TRACED TENSOR NORMS AND MULTIPLE SUMMING MULTILINEAR OPERATORS

P. RUEDA, E.A.ÁNCHEZ PÉREZ AND A. TALLAB

Abstract. Using a general tensor norm approach, our aim is to show that some distinguished classes of summing operators can be characterized by means of an "order reduction" procedure for multiple summing multilinear operators, which becomes the keystone of our arguments and can be considered our main result. We work in a tensor product framework involving traced tensor norms and the representation theorem for maximal operator ideals. Several applications are given not only to multi-ideals, but also to linear operator ideals. In particular, we get applications to multiple p-summing bilinear operators, (p,q)-factorable linear operators, $\tau(p)$-summing linear operators and absolutely p-summing linear operators, providing a characterization of this later class whenever the absolutely p-summing linear operators take values in an L^p-space.

Multilinear operator; Summing operator; Multiple summing operator; $\tau(p)$-summing operator; tensor norm.

1. Introduction and basic definitions

A considerable effort has been made in recent years to set a unified theory for multilinear operator ideals that are defined by summability properties. Generalizing the linear case, a broad family of multi-ideals that are defined by a vector norm inequality involving any kind of summability have been introduced in the mathematical literature (see [13, 16, 27] for early attempts). Some of them are for instance the p-summing multilinear operators, absolutely continuous multilinear operators, $(p_1, \ldots, p_n; p)$-dominated multilinear operators, multiple p-summing operators and factorable p-summing operators, among others ([1, 3, 4, 5, 6, 12, 17, 20, 21, 22, 23, 24]).

The tensor product point of view is a powerful approach for the study of operator ideals. In particular, the comparison of different topologies on tensor products allows to prove results on their structure and provides characterizations of the most common ideals. This methodological approach was presented in the classical work by Defant and Floret [8], and can also be found in [10, 28]. As far as we know, some beautiful ideas appearing in Ch.19 of [8] have not been used yet in the multilinear context. In particular, the advised reader may find there that a great part of the linear operator ideals can be described in terms of continuity of the canonically defined tensor product operator

$$id \otimes T : \ell^p \otimes_\beta E \to \ell^p \otimes_\alpha F \tag{1}$$

for adequate p and tensor norms β and α (see [8, Theorem 29.4]). For instance, the linear operator $T : E \to F$ is absolutely p-summing if and only if $id \otimes T : \ell^p \otimes_\varepsilon E \to \ell^p \otimes_{\Delta_p} F$ is continuous, where Δ_p satisfies $\Delta_p(\sum_{k=1}^n e_k \otimes x_k) = (\sum_{k=1}^n \|x_k\|^p)^{1/p}$

and ε is the injective tensor norm; or, $T : E \to F$ is p-dominated if and only if $id \otimes T : \ell^p \otimes_\varepsilon E \to \ell^p \otimes_\pi F$ is continuous, where π is the projective tensor norm.

Also, the so called calculus of traced tensor norms ([7]) will be a fundamental tool in this paper. It is a very useful tool when dealing with topological tensor products —and so with operator ideals— that surprisingly enough has not been used very often, although it provides a clear point of view for the study of composition and quotients of operator ideals. The reader can find in [9] one of the rare applications of this technique in a similar context. We use the presentation of this theory that is given in Ch.29 of [8].

In this paper we show that some multi-ideals can be understood using a unified tensor product point of view for the description of the ideals of multilinear operators in the same way that it is shown in the linear case. The main idea is to consider general classes of summing n-linear operators with respect to different tensor norms and characterize their summability properties in terms of an associated $n-1$-linear operator. This leads to an order reduction procedure that is faced in the first part of the paper. The order reduction theorem (Theorem 2.1) is the main result of the paper. Afterwards, we show how our general results look like when we restrict the attention to the bilinear case. Some applications are given to several ideals of summing operators, as multiple p-summing bilinear operators or p-factorable linear operators. These applications provide some tools that will be used in the last part where $\tau(p)$-summing linear operator are considered. As an example, we will use in this last part our ideas in the bilinear setting to get information for absolutely p-summing linear operators, showing that sometimes the multilinear point of view allows a better understanding of the linear problems: we will provide a —as far as we know— new characterization of absolutely p-summing linear operators having values in a subspace or a quotient of an L^p-space. In order to do that, we use our tools to describe the so called $\tau(p)$-summing linear operators and the corresponding "multiple version".

Let us give now some background information. Let E, F be (real or complex) Banach spaces and write F^* for the dual of F. Let $\mathbb{K}$ be the real or complex scalar field. As usual, $\mathcal{L}(E, F)$ denotes the space of continuous linear operators from E to F endowed with the sup norm. For $1 \leq p < \infty$, let p' be the conjugate of p, that is, $1/p + 1/p' = 1$. We write $\ell^p_w(E)$ and $\ell^p(E)$ for the spaces of sequences of vectors in E that are weakly p-summable and p-summable, respectively. If $(a_i)_{i=1}^\infty$ is such a sequence, we write $\|(a_i)_{i=1}^\infty\|_{w,p}$ and $\|(a_i)_{i=1}^\infty\|_p$ for the corresponding p-weak and p-strong sums. We denote by $\ell^{p,0}_w(E) := \{(x_i)_{i=1}^\infty \in \ell^p_w(E) : \|(x_i)_{i=N}^\infty\|_{w,p} \to 0$ as $N \to \infty\}$, and let ℓ^p_0 denote the subspace of ℓ^p of all sequences with only a finite number of nonzero "coordinates". The canonical unit vector basis of ℓ^p is denoted by $(e_i)_{i=1}^\infty$.

Our notation regarding tensor products and operator ideals is standard. We will use the term "tensor norm" in the sense that is used in [8], but including the norm Δ_p for the case of tensor products involving $L^p(\mu)$ spaces. As usual, $E \otimes_\alpha F$ denotes the tensor product $E \otimes F$ endowed with the tensor norm α and $E \widehat{\otimes}_\alpha F$ is its completion. The norm Δ_p on $L^p \otimes F$ is the one that comes from the Bochner space $L^p(F)$. We will consider several tensor norms on a tensor product $E \otimes F$. The most common ones are the projective norm π or the injective norm ε. It is well known that the space $\ell^{p,0}_w(F)$ is isometrically isomorphic to $\ell^p \widehat{\otimes}_\varepsilon F$ whereas $\ell^p(F)$ is isometrically isomorphic to $\ell^p \widehat{\otimes}_{\Delta_p} F$ (see [8, 12.9]).

If α is a tensor norm, we will write as usual α^t for its transpose, α' for its dual and α^* for its conjugate tensor norms ([8, Ch.12]). Moreover, we will write $\backslash\alpha$ and $/\alpha$ for the left injective and left projective associate tensor norms, respectively, and $\alpha\backslash$ and $\alpha/$ for the right projective and the right injective associate (see [8, Ch.20]). The related operator ideal notions are the injective hull $\mathcal{U}^{inj}$ and the surjective hull $\mathcal{U}^{sur}$ of a given operator ideal $\mathcal{U}$ ([8, Ch.9]). If $1 \leq p \leq \infty$ and $z \in E \otimes F$, recall that

$$g_p(z) := \inf\{\|(x_i)_i\|_p \|(y_i)_i\|_{w,p'} : z = \sum_{i=1}^n x_i \otimes y_i\}$$

$$d_p(z) := \inf\{\|(x_i)_i\|_{w,p'} \|(y_i)_i\|_p : z = \sum_{i=1}^n x_i \otimes y_i\}.$$

Our main reference for the theory of tensor norms is [8], and for operator ideals that book and also [11]. A lot of rather technical notions regarding tensor norms will be used in the paper; we have tried to explain all of them, but sometimes it is not easy in the context of a paper. By this reason, we will refer to the corresponding chapter of [8] when some new definition is introduced.

Let $T : E \to F$ be a continuous linear operator. We say that the bilinear operator $B_T : E \times F^* \to \mathbb{R}$ given by $B_T(a, b^*) := \langle T(a), b^* \rangle$, $(a, b^*) \in E \times F^*$, is the *bilinear map associated to* T. Given T, the operator $\widehat{T} : \ell^p_w(E) \to \ell^p_w(F)$ given by $\widehat{T}((a_i)_{i=1}^\infty) := (T(a_i))_{i=1}^\infty$, $(a_i)_{i=1}^\infty \in \ell^p_w(E)$ is well defined and continuous, and we refer to it as the *associated sequential operator* of T.

Let $\Pi_p(E, F)$ denote the space of all absolutely p-summing operators from E to F, endowed with its usual norm π_p. It is well known that T is p-summing if and only if $\widehat{T}$ takes its values in $\ell^p(F)$ and $\widehat{T} : \ell^p_w(E) \to \ell^p(F)$ is continuous.

