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Calculation of Winding Inductances via Magnetic Vector Potential, Discrete Convolution and Fast Fourier Transform

Abstract. — Fault analysis of induction motors with broken bars or inter-turn short circuits needs accurate machine models to correctly identify and quantify the magnitude of the fault. Winding inductances and their derivatives are a key component of these models. In this paper a new, very easy and extremely fast method to compute them, based on the Magnetic Vector Potential (MVP), the Discrete Circular Convolution and the Fast Fourier Transform (FFT), is presented.

*Streszczenie. Przedstawiono analizę uszkodzeń silnika indukcyjnego spowodowanych złamaniem pręta, lub zwarciem wewnętrznym. Oszacowano amplitudę wynikającą z uszkodzenia. Induktancja uzwojeń jest kluczowa w tym modelu. Zaproponowana metoda jest bardzo prosta i szybka. Zastosowano wektorowy potencjał magnetyczny oraz szybką transformatę Fouriera. (**Obliczenia induktancji uzwojeń metodą magnetycznego potencjału wektorowego, splotu dyskretnego i szybkiej transformaty Fouriera**)*

Keywords: Fourier Transform, Magnetic Vector Potential, Winding Inductances, Induction Motor.
Słowa kluczowe: Transformata Fouriera, magnetyczny potencjał wektorowy, silnik indukcyjny, induktancja uzwojeń.

Introduction

Transient analysis of rotating electrical machines by use of the well established d-q model neglects the harmonic contents generated by phase windings. In the Multiple Coupled Circuit Model (MCCM), by Fudeh and Ong, [1], the model of a symmetrical, general $m-n$ induction machine is established based on the phase self and mutual inductances, derived on harmonic bases. The accuracy of the analysis depends on the number of harmonics included in the calculation. In [2] the Winding Function Approach (WFA) for the calculation of inductances in machines with small and constant air gap, taking into account the space harmonics, was presented. This approach has been applied to the modelling of DC brushless motor [3] and induction machines in [4-11]. In [4] the effect of slot skewing and the linear rise of the air gap MMF across the slot are introduced in the WFA. Inductance evaluation under eccentricity conditions has been carried out for a three phase squirrel cage induction motors in [8] and [12], and for a single phase induction motor in [9]. The influence of slots geometry has been taken into account in a simplified way as air gap width variations in [10] and [11]. The consideration of these aspects improves the accuracy of the method, at the cost of a more complex and more time consuming winding computation process. As it is stated in [6], inductance computation typically consumes a high amount of time, so that only discrete curves of inductance versus rotor position are calculated and linear interpolation is applied at intermediate rotor positions. In this paper, a completely different way of attacking the problem is undertaken, developing a new method that is characterized by the following main points:

- The conductor, instead of the coil, is used as the basic element to compute both the MVP in the air gap and the flux linkage of a phase, so that windings in fault conditions can be easily modelled.
- Only the physical distribution of the conductors of the phases and a single function (the MVP of a conductor placed at the origin and fed with 1 A) are needed to derive a single resulting equation, which gives the mutual inductance of two phases for every relative position between them. This equation has the form of a discrete circular convolution, which can be computed in the spatial frequency domain in a very fast way using the FFT.

The mutual inductances of two phases corresponding to all of their relative angular positions, taking into account the spatial harmonics of the air gap MVP are obtained in the proposed method with a single equation. It is expressed in terms of the FFT of the conductors' distributions of the two phases and the FFT of a characteristic function of the machine, the MVP generated by a single conductor. Thanks to the effectiveness of the FFT, the process is extremely fast, even in the case of complex windings layouts (e.g. under fault conditions).

The structure of this paper is as follows: in section II the system equations for an induction machine are briefly reminded. In section III the proposed method for calculating the inductances is developed and explained in detail. Conclusions are presented in section IV.

