}, \Delta)$ if, for every $x' \in \mathcal{W}$ and $y \in \Delta$,

$$\sup_{x \in \Gamma} \|\Phi(x) + x'(x)y\| = 2.$$

The reader may note that this definition is related to that of the Daugavet center provided in [**4,** Definition 1.2] and that of the almost Daugavet property from [**9**].

We provide a concrete example of a function Φ and sets Γ, $\mathcal{W}$ and Δ for which every Banach space has the Φ-Daugavet property with respect to $(\Gamma, \mathcal{W}, \Delta)$.

Example 5.4. Let X be a real Banach space, and take $Y = X$. Consider the sets $\Gamma = B_X$,

$$\mathcal{W} = \{x' \in \ell_\infty(B_X) : |x'(x)| = 1 \text{ and } x'(x) = x'(-x) \text{ for all } x \in S_X\},$$

and $\Delta = S_X$. Let

$$\Phi : B_X \longrightarrow X$$

be a norm 1 function such that

$$\Phi(S_X) = S_X \quad \text{and} \quad \Phi(-x) = -\Phi(x) \quad \text{for all } x.$$

Take $\varepsilon > 0$. Fix a norm 1 function $x' \in \mathcal{W}$. If $y \in S_X$, take $x_0 \in S_X$ such that $\Phi(x_0) = y$. If $x'(x_0) = 1$, then

$$\sup_{x \in \Gamma} \|\Phi(x) + x'(x)y\| \geq \|\Phi(x_0) + x'(x_0)y\| \geq 2\|y\| = 2.$$

If $x'(x_0) = -1 = x'(-x_0)$, then $\Phi(-x_0) = -\Phi(x_0) = -y$, and thus,

$$\sup_{x \in \Gamma} \|\Phi(x) + x'(x)y\| \geq \|\Phi(-x_0) + x'(-x_0)y\| \geq \|-y - y\| = 2.$$

Therefore, X has the Φ-Daugavet property with respect to $(\Gamma, \mathcal{W}, \Delta)$.

The space ℓ^∞ and the function $\Phi(x) = x^3$ show an example of this situation, although ℓ^∞ does not have the Daugavet property.

The proof of the next result is a direct application of Proposition 5.1.

Corollary 5.5. *Let X and Y be Banach spaces, and consider Φ, Γ, $\mathcal{W}$ and Δ as in Definition 5.3. The following statements are equivalent.*

(1) *Y has the Φ-Daugavet property with respect to $(\Gamma, \mathcal{W}, \Delta)$.*
(2) *For every $y \in \Delta$, for every $x' \in \mathcal{W}$ of norm 1 and for every $\varepsilon > 0$, there are $\omega \in \mathbb{T}$ and an element $x \in S(\omega x', \varepsilon) \cap \Gamma$ such that*

$$\|\omega\Phi(x) + y\| \geq 2 - 2\varepsilon.$$

Remark 5.6. We show that, under the assumption that the function Φ maps B_X onto B_Y, the Daugavet property for Y implies the Φ-Daugavet property with respect to $\Gamma = B_X$,

$$\mathcal{W} = \{x' : X \longrightarrow \mathbb{R} : x' = y^* \circ \Phi,\ y^* \in S_{Y^*}\}$$

and $\Delta = S_Y$. This case is canonical, and in a sense also trivial, since the result is a consequence of simple computation. However, there are more examples which show that not all of the cases can be obtained in this way, i.e., there are families of functions $\mathcal{W}$ whose elements are not compositions of a given Φ and the functionals of S_{Y^*} for which Φ satisfies the Daugavet equation.

