Computing probabilistic solutions of the Bernoulli random differential equation

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Abstract

The random variable transformation technique is a powerful method to determine the probabilistic solution for random differential equations represented by the first probability density function of the solution stochastic process. In this paper, that technique is applied to construct a closed form expression of the solution for the Bernoulli random differential equation. In order to account for the general scenario, all the input parameters (coefficients and initial condition) are assumed to be absolutely continuous random variables with an arbitrary joint probability density function. The analysis is split into two cases for which an illustrative example is provided. Finally, a fish weight growth model is considered to illustrate the usefulness of the theoretical results previously established using real data.

Keywords: Bernoulli random differential equation, first probability density function, probabilistic solution, random variable transformation technique

1. Introduction and motivation

Ever since the early contributions by I. Newton, G. W. Leibniz, Jacob and Johann Bernoulli in the XVII century until now, differential equations have uninterruptedly demonstrated their capability to model successfully complex problems. There is virtually no applied scientific area where differential equations had not been used to deal with relevant problems. Numerous examples can be found in engineering, physics, chemistry, epidemiology, economics, etc. From a practical standpoint, the application of differential equations requires setting their inputs (coefficients, source term, initial and boundary conditions) using sampled data, thus containing uncertainty stemming from measurement errors. It has led to the consideration of randomness in the formulation of continuous models based on differential equations. In this regard, there are two main classes of equations, stochastic differential equations and random differential equations. In the former case, differential equations are forced by an irregular process such as a Wiener

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process or Brownian motion. This class of equations are usually written in terms of stochastic differentials, although they are interpreted as Itô stochastic integrals [1, 2]. Solutions of Itô-type stochastic differential equations typically exhibit nondifferentiability of their sample paths or trajectories due to the irregularity of the driving Brownian motion. The formulation of Itô-type stochastic differential equations from their deterministic counterpart can be justified by means of the perturbation of input parameters via white noise process, i.e., the formal derivative of a Brownian motion. This implicitly entails that Gaussian-type uncertainty is assumed for perturbed inputs. Although, stochastic differential equations have demonstrated to be powerful mathematical representations to model many problems, for instance in finance, engineering, biosciences, etc., [3, 4, 5, 6, 7, 8], clearly this approach does not cover important casuistries . A complementary approach to introduce uncertainty in differential equations is to allow the direct assignment of any probability distribution to input parameters, which is referred to as the *randomization process* of the deterministic or classical differential equation. If coefficients are assumed to be random variables (r.v.'s), then, for example, beta, exponential, Gaussian, etc., may be appropriate candidate probability distributions to account for model uncertainty. In principle, this approach is more flexible and it leads to random differential equations. Throughout this paper, random differential equations will considered only.

Similarly to deterministic case, the first goal when dealing with both stochastic and random differential equations is computing, exactly or approximately, the solution stochastic process (s.p.). Unlike deterministic context, now the determination of the main statistical functions associated to the solution s.p., such as the mean and variance functions, are also important goals to be achieved. In fact, the average behaviour of the solution as well as its variability around the mean are obtained from these two statistical moments. Although this information is valuable, and most contributions focus on the computation of the solution s.p. and its mean and variance/standard deviation functions, a more ambitious target includes the determination of the first probability density function (1-p.d.f.) of the solution. The 1-p.d.f. provides a full probabilistic description in each time instant of the solution s.p. Moreover, from the 1-p.d.f., both the mean and variance functions can be straightforwardly computed, but also asymmetry, kurtosis, and other higher unidimensional statistical moments. Even though in this paper we focus on the computation of the 1-p.d.f. of the solution, it is worth underlining that higher p.d.f.'s of the solution s.p. are also useful for giving further statistical characteristics. For example, from the second p.d.f. one can obtain the correlation function of the s.p. which gives a measure of the linear interdependence between the r.v.'s coming from evaluating the s.p. in two different time instants [4, p.39].

In order to determine the 1-p.d.f., the so-called random variable transformation (RVT) method will be applied throughout this paper. RVT method is a powerful technique that permits the computation of the p.d.f. of a r.v. which is obtained after mapping another r.v. whose p.d.f. is given [9]. A generalization of this method can be found in [10]. One of the most fruitful applications of RVT technique is getting the complete probabilistic description of the solution to random differential equations represented by the 1-p.d.f. of the solution s.p. Some recent contributions addressed to determine the 1-p.d.f. of the solution of particular random differential equations can be found, for example, in [11, 12, 13]. In [11] one provides a comprehensive study to compute the 1-p.d.f. of the solution s.p. of the random linear first-order differential equation. The study considers all possible cases with respect to the manner that randomness can appear either in the diffusion coefficient, source term or/and the initial condition. In [12] a logistic model where only the initial condition is random. Authors determine the 1-p.d.f. of the proportion of susceptibles of a SI-type epidemiological model. From the 1-p.d.f., a number of probabilistic properties of the solution s.p., such as the mean, the variance, the quartiles, etc., are given. The results obtained in this latter paper have been recently generalized in [13] by assuming that all input parameters are r.v.'s. In these three contributions on random ordinary differential equations, the RVT method constitutes the cornerstone to conduct their respective analyses. However, its applicability goes beyond random ordinary differential equations. For example, some interesting contributions deal with random partial differential equations [14, 15]; random integral-differential equations [16] and random difference equations [17]. Although, in all these contributions the 1-p.d.f of the solution s.p. of the corresponding problems is obtained in an exact way, the technique can be also applied to get numerical approximations, [18].