System equations

The equations system of an induction machine with m stator and n rotor phases with arbitrary layout (that is, even under winding fault conditions like inter-turn short circuits or broken bars) are:

$$(1) \quad \begin{cases} [U_S] = [R_S][I_S] + d[\Psi_S]/dt \\ [0] = [R_r][I_r] + d[\Psi_r]/dt \end{cases}$$

$$(2) \quad \begin{cases} [\Psi_s] = [L_{ss}][I_s] + [L_{sr}][I_r] \\ [\Psi_r] = [L_{sr}]^T[I_s] + [L_{rr}][I_r] \end{cases}$$

$$(3) \quad \begin{cases} [I_s] = [i_{s1} \ i_{s2} \ ... \ i_{sm}]^T \\ [I_r] = [i_{r1} \ i_{r2} \ ... \ i_{rn}]^T \end{cases}$$

$$(4) \quad [U_s] = [u_{s1} \ u_{s2} \ ... \ u_{sm}]^T$$

$$(5) \quad \begin{cases} T_e = [I_s]^T (\partial[L_{sr}]/\partial\theta)[I_r] \\ T_e - T_L = J \, d\Omega/dt = J \, d^2\theta/dt^2 \end{cases}$$

where $[U]$ is the voltage matrix, $[I]$ is the current matrix, $[R]$ is the resistance matrix, $[\Psi]$ is the flux linkage matrix and $[L]$ is the matrix of inductances. Subscripts s and r are assigned to the stator and rotor. Te is the electromechanical torque of the machine, T_L is the load torque, J is the rotor inertia, Ω is the mechanical speed, and θ is the mechanical angle.

To solve the system of equations (1)-(5), self and mutual phase inductance matrices must be calculated. Due to the presence of the derivatives in (1) and (5), it is necessary to achieve a very good accuracy in this process (especially if different fault conditions are to be detected and diagnosed in a sure way). To do so, the method developed in this paper takes into account the air gap MVP harmonics in the inductances calculation process. End turn and slot leakage inductances need to be pre-calculated, and are treated as constants in (2).

Proposed method for computing the mutual inductance between two phases via discrete circular convolutions and FFT

The inductance between two phases, I and J, is calculated in this paper through the following process:

1. Phase I is fed with a constant current of 1 A, and its MVP, $A_I(\varphi)$, is obtained, by the superposition of the MVP of the phase's conductors.
2. The flux linkage of phase J, due to the MVP generated by phase I, is determined, which gives the mutual inductance between both phases. If $J=I$, we get the phase magnetizing self inductance.

In [13] a similar method is presented, but it relies on the magnetomotive force instead of the MVP. The use of the MVP has the following key advantages [14-15]:

1. The MVP has a clear, physical meaning, and matches very well the use of the conductor (instead of the coil) as the basic winding unit.
2. The MVP produced by a phase can be obtained very quickly by circular convolution, via FFT.
3. The flux linkage of an arbitrary phase can be expressed very easily as the sum of the MVP at all the angular positions of its conductors.

MVP generated by a single conductor placed at the origin and fed with a constant 1A current

The following assumptions will be made in this section:

- constant permeability of the stator and rotor iron cores,
- uniform air gap,
- the conductor are parallel to the machine axis,
- no end-effects are considered,
- the winding conductors are placed in the air gap on the outer rotor surface and the inner stator one.

Under these conditions, the MVP has just one component A in the direction of the machine axis, given in cylindrical coordinates by the solution of the following equation:

$$(6) \quad \frac{\partial^2 A(r,\varphi)}{\partial r^2} + \frac{1}{r}\frac{\partial A(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A(r,\varphi)}{\partial \varphi^2} = -\mu J(r,\varphi)$$

where $J(r,\varphi)$ is the current density. Let consider a single conductor, located in the air-gap at coordinates $\varphi=0$ and $r=c$ (Fig. 1), and fed with a constant, unit current. In this simple case, (6) has been solved analytically using the method of separation of variables, giving