(1) First, we show the previous statement. Let Y be a Banach space with the Daugavet property, and let

$$\Phi : B_X \longrightarrow Y$$

satisfy $\Phi(B_X) = B_Y$. We then show that Y has the Φ-Daugavet property with respect to $(B_X, \mathcal{W}, S_Y)$, where

$$\mathcal{W} = \{x' : X \longrightarrow \mathbb{R} : x' = y^* \circ \Phi,\ \ y^* \in S_{Y^*}\}.$$

In order to see this, suppose that $\Phi : B_X \to Y$ satisfies $\Phi(B_X) = B_Y$. Then, we claim that, for each $\varepsilon > 0$, $y^* \in S_{Y^*}$ and $y \in S_Y$, there is an $x \in S(y^* \circ \Phi, \varepsilon)$ such that

$$\|\Phi(x) + y\| \geq 2 - 2\varepsilon.$$

Indeed, let $\varepsilon > 0$, $y \in S_Y$ and $y^* \in S_{Y^*}$. Then, by the Daugavet property for Y, there is an element $z \in S(y^*, \varepsilon)$ such that $\|z + y\| \geq 2 - 2\varepsilon$. Since Φ maps B_X onto B_Y, we find $x \in B_X$ such that $\Phi(x) = z \in S(y^*, \varepsilon)$, and thus,

$$\langle \Phi(x), y^* \rangle = y^* \circ \Phi(x) > 1 - \varepsilon \quad \text{and} \quad \|\Phi(x) + y\| \geq 2 - 2\varepsilon.$$

Application of Corollary 5.5 gives the result.

(2) There are other families of functions $\mathcal{W}$ for which the Daugavet equation is satisfied with a function Φ; however, they cannot be defined by composition as in (1). For example, take $X = Y = C(K)$, where K is a perfect compact Hausdorff space, and define $\mathcal{W}$ as the set of continuous linear functionals on $C(K)$. Consider the function

$$x \longmapsto \Phi(x) = x^3.$$

Clearly, a linear functional cannot be written as a composition of Φ and some other linear functional. However, for each norm 1 element $y \in S_{C(K)}$, we find an element $x \in S_{C(K)}$ such that $x^3 = y$. This, together with the Daugavet property of $C(K)$, implies (2) in Corollary 5.5. In order to see this, simply take into account that, by the Daugavet property of $C(K)$, for each $\varepsilon > 0$, $y \in S_{C(K)}$ and $y^* \in S_{C(K)^*}$, there is an $x \in S(y^*, \varepsilon/2)$ such that

$$\|x + y\| > 2 - 2(\varepsilon/2) = 2 - \varepsilon > 2 - 2\varepsilon.$$

Take $z \in S_{C(K)}$ such that $z^3 = x$, and thus, $\|z^3 + y\| > 2 - 2\varepsilon$. We show that $z \in S(y^*, \varepsilon)$ as well, that is, Corollary 5.5 (2) holds. Consider the measurable sets defined by setting

$$A^+ := \{w \in K : z(w) \geq 0\} \quad \text{and} \quad A^- := \{w \in K : z(w) < 0\}.$$

Take the decomposition of the measure μ which defines the functional y^* as a difference of positive disjointly supported measures $\mu = \mu^+ - \mu^-$. Then, using $|z^3| \leq |z|$, we obtain

$$\begin{aligned} 1 - \varepsilon/2 &\leq \int_K z^3 \, d\mu \\ &= \int_{A^+} |z^3| \, d\mu^+ + \int_{A^-} |z^3| \, d\mu^- \\ &\quad - \int_{A^+} |z^3| \, d\mu^- - \int_{A^-} |z^3| \, d\mu^+ \end{aligned}$$

$$\begin{aligned} &\leq \int_{A^+} |z|\, d\mu^+ + \int_{A^-} |z|\, d\mu^- \\ &\leq \mu^+(A^+) + \mu^-(A^-) \leq 1. \end{aligned}$$

Hence,

$$\mu^+(A^-) + \mu^-(A^+) \leq \varepsilon/2.$$

Consequently,

$$\begin{aligned} 1 &\geq \int_K z\, d\mu \\ &= \int_{A^+} |z|\, d\mu^+ + \int_{A^-} |z|\, d\mu^- \\ &\quad - \int_{A^+} |z|\, d\mu^- - \int_{A^-} |z|\, d\mu^+ \\ &\geq (1-\varepsilon/2) - (\mu^-(A^+) + \mu^+(A^-)) \\ &\geq 1 - 2(\varepsilon/2) = 1 - \varepsilon. \end{aligned}$$

Then, $z \in S(y^*, \varepsilon)$, and the result follows.

