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ON FINITE GROUPS WITH SQUARE-FREE CONJUGACY CLASS SIZES

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Abstract. We report on finite groups having square-free conjugacy class sizes, in particular in the framework of factorised groups.

1. Introduction

Only finite groups will be considered in this expository paper. Over the last decades, the influence of conjugacy class sizes on the structure of finite groups has been thoroughly investigated. For instance, a detailed account on this subject is collected in the survey of Camina and Camina [6]. We concentrate here on a particular arithmetical property of the conjugacy class sizes: when they are square-free. Since that survey, some new progress has been made in this direction. In particular, we are interested in such arithmetical property in the context of factorised groups. We will outline many of the known results in this line and we will provide references to the literature for their proofs.

First, in Section 2, we pay attention to conjugacy class sizes of some elements of a group not divisible by p^2, for some fixed prime p. Next, we consider square-free class sizes for all primes. Finally, in Section 4, the topic of conjugacy class sizes is combined with factorised groups, and some interesting achievements are shown.

We use the following notation: for a finite group G, let x^G be the *conjugacy class* of an element $x \in G$, and $\left|x^G\right|$ be its size. If p is a prime number, we say that $x \in G$ is a *p-regular* element if its

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order is not divisible by p, and that it is a *p-element* if its order is a power of p. The set of irreducible complex characters of G is $\mathrm{Irr}(G)$. If n is an integer, n_p is the highest power of p dividing n. The mth group of order n in the `SmallGroups` library [19] of GAP will be identified by $n\#m$. The remainder notation is standard, and it is taken mainly from [10].

2. Class sizes not divisible by p^2, for a prime p

The origin of this research could be located in the paper of Chillag and Herzog in 1990 ([8]). In fact, one of the first significant results in this framework is the following criterion of solubility which appears in that paper (see also [7, Proposition 4] for a proof avoiding the Classification of Finite Simple Groups (CFSG)):

Theorem 2.1. [8, Proposition 5] *If 4 does not divide any conjugacy class size of a finite group G, then it is soluble.*

Some years later, Cossey and Wang generalised the above result in [9] by considering a fixed prime p such that $(p-1, |G|) = 1$, instead of the prime $p = 2$ (note that the smallest prime dividing the group order always satisfies that condition):

Theorem 2.2. [9, Theorem 1] *Let G be a finite group, and p be a prime divisor of $|G|$ such that $\gcd(p-1, |G|) = 1$. If p^2 does not divide $\left|x^G\right|$ for all $x \in G$, then G is a soluble p-nilpotent group, and $G/\mathrm{O}_p(G)$ has Sylow p-subgroups of orders at most p. Further, if P is a Sylow p-subgroup of G, then P' has order at most p. Finally, if $P \neq \mathrm{O}_p(G)$, then $\mathrm{O}_p(G)$ is abelian.*

We point out that the above results use the Feit-Thompson theorem. Moreover, the asumption $(p-1, |G|) = 1$ is necessary, which can be easily seen by considering $G = A_5$, the alternating group of degree 5, and the prime $p = 3$.

The following example, which has been comunicated to us by Cossey and appears in [11], shows that the statement "$G/\mathrm{O}_p(G)$ has Sylow p-subgroups of orders at most p" in the above theorem is not true.

Example 2.3. *We consider the semidirect product $G = [C_5 \times C_5](\mathrm{Sym}(3) \times C_2)$ (where $\mathrm{Sym}(3)$ is the symmetric group of degree 3), and the prime $p = 2$. Then G satisfies the hypotheses of* Theorem 2.2 *but $\mathrm{O}_2(G) = 1$ and the Sylow 2-subgroup of G has order 4. The identification of that group in the* `SmallGroups` *library of* GAP *is* 300#25.