$$
A_0(r,\varphi) =
$$

$$(7) \quad
\begin{cases}
-\dfrac{\mu_0}{2\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n \cdot c^n} \dfrac{c^{2n}+b^{2n}}{b^{2n}-a^{2n}} \left(r^n + \dfrac{a^{2n}}{r^n} \right) \cos(n\varphi) + \dfrac{\mu_0 \ln(c)}{2\pi} \\
\hspace{10cm} r \leq c \\[4mm]
-\dfrac{\mu_0}{2\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n \cdot c^n} \dfrac{c^{2n}+a^{2n}}{b^{2n}-a^{2n}} \left(r^n + \dfrac{b^{2n}}{r^n} \right) \cos(n\varphi) + \dfrac{\mu_0 \ln(r)}{2\pi} \\
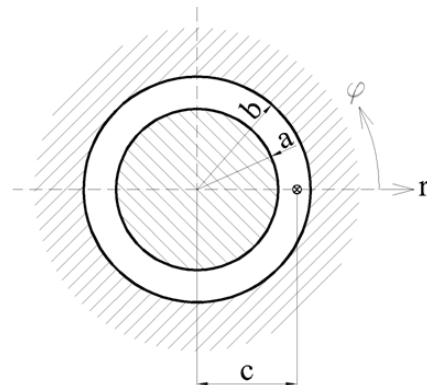\hspace{10cm} r > c
\end{cases}
$$

Fig. 1. Conductor located in the air gap at coordinates $\varphi=0$ and $r=c$.

Equation (8) has been represented in Fig. 2 for values a=1, b=1.5 and c=1.4.
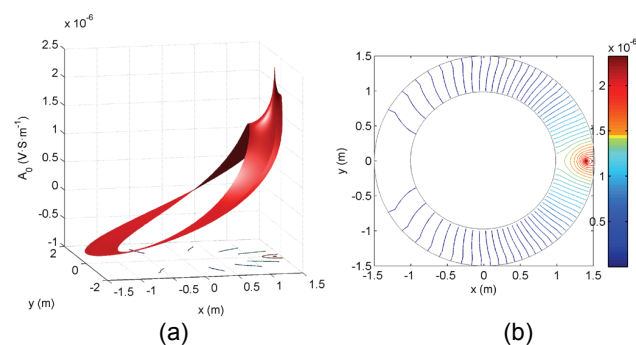

(a)   (b)

Fig. 2. MVP (a) and constant induction lines in the air-gap (b) that a conductor placed at $\varphi=0$ and $r=c$ produces when it is fed with a 1A current.

The conductors are considered to be placed either on the inner stator surface ($r=b$) or on the outer rotor surface ($r=a$), so (7) has to be evaluated only at these radial coordinates.
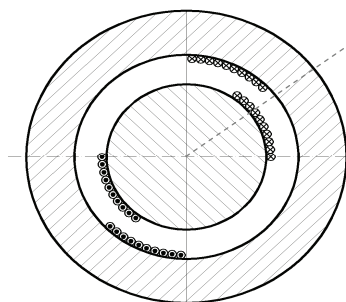

Fig. 3. Conductors located at the air gap on the inner stator surface ($r=b$) and on the outer rotor surface ($r=a$).

Besides, for any winding turn, the currents in the two opposite conductors has opposite signs, so the constant terms in (7) need not be considered. Under these assumptions, the MVP generated by a single conductor, at the same surface where it is located or at the opposite one, is given by

$$(8) \ A_0(\varphi) = \begin{cases} \sum_{n=1}^{\infty} \dfrac{\mu_0}{\pi n} \cdot \dfrac{b^{2n}+a^{2n}}{b^{2n}-a^{2n}} \cos(n\varphi) & same\ surface \\[4mm] \sum_{n=1}^{\infty} \dfrac{2\mu_0}{\pi n} \cdot \dfrac{a^n b^n}{b^{2n}-a^{2n}} \cos(n\varphi) & opposite\ surface \end{cases}$$