(3) Sometimes, surjectivity of Φ is unnecessary if the sets Γ, $\mathcal{W}$ and Δ are adequately chosen. Now, take $X = Y = C(K)$, $\Phi(x) = |x|^{1/4}$ and $\mathcal{W}$ the set of probability measures $\mathcal{P}(K) \subset C(K)^*$. Also, take $\Gamma = B_{C(K)^+}$ and $\Delta = S_{C(K)^+}$. Then, the Φ-Daugavet property with respect to $(\Gamma, \mathcal{W}, \Delta)$ is satisfied as a consequence of Corollary 5.5. In order to see this, note that, if $y \in S_{C(K)^+}$ and $\mu \in \mathcal{P}(K)$, then, for $\omega = 1$, we obtain by the Daugavet property of $C(K)$, given $\varepsilon > 0$, a (positive) function x of norm 1 in $S_{C(K)}$ such that

$$\int_K x\, d\mu \geq 1 - \varepsilon \quad \text{and} \quad \|x + y\| \geq 2 - 2\varepsilon.$$

Then, since $1 \geq x^{1/4} \geq x$, we obtain

$$\|x^{1/4} + y\| \geq \|x + y\| \geq 2 - 2\varepsilon,$$

i.e., the Φ-Daugavet property with respect to $(\Gamma, \mathcal{W}, \Delta)$ is satisfied. Again, the elements of $\mathcal{P}(K)$ cannot be factored through Φ.

The following result gives the main tool for extending the Daugavet equation to other functions not belonging to the set of products of scalar functions of $\mathcal{W}$ and elements of the unit sphere of Y. In particular, well-known arguments provide the condition of Theorem 5.7 concerning the inclusion of the image of a slice in a small ball, for the big class of the strong Radon-Nikodým operators. Notably, this class contains weakly compact operators (see, for example, the first part of [**10**], or [**6,** Theorem 1.1] for a version directly related to the present paper).

Theorem 5.7. *Let $\Psi : B_X \to Y$ be a norm 1 function. If the Banach space Y has the Φ-Daugavet property with respect to $(\Gamma, \mathcal{W}, \Delta)$ for $\mathcal{W} \subset \ell_\infty(B_X)$, and for all $\varepsilon > 0$ there are $x' \in \mathcal{W}$, $\delta > 0$ and $y \in \Delta$ such that, for all $\omega \in \mathbb{T}$, $\Psi(S(\omega x', \delta) \cap \Gamma) \subset B_\varepsilon(\overline{\omega} y)$, then*

$$\|\Phi + \Psi\|_\Gamma = 2.$$

Proof. Fix $\varepsilon > 0$. By the hypothesis, there are $x'_0 \in \mathcal{W}$ and $y \in \Delta$ such that, for every $\omega \in \mathbb{T}$, $\|\Psi(x) - \overline{\omega} y\| < \varepsilon$ for all $x \in S(\omega x'_0, \delta) \cap \Gamma$.

By Corollary 5.5, for x'_0 and y, there are $\omega_0 \in \mathbb{T}$ and $x_0 \in S(\omega_0 x'_0, \delta) \cap \Gamma$ such that $\|\omega_0 \Phi(x_0) + y\| \geq 2 - 2\varepsilon$. Then,

$$\begin{aligned} \|\Phi + \Psi\|_\Gamma &\geq \|\Phi(x_0) + \Psi(x_0)\| \\ &\geq \|\Phi(x_0) + \overline{\omega_0} y\| - \|\Psi(x_0) - \overline{\omega_0} y\| \\ &= \|\omega_0 \Phi(x_0) + y\| - \|\Psi(x_0) - \overline{\omega_0} y\| \\ &\geq 2 - 2\varepsilon - \varepsilon = 2 - 3\varepsilon. \end{aligned}$$

Since this occurs for each $\varepsilon > 0$, we obtain that $\|\Phi + \Psi\|_\Gamma = 2$. □

The proof of the next corollary is merely an application of Theorem 5.7 for $\mathcal{W} := \{x' = y^* \circ \Psi : y^* \in Y^*, \ \|y^* \circ \Psi\| = 1\}$, $\Gamma = B_X$ and $\Delta = S_X$, together with the argument in the proof of Theorem 3.11 regarding weakly compact sets that gives the condition for applying Theorem 5.7. The same comments regarding Radon-Nikodým functions given in Remark 3.12 apply in the present case.

