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Additional Information

Diagonal entries of the combined matrix of a totally negative matrix *

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Abstract

The combined matrix of a nonsingular matrix A is the Hadamard (entrywise) product $A \circ \left(A^{-1}\right)^T$. This paper deals with the characterization of the diagonal entries of a combined matrix $C(A)$ of a given nonsingular real matrix A. A partial answer describing the diagonal entries of $C(A)$ in the positive definite case was given by Fiedler in 1964. Recently in 2011, Fiedler and Markham characterized the sequence of diagonal entries of the combined matrix $C(A)$ for any totally positive matrix A when the size is 3. For this case, we characterize totally negative matrices and we find necessary and sufficient conditions for the sequence of diagonal entries of $C(A)$, in both cases, symmetric and nonsymmetric.

Keywords: Totally negative matrices, Hadamard product, relative gain array, combined matrix, stochastic matrices

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1 Preliminaries

This paper deals with the characterization of the diagonal entries of the combined matrix $C(A)$ of a given real matrix A. A partial answer describing the diagonal entries of $C(A)$ in the positive definite matrix case was given by Fiedler [9]. Recently, Fiedler and Markham [11] characterized the sequence of the diagonal entries of the combined matrix $C(A)$ for any totally positive matrix A of size 3. In this work, we study this problem for totally negative matrices.

All matrices in this paper are real. Let us recall the basic definitions and properties. Some recent works on combined matrices are given in [3] for positive and negative matrices, in [4] for sign regular matrices and in [5] for H–matrices. Test to recognize total positive matrices has been studied in [15].

We recall that the **combined matrix** $C(A)$ of a nonsingular matrix A is defined as $C(A) = A \circ (A^{-1})^T = A \circ A^{-T}$, where $\circ$ is the Hadamard (entry wise) product, $(A \circ B)_{ij} = a_{ij} b_{ij}$. The combined matrix $C(A)$ has the property that $\sum_j c_{ij} = \sum_i c_{ij} = 1$ for rows and columns. As a consequence, if $C(A) \geq 0$, then $C(A)$ is a doubly stochastic matrix. In [13] the interesting relationship among diagonal entries and eigenvalues of a diagonalizable matrix is given. Moreover, if A represents the gain of a control process, $C(A)$ represents the relative gain of the process [2]. This last representation can be applied to economic or chemical problems.

A property of combined matrices we use later is the following: If we multiply the matrix A by a nonsingular diagonal matrix from the left or from the right, its combined matrix $C(A)$ does not change.

We recall that a matrix A is **totally positive (negative)** if all its minors of any order are positive (negative). That is, if for every subsets $\alpha, \beta \subseteq N$: $\det(A[\alpha, \beta]) > 0$ ($\det(A[\alpha, \beta]) < 0$). They are denoted by TP (t.n.).

We use the special submatrices defined in [10]. A submatrix of A of size $r \times s$ is called **relevant** if has r consecutive rows and the first s columns or if it has the first r rows and s consecutive columns, where $r, s \in N$.

Theorem 1.1 ([6]) *Let A be a $n \times n$ matrix. A is totally negative if and only if the minor of every relevant submatrix of A is negative.*

In the next section we extend the results for totally positive matrices, given by Fiedler and Markham in [11], to the totally negative case. We find

necessary and sufficient conditions for the sequence of diagonal entries of $C(A)$ if A is a 3×3 totally negative matrix.

2   Results

Note that in case $n = 2$,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to A^{-1} = [\alpha_{ij}] = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

and

$$a_{11}\alpha_{11} = a_{22}\alpha_{22}, \quad a_{12}\alpha_{12} = a_{21}\alpha_{21}.$$

Therefore, we establish the following lemmas only for $n \geq 3$.

Lemma 2.1 *Let $A = [-a_{ij}]$ be an $n \times n$ totally negative matrix. Let $n \geq 3$ and B is a $k \times k$, $k \leq n$, submatrix of A. Then there exists a $k \times k$ matrix Q with the same entries of B but the entry $-a_{kk}$ is changed by $-\hat{a}_{kk}$ with $\hat{a}_{kk} > a_{kk}$ and such that $\det Q = 0$.*

Proof: Without loss of generality, let $k = n$ and then $B = A$. Denote the (i, j) cofactor of B by C_{ij}. Then,

$$-a_{1k}C_{1k} - a_{2k}C_{2k} - \cdots - \left(a_{kk} + \frac{\det B}{C_{kk}} \right) C_{kk} = 0. \tag{1}$$

