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Calculation of movements in an isostatic plane-framed structure with the Principle of Virtual Forces.

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1 Summary of key ideas

In this article we will explain the procedure to be followed to obtain a movement at a specific point of an isostatic plane-framed structure (displacement or rotation) with the Principle of Virtual Forces. We will illustrate it with an example.

2 Introduction

Sometimes, in structural analysis, we need to obtain the value of a movement at a specific point of the structure. Different methods can be used, but one of the most common is the Principle of Virtual Forces (PVF).

This method can be used either if the structure is isostatic (statically determinate) or hyperstatic (statically indeterminate), although, in the second case, the structure must be previously solved statically.

A virtual scenario with a virtual force applied at this specific point will be formulated. The virtual complementary work of the external force and the virtual complementary strain energy will be balanced. From this equation of energy balance, we will obtain the movement as it will be the only unknown.

In this document we will explain the method step by step, providing an example.

3 Objectives

After reading this document, the student will be able to:

- Formulate the corresponding virtual scenario according to the PVF
- Formulate the energy balance equation after obtaining the virtual complementary work of the external force and the virtual complementary strain energy
- Obtain the movement with the PVF

4 Calculation of a specific movement with the PVF

4.1 The principle of virtual forces

The real structure, in equilibrium and with a compatible kinematic configuration after the application of a system of external forces in a slow quasi-static loading process, is subjected to a virtual field of forces, applied instantaneously.

The virtual forces are arbitrary, that is, independent of the real external forces acting on the structure. They must be in equilibrium (statically admissible), that is, they must fulfill the equilibrium conditions. The equilibrium in the virtual forces scenario will be formulated with respect to the undeformed geometry (since the real structure fulfill the hypothesis of the theory of small movements)

The virtual external forces acting through the real displacements do an instantaneous virtual work called VIRTUAL COMPLEMENTARY WORK OF THE EXTERNAL FORCES (δW^*).

Internally, the virtual stresses and the real strains (induced by the real displacements) do an internal virtual complementary work which is stored as VIRTUAL COMPLEMENTARY STRAIN ENERGY (δU^*)

The compatibility conditions of the real structure remain unmodified; hence, the virtual complementary work of the external forces is equal to the virtual complementary strain energy. The equation of energy balance is as follows:

$$\delta W^* = \delta U^* \tag{1}$$

The virtual complementary work is the sum of the virtual complementary work done by all the external virtual forces/moments working with the corresponding real displacements/rotations.

The virtual complementary strain energy is the sum of the virtual complementary strain energy of all the members of the structure which deform in the virtual scenario. Being the structure isostatic (or statically solved in the case of an hyperstatic structure) there will be no unknowns in this expression.

4.2 Application of the PVF to the calculation of a specific movement. Procedure

As mentioned, to calculate a movement in a specific point of an isostatic plane-framed structure we can use the PVF.

The movement to be obtained will be in the expression of the virtual complementary work if we apply a virtual force/moment at the specific point and direction of the unknown movement (displacement/rotation).

Considering that we have only an equation of energy balance for each virtual force scenario, this movement must be the only unknown, thus, we will only apply a virtual force/moment. Being arbitrary we will assign the value 1 to the virtual force[1].

The procedure step by step will be as follows:

1. REAL STRUCTURE:
To fulfill the equilibrium conditions, we will formulate the equilibrium equations and will obtain the internal forces functions of all the members ($N(x)$ y $M(x)$).

2. VIRTUAL FORCES SCENARIO:
The virtual scenario will consist of a virtual force or moment (arbitrary value=1, unit load method) in the direction and point of application of the unknown movement.
To fulfill the equilibrium conditions, we will formulate the equilibrium equations and will obtain the internal forces functions of all the members ($\delta N(x)$ y $\delta M(x)$).

[1] The unit load method is a particular case of the Principle of Virtual Forces when there is only a virtual force/moment of value 1.

3. ENERGY BALANCE EQUATION ($\delta W^*=\delta U^*$):
We obtain the value of the virtual complementary work of the external forces and of the virtual complementary strain energy.