If the conductor is placed at a different angular position $\varphi = \alpha$ (instead of $\varphi = 0$), its MVP, $A_\alpha(\varphi)$, is obtained by shifting the curve $A_0(\varphi)$, to the new position,

$$(9) \qquad A_\alpha(\varphi) = A_0(\varphi - \alpha)$$

MVP Produced by a Phase with an Arbitrary Distribution of Conductors

The MVP $A_I(\varphi)$ that generates a phase I with an arbitrary distribution of conductors $Z_I(\alpha)$ (which includes the corresponding positive or negative sign, according to the current direction in each phase conductor), fed with a 1 A constant current, is obtained via linear superposition,

$$(10) \qquad A_I(\varphi) = \int_0^{2\pi} A_\alpha(\varphi) \cdot Z_I(\alpha) \cdot d\alpha$$

By means of (9), (10) can be expressed as a circular convolution (represented with the symbol $\otimes$):

$$(11) \quad A_I(\varphi) = \int_0^{2\pi} A_0(\varphi-\alpha) \cdot Z_I(\alpha) \cdot d\alpha = \left(A_0 \otimes Z_I \right)(\varphi)$$

Equation (11) must be evaluated at every angular coordinate φ, which can be cumbersome for a phase with complex layout. Nevertheless, there is a fast method to perform this calculation, based on the following property of the Fourier Transform (FT): the FT of the convolution of two functions is equal to the product of their respective FTs, FT($f \otimes g$)=FT(f) $\cdot$ FT(g), so

$$(12) \quad \left. \begin{array}{c} A_0(\varphi) \xrightarrow{FT} \hat{A}_0(\xi) \\[3mm] Z_I(\alpha) \xrightarrow{FT} \hat{Z}_I(\xi) \end{array} \right\} \ \hat{A}_0(\xi) \cdot \hat{Z}_I(\xi) = \hat{A}_I(\xi) \xrightarrow{IFT} A_I(\varphi)$$

where the FT of a function f is represented as $\hat{f}$, and:

- $\hat{A}_0(\xi)$ is the FT of the air gap MVP generated by a conductor placed at the origin, fed with a unit current. It must be computed only once, and it is a characteristic function of the machine.
- $\hat{Z}_I(\xi)$ is the FT of the distribution of the phase's conductors.

Equation (12) can be computed very effectively using the fast FT (FFT) and the inverse FT (IFFT), but this transforms can only be applied to discrete sequences. To obtain the discrete form of (12), the circular length of the machine air gap is divided into N equally spaced intervals, yielding an angular resolution of

$$(13) \qquad \Delta\alpha = 2\pi / N$$

For example, N=3600 yields a precision of 0.1°. The discrete sequences that result are N-column vectors. The sequence corresponding to the function $A_0(\varphi)$, $\mathbf{A_0}$, is

generated by sampling it at the beginning of each interval.

$$(14) \qquad \mathbf{A_0}[k] = A_0(k \cdot \Delta\alpha) \qquad k = 0..N-1$$

The sequence corresponding to the distribution of conductors $Z_I(\alpha)$, $\mathbf{Z_I}$, is generated by assigning to each interval the sum of all the conductors in that interval

$$(15) \qquad \mathbf{Z_I}[k] = \int_{k \cdot \Delta\alpha}^{(k+1) \cdot \Delta\alpha} Z_I(\alpha) \cdot d\alpha \qquad k = 0..N-1$$

Using the sequences (14) and (15), (12) can be expressed in vector form as:

$$(16) \qquad \mathbf{A_I} = \mathrm{IFFT}(\mathrm{FFT}(\mathbf{A_0}).*\mathrm{FFT}(\mathbf{Z_I}))$$

where the symbol .* denotes an element by element product of two vectors.