Recently, RVT technique has been applied by the authors to give a full probabilistic description to both, general linear first-order and Riccati random differential equations, represented by the 1-p.d.f. of their solutions [11, 19]. The aim of this paper is to continue extending this analysis to another important classical differential equations where probabilistic dependence among input r.v.'s will be assumed. In the following, we will consider the Bernoulli random initial value problem (IVP)

$$\left.\begin{array}{rcl} \dot{X}(t) & = & CX(t) + D(X(t))^{A}, \quad t \geq t_0\,, \\ X(t_0) & = & X_0\,, \end{array}\right\} \tag{1}$$

where t_0 denotes the initial time and all the input parameters, X_0, D, C and A, are assumed to be absolutely continuous r.v.'s defined in a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Hereinafter, $\mathcal{D}(X_0)$, $\mathcal{D}(D)$, $\mathcal{C}(C)$ and $\mathcal{D}(A)$ will denote their respectively domains. In order to provide as much generality as possible throughout our analysis, hereinafter we will assume that X_0, C, D and A are statistically dependent. In the following, $f_{X_0,D,C,A}(x_0, d, c, a)$ will denote their joint p.d.f.

The paper is organized as follows. In Section 2, some preliminaries and technical results about RVT technique that will be required throughout the paper, are included. Section 3 is addressed to determine the 1-p.d.f. of the solution s.p. to the Bernoulli random IVP (1) in the general scenario where all input parameters (X_0, D, C, A) are assumed to be r.v.'s. As it will be shown later, our approach requires splitting the analysis in two cases. For every case, an illustrative example is also provided. In Section 4, we take advantage of the ideas exhibited in Section 3 to illustrate the usefulness of computing the 1-p.d.f. to deal with a fish weight growth model. Conclusions are drawn in the last section.

2. Preliminaries

In this section, some technical results that will play a key role to solve the Bernoulli random IVP (1) are presented.

For the sake of clarity, we start by stating the Random Variable Transformation (RVT) method. This result permits the computation of the p.d.f. of a r.v. which is obtained after transforming another r.v. whose p.d.f. is known.

Theorem 1 (Multidimensional Random Variable Transformation method, [4]). *Let us consider* $\mathbf{U} = (U_1, \ldots, U_n)^{\mathsf{T}}$ *and* $\mathbf{V} = (V_1, \ldots, V_n)^{\mathsf{T}}$ *two n-dimensional absolutely continuous random vectors defined on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$*. Let* $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ *be a one-to-one deterministic transformation of* $\mathbf{U}$ *into* $\mathbf{V}$*, i.e.,* $\mathbf{V} = \mathbf{g}(\mathbf{U})$*. Assume that* $\mathbf{g}$ *is continuous in* $\mathbf{U}$ *and has continuous partial derivatives with respect to each* U_i*,* $1 \leq i \leq n$*. Then, if* $f_{\mathbf{U}}(\mathbf{u})$ *denotes the joint probability density function of vector* $\mathbf{U}$*, and* $\mathbf{h} = \mathbf{g}^{-1} = (h_1(v_1, \ldots, v_n), \ldots, h_n(v_1, \ldots, v_n))^{\mathsf{T}}$ *represents the* *inverse mapping of* $\mathbf{g} = (g_1(u_1,\dots,u_n),\dots,g_n(u_1,\dots,u_n))^{\mathsf{T}}$*, the joint probability density function of vector* $\mathbf{V}$ *is given by*

$$f_{\mathbf{V}}(\mathbf{v}) = f_{\mathbf{U}}\left(\mathbf{h}(\mathbf{v})\right)|J|, \tag{2}$$

where $|J|$*, which assumed to be different from zero, is the absolute value of the Jacobian defined by the determinant*

$$J = \det\left(\frac{\partial \mathbf{u}^{\mathsf{T}}}{\partial \mathbf{v}}\right) = \det\begin{pmatrix} \frac{\partial h_1(v_1,\dots,v_n)}{\partial v_1} & \cdots & \frac{\partial h_n(v_1,\dots,v_n)}{\partial v_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1(v_1,\dots,v_n)}{\partial v_n} & \cdots & \frac{\partial h_n(v_1,\dots,v_n)}{\partial v_n} \end{pmatrix}. \tag{3}$$

The following results are specializations of Theorem 1 that will be required later.

Lemma 2. *Let* $\mathbf{U} = (U_1, U_2)$ *be an absolutely continuous real random vector defined on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$*, with joint probability density function* $f_{\mathbf{U}}(u_1, u_2)$*. Assume that* $U_1(\omega) \neq 0$ *and* $U_2(\omega) \neq 1$ *for all* $\omega \in \Omega$*. Then, the probability density function* $f_W(w)$ *of the transformation* $W = (U_1)^{\frac{1}{1-U_2}}$ *is given by*

$$f_W(w) = \int_{\mathcal{D}(U_2)} f_{\mathbf{U}}(w^{1-u_2}, u_2)\left|(1-u_2)w^{-u_2}\right| \mathrm{d}u_2, \tag{4}$$

where $\mathcal{D}(U_2)$ *denotes the domain of* U_2*.*

Proof. Let us apply Theorem 1 to the transformation $\mathbf{v} = (v_1, v_2) = \mathbf{g}(\mathbf{u}) = ((u_1)^{\frac{1}{1-u_2}}, u_2)$. Its inverse mapping is given by $\mathbf{h}(\mathbf{v}) = ((v_1)^{1-v_2}, v_2)$, being its Jacobian

$$J = \det\left(\frac{\partial \mathbf{u}^T}{\partial \mathbf{v}}\right) = \det\begin{pmatrix} (1-v_2)(v_1)^{-v_2} & -(v_1)^{1-v_2}\log(v_1) \\ 0 & 1 \end{pmatrix} = (1-v_2)(v_1)^{-v_2}.$$

Notice that $J \neq 0$ because by hypotheses $U_1(\omega) \neq 0$ and $U_2(\omega) \neq 1$ with probability 1 (w.p. 1). Then, applying (2)–(3), the joint p.d.f. of $\mathbf{V} = ((U_1)^{\frac{1}{1-U_2}}, U_2)$ is given by

$$f_{\mathbf{V}}(v_1, v_2) = f_{\mathbf{U}}((v_1)^{1-v_2}, v_2)\left|(1-v_2)(v_1)^{-v_2}\right|.$$