On the other hand, the assertion that "P' has order at most p" in the previous result is related with the following well-known one due to Knoche:

Theorem 2.4. [12] *If P is a p-group, then there is not any conjugacy class size of P divisible by p^2 if and only if $|P'| \leq p$.*

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The previous development was enhanced, firstly by Li, and later on by Liu, Wang and Wei, by replacing conditions for all conjugacy classes by those referring only to a smaller set of elements in G:

Theorem 2.5. [15, Theorem 9] *Let G be a group, and let p be the smallest prime dividing the order of G. Assume that p^2 does not divide $\left|x^G\right|$ for each element $x \in G$ of q-power order, for q any prime not equal to p. Then G is a soluble p-nilpotent group.*

Theorem 2.6. *Let G be a finite group, and p be a prime divisor of $|G|$ such that* $\gcd(p-1, |G|) = 1$.

(1) [16, Theorem 6] *Suppose that no conjugacy class size of a p-regular element of G is divisible by p^2. Then G is a soluble p-nilpotent group, and $G/\operatorname{O}_p(G)$ has a Sylow p-subgroup of order at most p.*

(2) [16, Theorem 7] *Suppose that no conjugacy class size of a prime power order element of G is divisible by p^2. Then G is a soluble p-nilpotent group. Furthermore, if P is a Sylow p-subgroup of G, then P' is an elementary abelian or trivial group.*

We recall that the last assertion of the first part is not true, as we have shown in Example 2.3.

More recently, in 2014, Qian and Wang have gone a step further by considering just conjugacy class sizes of p-regular elements of prime power order, and complementing the above result of Li:

Theorem 2.7. [17, Theorem A] *Let G be a finite group, and p be a prime divisor of $|G|$ such that* $\gcd(p-1, |G|) = 1$. *If p^2 does not divide $\left|x^G\right|$ for all p-regular element $x \in G$ of prime power order, then G is a soluble p-nilpotent group, and $G/\operatorname{O}_p(G)$ has elementary abelian Sylow p-subgroups.*

Qian and Wang also showed that, under the weaker hypotheses of Theorem 2.7, the fact that "P' is elementary abelian or trivial" in Theorem 2.6(2) is not true. It is enough to consider the prime $p = 2$, and a 2-group G with non-abelian derived subgroup.

Another interesting and recent refinement is to consider only conjugacy class sizes of the so-called *vanishing* elements, that is, elements $g \in G$ such that there exists some $\chi \in \operatorname{Irr}(G)$ with $\chi(g) = 0$. In this context, Brough has proved in 2016 the next theorem:

Theorem 2.8. [5, Theorem A] *Let G be a finite group and p a prime dividing the size of G such that if q is any prime dividing the size of G, then q does not divide $p-1$. Suppose that no conjugacy class size of vanishing elements of G is divisible by p^2. Then G is a soluble group.*

On the other hand, a new approach in this context was presented by Beltrán and Guo in [4], dealing with coprime actions (and making use of the CFSG):

Theorem 2.9. [4, Theorem A] *Suppose that A is a group which acts coprimely on a group G. If the size of each A-invariant conjugacy class of G is not divisible by 4, then G is soluble.*

They also provide the following example to show that the above result can be false if the action is not coprime: consider $G = \operatorname{Alt}(6)$, the alternating group of 6 letters, and let A be the group of outer automorphisms of G, which is isomorphic to $C_2 \times C_2$.

To conclude this section, we briefly report on the natural dual problem to conjugacy classes: the irreducible complex character degrees. Recently, in [13, 14], it has been handled the case that the irreducible character degrees of a group G are not divisible by p^2, and some information about the orders of the Sylow p-subgroups of $G/\operatorname{O}_p(G)$ is gained. Observe that there are non-soluble groups having all the irreducible complex character degrees not divisible by 4. This is different from what occurs for conjugacy class sizes (see Theorem 2.1).