Constructing the matrix Q of the lemma choosing

$$-\hat{a}_{kk} = -a_{kk} - \frac{\det B}{C_{kk}},$$

we have $\hat{a}_{kk} > a_{kk}$, since B es totally negative, and $C_{kk} < 0$. Note that the left hand side of (1) is $\det Q$, therefore, $\det Q = 0$. $\qquad \square$

Lemma 2.2 *Let* $A = [-a_{ik}]$ *be an* $(n-1) \times n$ *totally negative matrix. If* $n \geq 3$ *and* $a_{11} = a_{12}$, *then*

$$
\det \begin{bmatrix} -a_{11} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n-1,1} & -a_{n-1,3} & \cdots & -a_{n-1,n} \end{bmatrix}
$$
$$
< \det \begin{bmatrix} -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{22} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n-1,2} & -a_{n-1,3} & \cdots & -a_{n-1,n} \end{bmatrix}. \tag{2}
$$

Proof: We use induction on n. If $n = 3$, the matrix A has the form

$$
A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}
$$

with $a_{11} = a_{12}$. Since A is totally negative, we have

$$
det \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} < 0
$$

and then $a_{22} < a_{21}$. Hence

$$
det \begin{bmatrix} -a_{11} & -a_{13} \\ -a_{21} & -a_{23} \end{bmatrix} < det \begin{bmatrix} -a_{12} & -a_{13} \\ -a_{22} & -a_{23} \end{bmatrix}.
$$

Suppose now that $n > 3$ and that for $(n-2) \times (n-1)$ size the result holds. We are going to apply Lemma 2.1 to obtain the submatrix Q such that

$$
\det Q = \det \begin{bmatrix} -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{22} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n-1,2} & -a_{n-1,3} & \cdots & -\hat{a}_{n-1,n} \end{bmatrix} = 0
$$

and $\hat{a}_{n-1,n} > a_{n-1,n}$. For that, given an $\epsilon > 0$, the matrix

$$
F_\epsilon = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n-1,1} & -a_{n-1,2} & \cdots & -\hat{a}_{n-1,n} + \epsilon \end{bmatrix}
$$

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has the same relevant submatrices that A except for the submatrix

$$
R_\epsilon = \begin{bmatrix} -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{22} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n-1,2} & -a_{n-1,3} & \cdots & -\hat{a}_{n-1,n} + \epsilon \end{bmatrix}.
$$

Observe that $\det R_\epsilon$ is equal to

$$
\det \begin{bmatrix} -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{22} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n-1,2} & -a_{n-1,3} & \cdots & -\hat{a}_{n-1,n} \end{bmatrix} + \epsilon \det \begin{bmatrix} -a_{12} & -a_{13} & \cdots & 0 \\ -a_{22} & -a_{23} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -a_{n-1,2} & -a_{n-1,3} & \cdots & 1 \end{bmatrix}
$$

where the first determinant is zero by Lemma 2.1 and the second one is negative because A is totally negative. Therefore, by Theorem 1.1, F_ϵ is totally negative. Then by continuity

$$
\det \begin{bmatrix} -a_{11} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n-1,1} & -a_{n-1,3} & \cdots & -\hat{a}_{n-1,n} \end{bmatrix} \leq 0. \tag{3}
$$

Denoting by C_1 and C_2 the cofactors of the element $-a_{n-1,n}$ in the left and right hand sides of equation (2), the left hand side of this equation can be written as

$$
\det \begin{bmatrix} -a_{11} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n-1,1} & -a_{n-1,3} & \cdots & (-a_{n-1,n} + \hat{a}_{n-1,n}) - \hat{a}_{n-1,n} \end{bmatrix}
$$

$$
= (-a_{n-1,n} + \hat{a}_{n-1,n})C_1 + \det \begin{bmatrix} -a_{11} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n-1,1} & -a_{n-1,3} & \cdots & -\hat{a}_{n-1,n} \end{bmatrix} \tag{4}
$$