We will also make use of the ideal of $\tau(p)$-summing linear operators. Since this operator ideal is not known for many researchers, let us explain its definition and its multiple version here. A linear operator $T : E \to F$ is *$\tau(p)$-summing* if there is a constant $C > 0$ such that

$$\left(\sum_{i=1}^n |\langle T(a_i), b_i^* \rangle|^p\right)^{1/p} \leq C \sup_{\|a^*\| \leq 1, \|b^{**}\| \leq 1} \left(\sum_{i=1}^n |\langle a_i, a^* \rangle \langle b_i^*, b^{**} \rangle|^p\right)^{1/p}$$

for all $a_1, \ldots, a_n \in E$, $b_1^*, \ldots, b_n^* \in F^*$, $n \in \mathbb{N}$. The infimum of all constants $C > 0$ is denoted by $\pi_{\tau(p)}(T)$.

Since the definition involves a vector norm inequality for the associate bilinear map $B_T : E \times F^* \to \mathbb{K}$ that cannot be reduced to a norm inequality for the linear map T, it seems natural to expect that the "multiple linear version" of this ideal provides a new class of operators. We say that a linear operator $T : E \to F$ is *multiple $\tau(p)$-summing* if for $a_1, \ldots, a_m \in E$ and $b_1^*, \ldots, b_n^* \in F^*$,

(2)
$$\left(\sum_{j=1}^n \sum_{i=1}^m |\langle T(a_i), b_j^* \rangle|^p\right)^{1/p} \leq C \sup_{\|a^*\| \leq 1,\, \|b^{**}\| \leq 1} \left(\sum_{j=1}^n \sum_{i=1}^m |\langle a_i, a^* \rangle \langle b_j^*, b^{**} \rangle|^p\right)^{1/p}.$$

Clearly, the above inequality is equivalent to:

$$\left(\sum_{j=1}^{n}\sum_{i=1}^{m}|\langle T(a_i), b_j^*\rangle|^p\right)^{1/p} \leq C\|(a_i)_{i=1}^m\|_{w,p}\|(b_j^*)_{j=1}^n\|_{w,p}.$$

The infimum of all constants $C > 0$ is denoted by $\pi_{m,\tau(p)}(T)$.

The class of $\tau(p)$-summing operators is a subclass of all p-summing operators and so a nicer behavior of the associated sequential operator is expected. Indeed, taking $a_i = a$ for all $i = 1, \ldots, n$ in the definition we get for any $\tau(p)$-summing operator $T : E \to F$ that

$$\sum_{j=1}^{n}|\langle T(a), b_j^*\rangle|^p \leq \pi_{\tau(p)}(T)^p \sup_{\|b^{**}\|\leq 1}\left(\sum_{j=1}^{n}|\langle b_j^*, b^{**}\rangle|^p\right)\|a\|^p.$$

If we apply such an inequality for $a_1, \ldots, a_m \in E$ and do the sum, it follows

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|\langle T(a_i), b_j^*\rangle|^p \leq \pi_{\tau(p)}(T)^p\sum_{i=1}^{m}\|a_i\|^p \sup_{\|b^{**}\|\leq 1}\left(\sum_{j=1}^{n}|\langle b_j^*, b^{**}\rangle|^p\right).$$

This shows that $\widehat{T}((a_i)_{i=1}^m) : F^* \to \ell^p$ is in fact a p-summing operator and that $\pi_p(\widehat{T}((a_i)_{i=1}^m)) \leq \pi_{\tau(p)}(T)\|(a_i)_{i=1}^m\|_p$. Consequently, $\widehat{T} : \ell^p(E) \to \Pi_p(F^*, \ell^p)$ is continuous and $\|\widehat{T}\| \leq \pi_{\tau(p)}(T)$.

A similar calculation easily shows that T is multiple $\tau(p)$-summing if and only if $\widehat{T}$ not only takes $\ell^p(E)$ to $\Pi_p(F^*, \ell^p)$ but also the bigger space $\ell^p_w(E)$ is mapped into $\Pi_p(F^*, \ell^p)$ in a continuous way, and that $\|\widehat{T}\| = \pi_{m,\tau(p)}(T)$. Although (multiple) $\tau(p)$-summing operators are defined in a linear context, their definition is essentially of a bilinear nature because the inequality (2) depends on the operator T via the bilinear map $B_T(a, b^*) = \langle T(a), b^*\rangle$, $(a, b^*) \in E \times F^*$, which clearly is absolutely p-summing. Let us pay attention to this bilinear map B_T. Note that T is nothing but the operator $(B_T)_2(a)(b^*) := B_T(a, b^*)$. Therefore, the summability of the bilinear map B_T can be characterized by means of the summability of the associated sequential operator $\widehat{T} = \widehat{(B_T)_2}$. Our main aim in this paper is to show that this is a particular case of a more general situation: several classes of distinguished summing bilinear (indeed, multilinear) operators can be characterize in terms of its associated sequential operator. Having in mind that $\ell^p(E) = \ell^p\hat{\otimes}_{\Delta_p}(E)$ and that $\ell^{p,0}_w(E) = \ell^p_0\hat{\otimes}_\varepsilon E$, these distinguished classes are determined by considering an arbitrary reasonable tensor norm γ: the (multiple) (γ, Δ_p, p)-summing multilinear operators. This is done in Section 2, where our main result (Theorem 2.1) is proved: it characterizes multiple (γ, Δ_p, p)-summing multilinear operators by means of the associated sequential operators. As a consequence of the Chevet-Persson-Saphar inequalities and some well known equivalences for different tensor norms, in Section 3 we provide the first examples where Theorem 2.1 applies: multiple p-summing bilinear operators and p-factorable linear operators. In Section 4 we get back on (multiple) $\tau(p)$-summing operators, that were our initial motivation. We analyze them and we end the paper with an application to tensor products involving L^p-spaces that will give the coincidence of p-summing operators and multiple $\tau(p)$-summing operators whenever they take values on a L^p-space.

2. Multilinear summing and multiple summing operators

In this paper we are interested in analyzing classes of ideals of multilinear operators that can be characterized by means of abstract summability properties associated to the continuity of the tensor product operators defined between spaces of vector valued sequence spaces. Our main reference is the class of (linear) p-summing operators between Banach spaces E and F, that can be described as the class of operators $T : E \to F$ such that the associated tensor product map $id \otimes T : \ell^p \otimes_\varepsilon E \to \ell^p \otimes_{\Delta_p} F$ is continuous. If we identify the elements of the algebraic tensor product $\ell_0^p \otimes E$ with finite sequences of elements of E via the identification $\sum_{i=1}^n e_i \otimes x_i \leftrightarrow (x_i)_{i=1}^n$ then, the restriction of the associated sequential operator $\widehat{T}$ of T to finite sequences can also be identified with the restriction $id \otimes T : \ell_0^p \otimes_\varepsilon E \to \ell^p \otimes_{\Delta_p} F$. Under these identifications, T is absolutely p-summing if and only if $\widehat{T} : \ell_0^p \otimes_\varepsilon E \to \ell^p \otimes_{\Delta_p} F$ is continuous. We will extend this idea to the case of multilinear operators. Two natural extensions of absolutely p-summing linear operators to the multilinear context are required for our purposes: p-summing and multiple p-summing multilinear operators. Both classes have been intensively studied and are well-known; classical references to these notions are [2] and [17, 24].

Let $1 \leq p \leq \infty$, $n \in \mathbb{N}$ and $E_1, \dots, E_n, F$ be Banach spaces. Recall that an n-linear operator $T : E_1 \times \cdots \times E_n \to F$ is said to be *p-summing* if there is a constant $C > 0$ such that for every choice of vectors $x_1^i, \dots, x_m^i \in E_i$, $i = 1, \dots, n$,

$$\Big(\sum_{i=1}^m \|T(x_i^1, \dots, x_i^n)\|^p \Big)^{1/p} \leq C \|(x_i^1)_i\|_{w,p} \cdots \|(x_i^n)_i\|_{w,p}.$$

It is said to be *multiple p-summing* if there is a constant $C > 0$ such that for $x_1^k, \dots, x_{m_k}^k \in E_k$, $k = 1, \dots, n$,

$$\Big(\sum_{i_1=1}^m \cdots \sum_{i_n=1}^m \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^p \Big)^{1/p} \leq C \|(x_{i_1}^1)_{i_1=1}^{m_1}\|_{w,p} \cdots \|(x_{i_n}^n)_{i_n=1}^{m_n}\|_{w,p}.$$

In terms of sequence spaces, these definitions can be rewritten as follows: an n-linear operator $T : E_1 \times \cdots \times E_n \to F$ is p-summing if the n-linear map

$$\overline{T} : (\ell_0^p \otimes_\varepsilon E_1) \times \cdots \times (\ell_0^p \otimes_\varepsilon E_n) \to \ell^p \otimes_{\Delta_p} F$$

given by

$$\overline{T}((\sum_{i=1}^m e_i \otimes x_i^1, \dots, \sum_{i=1}^m e_i \otimes x_i^n)) := \sum_{i=1}^m e_i \otimes T(x_i^1, \dots, x_i^n).$$

is continuous. On the other hand, T is multiple p-summing if the n-linear map

$$\widetilde{T} : (\ell_0^p \otimes_\varepsilon E_1) \times \cdots \times (\ell_0^p \otimes_\varepsilon E_n) \to \big(\ell^p \otimes_{\Delta_p} \cdots \otimes_{\Delta_p} \ell^p \big) \otimes_{\Delta_p} F$$

given by

$$((x_{i_1}^1)_{i_1}, \cdots, (x_{i_n}^n)_{i_n}) \rightsquigarrow \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes T(x_{i_1}^1, \cdots, x_{i_n}^n)$$

is continuous. Note that both $\overline{T}$ and $\widetilde{T}$ coincide with the restriction to finite sequences of the associated sequential operator $\widehat{T}$ whenever $n = 1$.