4. OBTAINING THE UNKNOWN MOVEMENT
Solving the energy balance equation, we obtain the movement

4.3 Worked example

We will calculate the horizontal displacement at joint A, in the structure shown in figure 1, considering and elastic and linear behavior and neglecting the strain energy associated with shear.

Data:
Members 1, 2 and 3: HEB 360 A = 180.6 cm^2 I = 43193 cm^4 E = 210000 N/mm^2

To simplify the problem, we will neglect the strain energy in members 1, 2 and 3 due to axial loading and we will consider that member 4 is a rigid body, consequently, it will not undergo either axial or bending deformation

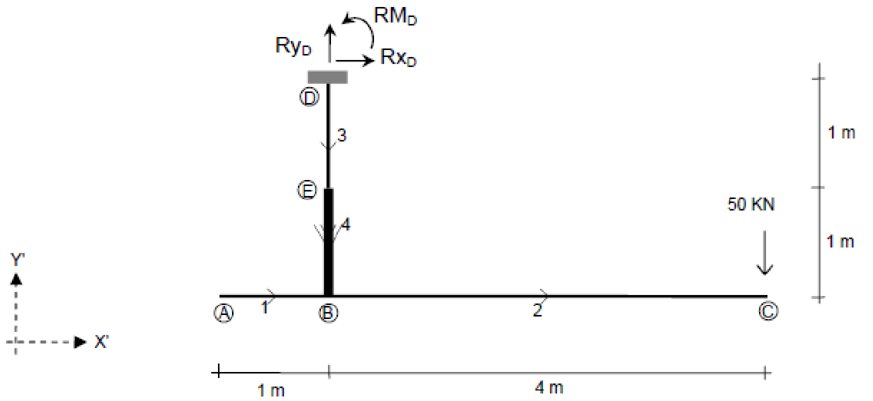

Figure 1. Example.

1. REAL STRUCTURE

The structure is isostatic; therefore, it can be solved statically with the equilibrium equations.

Reactions

With the overall equilibrium equations, we obtain the reactions (see figure 2):

$\sum F_x=0$ $Rx_D = 0$
$\sum F_y=0$ $Ry_D = 50$ KN
$\sum M_D=0$ $RM_D = 200$ KNm

Internal forces functions

For members 1, 2 and 3, we will obtain only the bending moment functions, because their strain energy due to axial loading shall be neglected, consequently we won't need the axial forces functions.

We will not obtain the internal forces functions in member 4, because being considered a rigid body it will not undergo either axial or bending deformation.

Member 1: $M1 = 0$
Member 2: $M_2(x) = -50x$
Member 3: $M3 = -200$

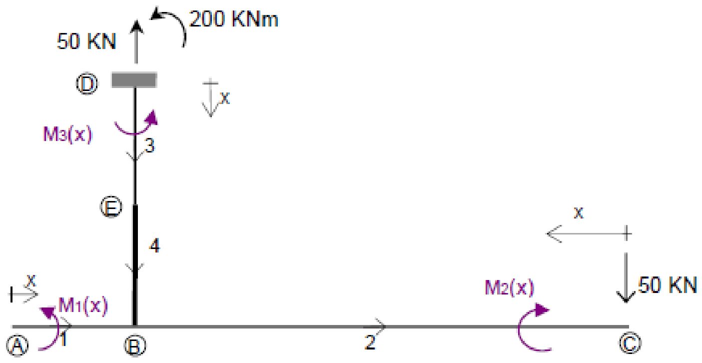

Figure 2. Equilibrium in real structure

2. <u>VIRTUAL FORCES SCENARIO</u>

To obtain the displacement, we will consider a virtual scenario with a horizontal unit virtual load in A (figure 3)