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and the right hand side can be written as

$$
\det \begin{bmatrix}
-a_{12} & -a_{13} & \ldots & -a_{1n} \\
-a_{22} & -a_{23} & \ldots & -a_{2n} \\
\vdots & \vdots & & \vdots \\
-a_{n-1,2} & -a_{n-1,3} & \ldots & (-a_{n-1,n} + \hat{a}_{n-1,n}) - \hat{a}_{n-1,n}
\end{bmatrix}
$$

$$
= (-a_{n-1,n} + \hat{a}_{n-1,n})C_2 + \det \begin{bmatrix}
-a_{12} & -a_{13} & \ldots & -a_{1n} \\
-a_{22} & -a_{23} & \ldots & -a_{2n} \\
\vdots & \vdots & & \vdots \\
-a_{n-1,2} & -a_{n-1,3} & \ldots & -\hat{a}_{n-1,n}
\end{bmatrix}. \quad (5)
$$

Recall that $a_{n-1,n} < \hat{a}_{n-1,n}$ by Lemma 2.1, $C_1 < C_2$ by induction hypothesis, the determinant in (4) is less than or equal to zero by (3) and the determinant in (5) is zero by Lemma 2.1. Then, the inequality (2) is true. $\square$

Note that the diagonal elements of $C(A) = A \circ A^{-T}$ and those of the matrix $A \circ A^{-1}$ are equal. Then, results referred to diagonal elements can be established for both matrices.

Theorem 2.1 *Let $A = [-a_{ij}]$ be an $n \times n$ totally negative matrix. If $n \geq 3$, the diagonal entries u_i of $C(A)$ satisfy: $u_i < 0$, $i = 1, 2, \ldots, n$ and*

$$
u_1 > u_2, \quad (6)
$$
$$
u_{n-1} < u_n. \quad (7)
$$

Proof: The negativity of the diagonal entries is evident, since A is totally negative and

$$
u_i = -a_{ii}\frac{A_{ii}}{\det A} < 0, \quad i = 1, 2, \ldots, n.
$$

Since $C(A)$ does not change if we multiply a row or a column by a positive number, we can assume that $a_{11} = a_{12} = a_{22} = 1$ and then our problem is to show inequalities (6) and (7) for the matrix

$$
A = \begin{bmatrix}
-1 & -1 & -A_{13} \\
-a_{21} & -1 & -A_{23} \\
-A_{31} & -A_{32} & -A_{33}
\end{bmatrix}
$$

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where A_{ij} are the appropriate submatrices of A. Since

$$u_1 = \frac{-1}{\det(A)} \det \begin{bmatrix} -1 & -A_{23} \\ -A_{32} & -A_{33} \end{bmatrix} \quad \text{and} \quad u_2 = \frac{-1}{\det(A)} \det \begin{bmatrix} -1 & -A_{13} \\ -A_{31} & -A_{33} \end{bmatrix}$$

the proof of (6) reduces to prove the inequality

$$\det \begin{bmatrix} -1 & -A_{13} \\ -A_{31} & -A_{33} \end{bmatrix} < \det \begin{bmatrix} -1 & -A_{23} \\ -A_{32} & -A_{33} \end{bmatrix}. \tag{8}$$

If we apply Lemma 2.2 to the following totally negative submatrices of A

$$A' = A[1,3|1,2,3] = \begin{bmatrix} -1 & -1 & -A_{13} \\ -A_{31} & -A_{32} & -A_{33} \end{bmatrix}$$

$$A'' = A[1,2,3|2,3] = \begin{bmatrix} -1 & -A_{13} \\ -1 & -A_{23} \\ -A_{32} & -A_{33} \end{bmatrix}$$

we have

$$\det \begin{bmatrix} -1 & -A_{13} \\ -A_{31} & -A_{33} \end{bmatrix} < \det \begin{bmatrix} -1 & -A_{13} \\ -A_{32} & -A_{33} \end{bmatrix}$$

and

$$\det \begin{bmatrix} -1 & -A_{13} \\ -A_{32} & -A_{33} \end{bmatrix} < \det \begin{bmatrix} -1 & -A_{23} \\ -A_{32} & -A_{33} \end{bmatrix}.$$

Then, the inequality (8) holds.

To prove inequality (7), we use the following: if $A = [-a_{ij}]$ is an $n \times n$ totally negative matrix and J is the $n \times n$ anti-identity matrix, the matrix JAJ is also totally negative. Then, since

$$\text{diag}(JAJ) = (-a_{nn}, -a_{n-1,n-1}, \ldots, -a_{11}),$$

applying (6) to JAJ we have $u_n > u_{n-1}$ and the proof is completed. □

Example 1 *Let the t.n. symmetric matrix*

$$A = \begin{bmatrix} -20 & -16 & -4 \\ -16 & -12.2 & -2.8 \\ -4 & -2.8 & -0.2 \end{bmatrix}.$$

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Then

$$A \circ A^{-1} = C(A) = \begin{bmatrix} -27 & 32 & -4 \\ 32 & -36.6 & 5.6 \\ -4 & 5.6 & -0.6 \end{bmatrix}$$

where $u_1 = -27$, $u_2 = -36.6$, $-u_3 = -0.6$ and the inequalities of Theorem 2.1 are satisfied.