These associated sequential multilinear operators are the key for the study, from an unified point of view, of those multilinear operators that improve the summability of sequences by means of all tensor tools described in the introduction. They are the inspiration for the following definitions.

2.1. (γ, α, p)**-summing multilinear operators.** Let α and γ be two reasonable tensor norms for the tensor product $\ell^p \otimes F$ —including the case $\alpha = \Delta_p$—, that is, $\varepsilon \leq \alpha, \gamma \leq \pi$, and let $T : E_1 \times \cdots \times E_n \to F$ be a continuous n-linear operator. Consider the n-linear map

$$\overline{T} : (\ell_0^p \otimes E_1) \times (\ell_0^p \otimes E_2) \times \cdots \times (\ell_0^p \otimes E_n) \to \ell^p \otimes F$$

defined as

$$\overline{T}((\sum_{i=1}^m e_i \otimes x_i^1, \ldots, \sum_{i=1}^m e_i \otimes x_i^n)) := \sum_{i=1}^m e_i \otimes T(x_i^1, \ldots, x_i^n).$$

We say that $T : E_1 \times \cdots \times E_n \to F$ is (γ, α, p)*-summing* if

$$\overline{T} : (\ell_0^p \otimes_\gamma E_1) \times (\ell_0^p \otimes_\gamma E_2) \times \cdots \times (\ell_0^p \otimes_\gamma E_n) \to \ell^p \otimes_\alpha F$$

is continuous.

2.2. **Multiple** (γ, α, p)**-summing multilinear operators.** The "multiple" case can be defined in an analogous way. An n-linear operator $T : E_1 \times \cdots \times E_n \to F$ is *multiple* (γ, α, p)*-summing* if the associated operator

$$\widetilde{T} : (\ell_0^p \otimes_\gamma E_1) \times (\ell_0^p \otimes_\gamma E_2) \times \cdots \times (\ell_0^p \otimes_\gamma E_n) \to \ell^p \otimes_{\Delta_p} \cdots \otimes_{\Delta_p} \ell^p \otimes_\alpha F$$

given by

$$\widetilde{T}\Big(\sum_{i_1=1}^{m_1} e_{i_1} \otimes x_{i_1}^1, \ldots, \sum_{i_n=1}^{m_n} e_{i_n} \otimes x_{i_n}^n\Big) := \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes T(x_{i_1}^1, \ldots, x_{i_n}^n)$$

is continuous.

Obviously, the notions of (γ, α, p)-summing and multiple (γ, α, p)-summing coincide for $n = 1$. Several well known multi-ideals are particular cases of these definitions. In the next sections we will analyze some of them, mainly the ones associated to important notions of summability. To start with, the class of (multiple) p-summing multilinear operators clearly coincides with the class of (resp. multiple) $(\varepsilon, \Delta_p, p)$-summing operators.

Let us write now two linear examples. The first one is given by the so called (q, p)-mixing linear operators. Recall that a linear operator $T : E \to F$ is (q, p)-*mixing* if for each Banach space valued q-summing operator S, the composition $S \circ T$ is p-summing. It is well-known that they can be characterized by means of the following summing inequality,

$$\Big(\sum_{j=1}^m \Big(\sum_{k=1}^n |\langle T(x_j), y_k' \rangle|^q\Big)^{p/q}\Big)^{1/p} \leq C \|(x_j)_{j=1}^m\|_{p,w} \cdot \|(y_k')_{k=1}^n\|_q$$

for a certain constant $C > 0$ and for all $x_1, \ldots, x_m \in E$ and $y_1', \ldots, y_n' \in F^*$ (see for example [8, Proposition 32.4]). In [8, Proposition 32.3], it can be found that this ideal can be characterized as follows. An operator $T : E \to F$ is (q, p)-mixing if and only if

$$id \otimes T : \ell^{q'} \otimes_{g_{p'}} E \to \ell^{q'} \otimes_{\Delta_{q'}} F$$

is continuous. In this case, $id \otimes T = \overline{T} = \widetilde{T}$ and so, $T : E \to F$ is (q,p)-mixing if and only if it is (multiple) $(g_{p'}, \Delta_{q'}, q')$-summing.

The second example is given by the so called p-dominated (linear) operators $T : E \to F$, that are defined by the inequalities

$$\sum_{i=1}^{n} |\langle T(x_i), y_i' \rangle| \leq C \, \|(x_i)_{i=1}^n\|_{p,w} \cdot \|(y_i')_{i=1}^n\|_{p',w}$$

for a certain constant $C > 0$ and for all $x_1, \ldots, x_m \in E$ and $y_1', \ldots, y_n' \in F^*$. By [8, Proposition 29.5], an operator T is p-dominated if and only if the map

$$id \otimes T : \ell^p \otimes_\varepsilon E \to \ell^p \otimes_\pi F$$

is continuous. That is, if and only if it is (ε, π, p)-summing, or multiple (ε, π, p)-summing.

The next theorem is the main result in this paper. It gives some sort of *order reduction procedure* for factorizations of multilinear maps. We define the tensor norm α appearing in it using the so called "calculus of traced tensor norms", that can be found in [8, Ch.29]. Let us explain first this construction in the particular case that we are dealing with.

Let β and γ be tensor norms on $E \otimes \ell^{p'}$ and $\ell^p \otimes F$, respectively, and write $C : (E \otimes \ell^{p'}) \otimes (\ell^p \otimes F) \to E \otimes F$ for the tensor contraction given by

$$C\big((a \otimes u) \otimes (v \otimes b)\big) = \langle u, v \rangle a \otimes b.$$

The *traced tensor norm* $\gamma \otimes_{\ell^p} \beta$ is defined as the quotient norm defined by the projective tensor product, that is, for $z \in E \otimes F$,

$$\gamma \otimes_{\ell^p} \beta(z) = \inf \big\{ \pi(w) : w \in (E \otimes_\beta \ell^{p'}) \otimes_\pi (\ell^p \otimes_\gamma F) \text{ such that } C(w) = z \big\}.$$

(In this notation, notice the change of the order of β and γ with respect to the original order in the Cartesian product). In the particular case of the usual tensor norms (as Δ_p and ε), it is easy to prove that for example $d_p = \Delta_p \otimes_{\ell^p} \varepsilon$ (see Proposition 1 in [8, 29.3]).

Theorem 2.1. *Let $n \geq 2$, $1 \leq p < \infty$ and let γ be a tensor norm. Let $\alpha = \gamma \otimes_{\ell^p} \Delta_{p'}^t$. If $T : E_1 \times \cdots \times E_n \to F$ is an n-linear operator then, the following assertions are equivalent.*

(i) *T is multiple (γ, Δ_p, p)-summing.*

(ii) *The $(n-1)$-linear map*

$$\widetilde{T}_n : (\ell_0^p \otimes_\gamma E_1) \times \cdots \times (\ell_0^p \otimes_\gamma E_{n-1}) \to \Big(\big(\ell_0^{p'} \otimes_{\Delta_{p'}} \overset{n-1}{\cdots} \otimes_{\Delta_{p'}} \ell_0^{p'} \otimes_{\Delta_{p'}} F^* \big) \otimes_\alpha E_n \Big)^*$$

given by

$$\Big\langle \widetilde{T}_n\big((a_{i_1}^1)_{i_1=1}^{m_1}, \ldots, (a_{i_{n-1}}^{n-1})_{i_{n-1}=1}^{m_{m-1}}\big), \sum_{i_n=1}^{m_n} \Big(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b_{i_1,\ldots,i_n}^* \Big) \otimes a_{i_n}^n \Big\rangle$$

$$:= \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} \big\langle T(a_{i_1}^1, \cdots, a_{i_n}^n), b_{i_1,\ldots,i_n}^* \big\rangle$$

is well-defined and continuous.