3. Square-free class sizes

In contrast to the last section, where we concentrated on a fixed prime p, in this section we consider all prime divisors of $|G|$. For our purposes, and for the sake of completeness, it is convenient to collect here some results on square-free class sizes, even though the majority of them are already gathered in [6]. In this line, the first substantial result can be found in [8]:

Theorem 3.1. [8, Theorem 1] *Suppose that $\left|x^G\right|$ is a square-free number for each element $x \in G$. Then G is supersoluble, and both $|G/\operatorname{F}(G)|$ and $|\operatorname{F}(G)'|$ are square-free numbers. In particular, $G' \leqslant \operatorname{F}(G)$ and $G/\operatorname{F}(G)$ is cyclic.*

On the other hand, Cossey and Wang strengthened this result in 1999, avoiding the CFSG:

Theorem 3.2. [9, Theorem 2] *Let G be a finite group and assume that all $\left|x^G\right|$ are square-free, for every $x \in G$. Then G is supersoluble, and both $G/\operatorname{F}(G)$ and G' are cyclic groups with square-free orders. The class of $\operatorname{F}(G)$ is at most 2, and G is metabelian.*

It is reasonable to examine whether the total data of the conjugacy class sizes is a requirement to obtain such structural results. Indeed, in the same year, Li restricted the previous hypotheses to only prime power order elements (see also [16, Lemma 8]), proving the following:

Theorem 3.3. [15, Theorem 10] *Suppose that all class sizes of elements of prime power order in the group G are square-free. Then G is supersoluble, the derived length of G is bounded by 3, $G/\operatorname{F}(G)$ is a direct product of elementary abelian groups, and $|\operatorname{F}(G)'|$ is a square-free number.*

Observe that there exist groups G (like $\operatorname{Sym}(3) \times D_{10}$) satisfying the above hypothesis, in which $G/\operatorname{F}(G)$ neither is cyclic nor has square-free order, in contrast to Theorem 3.2 (the same happens with G', for instance with $G = D_{14} \times [C_7]C_3$).

Finally, we mention that Brough was also concerned with square-free class sizes for vanishing elements.

Theorem 3.4. [5, Theorem B] *Let G be a finite group, and suppose that every conjugacy class size of vanishing elements of G is square-free. Then G is supersoluble.*

4. The case of factorised groups

In parallel to the research on the effect of square-free class sizes on a group structure, the study of groups factorised as the product of subgroups was also gaining an increasing interest, especially when they are connected by certain permutability properties. A new research line arises when considering both perspectives simultaneously, that is, to analyse factorised groups in which the conjugacy class sizes of some elements in the factors are square-free. This new tendency has been little investigated, and perhaps the first authors that tested some preliminary results were Liu, Wang and Wei:

Theorem 4.1. [16, Proposition 9] *Let A and B be normal subgroups of G such that $G = AB$. Suppose that $\left|x^G\right|$ is square-free for every element $x \in A \cup B$ of prime power order. Then G is supersoluble.*

In the same article, they also weakened the hypothesis of normality of the factors by assuming that they are *permutable* in G, but considering all elements in the factors. Recall that a subgroup H of G is said to be permutable in G if $HK = KH$, for every subgroup K of G.

Theorem 4.2. [16, Theorem 10] *Let A and B be permutable subgroups of G such that $G = AB$. Suppose that $\left|x^G\right|$ is square-free for every element $x \in A \cup B$. Then G is supersoluble.*

Other authors have also looked into this fusion of square-free class sizes and factorised groups, imposing only permutability conditions over one of the factors (see for instance [18]), such as: A is subnormal in G, $\langle A, A^g\rangle$ is nilpotent for every $g \in G$, A is S-permutable in G, etc. Note that each one of these properties implies that A is subnormal in G, so every conjugacy class size in A divides its corresponding class size in the whole group.

In general, the product of two supersoluble normal subgroups of a finite group needs not to be supersoluble. This fact originated the inquire of new permutability conditions (stronger than the normality of the factors) which supply the supersolubility of that kind of products. In this direction, Asaad and Shaalan proved in [1], among other results, that if $G = AB$ with A and B supersoluble, then G is supersoluble if the derived subgroup G' of G is a nilpotent group and, moreover, A permutes with every subgroup of B and B permutes with every subgroup of A. Subgroups A and B satisfying the latter condition are called *mutually permutable* (see [3]). Notice that if $G = AB$ with A and B normal subgroups of G, then G is the product of two mutually permutable subgroups.