Example 2 *Given the t.n. nonsymmetric matrix*

$$B = \begin{bmatrix} -1 & -2 & -3 & -4 \\ -15 & -15 & -15 & -15 \\ -15 & -14 & -12 & -8 \\ -30 & -27 & -20 & -3 \end{bmatrix}$$

then

$$\mathrm{diag}(B \circ B^{-1}) = \mathrm{diag}(C(B)) = (-3, -110.5, -108, -1.5)$$

and so, the inequalities of Theorem 2.1 hold.

Note that this example gives also examples of the same result for different 3×3 t.n. matrices.

3   Sequence of diagonal entries

Now let us formulate our problem: find necessary and sufficient conditions for an ordered n-tuple of real numbers to be the ordered n-tuple of the diagonal entries of $C(A)$ if A is a totally negative $n \times n$ matrix. Let us proceed in the same way as in [11], so we shall study and prove the result in the 3×3 symmetric and nonsymmetric cases.

3.1   Symmetric case

Lemma 3.1 *A 3×3 symmetric matrix is totally negative if and only if it is positively diagonally congruent to the matrix*

$$T = \begin{bmatrix} -1 & -x_3 & -x_2 \\ -x_3 & -1 & -x_1 \\ -x_2 & -x_1 & -1 \end{bmatrix}, \tag{9}$$

where x_1, x_2, x_3 are positive numbers satisfying

$$x_i > 1, \quad i = 1, 2, 3, \tag{10}$$

$$x_2 > x_1 x_3 \quad and \tag{11}$$

$$-1 - 2x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 = \det T < 0. \tag{12}$$

Proof: ($\Rightarrow$) Let $A = [-a_{ij}]$ be a 3×3 symmetric totally negative matrix and let D be the positive diagonal matrix

$$D = \begin{bmatrix} \sqrt{a_{11}} & 0 & 0 \\ 0 & \sqrt{a_{22}} & 0 \\ 0 & 0 & \sqrt{a_{33}} \end{bmatrix}.$$

Then, $A = DTD$, where

$$T = \begin{bmatrix} -1 & -\frac{a_{12}}{\sqrt{a_{11}a_{22}}} & -\frac{a_{13}}{\sqrt{a_{11}a_{33}}} \\ -\frac{a_{12}}{\sqrt{a_{11}a_{22}}} & -1 & -\frac{a_{23}}{\sqrt{a_{22}a_{33}}} \\ -\frac{a_{13}}{\sqrt{a_{11}a_{33}}} & -\frac{a_{23}}{\sqrt{a_{22}a_{33}}} & -1 \end{bmatrix}.$$

Therefore, A is positively diagonally congruent to a matrix T as in (9), where

$$x_1 = \frac{a_{23}}{\sqrt{a_{22}a_{33}}}, \qquad x_2 = \frac{a_{13}}{\sqrt{a_{11}a_{33}}}, \qquad x_3 = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}$$

are positive numbers.

Since A is totally negative $A_{11} = a_{22}a_{33} - (a_{23})^2 < 0$, then $a_{23} > \sqrt{a_{22}a_{33}}$ and $x_1 > 1$. Similarly, we conclude that $x_2 > 1$ and $x_3 > 1$ since $A_{22} < 0$ and $A_{33} < 0$ respectively. Then condition (10) holds.

On the other hand, $A_{31} = a_{12}a_{23} - a_{22}a_{13} < 0$ implies that $a_{13} > \frac{a_{12}a_{23}}{a_{22}}$ and $\frac{a_{13}}{\sqrt{a_{11}a_{33}}} > \frac{a_{12}a_{23}}{a_{22}\sqrt{a_{11}a_{33}}}$. As a consequence

$$\frac{a_{13}}{\sqrt{a_{11}a_{33}}} > \frac{a_{23}}{\sqrt{a_{22}a_{33}}} \cdot \frac{a_{12}}{\sqrt{a_{11}a_{22}}},$$

then $x_2 > x_1 x_3$ and condition (11) is true.