Finally, by marginalizing with respect to r.v. $V_2 = U_2$, one obtains the p.d.f. of $W = V_1 = (U_1)^{\frac{1}{1-U_2}}$ given by (4). □

Lemma 3. *Let* $Z : \Omega \to \mathbb{R}$ *be an absolutely continuous real random variable defined on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$*, with probability density function* $f_{\mathbf{Z}}(z)$*. Assume that* $Z(\omega) \neq 0$ *for all* $\omega \in \Omega$*. Then, the probability density function* $f_W(w)$ *of the transformation* $W = Z^3$ *is given by*

$$f_W(w) = \frac{1}{3} f_{\mathbf{Z}}\left(\sqrt[3]{w}\right)|w|^{-2/3}. \tag{5}$$

Proof. This result is a direct consequence of Theorem 1 for $n = 1$, $\mathbf{U} = Z$, $\mathbf{V} = W$ and $W = g(Z) = Z^3$. Notice that the inverse transformation of g is $h(w) = \sqrt[3]{w}$ and its Jacobian is given by $h'(w) = 1/3w^{-2/3}$. It is well-defined because $Z(\omega) \neq 0$ w.p. 1, by hypothesis. Then, applying (3) one obtains directly expression (5). This proves the result. □

3. Study of the Bernoulli random differential equation

In this section, the 1-p.d.f., $f_1(x,t)$, of the solution s.p. $X(t)$ to the Bernoulli random IVP (1) will be determined. First, it is important to point out that the following cases have been treated in earlier contributions:

- The case where coefficient D is zero w.p. 1, i.e., $\mathbb{P}[\{\omega \in \Omega : D(\omega) = 0\}] = 1$, or A is one w.p. 1, i.e., $\mathbb{P}[\{\omega \in \Omega : A(\omega) = 1\}] = 1$, which correspond to the random first-order homogeneous linear differential equation, has been studied in [11].
- The case where coefficient A is zero w.p. 1, i.e., $\mathbb{P}[\{\omega \in \Omega : A(\omega) = 0\}] = 1$, which corresponds to the random first-order non-homogeneous linear differential equation, has been studied in [11].
- The case where coefficient A is two w.p. 1, i.e., $\mathbb{P}[\{\omega \in \Omega : A(\omega) = 2\}] = 1$, which corresponds to the Riccati random homogeneous differential equation, has been studied in [19].

Notice that a comprehensive analysis, similar to one exhibited in [11], of all the possible casuistries depending on the deterministic and random nature of the four input parameters, X_0, D, C and A, will involve 15 cases (obviously, excluding the full deterministic case where all inputs are constants). In this paper we will focus on the case where all inputs parameters are r.v.'s with a joint p.d.f. $f_{X_0,D,C,A}(x_0,d,c,a)$. In order to conduct the analysis of the Bernoulli random IVP (1) it is convenient to distinguish the Cases I–II listed in Table 1. The distinction between these two cases helps to apply RVT technique as it will apparent later (see Eq. (8)). Hereinafter, it is assumed that

$$\mathbb{P}[\{\omega \in \Omega : A(\omega) \neq 1\}] = 1, \quad \mathbb{P}[\{\omega \in \Omega : X_0(\omega) \neq 0\}] = 1. \tag{6}$$

Case I	$\mathbb{P}[\{\omega \in \Omega : C(\omega) \neq 0\}] = 1$
Case II	$\mathbb{P}[\{\omega \in \Omega : C(\omega) = 0\}] = 1$

Table 1: List of the two cases considered to compute the 1-p.d.f. of the solution s.p. to the Bernoulli random IVP (1) under assumptions (6).

Notice that in the case $X_0 = 0$ w.p. 1, $X(t) \equiv 0$ is clearly the unique solution of IVP (1).

Before presenting the study, it is important to emphasize that as the RVT method constitutes the unifying technique to conduct our analysis in the two cases listed in Table 1. With the aim of facilitating our exposition, in the subsequent subsections the results are presented following a common structure.

In each one of the Cases I–II, we will present a numerical example to illustrate the theoretical results established. In these examples statistical independence among inputs will be assumed to facilitate computations. Nevertheless, in the next section a full example involving statistical dependence will be exhibited.

3.1. Case I

Let us assume that the linear coefficient C is different from zero w.p. 1. In the following, $f_{X_0,D,C,A}(x_0,d,c,a)$ will denote the joint p.d.f. of random vector (X_0,D,C,A).

In order to determine the 1-p.d.f. of the solution s.p., $X(t)$, of the IVP (1), it is convenient to consider the following change of variable

$$X(t) = (Z(t))^{\frac{1}{1-A}}. \tag{7}$$

This permits the transformation of (1) into the following linear IVP

$$\left.\begin{array}{rcl} \dot{Z}(t) & = & (1-A)CZ(t)+(1-A)D, \quad t \geq t_0\,, \\ Z(t_0) & = & (X_0)^{1-A}\,, \end{array}\right\}$$

whose exact closed form solution s.p. is given by

$$Z(t) = (X_0)^{1-A}\, \mathrm{e}^{(1-A)C(t-t_0)} + \frac{D}{C}\, \mathrm{e}^{(1-A)C(t-t_0)} - \frac{D}{C}. \tag{8}$$

Let us fix $t \geq t_0$ and denote $Z = Z(t)$. Let us consider the mapping $r : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ such that $r(x_0,d,c,a) = (r_1(x_0,d,c,a), r_2(x_0,d,c,a), r_3(x_0,d,c,a), r_4(x_0,d,c,a))$ being

$$z_1 = r_1(x_0,d,c,a) = z,\;\; z_2 = r_2(x_0,d,c,a) = d,\;\; z_3 = r_3(x_0,d,c,a) = c,\;\; z_4 = r_4(x_0,d,c,a) = a.$$