Corollary 5.8. *Let $\Phi : B_X \to Y$ be a norm 1 function such that $\Phi(B_X) = B_Y$, and let*

$$\Psi : B_X \longrightarrow Y$$

be a norm 1 weakly compact function. Suppose that Y has the Φ-Daugavet property. Then, $\|\Phi + \Psi\| = 2$.

We conclude by giving some special new tools for obtaining applications in the case of $C(K)$- and $L^1(\mu)$-spaces.

5.1. A general test for the Φ-Daugavet property: The case of functions on $C(K)$-spaces. The requirement

$$\Psi(S(\omega x', \delta) \cap \Gamma) \subset B_\varepsilon(\overline{\omega} y)$$

in Theorem 5.7 seems to be a difficult property to directly verify. The next result provides a simpler test which can be used in some cases. We use this new tool to analyze the Daugavet equation for functions on $C(K)$-spaces.

Proposition 5.9. *Let X be a Banach space. Let $z \in S_X$, $K > 0$, and let $\Phi, \Psi : B_X \to X$ be norm 1 functions. Take a subset $B \subset B_X$. The following statements are equivalent.*

(1) *There is a w^*-compact convex set $V \subset X^*$ such that, for all finite sequences $x_1, \dots, x_n \in B$ and positive scalars $\alpha_1, \dots, \alpha_n$ such that*

$$\sum_{i=1}^{n} \alpha_i = 1,$$

we have

$$\sum_{i=1}^{n} \alpha_i \|\Psi(x_i) - z\| \leq K \sup_{x^* \in V} \left(1 - \left\langle \sum_{i=1}^{n} \alpha_i \Phi(x_i), x^* \right\rangle\right).$$

(2) *For each $\varepsilon > 0$, there exists an $x_0^* \in V$ such that*

$$\|\Psi(x) - z\| \leq K(1 - \langle \Phi(x), x_0^* \rangle) \quad \textit{for all } x \in B_X.$$

These equivalent properties imply that, for each $\varepsilon > 0$, there exists an $x_0^* \in V$ such that $\Psi(S(x_0^* \circ \Phi, \varepsilon) \cap B) \subset B_{K\varepsilon}(z)$.

Proof. We obtain this result as a consequence of Ky Fan's lemma (see [**13,** page 40]); thus, it is, in essence, a consequence of the Hahn-Banach theorem.

Here, we only sketch the proof. Consider the concave set of convex functions

$$\Upsilon : V \longrightarrow \mathbb{R},$$

defined by

$$\Upsilon(x^*) := \sum_{i=1}^n \alpha_i \|\Psi(x_i) - z\| - K\bigg(1 - \bigg\langle \sum_{i=1}^n \alpha_i \Phi(x_i), x^* \bigg\rangle\bigg),$$

where $\alpha_i > 0$,

$$\sum_{i=1}^n \alpha_i = 1$$

and $x_1, \dots, x_n \in B$. The inequality in (1) provides for Υ an element $x^*_\Upsilon \in V$ such that $\Upsilon(x^*_\Upsilon) \le 0$. Ky Fan's lemma gives an element $x^*_0 \in V$ such that $\Upsilon(x^*_0) \le 0$ for all functions Υ in the family. This proves (1) $\Rightarrow$ (2), and the converse is obvious.

On the other hand, if $x \in S(x^*_0 \circ \Phi, \varepsilon) \cap B$, then

$$\|R(x) - z\| \le K(1 - \langle \Phi(x), x^*_0 \rangle) \le K\varepsilon.$$

This proves the final statement. $\square$

Example 5.10. Now, we give an application of the criterion provided in Proposition 5.9. Let

$$X = C(K) \quad \text{and} \quad V = B_{C(K)^*}.$$

Take a positive norm 1 function f in $C(K)$. Define the class of functions C by

$$C = \{g \in B_{C(K)} : g^2 \le f \le |g|\}.$$

We see that the requirements of Proposition 5.9 are satisfied for $B = C$, and Φ and Ψ are defined by $\Phi(g) = g^2$ and $\Psi(g) = |g|$. Note that, for all positive functions $h \in B_{C(K)}$, $\mathbf{1} - h \le \mathbf{1} - h^2$. Then, for all $g_1, \dots, g_n \in C$ and positive $\alpha_1, \dots, \alpha_n$ such that