Finally, inequality (12) holds because $\det T = \frac{\det A}{(\det D)^2}$ and A is totally negative.

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($\Leftarrow$) Since we have to prove that DTD is t.n. for a positive diagonal matrix D, it is enough to prove that the matrix T given in (9) satisfying conditions (10) - (12) is a t.n. matrix. All 1×1 and 3×3 minors are negative by (10) and (12) inequalities. Denote by T_{ij} the corresponding 2×2 minors of T. The 2×2 minors corresponding to diagonal entries satisfy $T_{ii} = 1 - x_i{}^2 < 0$ by (10). Now we need only to show that 2×2 minors T_{ij} for $i \neq j$ are negative. Since T is symmetric, $T_{13} = T_{31} = x_1 x_3 - x_2 < 0$ by (11). The minors $T_{12} = T_{21} = x_3 - x_1 x_2$ are negative since

$$x_2 > x_1 x_3 \quad \text{and} \quad x_1 > 1 \quad \Rightarrow \quad x_1 x_2 > x_1^2 x_3 > x_3.$$

Finally, $T_{23} = T_{32} = x_1 - x_2 x_3 < 0$ can be obtained by multiplying (11) by $x_3 > 1$. $\qquad \square$

Note that inequality (11) in Lemma 3.1 can be replaced by $x_1 x_2 > x_3$ or by $x_2 x_3 > x_1$.

Theorem 3.1 *The necessary and sufficient condition for three negative numbers u_1, u_2 and u_3, be the diagonal entries of the combined matrix of a 3×3 symmetric totally negative matrix A is*

$$u_1 + u_3 - u_2 - 1 > 0. \tag{13}$$

Proof: Let A be a 3×3 symmetric totally negative matrix. By Lemma 3.1 A is positively diagonally congruent to the matrix T of (9). Since $C(DA) = C(AD) = C(A)$ for any nonsingular diagonal matrix D, it follows that $C(A) = C(T)$. Thus we can rewrite our theorem in terms of T. Note that the diagonal entries of $C(T)$ are

$$u_1 = \frac{-(1 - x_1{}^2)}{\det T}, \quad u_2 = \frac{-(1 - x_2{}^2)}{\det T}, \quad u_3 = \frac{-(1 - x_3{}^2)}{\det T}, \tag{14}$$

and then

$$u_1 + u_3 - 1 - u_2 = \frac{1}{\det T}(2x_2)(x_1 x_3 - x_2). \tag{15}$$

($\Rightarrow$) By equation (15) we have $u_1 + u_3 - 1 - u_2 > 0$ since $\det T < 0$, $x_1 x_3 < x_2$ and $x_2 > 0$.

($\Leftarrow$) Note that $x_i{}^2 = 1 + u_i \det T$, $i = 1, 2, 3$ by equations (14). Adding these values on equation (12), we have

$$\det T = -1 - 2\sqrt{(1 + u_1 \det T)(1 + u_2 \det T)(1 + u_3 \det T)}$$
$$+ (1 + u_1 \det T) + (1 + u_2 \det T) + (1 + u_3 \det T),$$

which can be written as

$$4(\det T)^2 u_1 u_2 u_3 + \det T[4(u_1 u_2 + u_1 u_3 + u_2 u_3)$$
$$- (u_1 + u_2 + u_3 - 1)^2] + 4 = 0. \qquad (16)$$

Let us see that this second degree equation written in brief as $a(\det T)^2 + b(\det T) + c = 0$ has a negative root. Its discriminant satisfies $b^2 - 4ac > b^2$, since $-4ac = -64u_1 u_2 u_3 > 0$. Therefore always $-b + \sqrt{b^2 - 4ac} > 0$. Then

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0,$$

since $a = 4u_1 u_2 u_3 < 0$. Hence there exists a solution satisfying $\det T < 0$.

Coefficients x_i satisfy $x_i > 1$ since $\det T < 0$ and $u_i < 0$. Then inequalities in (10) hold.