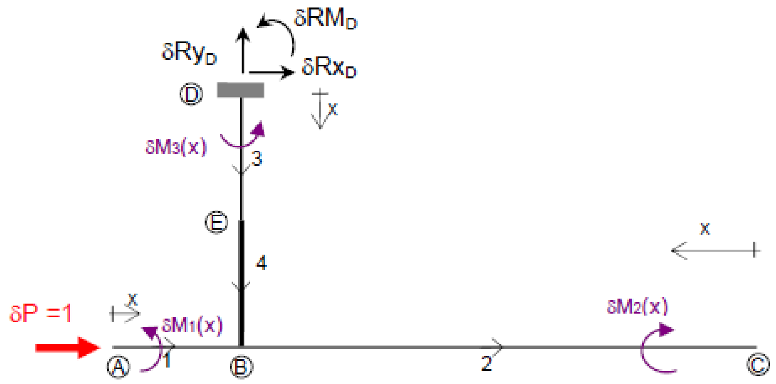

Figure 3. Virtual forces scenario

<u>Reactions</u>

With the overall equilibrium equations, we obtain the reactions:

$\sum F_x=0$ $\delta Rx_D = -1$
$\sum F_y=0$ $\delta Ry_D = 0$
$\sum M_D=0$ $vRM_D = -2$

Internal forces functions

Member 1: $\delta M_1 = 0$
Member 2: $\delta M2 = 0$
Member 3: $\delta M3(x) = 2 - x$

3. ENERGY BALANCE EQUATION ($\delta W^* = \delta U^*$):

δW^*: virtual complementary work of the external forces

$$\delta W^* = 1 \cdot dx_A$$

δU^*: virtual complementary strain energy

$$\delta U^* = \delta U^*_{f3} = \int_0^{L_3} \frac{M_3(x)}{EI_3} \delta M_3(x)\,dx = \int_0^1 \frac{(-200)}{2.1 \cdot 10^8 \; 43193 \cdot 10^{-8}} (2-x)\,dx = -3.307 \cdot 10^{-3}$$

4. OBTAINING THE UNKNOWN MOVEMENT

Substituting virtual complementary work and virtual complementary strain energy in the energy balance equation, we obtain the value of the horizontal displacement of joint A:

$$dx_A = -3.307 \cdot 10^{-3} \text{ m} \quad (\longleftarrow)$$

The negative sign implies that the movement is to the left

5 Closing

In this document we have obtained the movement at a specific point of an isostatic plane-framed structure with the Principle of Virtual Forces.

As a practical application and self-training, we propose the student to obtain the vertical displacement at joint D of the structure shown in figure 4.

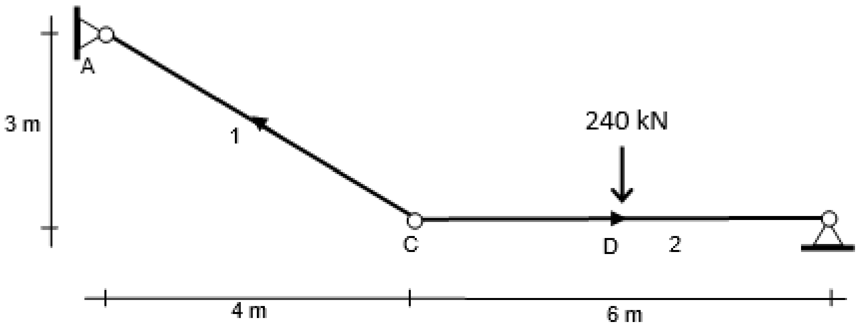

Figure 4. Self-training example

DATA: EA_1=546000 kN EA_2=4153800 kN EI_2=121128 kNm^2

(Results: $dy_D = -1.06 \cdot 10^{-2}$ m)

6 Bibliography

6.1 Books:

[1] Abdilla E. “Fundamentos energéticos de la Teoría de Estructuras. Segunda parte-Aplicaciones. Volumen 1”. Editorial UPV, ref.: 2003.718, 2003

[2] Basset, L. “Aplicación del Principio de Conservación de la Energía y del Teorema de la Carga Unidad para la obtención de movimientos”, Artículo Docente ETSA, Artículo Docente ETSA, 2013. Disponible en Riunet: http://hdl.handle.net/10251/30429

[3] Basset, L. “El Principio de las Fuerzas Virtuales” Artículo Docente ETSA, 2017. Disponible en Riunet: http://hdl.handle.net/10251/84061

6.2 Figures:

Figure 1. Example.
Figure 2. Equilibrium in real structure
Figure 3. Virtual forces scenario
Figure 4. Self-training example

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