Proof. First let us show that the map $\widetilde{T}_n$ is well-defined. Note that each element of the tensor product $\big(\ell_0^{p'} \otimes_{\Delta_{p'}} \overset{n-1}{\cdots} \otimes_{\Delta_{p'}} \ell_0^{p'} \otimes_{\Delta_{p'}} F^*\big)$ can be uniquely represented as $\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b^*_{i_1,\ldots,i_{n-1}}$. Indeed, clearly each element of the tensor product can be written in this way. Take now two representations

$$\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b^*_{i_1,\ldots,i_{n-1}}$$

and

$$\sum_{i_1=1}^{m'_1} \cdots \sum_{i_{n-1}=1}^{m'_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes (b^*)'_{i_1,\ldots,i_{n-1}}$$

of the same tensor. Completing with zeros when necessary, we can assume that $m_k = m'_k$ for all $k = 1, \ldots, n-1$. For each $j_1, \ldots, j_{n-1}$ with $1 \le j_k \le m_k, k = 1, \ldots, n-1$, and $b \in F$ consider $S_{j_1,\ldots,j_{n-1};b} := e_{j_1} \otimes \cdots \otimes e_{j_{n-1}} \otimes b \in \ell_0^p \otimes \overset{n-1}{\cdots} \otimes \ell_0^p \otimes F$ as an element in the dual $\big(\ell_0^{p'} \otimes_{\Delta_{p'}} \overset{n-1}{\cdots} \otimes_{\Delta_{p'}} \ell_0^{p'} \otimes_{\Delta_{p'}} F^*\big)^*$. Then,

$$S_{j_1,\ldots,j_{n-1};b}\Big(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b^*_{i_1,\ldots,i_{n-1}}\Big)$$

$$= \sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} S_{j_1,\ldots,j_{n-1};b}\big(e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b^*_{i_1,\ldots,i_{n-1}}\big) = \langle b, b^*_{j_1,\ldots,j_{n-1}}\rangle.$$

Then

$$S_{j_1,\ldots,j_{n-1};b}\Big(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes \big(b^*_{i_1,\ldots,i_{n-1}} - (b^*)'_{i_1,\ldots,i_{n-1}}\big)\Big)$$

$$= \langle b, b^*_{j_1,\ldots,j_{n-1}} - (b^*)'_{j_1,\ldots,j_{n-1}}\rangle = 0.$$

Since this can be done for every $b \in E$, this gives that $b^*_{j_1,\ldots,j_{n-1}} = (b^*)'_{j_1,\ldots,j_{n-1}}$ for every set of indexes $j_1, \ldots, j_{n-1}$.

Identifying $\ell_0^p \otimes E_j$ with a linear subspace of $\ell^p(E_j)$, fix $(a^j_{i_j})_{i_j=1}^{m_j}$ in $\ell^p \otimes E_j$, $j = 1, \ldots, n-1$, and write $\mathbf{a} := \big((a^1_{i_1})_{i_1=1}^{m_1}, \ldots, (a^{n-1}_{i_{n-1}})_{i_{n-1}=1}^{m_{n-1}}\big)$ for short. Let us see that $\widetilde{T}_n(\mathbf{a}) = \widetilde{T}_n\big((a^1_{i_1})_{i_1=1}^{m_1}, \ldots, (a^{n-1}_{i_{n-1}})_{i_{n-1}=1}^{m_{n-1}}\big)$ defines an element of the dual space $\Big(\big(\ell_0^{p'} \otimes_{\Delta_{p'}} \overset{n-1}{\cdots} \otimes_{\Delta_{p'}} \ell_0^{p'} \otimes_{\Delta_{p'}} F^*\big) \otimes_\alpha E_n\Big)^*$. Observe that given an element of the product $\Big(\ell_0^{p'} \otimes_{\Delta_{p'}} \overset{n-1}{\cdots} \otimes_{\Delta_{p'}} \ell_0^{p'} \otimes_{\Delta_{p'}} F^*\Big) \times E_n$ of the form $\Big(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b^*_{i_1,\ldots,i_{n-1}}, a^n\Big)$, the formula

$$B_{\widetilde{T}_n(\mathbf{a})}\Big(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b^*_{i_1,\ldots,i_{n-1}}, a^n\Big)$$

$$= \sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \langle T(a^1_{i_1}, \cdots, a^{n-1}_{i_{n-1}}, a^n), b^*_{i_1,\ldots,i_{n-1}}\rangle$$

defines a bilinear form whose linearization takes the same values as does the form $\widetilde{T}_n(\mathbf{a})$. Since the linearization of $B_{\widetilde{T}_n(\mathbf{a})}$ is defined on the tensor product space $\Big(\ell_0^{p'} \otimes_{\Delta_{p'}} \overset{n-1}{\cdots} \otimes_{\Delta_{p'}} \ell_0^{p'} \otimes_{\Delta_{p'}} F^*\Big) \otimes E_n$, the form $\widetilde{T}_n(\mathbf{a})$ can be considered an element of its algebraic dual space. Assuming (i), let us prove that it actually belongs to the topological dual when we consider the norm $\alpha = \gamma \otimes_{\ell^p} \Delta_{p'}^t$, that is, $\widetilde{T}_n(\mathbf{a}) \in \Big(\big(\ell_0^{p'} \otimes_{\Delta_{p'}} \overset{n-1}{\cdots} \otimes_{\Delta_{p'}} \ell_0^{p'} \otimes_{\Delta_{p'}} F^* \big) \otimes_\alpha E_n \Big)^*$. Take

$$\sum_{i_n=1}^{m_n} \Big(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b^*_{i_1,\dots,i_n} \otimes a^n_{i_n} \Big)$$

in $\big(\ell_0^{p'} \otimes_{\Delta_{p'}} \overset{n-1}{\cdots} \otimes_{\Delta_{p'}} \ell_0^{p'} \otimes_{\Delta_{p'}} F^* \big) \otimes E_n$ and define

$$S := \sum_{j_1=1}^{m_1} \cdots \sum_{j_n=1}^{m_n} e_{j_1} \otimes \cdots \otimes e_{j_{n-1}} \otimes T(a^1_{i_1}, \dots, a^n_{i_n}) \in \ell^p \otimes \cdots \otimes \ell^p \otimes F$$

as an element in $\big(\ell_0^{p'} \otimes_{\Delta_{p'}} \overset{n-1}{\cdots} \otimes_{\Delta_{p'}} \ell_0^{p'} \otimes_{\Delta_{p'}} F^* \big)^*$. Then,

$$\begin{aligned}
&\Big| \Big\langle \widetilde{T}_n(\mathbf{a}), \sum_{i_n=1}^{m_n} \Big(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b^*_{i_1,\dots,i_n} \otimes a^n_{i_n} \Big) \Big\rangle \Big| \\
&\qquad = \Big| \sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \sum_{i_n=1}^{m_n} \Big\langle T(a^1_{i_1}, \cdots, a^{n-1}_{i_{n-1}}, a^n_{i_n}), b^*_{i_1,\dots,i_n} \Big\rangle \Big| \\
&\qquad = \Big| \Big\langle \sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \sum_{i_n=1}^{m_n} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b^*_{i_1,\dots,i_n}, S \Big\rangle \Big| \\
&\qquad \le \|S\| \Delta_{p'} \Big(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \sum_{i_n=1}^{m_n} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b^*_{i_1,\dots,i_n} \Big). \qquad (3)
\end{aligned}$$

As we are assuming that T is multiple (γ, Δ_p, p)-summing we have that

$$\begin{aligned}
\|S\| &\le \Big\| \big(T(a^1_{i_1}, \dots, a^n_{i_n}) \big)_{i_1,\dots,i_n} \Big\|_p \\
&= \Delta_p \Big(\sum_{i_1=1}^{m_1} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \sum_{i_n=1}^{m_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes T(a^1_{i_1}, \dots, a^n_{i_n}) \Big) \\
&\le \|\widetilde{T}\| \prod_{1 \le k \le n} \gamma \Big(\sum_{i_k=1}^{m_k} e_{i_k} \otimes a^k_{i_k} \Big). \qquad (4)
\end{aligned}$$

Combining (3) with (4) we get $\widetilde{T}_n(\mathbf{a}) \in \Big(\big(\ell_0^{p'} \otimes_{\Delta_{p'}} \overset{n-1}{\cdots} \otimes_{\Delta_{p'}} \ell_0^{p'} \otimes_{\Delta_{p'}} F^* \big) \otimes_\alpha E_n \Big)^*$ and

$$\big\| \widetilde{T}_n(\mathbf{a}) \big\| \le \big\| \widetilde{T} \big\| \prod_{1 \le k \le n-1} \gamma \Big(\sum_{i_k=1}^{m_k} e_{i_k} \otimes a^k_{i_k} \Big)$$

that is, $\widetilde{T}_n$ is continuous and $\big\| \widetilde{T}_n \big\| \le \big\| \widetilde{T} \big\|$.

We now assume (ii) and let us prove that T is multiple (γ, Δ_p, p)-summing.