$$\sum_{i=1}^n \alpha_i = 1,$$

we obtain

$$\begin{aligned}\sum_{i=1}^n \alpha_i \||g_i| - \mathbf{1}\| &\le \sum_{i=1}^n \alpha_i \|\mathbf{1} - f\| = \|\mathbf{1} - f\| \\ &\le \sum_{i=1}^n \alpha_i \|\mathbf{1} - g_i^2\|\end{aligned}$$

$$\leq \sup_{x^* \in B_{C(K)^*}} \left(1 - \left\langle \sum_{i=1}^{n} \alpha_i g_i^2, x^* \right\rangle\right).$$

Consequently, an application of Proposition 5.9 shows that, for each $\varepsilon > 0$, there exists an $x_0^* \in C(K)^*$ such that

$$\Psi(S(x_0^* \circ \Phi, \varepsilon) \cap C) \subset B_{K\varepsilon}(\mathbf{1}).$$

Note that, in order to be able to apply Proposition 5.9 in a nontrivial way, it must be assumed that $S(x^* \circ \Phi, \varepsilon) \cap B \neq \emptyset$. For example, in the next corollary, the requirement is satisfied since $B = B_X$. Note also that the requirement on Φ of being surjective from B_X to B_X ensures that the slices $S(x^* \circ \Phi, \varepsilon)$ are not themselves empty.

Corollary 5.11. *Let $\Phi, \Psi : B_X \to X$ be norm 1 functions. Assume that there exist $z \in S_X$ and $K > 0$ such that, for all $x_1, \dots, x_n \in B_X$ and $\alpha_1, \dots, \alpha_n \geq 0$ satisfying*

$$\sum_{i=1}^{n} \alpha_i = 1,$$

there is an element $x \in B_X$ such that the inequality

$$\sum_{i=1}^{n} \alpha_i \|\Psi(x_i) - z\| \leq K \left\| x - \sum_{i=1}^{n} \alpha_i \Phi(x_i) \right\|$$

holds. Then, for each $\varepsilon > 0$, there exist $\delta > 0$ and $x_0^* \in S_{X^*}$ such that

$$\Psi(S(x_0^* \circ \Phi, \delta)) \subset B_{K\varepsilon}(z).$$

Proof. Fix $x_1, \dots, x_n \in X$ and $\alpha_1, \dots, \alpha_n$, and consider the element $x \in B_X$ given in the statement. Using the inequality, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i \|\Psi(x_i) - z\| &\leq K \sup_{x^* \in B_{X^*}} \left\langle x - \sum_{i=1}^{n} \alpha_i \Phi(x_i), x^* \right\rangle \\ &\leq K \sup_{x^* \in B_{X^*}} \left(1 - \left\langle \sum_{i=1}^{n} \alpha_i \Phi(x_i), x^* \right\rangle\right). \end{aligned}$$

Application of Proposition 5.9 gives the result. $\square$

Example 5.12. Take $X = C(K)$ for a perfect K, $\Phi(x) = x^2$ and

$$\Psi(x) = \Big(\int_K x^2\, d\mu\Big) y$$

for a probability measure on K and a fixed function $y \in S_{C(K)}$. Then, taking $z = y$, we obtain

$$\begin{aligned}\sum_{i=1}^n \alpha_i \Big\| \Big(\int_K x_i^2\, d\mu\Big) y - z \Big\| &\leq \sum_{i=1}^n \alpha_i \Big(1 - \int_K x_i^2\, d\mu\Big) \|z\| \\ &= \int_K d\mu - \sum_{i=1}^n \alpha_i \int_K x_i^2\, d\mu \\ &\leq \Big\| \mathbf{1} - \sum_{i=1}^n \alpha_i x_i^2 \Big\|\end{aligned}$$

for each finite set of functions $x_1, \ldots, x_n \in B_{C(K)}$ and $0 \leq \alpha_1, \ldots, \alpha_n$ such that