Theorems 4.1 and 4.2 are related with the previous paragraph, because they also provide conditions for a factorised group to be supersoluble. Motivated by them, Ballester-Bolinches, Cossey and Li extended this development of Liu, Wang and Wei, through products of mutually permutable subgroups. Firstly, they paid attention to the case of a fixed prime p:

Theorem 4.3. [2, Theorem 1.3] *Let the group $G = AB$ be the product of the mutually permutable subgroups A and B. Suppose that for every p-regular element $x \in A \cup B$, $\left|x^G\right|$ is not divisible by p^2.*

Then the order of a Sylow p-subgroup of every chief factor of G is at most p. In particular, if G is p-soluble, we have that G is p-supersoluble.

Moreover, if the above conditions hold for every prime, they got the supersolubility of G:

Corollary 4.4. [2, Corollary 1.5] *Let the group $G = AB$ be the product of the mutually permutable subgroups A and B. Suppose that for every prime p and every p-regular element $x \in A \cup B$, $\left|x^G\right|$ is not divisible by p^2. Then G is supersoluble.*

More recently, we have gone ahead in this research. We have proved in [11] the following result, which is related to Theorem 2.7:

Theorem 4.5. [11, Theorem A] *Let $G = AB$ be the product of the mutually permutable subgroups A and B, and p be a prime divisor of $|G|$ with $\gcd(p-1, |G|) = 1$. If p^2 does not divide $\left|x^G\right|$ for any p-regular element $x \in A \cup B$ of prime power order, then we have:*

(1) *G is soluble.*
(2) *G is p-nilpotent.*
(3) *The Sylow p-subgroups of $G/\operatorname{O}_p(G)$ are elementary abelian.*

In contrast to what happens in Theorems 4.1 and 4.2 (and also in the corresponding ones in [18]), the hypotheses in Theorem 4.5 for the conjugacy class sizes of the elements $x \in A \cup B$ are not automatically acquired by the factors. For instance, the group G of Example 2.3 can be factorised as the mutually permutable product of $A = D_{10} \times D_{10}$ and $B = [C_5 \times C_5]C_3$ (we checked this using `GAP`, [19]). It is clear that $G = AB$ satisfies the hypotheses of the last theorem for $p = 2$, but there are elements $x \in A$ with $\left|x^A\right|$ divisible by 4.

We have also targeted the case of p-groups, spreading out Theorem 2.4:

Proposition 4.6. [11, Proposition 2] *Let $P = AB$ be a p-group such that p^2 does not divide $\left|x^P\right|$ for all $x \in A \cup B$. Then $P' \leqslant \Phi(P) \leqslant \operatorname{Z}(P)$, P' is elementary abelian and $|P'| \leq p^2$.*

It has been also shown in [11, Theorem B] that the hypotheses of Theorem 4.3 can be weakened to p-regular elements of prime power order, in order to get the last assertion regarding p-supersolubility.

On the other hand, when considering all prime divisors of $|G|$, we gained the next result:

Theorem 4.7. [11, Corollary F] *Let $G = AB$ be the product of the mutually permutable subgroups A and B. Suppose that $\left|x^G\right|$ is square-free for each element $x \in A \cup B$. Then we have:*

(1) *G is supersoluble.*
(2) *$G/\operatorname{F}(G)$ is abelian with elementary abelian Sylow p-subgroups of order at most p^2, for each prime p.*
(3) *G' is abelian with elementary abelian Sylow subgroups.*
(4) *$\operatorname{F}(G)'$ has Sylow p-subgroups of order at most p^2, for each prime p.*

The last result generalises Corollary 4.4. Finally, we highlight that some of the above assertions can be proved under weaker conditions (see [11]), for instance by restricting to either p-regular elements or prime power order elements.

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