Fix sequences $(a^1_{i_1})_{i_1=1}^{m_1}, \dots, (a^n_{i_n})_{i_n=1}^{m_n}$ in the corresponding spaces and take an element

$$\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} e_{i_1} \otimes \cdots \otimes e_{i_{n-1}} \otimes b^*_{i_1,\dots,i_n}$$

in the unit ball of $\ell_0^{p'} \otimes_{\Delta_{p'}} \overset{n-1}{\cdots} \otimes_{\Delta_{p'}} \ell_0^{p'} \otimes_{\Delta_{p'}} F^*$ satisfying

$$\Delta_p\Big(\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} e_{i_1}\otimes\cdots\otimes e_{i_n}\otimes T(a^1_{i_1},\dots,a^n_{i_n})\Big)=\Big(\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n}\|T(a^1_{i_1},\dots,a^n_{i_n})\|^p\Big)^{1/p}$$

$$=\Big|\Big\langle\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} e_{i_1}\otimes\cdots\otimes e_{i_n}\otimes T(a^1_{i_1},\dots,a^n_{i_n}),\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} e_{i_1}\otimes\cdots\otimes e_{i_{n-1}}\otimes b^*_{i_1,\dots,i_n}\Big\rangle\Big|$$

$$=\Big|\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n}\Big\langle T(a^1_{i_1},\cdots,a^{n-1}_{i_{n-1}},a^n_{i_n}),b^*_{i_1,\dots,i_n}\Big\rangle\Big| \tag{5}$$

Using the continuity of $\widetilde{T}_n$ in the second inequality, we get

$$(5)=\Big|\Big\langle\widetilde{T}_n\big((a^1_{i_1})_{i_1},\dots,(a^{n-1}_{i_{n-1}})_{i_{n-1}}\big),\sum_{i_n=1}^{m_n}\Big(\sum_{i_1=1,\dots,i_{n-1}=1}^{m_1,\dots,m_{n-1}} e_{i_1}\otimes\cdots\otimes e_{i_{n-1}}\otimes b^*_{i_1,\dots,i_n}\otimes a^n_{i_n}\Big)\Big\rangle\Big|$$

$$\leq\|\widetilde{T}_n\|\prod_{1\leq k\leq n-1}\gamma\Big(\sum_{i_k=1}^{m_k} e_{i_k}\otimes a^k_{i_k}\Big)\alpha\left(\sum_{i_n=1}^{m_n}\Big(\sum_{i_1=1,\dots,i_{n-1}=1}^{m_1,\dots,m_{n-1}} e_{i_1}\otimes\cdots\otimes e_{i_{n-1}}\otimes b^*_{i_1,\dots,i_n}\otimes a^n_{i_n}\Big)\right)$$

$$\leq\|\widetilde{T}\|\prod_{1\leq k\leq n}\gamma\Big(\sum_{i_k=1}^{m_k} e_{i_k}\otimes a^k_{i_k}\Big)\Delta_{p'}\Big(\sum_{i_1=1,\dots,i_n=1}^{m_1,\dots,m_n} e_{i_1}\otimes\cdots\otimes e_{i_{n-1}}\otimes b^*_{i_1,\dots,i_n}\Big). \tag{6}$$

This proves that T is multiple (γ,Δ_p,p)-summing.

□

3. Bilinear multiple (γ,Δ_p,p)-summing operators

In this section several applications of Theorem 2.1 are provided for multiple (γ,Δ_p,p)-summing bilinear operators, with respect to classical tensor norms γ that are relevant for the general theory of operator ideals. The aim is to characterize the classes formed by these bilinear operators by means of the associated sequential operator.

Let $1\leq p<\infty$. A bilinear operator $T:E_1\times E_2\to F$ is *multiple p-summing* if there is a constant $C>0$ such that for $x_1,\dots,x_m\in E_1$ and $y_1,\dots,y_n\in E_2$,

$$\Big(\sum_{j=1}^m\sum_{i=1}^n\|T(x_j,y_i)\|^p\Big)^{1/p}\leq C\,\|(x_j)\|_{w,p}\,\|(y_i)\|_{w,p}$$

$$=C\sup_{x^*\in B_{E_1^*},y^*\in B_{E_2^*}}\Big(\sum_{j=1}^m\sum_{i=1}^n|\langle x_j,x^*\rangle|^p|\langle y_i,y^*\rangle|^p\Big)^{1/p}.$$

We will write $\pi_{m,p}(T)$ for the infimum of all constants satisfying the inequality above. As we said before, multiple p-summing operators coincide with mutiple (ε,Δ_p,p)-summing operators. We write now the characterization of this class of bilinear maps in terms of sequential properties.

Given a continuous bilinear operator $T:E_1\times E_2\to F$, the associated sequential operator associated to the linear operator $T_2:E_1\to\mathcal{L}(E_2,F)$, $T_2(x_1)(x_2):=T(x_1,x_2)$, is the operator $\widehat{T_2}:\ell^p(E_1)\to\mathcal{L}(E_2,\ell^p(F))$ given by $\widehat{T_2}((x_i)_i)(y)=(T(x_i,y))_i\in\ell^p(F)$, $y\in E_2$, $(x_i)_i\in\ell^p(E_1)$. It is easy to see that it is continuous, but this continuity does not characterize to be multiple p-summing. Next result does it.

Corollary 3.1. *Let $T : E_1 \times E_2 \to F$ be a continuous bilinear operator. The following statements are equivalent.*

(i) *T is multiple p-summing, that is, T is multiple $(\varepsilon, \Delta_p, p)$-summing.*

(ii) *$\widehat{T_2}$ maps $\ell_w^{p,0}(E_1)$ into $\Pi_p(E_2, \ell^p(F))$ continuously.*

Moreover, $\pi_{m,p}(T) = \|\widehat{T_2}\|$.

Proof. Recall that the bilinear map $T : E_1 \times E_2 \to F$ is multiple $(\varepsilon, \Delta_p, p)$-summing if

$$\widetilde{T} : (\ell_0^p \otimes_\varepsilon E_1) \times (\ell_0^p \otimes_\varepsilon E_2) \to (\ell^p \otimes_{\Delta_p} \ell^p) \otimes_{\Delta_p} F$$

is continuous. Consider

$$\alpha = \gamma \otimes_{\ell^p} \Delta_{p'}^t = \varepsilon \otimes_{\ell^p} \Delta_{p'}^t = g_{p'}$$

(see [8, S.29.3] for the last equality). Taking $\gamma = \varepsilon$ in Theorem 2.1, we get that the continuity of $\widetilde{T}$ is equivalent to the continuity of

$$\widetilde{T}_2 : \ell_0^p \otimes_\varepsilon E_1 \to \Big(\big(\ell_0^{p'} \otimes_{\Delta_{p'}} F^* \big) \otimes_{g_{p'}} E_2 \Big)^*.$$

Since

$$\Big(\big(\ell_0^{p'} \otimes_{\Delta_{p'}} F^*\big) \otimes_{g_{p'}} E_2 \Big)^* = \Big(\ell_0^{p'}(F^*) \otimes_{g_{p'}} E_2 \Big)^* = \Big(E_2 \otimes_{(g_{p'}^*)'} \ell_0^{p'}(F^*) \Big)^* = \Pi_p(E_2, (\ell_0^{p'}(F^*))^*)$$

(see [8, S.17.12] for the last equality) and taking into account that by its definition the evaluations of $\widehat{T_2} = \widetilde{T}_2$ takes values in $\ell_0^p(F)$, we get the result.

□

Inspired by the Chevet-Persson-Saphar inequalities

$$d_{p'}^* \le d_p \le \Delta_p \le g_{p'}^* \le g_p \tag{7}$$

on $\ell^p \otimes E$ (see 25.10 in [8]), let us write the results for the natural tensor norms associated to $\Delta_{q'}$ that provide weaker topologies, that are d_q^* and $d_{q'}$. We start giving the specific definitions of multiple (γ, Δ_p, p)-summing operators for $\gamma = d_q^*, d_{q'}$ in terms of inequalities. The continuous bilinear map $T : E_1 \times E_2 \to F$ is

- multiple (d_q^*, Δ_p, p)-summing if there is a constant $k > 0$ such that for all finite sequences $(x_j)_{j=1}^m$ and $(y_i)_{i=1}^n$ in E_1 and E_2, respectively, we have

$$\Big(\sum_{j=1}^m \sum_{i=1}^n \|T(x_j, y_i)\|^p \Big)^{1/p} \le k \, d_q^*\big((x_j)_{j=1}^m\big) \, d_q^*\big((y_i)_{i=1}^n\big),$$

  The norm of T as a multiple (d_q^*, Δ_p, p)-summing operator is the infimum of all such constants $k > 0$.
- multiple $(d_{q'}, \Delta_p, p)$-summing if there is a constant $k > 0$ such that for all finite sequences $(x_j)_{j=1}^m$ and $(y_i)_{i=1}^n$ in E_1 and E_2, respectively, we have

$$\Big(\sum_{j=1}^m \sum_{i=1}^n \|T(x_j, y_i)\|^p \Big)^{1/p} \le k \, d_{q'}\big((x_j)_{j=1}^m\big) \, d_{q'}\big((y_i)_{i=1}^n\big).$$

  The norm of T as a multiple $(d_{q'}, \Delta_p, p)$-summing operator is the infimum of all such constants $k > 0$.

As a consequence of (7) we have that multiple p-summing bilinear operators are multiple (d_q^*, Δ_p, p)-summing, and these ones are $(d_{q'}, \Delta_p, p)$-summing.