It is straightforward to check that the inverse mapping of r is defined by $s : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$, $s(z_1,z_2,z_3,z_4) = (s_1(z_1,z_2,z_3,z_4), s_2(z_1,z_2,z_3,z_4), s_3(z_1,z_2,z_3,z_4), s_4(z_1,z_2,z_3,z_4))$, where

$$\begin{array}{rclcl} x_0 & = & s_1(z_1,z_2,z_3,z_4) & = & \left(\dfrac{-z_2+\mathrm{e}^{-z_3(t-t_0)(1-z_4)}(z_1z_3+z_2)}{z_3}\right)^{\frac{1}{1-z_4}}, \quad d = s_2(z_1,z_2,z_3,z_4) = z_2, \\ c & = & s_3(z_1,z_2,z_3,z_4) & = & z_3, \qquad a = s_4(z_1,z_2,z_3,z_4) = z_4. \end{array}$$

Moreover, the Jacobian of s mapping is given by

$$J = \frac{\mathrm{e}^{-z_3(t-t_0)(1-z_4)}}{1-z_4}\left(\frac{-z_2+\mathrm{e}^{-z_3(t-t_0)(1-z_4)}(z_1z_3+z_2)}{z_3}\right)^{\frac{z_4}{1-z_4}}. \tag{9}$$

Therefore, by Theorem 1, the joint p.d.f. of random vector (Z_1,Z_2,Z_3,Z_4) can be written as

$$\begin{array}{rcl} f_{Z_1,Z_2,Z_3,Z_4}(z_1,z_2,z_3,z_4) & = & f_{X_0,D,C,A}\left(\left(\dfrac{-z_2+\mathrm{e}^{-z_3(t-t_0)(1-z_4)}(z_1z_3+z_2)}{z_3}\right)^{\frac{1}{1-z_4}}, z_2, z_3, z_4\right) \\ & \times & \left|\dfrac{\mathrm{e}^{-z_3(t-t_0)(1-z_4)}}{1-z_4}\left(\dfrac{-z_2+\mathrm{e}^{-z_3(t-t_0)(1-z_4)}(z_1z_3+z_2)}{z_3}\right)^{\frac{z_4}{1-z_4}}\right|. \end{array}$$

Taking into account (7), $Z = Z_1$, $A = Z_4$, hence the p.d.f. of random vector (Z,A) is obtained as the marginal p.d.f. of $f_{Z_1,Z_2,Z_3,Z_4}(z_1,z_2,z_3,z_4)$ with respect to $Z_2 = D$ and $Z_3 = C$, that is,

$$\begin{array}{rcl} f_{Z,A}(z,a) & = & \displaystyle\int_{\mathcal{D}(D)}\int_{\mathcal{D}(C)} f_{X_0,D,C,A}\left(\left(\frac{-d+\mathrm{e}^{-c\,(t-t_0)(1-a)}(zc+d)}{c}\right)^{\frac{1}{1-a}}, d, c, a\right) \\ & \times & \left|\dfrac{\mathrm{e}^{-c(t-t_0)(1-a)}}{1-a}\left(\dfrac{-d+\mathrm{e}^{-c(t-t_0)(1-a)}(zc+d)}{c}\right)^{\frac{a}{1-a}}\right| \mathrm{d}c\, \mathrm{d}d. \end{array}$$

Now, using the change of variable (7), applying Lemma 2 with the following identification: $U_1 = Z$, $U_2 = A$ and $W = X$, and considering $t \geq t_0$ arbitrary, the 1-p.d.f. of the solution s.p. $X(t) = (Z(t))^{\frac{1}{1-A}}$ is given by

$$\begin{aligned} f_1(x,t) &= \int_{\mathcal{D}(A)} \int_{\mathcal{D}(D)} \int_{\mathcal{D}(C)} f_{X_0,D,C,A}\left(\left(\frac{-d + \mathrm{e}^{-c\,(t-t_0)(1-a)}(x^{1-a}\,c+d)}{c}\right)^{\frac{1}{1-a}}, d, c, a\right) \\ &\times \left|\frac{\mathrm{e}^{-c(t-t_0)(1-a)}}{x^a}\left(\frac{-d + \mathrm{e}^{-c(t-t_0)(1-a)}(x^{1-a}c+d)}{c}\right)^{\frac{a}{1-a}}\right| \mathrm{d}c\,\mathrm{d}d\,\mathrm{d}a\,. \end{aligned} \tag{10}$$

Example 1. *Let us consider the random IVP* (1) *with* $t_0 = 0$ *and assume the following probability distributions for its inputs parameters: the nonlinear coefficient, D, is a uniform r.v. on the interval* $[0,1]$*,* $D \sim \mathrm{U}([0,1])$*; the linear coefficient, C, is an exponential r.v. of mean* $1/35$*,* $C \sim \mathrm{Exp}(35)$*; the exponent A is a beta r.v. with parameters* $(2,3)$*,* $A \sim \mathrm{Be}(2;3)$*, and the initial condition is an exponential r.v. with mean* 2.5*,* $X_0 \sim \mathrm{Exp}(1/2.5)$*. In Figure 1, the 1-p.d.f.* $f_1(x,t)$ *of the solution s.p. to the random IVP* (1) *has been plotted. Computations to perform this graphical representation have been carried out using expression* (10) *on the time interval* $[0,5]$*. Notice that the joint p.d.f. of the input data* (X_0, D, C, A) *is given by*

$$\begin{aligned} f_{X_0,D,C,A}(x_0,d,c,a) &= f_{X_0}(x_0) f_D(d) f_C(c) f_A(a) \\ &= \left(\frac{1}{2.5}\,\mathrm{e}^{-\frac{1}{2.5}x_0}\right) \times (1) \times \left(35\,\mathrm{e}^{-35c}\right) \times \left(12d(1-d)^2\right) \\ &= 168\,\mathrm{e}^{-(0.4x_0+35c)}\, d(1-d)^2 \quad \text{if} \quad (x_0,d,c,a) \in \mathcal{D}(X_0)\times\mathcal{D}(D)\times\mathcal{D}(C)\times\mathcal{D}(A), \end{aligned}$$