Moreover, by (15), we obtain $x_1 x_3 - x_2 < 0$ since $\det T < 0$ and $x_2 > 0$. Then (11) is true. $\qquad \square$

Example 3 *It is easy to see that for the following 3×3 t.n. symmetric matrix A, the diagonal entries of $C(A)$ satisfy equation (13):*

$$A = \begin{bmatrix} -1 & -3 & -8 \\ -3 & -1 & -2 \\ -8 & -2 & -1 \end{bmatrix} \rightarrow C(A) = \begin{bmatrix} -0.15 & 1.95 & -0.8 \\ 1.95 & -3.25 & 2.2 \\ -0.8 & 2.2 & -0.4 \end{bmatrix}$$

where $(u_1 + u_3 - 1) = -1.55 > -3.15 = u_2$.

Now let us see a converse example.

Example 4 *Given $u_1 = -2$, $u_2 = -8$, $u_3 = -4$ such that $u_1 + u_3 - u_2 - 1 = 1 > 0$, we will find a symmetric t.n. matrix T such that $\mathrm{diag}(C(T)) = (u_1, u_2, u_3)$.*

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Replacing the given values in (16) it is obtained

$$-256(\det T)^2 - \det T + 4 = 0.$$

This equation has a solution $\det T < 0$, required by the theorem,

$$\det T = -\frac{1 + \sqrt{4097}}{512} \approx -0.1269684.$$

Using equations $x_i = \sqrt{1 + u_i \det T}$, $i = 1, 2, 3$ we have

$$x_1 \approx 1.1197932, \quad x_2 \approx 1.4197701, \quad x_3 \approx 1.227955,$$

which satisfy $x_1 x_3 - x_2 < 0$. Then, the conditions of Lemma 3.1 are satisfied and T is a t.n. matrix. The combined matrix of T is

$$C(T) = \begin{bmatrix} -2 & 3.5 & -0.5 \\ 3.5 & -8 & 5.5 \\ -0.5 & 5.5 & -4 \end{bmatrix}.$$

where $\operatorname{diag}(C(T)) = (-2, -8, -4)$, just as we expected.

3.2 Nonsymmetric case

Let us turn now to the case of a general totally negative 3×3 matrix. First we prove the next lemma.

Lemma 3.2 *A 3×3 matrix A is totally negative if and only if it is positively diagonally equivalent to the matrix*

$$T = \begin{bmatrix} -\frac{v_1}{v_1 + 1} & -1 & -p \\ -1 & -1 & -1 \\ -q & -1 & -\frac{v_3}{v_3 + 1} \end{bmatrix}, \tag{17}$$

where

$$v_i > 0, \quad i = 1, 3, \tag{18}$$

$$p, q > 1, \tag{19}$$

$$\det T = \frac{(p-1)(q-1)(v_1 + 1)(v_3 + 1) - 1}{(v_1 + 1)(v_3 + 1)} < 0. \tag{20}$$

Proof: ($\Rightarrow$) Let $A = [-a_{ij}]$ a 3×3 totally negative matrix. Constructing

$$T = \begin{bmatrix} \frac{-a_{11}a_{22}}{a_{12}a_{21}} & -1 & \frac{-a_{13}a_{22}}{a_{12}a_{23}} \\ -1 & -1 & -1 \\ \frac{-a_{31}a_{22}}{a_{32}a_{21}} & -1 & \frac{-a_{33}a_{22}}{a_{32}a_{23}} \end{bmatrix}$$

and

$$D_1 = \begin{bmatrix} a_{12} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{32} \end{bmatrix}, \quad D_2 = \begin{bmatrix} \frac{a_{21}}{a_{22}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{a_{23}}{a_{22}} \end{bmatrix},$$

we have that $A = D_1 T D_2$. Taking

$$v_1 = \frac{a_{11}a_{22}}{a_{12}a_{21} - a_{11}a_{22}}, \quad v_3 = \frac{a_{33}a_{22}}{a_{32}a_{23} - a_{33}a_{22}}, \quad p = \frac{a_{13}a_{22}}{a_{12}a_{23}} \text{ and } q = \frac{a_{31}a_{22}}{a_{21}a_{23}},$$

the matrix T has the form (17).

As A is a totally negative matrix, $A_{33} < 0$. This implies $a_{11}a_{22} - a_{21}a_{12} < 0$ thus $v_1 > 0$. In a similar way, $A_{11} < 0$ implies $a_{33}a_{22} - a_{32}a_{23} < 0$, then $v_3 > 0$. Moreover, since $A_{31} = a_{12}a_{23} - a_{13}a_{22} < 0$ we have $p > 1$. From $A_{13} < 0$ we obtain $q > 1$. Then, conditions (18) and (19) hold.