Corollary 3.2. *Let $T : E_1 \times E_2 \to F$ be a continuous bilinear operator and $1 \leq q < \infty$. The following statements are equivalent.*

(i) *T is multiple (d_q^*, Δ_p, p)-summing.*

(ii) *$\widetilde{T}_2$ maps $\ell_0^p \otimes_{d_q^*} E_1$ into $\mathcal{L}_{p,q}^{inj}(E_2, \ell^p(F))$ continuously.*

Moreover, the norm as multiple (d_q^*, Δ_p, p)-summing operator coincides with $\|\widetilde{T}_2\|$.

Proof. The proof is again an application of Theorem 2.1, but in this case $\gamma = d_q^*$, and

$$\widetilde{T}_2 : \ell_0^p \otimes_{d_q^*} E_1 \to \big((\ell_0^{p'} \otimes_{\Delta_{p'}} F^*) \otimes_\alpha E_2\big)^*$$

for

$$\alpha = d_q^* \otimes_{\ell^p} \Delta_{p'}^t = \backslash\alpha_{p,q}^*,$$

where the last equality follows from [8, Proposition 29.9]. There, it is also shown that $\alpha_{p,q}\backslash \sim \mathcal{L}_{p,q}^{inj}$, that is, $\alpha_{p,q}\backslash$ is the associated tensor norm to the operator ideal of the injective hull of the (p,q)-factorable operators. Since $\backslash\alpha_{p,q}^* = (\alpha_{p,q}\backslash)^* = \big((\alpha_{p,q}\backslash)'\big)^t$, by the representation theorem for the injective hull of the p,q-factorable operators (see [8, Theorem 17.5]) we get

$$\Big(\ell_0^{p'}(F^*) \otimes_{(\alpha_{p,q}\backslash)^*} E_2\Big)^* = \Big(E_2 \otimes_{(\alpha_{p,q}\backslash)'} \ell_0^{p'}(F^*)\Big)^* = \mathcal{L}_{p,q}^{inj}\big(E_2, (\ell_0^{p'}(F^*))^*\big),$$

which, again by the defintion of $\widehat{T_2} = \widetilde{T}_2$, gives the result. $\square$

Let us show now the case $\gamma = d_{q'}$. In this case $\alpha = d_{q'} \otimes_{\ell^p} \Delta_{p'}^t = \backslash\alpha_{p,q}^*/$, where the last equality follows from [8, Proposition 29.9]. The surjective hull of the injective hull of the ideal of p,q-factorable operators is the associated operator ideal to $(\backslash\alpha_{p,q}^*/)^*$, that is, $\mathcal{L}_{p,q}^{inj\,sur} \sim /\alpha_{p,q}\backslash = (\backslash\alpha_{p,q}^*/)^*$ (see [8, S.29.9]). Since $\backslash\alpha_{p,q}^*/$ is the projective associate to $\alpha_{p,q}$, it is finitely generated ([8, Corollary 20.6.2]) and so $\backslash\alpha_{p,q}^*/ = (\backslash\alpha_{p,q}^*/)''$ by [8, S. 15.3]. Then, by the representation theorem [8, Theorem 17.5] we get

$$\mathcal{L}_{p,q}^{inj\,sur}\big(E_2, (\ell_0^{p'}(F^*))^*\big) = \Big(E_2 \otimes_{(\backslash\alpha_{p,q}^*/^*)'} \ell_0^{p'}(F^*)\Big)^* = \Big(\ell_0^{p'}(F^*) \otimes_{\backslash\alpha_{p,q}^*/} E_2\Big)^*.$$

This gives the following corollary of Theorem 2.1.

Corollary 3.3. *Let $T : E_1 \times E_2 \to F$ be a continuous bilinear operator and $1 \leq q < \infty$. The following statements are equivalent.*

(i) *T is multiple $(d_{q'}, \Delta_p, p)$-summing.*

(ii) *$\widetilde{T}_2$ maps $\ell_0^p \otimes_{d_{q'}} E_1$ into $\mathcal{L}_{p,q}^{inj\,sur}(E_2, \ell_0^p(F))$ continuously.*

Moreover, the norm as multiple $(d_{q'}, \Delta_p, p)$-summing operator coincides with $\|\widetilde{T}_2\|$.

For the simplest case $q = p'$ the operator ideals involved are injective and surjective hulls of the ideal of the p-factorable operators ([8, Ch.18]). In particular, an operator belongs to $\mathcal{L}_p^{inj}$ if and only if it factors through a subspace of some $L^p(\mu)$, and the factorization norm given by the infimum of the products of the norms of the operators involved in each factorization for each suitable subspace coincides with the ideal norm (see the comments after Proposition 25.9 in [8]). In the same way, the ideal $\mathcal{L}_p^{inj\,sur}$ is characterized by factorizations through a subspace of a quotient of some L^p-space. This provides the following results.

- T is multiple $(d^*_{p'}, \Delta_p, p)$-summing if and only if there is a constant $k > 0$ such that for each sequence (x_i) in $\ell_0^p \otimes E_1$, the factorization norm $\inf \|A\| \cdot \|B\|$ of $\widetilde{T}_2((x_i)) = A \circ B$ through a subspace of an L^p-space satisfies
$$\inf_{\widetilde{T}_2((x_i))=A\circ B} \|A\| \cdot \|B\| \leq k\, d^*_{p'}((x_i)).$$
- T is multiple (d_p, Δ_p, p)-summing if and only if there exists a constant $k > 0$ such that for each sequence (x_i) in $\ell_0^p \otimes E_1$, the factorization norm $\inf \|A\| \cdot \|B\|$ of $\widetilde{T}_2((x_i)) = A \circ B$ through a subspace of a quotient of an L^p-space (equivalently, a quotient of a subspace of some L^p-space) satisfies
$$\inf_{\widetilde{T}_2((x_i))=A\circ B} \|A\| \cdot \|B\| \leq k\, d_p((x_i)).$$

4. Applications. The linear case again: multiple $\tau(p)$-summing and p-summing linear operators

In this section we will analyze with our tools the case of the so called $\tau(p)$-summing operators and their "multiple version". As we explained in Section 2.2, the multiple-type ideals of the usual classes of summing operators make sense in the multilinear case rather than in the linear one, since *linear* p-summing operators and *linear* multiple p-summing operators are the same thing. However, we will see in this section that $\tau(p)$-summing (linear) operators are naturally defined in the multilinear context rather than in the linear one, as happens for instance in the case of linear (p, q)-dominated operators (see for example [8, Ch.19]). This is so because the inequality in the definition involves the evaluation of the associated bilinear form $B_T(a, b') = \langle T(a), b'\rangle$ instead of $\|T(a)\|$. In fact, multiple $\tau(p)$-summing linear operators are the ones that satisfy that this bilinear form is p-summing. This makes natural to analyze both the original and the multiple cases. We will finish the section —and the paper— by showing some applications to summing operators on L^p-spaces.

4.1. $\tau(p)$-summing operators.

Let us consider now the $\tau(p)$-summing operators that were presented in the introduction. Mujica extended in [19] the τ-summing linear operators introduced by Pietsch in [26] to the context of the multilinear operators, generalizing also the class to the corresponding case $1 \leq p < \infty$. We shall consider only the linear case.

There are two known domination theorems for $\tau(p)$-summing operators. We will use the following one, that is a particular case of the characterization of Mujica for multilinear operators given in [19, Theorem 2.1] when $1 \leq p < \infty$. Mezrag and Tallab showed it for the Lipschitz case in [18].

Theorem 4.1. *(Theorem 2.1 in [19]). Let $T \in \mathcal{L}(E, F)$. Then there is a positive constant C such that the following assertions are equivalent.*

(1) *The operator T is $\tau(p)$-summing and $\pi_{\tau(p)}(T) \leq C$.*

(2) *There exist Borel probability measures μ on B_{E^*} and ν on $B_{F^{**}}$, such that for all a in E and b^* in F^*, we have*

$$|\langle T(a), b^* \rangle| \leq C\Big(\int_{B_{E^*}} \int_{B_{F^{**}}} |\langle a, a^* \rangle \langle b^*, b^{**} \rangle|^p \, d\nu(b^{**}) d\mu(a^*) \Big)^{\frac{1}{p}}. \tag{8}$$

Moreover, $\pi_{\tau(p)}(T) = \inf C$, *where the infimum is taken over all constants* $C > 0$ *fulfilling (8).*

If $T : E \to F$ is a continuous linear operator, its transpose is denoted by $T^* : F^* \to E^*$. The proof of the following lemma is straightforward.

Lemma 4.2. *Let* $T : E \to F$ *be a continuous linear operator. If* T *is* $\tau(p)$*-summing then* T *and* T^* *are* p*-summing.*

Consider the weak star topology on B_{X^*} and a probability Radon measure μ on B_{X^*}. Given the canonical evaluation mapping $e_X : X \to C(B_{X^*})$ and the formal identity mapping $j_p : C(B_{X^*}) \to L^p(\mu)$, define the linear subspace $S_E^\mu := j_p(e_X(X)) \subset L^p(\mu)$. For short we just denote $i_E^\mu : E \to S_E^\mu$ the composition $j_p \circ e_X : X \to S_E^\mu$.