$$\sum_{i=1}^n \alpha_i = 1.$$

Consequently, the result holds and, for each $\varepsilon > 0$, there is a slice $S(x_0^* \circ \Phi, \delta)$ such that

$$\Psi(S(x_0^* \circ \Phi, \delta)) \subset B_\varepsilon(z).$$

However, observe that the slices $S(x_0^* \circ \Phi, \delta)$ can be empty in this case, and thus, the Daugavet equation cannot be assured in general by applying Remark 5.6 (1). In fact, the equation does not hold when using, for example, $y = -\mathbf{1}$; in this case,

$$\sup_{x \in B_{C(K)}} \Big\| x^2 + \Big(\int_K x^2\, d\mu\Big)(-\mathbf{1}) \Big\| \leq 1.$$

However, if we take $y = \mathbf{1}$, we obtain

$$\sup_{x \in B_{C(K)}} \|x^2 + \Big(\int_K x^2\, d\mu\Big)\mathbf{1}\| = 2,$$

and the Daugavet equation holds.

Note that Remark 5.6 (1) provides the Daugavet equation for the "order 3 version" of this result since $\Phi(x) = x^3$ satisfies $\Phi(B_{C(K)}) =$

$B_{C(K)}$. Therefore, due to the Daugavet property of $C(K)$, for every $\mu \in S_{C(K)^*}$ and $y \in S_{C(K)}$, we have

$$\sup_{x \in B_{C(K)}} \left\| x^3 + \left(\int_K x^3 \, d\mu \right) y \right\| = 2.$$

5.2. The case of $L^1(\mu)$-spaces for non-atomic measures μ. In this subsection, we analyze several functions Φ which are natural candidates for being functions Φ on (the unit ball of) L^1 in the results previously discussed.

Some cases that are, in a sense, canonical for application of our results are the following. The first is given by the function $\Phi_0(f) := |f|$, $f \in L^1(\mu)$. The second case is the function

$$\Phi_* := B_{L^1[0,1]} \longrightarrow B_{L^1[0,1]},$$

given by the expression $\Phi_*(f) = |f| * |f|$, where $*$ denotes the convolution in $L^1[0,1]$. The third is given by the formula

$$\Phi_2(f) := \left(\int_\Omega |f| \, d\mu \right) \cdot f.$$

Adapting the proofs of [**12,** Theorem 2.6 and Proposition 2.7] which are based on some classical arguments for the Daugavet property in $L^1(\mu)$, we obtain the following results which may be applied to these examples.

Lemma 5.13. *Let (Ω, Σ, μ) be a non-atomic measure space. Let $\mathcal{W}$ be a set of norm 1 scalar functions in $\ell_\infty(B_{L^1(\mu)})$. Let*

$$\Phi : B_{L^1(\mu)} \longrightarrow L^1(\mu)$$

be a norm 1 function such that $\|\Phi(z)\| = 1$ for each $z \in S_{L^1(\mu)}$ and also satisfying that, for each δ, $\varepsilon > 0$ and $x' \in \mathcal{W}$, a norm 1 simple function z may be found such that $\mu(\operatorname{supp} \Phi(z)) < \delta$ and $|x'(\Phi(z))| > 1 - \varepsilon$. Then,

$$\|\Phi + x' \otimes y\| = 2$$

for all $x' \in \mathcal{W}$, $y \in S_{L^1(\mu)}$.

Proof. Using Proposition 5.1, let $\varepsilon > 0$, $x' \in \mathcal{W}$ and $y \in S_{L^1(\mu)}$. We show that we can find ω and an element $x \in S(\omega x', \varepsilon)$ such that

$$\|\omega\Phi(x) + y\| > 2 - 2\varepsilon.$$

First, note that there exists a $\delta > 0$ such that

$$\int_A |y|\, d\mu < \varepsilon \quad \text{for each } A \in \Sigma,\ \mu(A) < \delta.$$