Finally, it is easy to check that condition (20) corresponds to $\det T$ and $\det T < 0$ since $\det A = \det D_1 \det T \det D_2 < 0$ and A is t.n.

($\Leftarrow$) We need to show that if A is positively diagonally equivalent to T satisfying the corresponding conditions, then A is totally negative. Our proof reduces to show that T is totally negative.

By conditions (18) to (20), entries t_{ij} and $\det T$ are negative. If we define

$$\frac{v_1}{v_1 + 1} = r \quad \text{and} \quad \frac{v_3}{v_3 + 1} = s,$$

then matrix T has the structure

$$T = \begin{bmatrix} -r & -1 & -p \\ -1 & -1 & -1 \\ -q & -1 & -s \end{bmatrix},$$

where $0 < r, s < 1$ and $p, q > 1$. From here is clear that all minors of T of order 2 are negative and then T is t.n. $\square$

Theorem 3.2 *The necessary and sufficient condition for three negative numbers u_1, u_2 and u_3, be the diagonal entries of the combined matrix of a 3×3 totally negative matrix A is*

$$u_1 + u_3 - u_2 - 1 > 0. \tag{21}$$

Proof: Let A be a 3×3 totally negative matrix. By Lemma 3.1 A is positively diagonally equivalent to a matrix T as in (17). Considering that $C(DA) = C(AD) = C(A)$ for all nonsingular diagonal matrix D, it follows that $C(A) = C(T)$. Then, we assume that $A = T$.

($\Rightarrow$) With the above consideration we will prove that given T defined by (17) and satisfying conditions (18), (19), (20) and $\mathrm{diag}(C(T)) = (u_1, u_2, u_3)$, then $u_1 + u_3 - u_2 - 1 > 0$.

First, we denote the diagonal entries of T^{-1} by $(\hat{a}_{11}, \hat{a}_{22}, \hat{a}_{33})$ and by d the numerator of the fraction (20), Then

$$\hat{a}_{11} = \frac{1}{\det T} \left(\frac{v_3}{v_3 + 1} - 1 \right) = \frac{1}{\det T} \left(\frac{-1}{v_3 + 1} \right) = \frac{-(v_1 + 1)}{d},$$

$$\hat{a}_{22} = \frac{1}{\det T} \left(\frac{v_1 v_3}{(v_1 + 1)(v_3 + 1)} - pq \right) = \frac{v_1 v_3 - pq(v_1 + 1)(v_3 + 1)}{d},$$

$$\hat{a}_{33} = \frac{1}{\det T} \left(\frac{v_1}{v_1 + 1} - 1 \right) = \frac{-(v_3 + 1)}{d}.$$

Thus, the diagonal entries of $C(A) = A \circ (A^{-T})$ are

$$u_1 = \frac{v_1}{d}, \quad u_2 = -\frac{v_1 v_3 - pq(v_1 + 1)(v_3 + 1)}{d}, \quad u_3 = \frac{v_3}{d}. \tag{22}$$

Replacing equations (22) in the left part of condition (21), we have

$$u_1 + u_3 - u_2 - 1 = \frac{1}{d}(v_1 + v_3 + v_1 v_3 - pq(v_1 + 1)(v_3 + 1) - d).$$

Taking into account $v_1 + v_3 + v_1 v_3 = (v_1 + 1)(v_3 + 1) - 1$, we have

$$u_1 + u_3 - u_2 - 1 = \frac{1}{d}(v_1 + 1)(v_3 + 1)(1 - pq - (p - 1)(q - 1))$$

is a positive number by conditions (18)–(20).

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($\Leftarrow$) Now we will prove that if u_1, u_2, u_3 are negative numbers such that

$$u_1 + u_3 - u_2 - 1 > 0,$$

then we can built a matrix T with the structure (17) satisfying (18)–(20) such that u_1, u_2, u_3 are the diagonal entries of $C(T)$. That is, we are going to prove that the system

$$u_1 = \frac{v_1}{d}, \quad u_2 = -\frac{v_1 v_3 - pq(v_1 + 1)(v_3 + 1)}{d}, \quad u_3 = \frac{v_3}{d}, \tag{23}$$

has the expected solutions.