Theorem 4.3. *Let* $T : E \to F$ *be a continuous linear operator. Let* $B_T : E \times F^* \to \mathbb{K}$ *be its associated bilinear map. The following statements are equivalent.*

(i) *T is $\tau(p)$-summing.*
(ii) *The (non-linear, non-bilinear) map* $T_\times : \ell_w^{p,0}(E \times F^*) \to \ell^p$ *given by* $T_\times((a_i, b_i^*)_i) := (\langle T(a_i), b_i^* \rangle)_i$, $(a_i, b_i^*) \in E \times F^*$ *for all* i, *is bounded with bound* C.
(iii) *There are Borel probability measures* η *and* ν *on* B_{E^*} *and* $B_{F^{**}}$ *respectively and a continuous bilinear form* $S : S_E^\mu \times S_{F^*}^\nu \to \mathbb{R}$ *such that the associated bilinear map* $B_T : E \times F^* \to \mathbb{K}$ *factors as*

$$\begin{array}{ccc} E \times F^* & \xrightarrow{\quad B_T \quad} & \mathbb{K}. \\ \Big\downarrow {\scriptstyle i_E^\mu \times i_{F^*}^\nu} & \nearrow_{S} & \\ S_E^\mu \times S_{F^*}^\nu \subseteq L^p(\eta) \times L^p(\nu) & & \end{array}$$

Moreover, $\pi_{\tau(p)}(T)$ *coincides with the infimum of the bounds* $C > 0$.

Proof. The equivalence between (i) and (ii) is clear: it follows from the inequality

$$\|(\langle T(a_i), b_i^* \rangle)_{i=1}^n\|_{\ell^p} = \left(\sum_{i=1}^n |\langle T(a_i), b_i^* \rangle|^p \right)^{\frac{1}{p}}$$

$$\leq C \sup_{\substack{\|a^*\|_{E^*} \leq 1 \\ \|b\|_F \leq 1}} \left(\sum_{i=1}^n |\langle a_i, a^* \rangle \langle b_i^*, b \rangle|^p \right)^{\frac{1}{p}} = C\|((a_i, b_i^*))_{i=1}^n\|_{p,w},$$

where $(a_i)_{i=1}^n$ is a sequence in E and $(b_i^*)_{i=1}^n$ is a sequence in F^*.

(i) $\Rightarrow$ (iii). Using Lemma 4.2, by the classical Pietsch factorization theorem there are Borel probabilities measures μ and ν on B_{E^*} and $B_{F^{**}}$ respectively and continuous linear operators $u : S_E^\mu \to F$ and $v : S_{F^*}^\nu \to E^*$such that $T = u \circ i_E^\mu$ and $T^* = v \circ i_{F^*}^\nu$. The bilinear mapping $R : u(S_E^\mu) \times v(S_{F^*}^\nu) \to \mathbb{K}$ given by $R(T(a), T^t(b^*)) = \langle T(a), b^* \rangle$ is well defined and the composition $S := R \circ (u \times v) :$

$S_E^{\mu} \times S_{F^*}^{\nu} \to \mathbb{K}$ is a bilinear form. The continuity of S follows from Theorem 4.1. A simple calculation shows that $S \circ (i_E^{\mu} \times i_{F^*}^{\nu}) = B_T$. Conversely, the continuity of S gives the inequality (2) in the domination Theorem 4.1, which finishes the proof. □

4.2. **Multiple $\tau(p)$-summing operators.** As it happens with the examples in Section 3, multiple $\tau(p)$-summing operators are closely related to $\tau(p)$-summing operators, but the inequality is required for matrices $(a_j)_{j=1}^m \times (b_i^*)_{i=1}^n$ and not just for the diagonals $((a_i, b_i^*))_i \subseteq E \times F^*$. This class can be defined just by considering the linear operators that satisfy that the associated bilinear form is multiple p-summing.

Let $1 \leq p < \infty$. We will say that a linear operator $T : E \to F$ is *multiple $\tau(p)$-summing* if there is a constant $C > 0$ such that for $a_1, \dots, a_m \in E$ and $b_1^*, \dots, b_n^* \in F^*$,

$$\sum_{j=1}^{m} \sum_{i=1}^{n} |\langle T(a_j), b_i^* \rangle|^p \leq C^p \sup_{a^* \in B_{E^*}, b^{**} \in B_{F^{**}}} \sum_{j=1}^{m} \sum_{i=1}^{n} |\langle a_j, a^* \rangle|^p |\langle b_i^*, b^{**} \rangle|^p.$$

We will write $\pi_{m,\tau(p)}(T)$ for the infimum of all constants satisfying the inequality above. As the reader can notice, multiple $\tau(p)$-summing operators are just operators that satisfy that its associated bilinear form $(a, b^*) \mapsto \langle T(a), b^* \rangle$ is multiple p-summing.

Note also that in this case, given a continuous linear operator $T : E \to F$, the associated sequential operator $\widehat{T}$ can be defined from $\ell_w^p(E)$ to $\ell_w^p(F)$. In this case, Corollary 3.1 reads as follows, which is an alternative straightforward way of getting the characterization obtained at the end of Section 1.

Corollary 4.4. *Let $T : E \to F$ be a continuous linear operator. Let $\widehat{T} : \ell_w^p(E) \to \ell_w^p(F)$ be the associated sequential operator induced by T. The following statements are equivalent.*

(i) *T is multiple $\tau(p)$-summing.*
(ii) *$\widehat{T}$ maps $\ell_w^{p,0}(E)$ into $\Pi_p(F^*, \ell^p)$ continuously.*

Moreover, $\pi_{m,\tau(p)}(T) = \|\widehat{T}\|$.

This result allows the comparison of our new space of operators with some classical ones. Let us show some direct consequences, all of them based in the continuous inclusions

$$\ell^p(F) = F \widehat{\otimes}_{\Delta_p} \ell^p \hookrightarrow F \widehat{\otimes}_{d'_{p'}} \ell^p \hookrightarrow F \widehat{\otimes}_{\varepsilon} \ell^p = \ell_w^{p,0}(F),$$

that we explain in what follows.

First of all, note that $\widehat{T}$ takes finite sequences to elements in the space $F \otimes \ell^p$. In the following formulas, the space $\ell^{p'}$ should be replaced by c_0 whenever $p' = \infty$. Since the tensor norm $d'_{p'}$ is totally accesible (see Theorem 21.5 in [8]), the natural map

$$F \widehat{\otimes}_{d'_{p'}} \ell^p \hookrightarrow \left(F^* \otimes_{d_{p'}} \ell^{p'}\right)^*$$

is in fact a metric injection by the duality theorem for tensor norms (see also 15.7 in [8], or [29]). Therefore, we have that

$$\Pi_p^0(F^*, \ell^p) := F \widehat{\otimes}_{d'_{p'}} \ell^p \hookrightarrow \left(F^* \otimes_{d_{p'}} \ell^{p'}\right)^* = \Pi_p(F^*, \ell^p). \tag{9}$$

Since $\varepsilon \leq d'_{p'}$ —it is a reasonable tensor norm— and $F\widehat{\otimes}_{\varepsilon}\ell^p = \ell^{p;0}_w(F)$ we have that the identification map i is continuous. This closes the factorization. The converse use the same identifications and is also easy to see. This proves the following

Corollary 4.5. *Let $1 \leq p < \infty$. The operator T is multiple $\tau(p)$-summing if and only if there is a factorization for the associated sequential operator $\widehat{T}$ as*

$$\begin{tikzcd} \ell^p_w(E) \arrow[rr, "\widehat{T}"] \arrow[dr, "\widehat{T}"'] & & \ell^p_w(F), \\ & \Pi^0_p(F^*, \ell^p) \arrow[ur, "i"'] & \end{tikzcd}$$

where the space $\Pi^0_p(F^*, \ell^p) := F\widehat{\otimes}_{d'_{p'}}\ell^p$ is (isometrically isomorphic to) a subspace of $\Pi_p(F^*, \ell^p)$.

We can also see that *if an operator $T : E \to F$ is p-summing, then it is multiple $\tau(p)$-summing* as a consequence of the factorization; it can also be proved by a direct calculation. Indeed, if T is p-summing $\widehat{T}$ is defined from $\ell^{p;0}_w(E)$ to $\ell^p(F)$. By the Chevet-Persson-Saphar inequalities (see 15.10 in [8]), we have that $d^*_{p'} \leq \Delta_p$ on $\ell^p \otimes F$, and so the continuity of the identification map

$$\ell^p(F) = \ell^p\widehat{\otimes}_{\Delta_p}F \to \ell^p\widehat{\otimes}_{d^*_{p'}}F = F\widehat{\otimes}_{d'_{p'}}\ell^p = \Pi^0_p(F^*, \ell^p).$$

Corollary 4.4 gives that T is multiple $\tau(p)$-summing.