being $\mathcal{D}(X_0) = \mathcal{D}(C) =]0,+\infty[$*,* $\mathcal{D}(D) = \mathcal{D}(A) =]0,1[$*, and* $f_{X_0,D,C,A}(x_0,d,c,a) = 0$ *otherwise. Notice that the hypothesis of pairwise independence of r.v.'s* X_0*, A, C and D has been used in the above expression for* $f_{X_0,D,C,A}(x_0,d,c,a)$ *(see comment before Section 3.1). From a computational point of view it is worth pointing out that the graphical representation shown in Figure 1 has been built fixing a value of time* $t = \hat{t} \in \{0, 0.5, 1, 1.5, 2, \ldots, 5\}$ *and then, first checking that* $\int_{-\infty}^{\infty} f_1(x,\hat{t})\,\mathrm{d}x \approx 1$*, and secondly, using the software Mathematica® [20] to calculate the three-dimensional integral given by* (10)*. From this graph one observes that the variability of the solution* $X = X(t)$ *increases rapidly over the time.*

3.2. Case II

Now, we address the case where C is 0 w.p. 1. This corresponds to the following particular case of IVP (1)

$$\left.\begin{aligned} \dot{X}(t) &= D(X(t))^A, \quad t \geq t_0\,, \\ X(t_0) &= X_0\,, \end{aligned}\right\} \tag{11}$$

whose solution is given by

$$X(t) = \left(D(t-t_0)(1-A) + (X_0)^{1-A}\right)^{\frac{1}{1-A}}.$$

Let us fix $t \geq t_0$. Unlike Case I, now transformation (7), involving r.v. Z, is not needed to compute the solution of IVP (11). Hence, we will compute directly the 1-p.d.f. of $X(t)$ without computing previously the p.d.f. of r.v. Z. With this end, let us define the mapping $r : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $r(x_0,d,a) = (r_1(x_0,d,a), r_2(x_0,d,a), r_3(x_0,d,a))$ where

$$x_1 = r_1(x_0,d,a) = \left(d(t-t_0)(1-a) + (x_0)^{1-a}\right)^{\frac{1}{1-a}}, \quad x_2 = r_2(x_0,d,a) = d, \quad x_3 = r_3(x_0,d,a) = a.$$

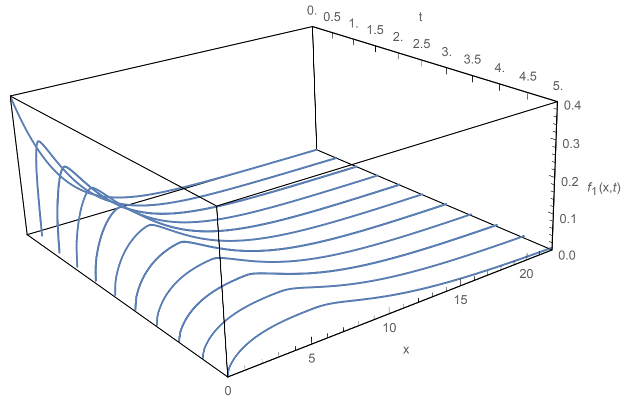

Figure 1: Plot of the 1-p.d.f. $f_1(x,t)$ given by (10) in the Example 1 at the following values of $t \in \{0, 0.5, 1, 1.5, \dots, 5\}$.

The inverse mapping s of is given by $s : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, whose components $s_i(x_1, x_2, x_3)$, $1 \le i \le 3$ are

$$x_0 = s_1(x_1, x_2, x_3) = \left((x_1)^{1-x_3} - x_2(t-t_0) + x_2x_3(t-t_0)\right)^{\frac{1}{1-x_3}}, \; d = s_2(x_1, x_2, x_3) = x_2, \; a = s_3(x_1, x_2, x_3) = x_3.$$

In accordance with Theorem 1, the joint p.d.f. of random vector (X_1, X_2, X_3) is given by

$$\begin{aligned} f_{X_1,X_2,X_3}(x_1, x_2, x_3) &= f_{X_0,D,A}\left(\left((x_1)^{1-x_3} - x_2(t-t_0) + x_2x_3(t-t_0)\right)^{\frac{1}{1-x_3}}, x_2, x_3\right) \\ &\times \left| (x_1)^{-x_3} \left((x_1)^{1-x_3} - x_2(t-t_0) + x_2x_3(t-t_0)\right)^{\frac{x_3}{1-x_3}} \right|, \end{aligned} \tag{12}$$

where the factor in absolute value corresponds to the Jacobian of s mapping.

Finally considering $t \ge t_0$ arbitrary, the 1-p.d.f. of the solution s.p. $X(t)$ to IVP (11) is obtained as the (X_2, X_3)-marginal of the p.d.f. (12)

$$f_1(x,t) = \int_{\mathcal{D}(D)} \int_{\mathcal{D}(A)} f_{X_0,D,A}\left(\left(x^{1-a} - d(t-t_0) + ad(t-t_0)\right)^{\frac{1}{1-a}}, d, a\right) \left| x^{-a}\left(x^{1-a} - d(t-t_0) + ad(t-t_0)\right)^{\frac{a}{1-a}} \right| \mathrm{d}a\, \mathrm{d}d. \tag{13}$$

Remark 1. Notice that expression (13) is not a particular case of expression (10) since in the context of Case II we are assuming that $C = 0$ w.p. 1. If we revise carefully the application of Theorem 2 within our discussion of Case I, the hypothesis $C = 0$ w.p. 1 is required to guarantee the jacobian $J \neq 0$ (see expression (9) where $z_3 = c$). Moreover, expression (13) cannot be obtained as a limit of expression (10) as $c \to 0$. These facts have motivated the distinction of the two cases listed in Table 1.