By the requirement on Φ for $\delta > 0$ and $\varepsilon > 0$, and choosing $\omega \in \mathbb{T}$ such that $\omega x'(z) = |x'(z)|$, we have $z \in S(\omega x', \varepsilon)$. Thus, we obtain

$$\begin{aligned} \|y + \omega\Phi(z)\| &= \int_{\Omega\setminus \text{supp } \Phi(z)} |y|\, d\mu + \int_{\text{supp } \Phi(z)} |y + \omega\Phi(z)|\, d\mu \\ &\geq \|y\| - \int_{\text{supp } \Phi(z)} |y|\, d\mu + \|\Phi(z)\| - \int_{\text{supp } \Phi(z)} |y|\, d\mu \\ &> 2 - 2\varepsilon. \end{aligned}$$

Proposition 5.1 gives the result. $\square$

Lemma 5.14. *Let (Ω, Σ, μ) be a non-atomic measure space. Let $\mathcal{W}$ be a set of norm 1 scalar functions from $L^1(\mu)$ that are weakly sequentially continuous. Let*

$$\Phi : B_{L^1(\mu)} \longrightarrow L^1(\mu)$$

be a norm 1 map which maps $S_{L^1(\mu)}$ onto $S_{L^1(\mu)}$. Then,

$$\|\Phi + x' \otimes y\| = 2$$

for all $x' \in \mathcal{W}$, $y \in S_{L^1(\mu)}$.

Proof. Let $x' \in \mathcal{W}$ and δ, $\varepsilon > 0$. Since it is weakly sequentially continuous, by [**12,** Lemma 2.5], we can find a norm 1 simple function x such that $\mu(\text{supp } x) < \delta$ and $|x'(\Phi(x))| > 1 - \varepsilon$. The surjectivity of Φ provides an element $z \in S_{L^1(\mu)}$ such that $\Phi(z) = x$. The requirement of z for Φ in Lemma 5.13 has been obtained; hence, the result holds. $\square$

In order to adapt the results on weak sequential continuity, shown to be useful in the case of the polynomial Daugavet property for $L^1(\mu)$, see [**12**], there are two requirements on Φ which are useful and are included as follows.

In the next proposition, we call a function

$$\Phi : B_{L^1(\mu)} \longrightarrow L^1(\mu)$$

admissible if the following requirements are satisfied:

(i) Φ must send functions of small support to functions of small support, i.e., for each $\delta > 0$, there is a $\delta' > 0$ such that, for a function $f \in L^1(\mu)$ with support satisfying $\mu(\text{supp } f) < \delta'$, we have that $\mu(\text{supp}_{\Phi(f)}) < \delta$.

(ii) For all $f \in S_{L^1(\mu)}$, $\|\Phi(f)\| = 1$.

Note that the mappings Φ_0, Φ_* and Φ_2 mentioned at the beginning of this subsection are admissible.

Proposition 5.15. *Let (Ω, Σ, μ) be a non-atomic measure space. Let*

$$\Phi : B_{L^1(\mu)} \longrightarrow L^1(\mu)$$

be a norm 1 admissible function. Let

$$\mathcal{W} \subset \ell_\infty(B_{L^1(\mu)})$$

be a set of norm 1 scalar functions from $B_{L^1(\mu)}$ to $\mathbb{K}$ such that $x' \circ \Phi$ is norm 1 and weakly sequentially continuous for each $x' \in \mathcal{W}$. Then,

$$\|\Phi + x' \otimes y\| = 2 \quad \textit{for all } x' \in \mathcal{W},\ y \in S_{L^1(\mu)}.$$

Proof. Using Lemma 5.13, let ε, $\delta > 0$ and $p \in \mathcal{W}$. Note that, since Φ is admissible, there is a $\delta' > 0$ such that, if $f \in L^1(\mu)$ and $\mu(\text{supp } f) < \delta'$, we have that $\mu(\text{supp } \Phi(f)) < \delta$.

Since $x' \circ \Phi$ is weakly sequentially continuous, by **[12,** Lemma 2.5], we can find a norm 1 simple function z such that

$$\mu(\text{supp } z) < \delta' \quad \text{and} \quad |x'(\Phi(z))| > 1 - \varepsilon.$$

Finally, note that, we also have

$$\mu(\text{supp } \Phi(z)) < \delta,$$

by the admissibility of Φ. Lemma 5.13 gives the result. □

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