4.3. **p-summing and multiple $\tau(p)$-summing operators on L^p-spaces.** Let us finish the paper with some applications to the representation of ideals of summing operators into L^p-spaces. One of the former papers that contributed to the success of the operator ideal theory and showed the central role of the p-summing operators in the modern functional analysis was the paper [14] by Lindenstrauss and Pełczynski. Closely related to the applications of Grothendieck's inequality, the results on the coincidence of p-summing operators with other operator ideals for particular —but relevant— spaces opened the door to some fruitful applications that are nowadays well-known (see for example Ch.1 in [11] and the references therein). In the same direction and from the tensorial point of view, the Chevet-Persson-Saphar inequalities (see 15.10 in [8]) provide different descriptions of the ideal of p-summing operators when the spaces involved are L^p-spaces (see [29]). In what follows we show one of them, that holds as a consequence of the results of the previous section when F is an L^p-space.

Corollary 4.6. *Let $1 \leq p \leq \infty$. Let μ be a measure and let $T : E \to L^p(\mu)$ be a continuous linear operator. Then T is multiple $\tau(p)$-summing if and only if T is p-summing.*

Proof. This is a consequence of Theorem 4.4 when $F = L^p(\mu)$. The right-to-left implication is a direct consequence of the comments at the end of the last subsection, and is true for any range space F. Assume now that T is multiple $\tau(p)$-summing. Note that $\ell^p\widehat{\otimes}_{\Delta_p}L^p(\mu) = \ell^p(L^p(\mu))$ and $\Delta_p = d^*_{p'}$ on $\ell^p \otimes L^p(\mu)$ (see Corollary 2 in 15.10 of [8], see also [8, 20.5] and [8, 25.10]). Thus,

$$\Pi^0_p(L^{p'}(\mu), \ell^p) = L^p(\mu)\widehat{\otimes}_{d'_{p'}}\ell^p = \ell^p\widehat{\otimes}_{d^*_{p'}}L^p(\mu) = \ell^p\widehat{\otimes}_{\Delta_p}L^p(\mu) = \ell^p(L^p(\mu)).$$

Corollary 4.5 yields then the following diagram

$$\begin{tikzcd}[column sep=small] \ell_w^{p,0}(L^p(\mu)) \arrow[rr, "\widehat{T}"] \arrow[dr, "\widehat{T}"'] & & \ell_w^p(L^p(\mu)), \\ & \Pi_p^0(L^{p'}(\mu), \ell^p) = \ell^p(L^p(\mu)) \arrow[ur, "i"'] & \end{tikzcd}$$

Therefore, T is p-summing. □

Another direct application can be obtained for the multilinear versions of $\tau(p)$-summing operators and multiple $\tau(p)$-summing operators by using the arguments of Ch.25 of [8]. In [8, Ch.25.10] coincidence results of the form $\ell^p \widehat{\otimes}_\alpha F = \ell^p(F)$ are provided for some tensor norms α whenever F is a subspace of a quotient of some L^p-space. Thus, we can extend the previous arguments for subspaces and quotients of L^p-spaces and for (γ, α, p)-summing operators and their multiple version to our broader class of multilinear operators. Let us write the (γ, α, p)-summing case; the result for the multiple case is similar. Recall that an n-linear operator $T : E_1 \times \cdots \times E_n \to F$ is (γ, α, p)-summing if the associated n-linear operator

$$\overline{T} : (\ell_0^p \otimes_\gamma E_1) \times (\ell_0^p \otimes_\gamma E_2) \times \cdots \times (\ell_0^p \otimes_\gamma E_n) \to \ell^p \otimes_\alpha F$$

defined in Section 2.1, is continuous.

As an application of [8, Proposition 25.10] we obtain the following result.

Corollary 4.7. *Let $E_1, \dots, E_n$ and F be Banach spaces, let $1 \leq p < \infty$, and consider an n-linear operator $T : E_1 \times \cdots \times E_n \to F$. If*

(i) *F is isomorphic to a subspace of some $L^p(\mu)$, and $\alpha = d_{p'}^*$ or $\alpha = g_{p'}^*$, or*

(ii) *F^{**} (or F if $1 < p < \infty$) is isomorphic to a quotient of some $L^p(\mu)$ and $\alpha = d_p$ or $\alpha = g_p$,*

then T is (γ, α, p)-summing if and only if it is (γ, Δ_p, p)-summing.

References

[1] D. Achour, E. Dahia, P. Rueda and E.A. Sánchez-Pérez, Factorization of strongly (p, σ)-continuous multilinear operators, Linear Multilinear Algebra 62 (2014), 1649–1670.

[2] R. Alencar and M. C. Matos, Some classes of multilinear mappings between Banach spaces, Publicaciones del Departamento de Análisis Matemático 12, Universidad Complutense Madrid, (1989).

[3] G. Botelho, H.-A. Braunss, H. Junek and D. Pellegrino, Inclusions and coincidences for multiple summing multilinear mappings, Proc. Amer. Math Soc. 137 (2009), 991–1000.

[4] G. Botelho, and D. Pellegrino, When every multilinear mapping is multiple summing. Math. Nach. 282(10) (2009), 1414–1422.

[5] G. Botelho, D. Pellegrino and P. Rueda, A unified Pietsch Domination Theorem, J. Math. Anal. Appl. 365 (2010), 269–276.

[6] E. Dahia, D. Achour and E.A. Sánchez-Pérez, Absolutely continuous multilinear operators, J. Math. Anal. Appl. 397 (2013), 205–224.

[7] A. Defant, Produkte von Tensornormen, Habilitationsschrift, U. Oldenburg, 1986.

[8] A. Defant and K. Floret, Tensor norms and operator ideals. North-Holland Mathematics Studies, 176. North-Holland Publishing Co., Amsterdam, 1993.

[9] A. Defant and D. Pérez-García, A tensor norm preserving unconditionality for L_p-spaces. Trans. Amer. Math. Soc. 360(6), (2008) 3287–3306.

[10] J. Diestel, J.H. Fourie and J. Swart, The metric theory of tensor products. Grothendieck's résumé revisited. American Mathematical Society, Providence, RI, 2008.
[11] J. Diestel, H. Jarchow, A. Tonge, Absolutely summing operators. Cambridge University Press, Cambridge, 1995.
[12] V. Dimant, Strongly p-summing multilinear operators, J. Math. Anal. Appl. 278 (2003), 182–193.
[13] S. Geiss, Ideale multilinearer Abbildungen, Diplomarbeit, Brandenburgische Landeshochschule, 1985.
[14] J. Lindenstrauss and A. Pełczynski, Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications, Studia Math. T.XXIX (1968), 275–326.
[15] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces II, Springer, Berlin, 1979.
[16] M. C. Matos, On multilinear mappings of nuclear type, Rev. Mat. Complut. 6 (1993), 61–81.
[17] M.C. Matos, Fully absolutely summing and Hilbert-Schmidt multilinear mappings. Collect. Math. 54 (2003), 2, 111–136.
[18] L. Mezrag and A. Tallab, On the Lipschitz $\tau(p)$-summing operators. Preprint.
[19] X. Mujica, $\tau(p;q)$-summing mappings and the domination theorem, Port. Math. (N. S.) 65 (2) (2008), 211–226.
[20] D. Pellegrino; P. Rueda and E. A. Sánchez Pérez, Surveying the spirit of absolute summability on multilinear operators and homogeneous polynomials. RACSAM. 110(1) (2016) 285–302.
[21] D. Pellegrino and J. Santos, Absolutely summing multilinear operators: a panorama. Quaest. Math., 34(4) (2011), 447–478.
[22] D. Pellegrino and J. Santos, A general Pietsch Domination Theorem, J. Math. Anal. Appl. 375 (2011), 371–374.
[23] D. Pellegrino, J. Santos and J. Seoane-Sepúlveda, Some techniques on nonlinear analysis and applications. Adv. Math. 229 (2012), 1235–1265.
[24] D. Pérez García and I. Villanueva, Multiple summing operators on Banach spaces. J. Math. Anal. Appl. 285(1) (2003) 86–96.
[25] A. Pietsch, Absolute p-summierende Abbildungen in normierten Räumen, Studia Math. 28 (1967), 333–353.
[26] A. Pietsch, Operator Ideals, North Holland, Amsterdam, 1980.
[27] A. Pietsch, Ideals of multilinear functionals (designs of a theory). Proceedings of the Second International Conference on Operator Algebras, Ideals, and their Applications in Theoretical Physics (Leibzig). Teubner-Texte (1983), 185–199.
[28] R.A. Ryan, Introduction to tensor products of Banach spaces. Springer Monographs in Mathematics, Springer, London, 2002.
[29] P. Saphar, Produits tensoriels d'espaces de Banach et classes d'applications linaires, Studia Math. 38 (1970) 71–100.

P. Rueda. Departamento de Análisis Matemático, Universidad de Valencia, C/ Dr. Moliner 50, 46100 Burjassot (Valencia). Spain., E.A. Sánchez Pérez. Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n, 46071 Valencia. Spain., H. Tallab. University of Med Boudiaf-M'sila, Laboratoire d'Analyse Fonctionnelle et Géométrie des Espaces, 28000 M'sila, Algeria.