Example 2. *In order to illustrate the theoretical results previously established, let us consider the random IVP* (11) *being the initial condition X_0 an exponential r.v. of mean 1, $X_0 \sim \mathrm{Exp}(1)$ at the initial time $t_0 = 0$; the nonlinear coefficient D a beta r.v. $D \sim \mathrm{Be}(2;3)$ and the exponent A a* *standard Gaussian r.v., $A \sim \mathrm{N}(0;1)$, truncated at the domain $]-\infty, 1]$. Figure 2 shows the 1-p.d.f. of the solution s.p. to IVP (11) on the time interval $[0,6]$. It has been computed according to expression (13) and taking into account that*

$$\begin{aligned} f_{X_0,D,A}(x_0,d,a) &= f_{X_0}(x_0) f_D(d) f_A(a) \\ &= \left(\mathrm{e}^{-x_0}\right) \times \left(12d(1-d)^2\right) \times \frac{\mathrm{e}^{-\frac{a^2}{2}}}{\int_{-\infty}^{1} \mathrm{e}^{-\frac{z^2}{2}}\,\mathrm{d}z} \\ &= \sqrt{\frac{2}{\pi}} \frac{12}{1+erf\left(\frac{\sqrt{2}}{2}\right)} \mathrm{e}^{-\left(x_0+\frac{a^2}{2}\right)} d(1-d)^2 \quad \text{if} \quad (x_0,d,a) \in \mathcal{D}(X_0) \times \mathcal{D}(D) \times \mathcal{D}(A), \end{aligned}$$

being $\mathcal{D}(X_0) =]0,+\infty[$, $\mathcal{D}(D) =]0,1[$, $\mathcal{D}(A) =]-\infty,1[$, and $f_{X_0,D,A}(x_0,d,a) = 0$ otherwise. Here $erf(t) = 2/\sqrt{\pi}\int_0^t \mathrm{e}^{-z^2}\,\mathrm{d}z$ stands for the error function. Similar comments to the ones exhibited in Example 1 regarding the computations carried out using the software Mathematica® to plot the 1-p.d.f. $f_1(x,t)$ can be made. From this representation, one observes that variance increases as time goes on.

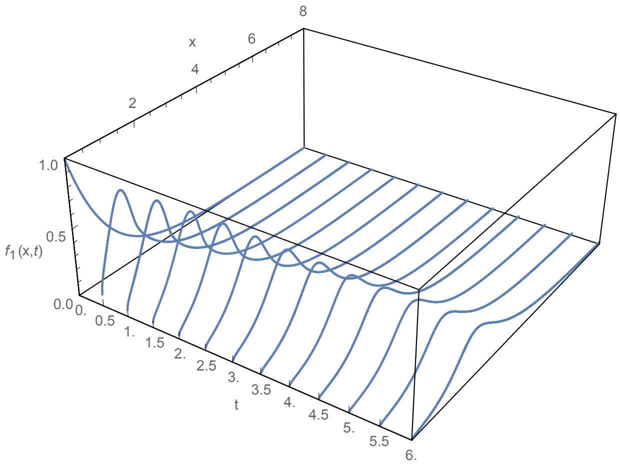

Figure 2: Plot of the 1-p.d.f. $f_1(x,t)$ given by (13) in the Example 2 at the following values of $t \in \{0, 0.5.1, 1.5, \dots, 6\}$.

4. An application of the Bernoulli random differential equation to modelling

On the one hand, so far closed expressions for the 1-p.d.f. of the solution s.p. to the Bernoulli random IVP (1) have been provided for the two cases listed in Table 1. On the other hand, as it has been highlighted in Section 1, random differential equations are very useful in modelling. Next, we will illustrate this claim by considering a deterministic model, formulated by a Bernoulli differential equation, that describes the fish weight growth over the time. In a first step, we assume that input parameters (coefficients and initial condition) are r.v.'s rather than deterministic constants. Secondly, using real data we will assign a reliable probabilistic distribution to random inputs using an inverse frequentist technique. Then, we will take advantage of RVT technique to determine the 1-p.d.f. of the solution s.p. to the model. Finally, from this important deterministic function, both punctual and probabilistic predictions based on confidence intervals will be constructed.

The following IVP, usually referred to as Bertalanffy model [21, 22], has been applied extensively to describe the fish weight growth, $W(t)$, at the time instant t, (see for example, [23, 24])

$$\left.\begin{array}{rcl} \dot{W}(t) & = & -\lambda W(t) + \eta (W(t))^{2/3}, \quad t \geq t_0\,, \\ W(t_0) & = & W_0\,. \end{array}\right\} \tag{14}$$

It is worth to point out that, recently, some authors have considered stochastic versions of this model following an based on Itô-type stochastic differential equations, [25, 26].

Under our approach, let us assume that the initial condition, W_0, and the coefficients, η and λ of IVP (14) are absolutely continuous r.v.'s defined on a common probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, being $f_{W_0,\eta,\lambda}(w_0, \eta, \lambda)$ their joint p.d.f. Moreover, let us suppose that

$$\mathbb{P}\left[\{\omega \in \Omega : W_0(\omega) \neq 0\}\right] = 1, \quad \mathbb{P}\left[\{\omega \in \Omega : \lambda(\omega) \neq 0\}\right] = 1.$$

Considering the identification $X(t) = W(t)$, $C = -\lambda$, $D = \eta$, $A = 2/3$ and $X_0 = W_0$, the initial value problem (14) can be treated as a particular case of (1). Hence, one can obtain the 1-p.d.f. $f_1(w, t)$ of the solution s.p. $W(t)$ of (14) introducing the delta Dirac function, $\delta(a - 2/3)$, in expression (10). An alternative way to do that is to treat the problem (14) directly depending on the three r.v.'s W_0, η, λ, in an similar way as in Case I. With this aim, let us consider the following change of variable

$$W(t) = (Z(t))^3. \tag{15}$$

This permits the transformation of (14) into the following random linear IVP

$$\left.\begin{array}{rcl} \dot{Z}(t) & = & -(1/3)\lambda Z(t) + (1/3)\eta, \quad t \geq t_0\,, \\ Z(t_0) & = & (W_0)^{1/3}\,, \end{array}\right\}$$

whose solution s.p. is given by

$$Z(t) = (W_0)^{1/3}\, \mathrm{e}^{-(1/3)\lambda(t-t_0)} - \frac{\eta}{\lambda}\, \mathrm{e}^{-(1/3)\lambda(t-t_0)} + \frac{\eta}{\lambda}. \tag{16}$$