Observe that $u_2 d = pq(v_1 + 1)(v_3 + 1) - v_1 v_3$ implies

$$pq = \frac{u_2 d + u_1 u_3 d^2}{(u_1 d + 1)(u_3 d + 1)},$$

and this implies

$$\frac{1}{p}\frac{1}{q} = \frac{(u_1 d + 1)(u_3 d + 1)}{u_2 d + u_1 u_3 d^2}. \tag{24}$$

From definition of d, it follows

$$(p - 1)(q - 1) = \frac{(d + 1)}{(u_1 d + 1)(u_3 d + 1)}. \tag{25}$$

From equation (24) it follows

$$(u_1 d + 1)(u_3 d + 1) = \frac{u_2 d + u_1 u_3 d^2}{pq}. \tag{26}$$

Replacing (26) in (25), we have

$$(p - 1)(q - 1) = \frac{(d + 1)pq}{u_2 d + u_1 u_3 d^2}.$$

Finally

$$(1 - \frac{1}{p})(1 - \frac{1}{q}) = \frac{d + 1}{u_2 d + u_1 u_3 d^2}. \tag{27}$$

Our system of equations (23) has been reduced to equations (24) and (27).

It is well known that a necessary and sufficient condition for the system of equations $xy = u$, $(1 - x)(1 - y) = v$ has solutions $x, y \in]0, 1[$ is that

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$\sqrt{u} + \sqrt{v} \leq 1$. Using this equivalence with $x = \dfrac{1}{p}$, $y = \dfrac{1}{q}$, the system given by equations (24) and (27) has solutions in $]0, 1[$ if and only if

$$\sqrt{\frac{(u_1 d + 1)(u_3 d + 1)}{u_2 d + u_1 u_3 d^2}} + \sqrt{\frac{d + 1}{u_2 d + u_1 u_3 d^2}} \leq 1. \qquad (28)$$

If we replace $d = -1$ in left side of (28) and use equation (23) we have

$$\sqrt{\frac{(-u_1 + 1)(-u_3 + 1)}{-u_2 + u_1 u_3}} = \sqrt{\frac{(v_1 + 1)(v_3 + 1)}{pq(v_1 + 1)(v_3 + 1) - v_1 v_3 + v_1 v_3}} = \sqrt{\frac{1}{pq}} < 1.$$

This inequality is strict since $u_1 + u_3 - u_2 - 1 > 0$ implies $(1 - u_1)(1 - u_3) < -u_2 + u_1 u_3$. Thus, there exists an small enough $0 < \varepsilon < 1$ such that for $d = -1 + \varepsilon$ condition (28) holds. Then, there exist $p, q > 1$ satisfying the system given by (24) and (27). This implies $v_1 = du_1 > 0$, $v_3 = du_3 > 0$ and $\det T = \dfrac{d}{(v_1 + 1)(v_3 + 1)} < 0$. $\qquad \square$

The following example illustrates this last theorem.

Example 5 *Let A be the 3×3 nonsingular nonsymmetric totally negative matrix and A^{-1} its inverse*

$$A = \begin{bmatrix} -8 & -14 & -19 \\ -13 & -20 & -27 \\ -17 & -26 & -35 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 21.5 & -15.5 \\ 1 & -15 & 11 \end{bmatrix}$$

The combined matrix of A is

$$C(A) = \begin{bmatrix} -8 & 28 & -19 \\ 26 & -430 & 405 \\ -17 & 403 & -385 \end{bmatrix}.$$

As we can see the diagonal entries are negative numbers and satisfy the required condition.

Now let us see a converse example.

Example 6 *Given negative numbers $u_1 = -2$, $u_2 = -8$, $u_3 = -3$ satisfying condition $u_1 + u_3 - u_2 - 1 > 0$, we will find a matrix T such that $\mathrm{diag}(C(T)) = (u_1, u_2, u_3)$.*

Replacing the given values in equation (28) the value $d = -\frac{3-\sqrt{201}}{12} \approx -0.931454$ gives the equality. So, taking a smaller value for example, $d = -0.94$ we obtain v_1 and v_3. Using the equation (24) with the condition $p, q > 1$ we can obtain p and q, for instance, $p = 1.00527$ and $q = 1.1604$.

With these values we built the matrix

$$T = \begin{bmatrix} -0.66443 & -1 & -1.00527 \\ -1 & -1 & -1 \\ -1.1604 & -1 & -0.748111 \end{bmatrix}$$

and its combined matrix

$$C(T) = \begin{bmatrix} -2 & 4.92693 & -1.92693 \\ 3.07307 & -8 & 5.92693 \\ -0.0730739 & 4.07307 & -3 \end{bmatrix}$$

has the given diagonal entries.

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