Fig. 4 shows the application of (16) to compute the MVP generated by an example winding phase with three coils, fed with a constant current of 1A.
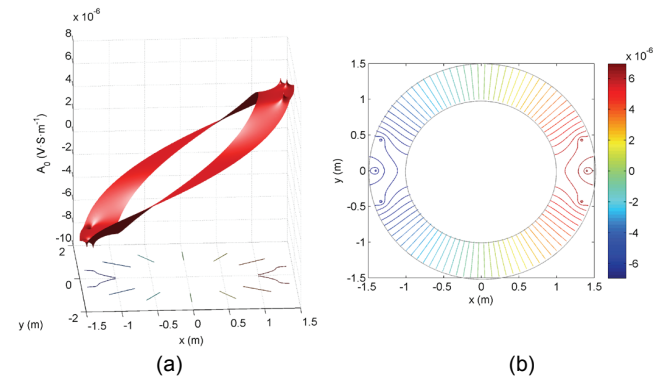

Fig. 4. MVP in the air gap (a) and constant induction lines (b) produced by a phase with three coils.

Flux Linkage of a Phase with an Arbitrary Distribution of Conductors

The flux linkage of a phase with an arbitrary distribution of conductors is obtained by simply adding up the values of the MVP at the positions corresponding to each one of its conductors. The flux linkage of a coil with a single turn, under the most general assumption of an arbitrary air gap length, (as represented in Fig. 5), can be obtained directly from the values of the MVP at the location of each coil conductor and their orientation as:

$$(17) \ \Psi_{aa'} = \iint_S \vec{B}dS = \iint_S \left(\nabla \times \vec{A} \right) dS = \oint_{1234} \vec{A}dl = (A_a - A_{a'})\ell$$

The flux linkage Ψ_{IJ} of a phase J with an arbitrary conductors' distribution, $Z_J(\alpha)$, due to the MVP generated by another phase I, when their relative angle position is an arbitrary angle ε, is obtained as:

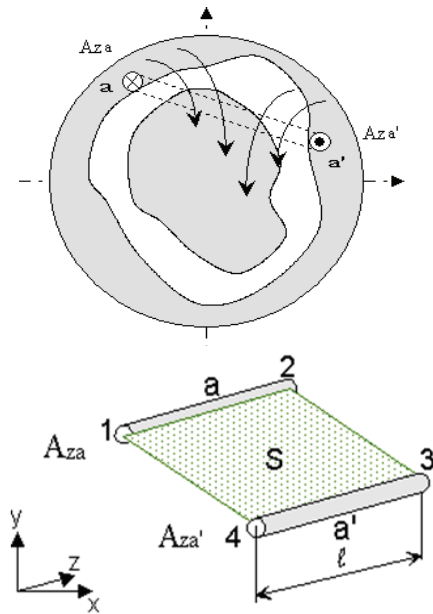$$(18) \qquad \psi_{IJ}(\varepsilon) = \int_0^{2\pi} A_I(\alpha) \cdot Z_J(\alpha - \varepsilon) \cdot d\alpha$$

Fig. 5. Flux linkage of a coil with a single turn (top) computed from the values of the MVP at the position of its axial conductors (bottom).

If phase I is fed with a constant current of 1 A, **Błąd! Nie można odnaleźć źródła odwołania.** is the mutual inductance between phases I and J for an angular relative position ε. The integral in **Błąd! Nie można odnaleźć źródła odwołania.** can be computed with an expression similar to (16)

$$(18) \qquad M_{IJ} = IFFT(FFT(A_I).*FFT(Z_J)^*)$$

where the superscript * stands for complex conjugate. L_{IJ} is an N-column vector, whose kth element is the mutual inductance of phases I and J, for a relative angle between them equal to $k \cdot 2\pi/N$. Equations (16) and (18) can be further combined, yielding a very simple final expression:

$$(19) \qquad M_{IJ} = IFFT(FFT(Z_I).*FFT(A_0).*FFT(Z_J)^*)$$

Using (19) with the appropriate winding conductors' distribution, the mutual inductance between two stator phases, two rotor phases, or one stator and one rotor phases, for all the values of their relative angular displacements, can be easily obtained

$$(20) \quad \begin{cases} M_{ss} = IFFT(FFT(Z_s).*FFT(A_0).*FFT(Z_s)^*) \\ M_{rr} = IFFT(FFT(Z_r).*FFT(A_0).*FFT(Z_r)^*) \\ M_{sr} = IFFT(FFT(Z_s).*FFT(A_0).*FFT(Z_r)^*) \end{cases}$$