Following an analogous reasoning we have exhibited in Case I, based on the application of RVT technique, we obtain the p.d.f. of r.v. $Z = Z(t)$ with $t \geq t_0$ fixed,

$$\begin{array}{rcl} f_Z(z) & = & 3 \displaystyle\int_{\mathcal{D}(\eta)} \int_{\mathcal{D}(\lambda)} f_{W_0,\eta,\lambda}\left(\left(\frac{\mathrm{e}^{(1/3)\lambda(t-t_0)}\, \lambda z + \eta - \mathrm{e}^{(1/3)\lambda(t-t_0)}\, \eta}{\lambda} \right)^3, \eta, \lambda \right) \\ & \times & \left(\dfrac{\mathrm{e}^{(1/3)\lambda(t-t_0)}\, \lambda z + \eta - \mathrm{e}^{(1/3)\lambda(t-t_0)}\, \eta}{\lambda} \right)^2 \mathrm{e}^{(1/3)\lambda(t-t_0)}\, \mathrm{d}\lambda\, \mathrm{d}\eta. \end{array}$$

Now, using the change of variable (15), for every $t \geq t_0$, the 1-p.d.f. of the solution s.p. $W = W(t)$ that represents the fish weight growth is obtained by applying Lemma 3

$$\begin{array}{rcl} f_1(w,t) & = & \dfrac{1}{3} f_Z\left(w^{1/3}\right) |w|^{-2/3} \\ & = & \displaystyle\int_{\mathcal{D}(\eta)} \int_{\mathcal{D}(\lambda)} f_{W_0,\eta,\lambda}\left(\left(\frac{\mathrm{e}^{(1/3)\lambda(t-t_0)}\, \lambda w^{1/3} + \eta - \mathrm{e}^{(1/3)\lambda(t-t_0)}\, \eta}{\lambda} \right)^3, \eta, \lambda \right) \\ & \times & \left(\dfrac{\mathrm{e}^{(1/3)\lambda(t-t_0)}\, \lambda w^{1/3} + \eta - \mathrm{e}^{(1/3)\lambda(t-t_0)}\, \eta}{\lambda} \right)^2 \mathrm{e}^{(1/3)\lambda(t-t_0)}\, |w|^{-2/3} \mathrm{d}\lambda\, \mathrm{d}\eta. \end{array} \tag{17}$$

Table 2 shows data about fish weights in lbs for *walleye* species over the period 1–33 in years [27]. Notice that the sample size is $N = 33$.

t_i (years)	1	2	3	4	5	6	7	8	9	10	11
w_i (lbs)	0.2	0.4	0.6	0.9	1	1.3	1.6	1.8	2.3	2.6	2.9
t_i (years)	12	13	14	15	16	17	18	19	20	21	22
w_i (lbs)	3.1	3.4	3.7	4.5	5.2	5.7	6.2	6.5	6.7	6.8	7.2
t_i (years)	23	24	25	26	27	28	29	30	31	32	33
w_i (lbs)	8.2	9	9.5	10	10.5	11	11.5	12	12.5	13	14

Table 2: Fish weights w_i for *walleye* species in lbs every year t_i, $1 \leq i \leq 33 = N$, [27].

Next, we are going to apply the random Bertalanffy model (14) with $t_0 = 1$ to describe the evolution of fish weight over the time. For that purpose, we first need to assign a reliable probabilistic distribution to input random vector $\mathbf{Q} = (W_0, \eta, \lambda)$. To this end, several methods and techniques are available such as frequentist and bayessian techniques [28, 29, 30]. In this paper, this will be done applying an inverse frequentist technique for parameter estimation, which is a particular *Inverse Uncertainty Quantification* technique. As the results established in Section 3 are valid for dependent r.v.'s, and accounting for statistical dependence among input parameters (W_0, η, λ) is an important issue, we have chosen an inverse frequentist approach because it allows us to consider statistical dependence in a very flexible manner. Under inverse frequentist technique, it is assumed that the measured quantity of interest, i.e., fish weights w_i of our observations (t_i, w_i), $1 \leq i \leq N = 33$ are corrupted by measurement errors ϵ_i, i.e.,

$$w_i = W(t_i; \mathbf{q}) = W(t_i; w_0, \eta, \lambda) + \epsilon_i, \quad 1 \leq i \leq 33 = N. \tag{18}$$

As usual, notice that we now use lower-case letters to emphasize that the model is being evaluated at specific numerical values (t_i, w_i).

The mathematical inverse problem associated with parameter estimation can then be stated as follows: to quantify uncertainty associated with $\mathbf{q} = (w_0, \eta, \lambda)$ from the measurement errors ϵ_i, and then assigning a probabilistic distribution to random vector $\mathbf{Q} = (W_0, \eta, \lambda)$. A basic tenet of inverse frequentist approach, that will be checked later, is that errors are i.i.d. (independent and identically distributed) and $\epsilon_i \sim \mathrm{N}(0; \sigma^2)$, being $\sigma > 0$ fixed but unknown. As a consequence of this assignment of uncertainty to model parameters through measurement errors according to formulation (18), the probabilistic distribution for the random vector $\mathbf{Q} = (W_0, \eta, \lambda)$ is assumed to be a multivariate Gaussian distribution

$$\mathbf{Q} = (W_0, \eta, \lambda) \sim \mathrm{N}_3(\mu_{\mathbf{Q}}; \Sigma_{\mathbf{Q}}),$$

where the mean vector $\mu_{\mathbf{Q}} = (\hat{w}_0, \hat{\eta}, \hat{\lambda})$ is defined from appropriate estimates of (w_0, η, λ) and $\Sigma_{\mathbf{Q}}$ represents the variance-covariance matrix that will be determined below.