The derivative with respect to the angle of the mutual phase inductances can be easily obtained from (20) by numerical differentiation. Using a centered differences schema, and applying (13), gives:

$$(21) \qquad \frac{dM_{sr}[k]}{d\theta} = M'_{sr}[k] \approx \frac{M_{sr}[k+1] - M_{sr}[k+1]}{2\Delta\alpha}$$

Assembly of the matrices of inductances

In the case of a machine with n_S stator phases and n_r rotor phases, the inductance matrices Lss, Lrr and Lsr that

appear in (2), must be assembled, using (20). Lss and Lrr elements are independent of the rotor angular position. The value of their i, j th element is given by

$$(22) \quad L_{ss}[i,j] = \begin{cases} L\sigma_s + M_{ss}[0] & i = j \\ M_{ss}\left[(i-j)\dfrac{N}{p \cdot n_s}\right] & i \neq j \end{cases} \quad i,j = 0..n_s - 1$$

$$(23) \quad L_{rr}[i,j] = \begin{cases} L\sigma_r + M_{rr}[0] & i = j \\ M_{rr}\left[(i-j)\dfrac{N}{p \cdot n_r}\right] & i \neq j \end{cases} \quad i,j = 0..n_r - 1$$

where $L\sigma_s$ and $L\sigma_r$ are the stator and rotor phase leakage inductances, and all the indices are taken as modulo-N values. On the contrary, the i, j th element of matrix Lsr, which corresponds to the mutual inductance between the stator phase i and the rotor phase j, depends on the rotor angular position θ. As the angular coordinate has been discretized using N intervals, the values of θ must be restricted to integer multiples of $2\pi/N$

$$(24) \qquad \theta_k = k\frac{2\pi}{N} = k \cdot \Delta\alpha \qquad k = 0..N-1$$

So, the matrix Lsr is given by

$$(25) \quad L_{sr}[i,j](\theta_k) = M_{sr}\left[\frac{i \cdot N}{p \cdot n_s} - \left(\frac{j \cdot N}{p \cdot n_r} + k\right)\right] \quad \begin{matrix} i = 0..n_s - 1 \\ j = 0..n_r - 1 \\ k = 0..N-1 \end{matrix}$$

The same result can be used to obtain de derivative of the matrix Lsr with respect to the angular coordinate, used in (5). It suffices to replace in (25) the matrix Msr with its angular derivative, given by (21):

$$(26) \qquad \frac{dL_{sr}[i,j](\theta_k)}{d\theta} = M'_{sr}\left[\frac{i \cdot N}{p \cdot n_s} - \left(\frac{j \cdot N}{p \cdot n_r} + k\right)\right]$$

In the case of a rotor position θ that does not coincide with an integer multiple of $\Delta\alpha$, the value of the mutual inductances between stator and rotor phases, and its derivative, are obtained with a linear interpolation with the nearest values of Lsr and Msr, respectively.

Conclusions

A new and completely different approach for the calculation of winding inductances in rotating electrical machines has been presented in this paper. The election of the conductor as the winding basic element, the use of the MVP as the main magnetic quantity, and the formulation of inductances in terms of a discrete circular convolution, computed with the FFT, are the key elements of the new method. After discretization of the air gap into N equally spaced intervals, the mutual inductances of two phases corresponding to N relative angular positions, taking into account the first $N/2$ air gap MVP spatial harmonics, are obtained simultaneously with a single equation, solved via FFT. The method involves only three discrete sequences, namely the distributions of the conductors of the two phases and a characteristic function of the machine: the MVP generated by a conductor placed at the origin with a unit

current flowing through it. Arbitrary, complex winding layouts, the slot width, and skew effects can be easily modeled with the proposed approach. As the method can handle arbitrary phase conductor distributions, it is highly suitable to the analysis of machines with stator or rotor faults, such as inter-turn short circuits or broken bars.

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