In order to achieve this goal, first notice that from (15) and (16) , it is clear that the dependence of the weight $W(t; \mathbf{q})$ on $\mathbf{q}$ is nonlinear. A least squares fit to the data yields the following parameter estimates

$$\mu_{\mathbf{Q}} = (\hat{w}_0, \hat{\eta}, \hat{\lambda}), \qquad \hat{w}_0 = 0.365934, \quad \hat{\eta} = 0.305461, \quad \hat{\lambda} = 0.0880184. \tag{19}$$

The residuals of the fitting are,

$$\epsilon_i = W(t_i; \hat{w}_0, \hat{\eta}, \hat{\lambda}) - w_i, \quad 1 \le i \le 33 = N, \tag{20}$$

where, $W(t_i; \hat{w}_0, \hat{\eta}, \hat{\lambda})$ is the solution of IVP (14) with $t_0 = 1$ evaluated at every year $t_i = i \in \{1, 2, \ldots, 33\}$ with model parameters given by (19) and, w_i are the fish weight data collected in Table 2. Notice that $W(t; \hat{w}_0, \hat{\eta}, \hat{\lambda}) = W(t)$ is obtained from (15) and (16). The model fitting and the residuals are shown in Figure 3 (left) and Figure 3 (right), respectively. From this latter graphical representation, one observes that residuals do not exhibit discernible pattern, thus motivating the assumption that errors are independent and identically distributed. To check that errors are normally distributed (null hypothesis), a Shapiro-Walk test has been applied. Fixed a confidence level, say α, null hypothesis is rejected when the p-value is smaller than α; otherwise the normality of the residuals is accepted (i.e., cannot be rejected) [31]. In our case, Shapiro-Walk test has been applied taking $\alpha = 0.05$, showing the normality of the residuals (see Table 3). This conclusion has been reinforced by means a Q-Q plot (see Figure 4).

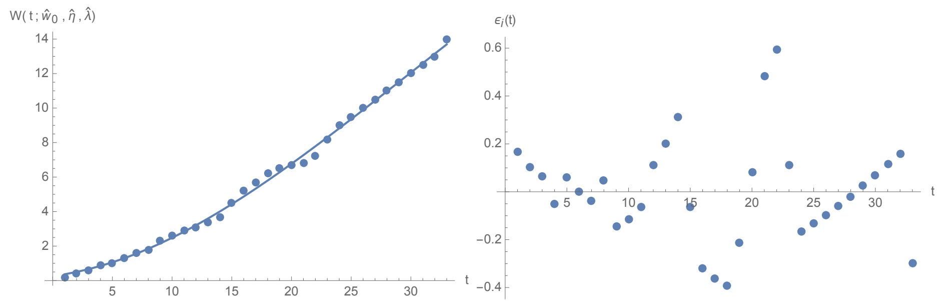

Figure 3: Left: Model fitting to the fish weights using least mean squares method. Right: Residuals at the $N = 33$ fish weights data.

Normality Test	Statistic	p-value
Shapiro-Walk Test	0.958995	0.242077

Table 3: Shapiro-Walk test to check the normality of the residuals.

From the least mean square fitting, the error standard deviation estimate is given by

$$\sigma = \sqrt{\sum_{i=1}^{33} (\epsilon_i)^2} = 0.214435, \tag{21}$$

where the residuals ϵ_i, are defined by (20).

According to frequentist parameter estimation method described in [30], to account for vari-

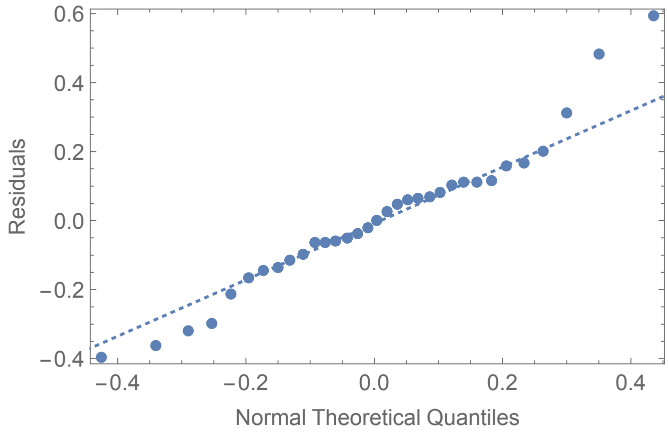

Figure 4: Q-Q plot for the least mean squares residuals.

ability of the $p = 3$ model parameters, $\mathbf{Q} = (W_0, \eta, \lambda)$, the $N \times p = 33 \times 3$ sensitivity matrix

$$\chi(\mathbf{Q}) = \left.\begin{bmatrix} \dfrac{\partial W(t_1;\mathbf{Q})}{\partial W_0} & \cdots & \dfrac{\partial W(t_{33};\mathbf{Q})}{\partial W_0} \\ \dfrac{\partial W(t_1;\mathbf{Q})}{\partial \eta} & \cdots & \dfrac{\partial W(t_{33};\mathbf{Q})}{\partial \eta} \\ \dfrac{\partial W(t_1;\mathbf{Q})}{\partial \lambda} & \cdots & \dfrac{\partial W(t_{33};\mathbf{Q})}{\partial \lambda} \end{bmatrix}^{\mathrm{T}}\right|_{\mathbf{Q}=(\hat{w}_0,\hat{\eta},\hat{\lambda})} \tag{22}$$

has been computed. A graphical representation of the entries of this matrix by files has been plotted in Figure 5. From these representations, and taking into account their vertical scales, one deduces that the critical model input parameter with respect to the sensitivity analysis is λ.

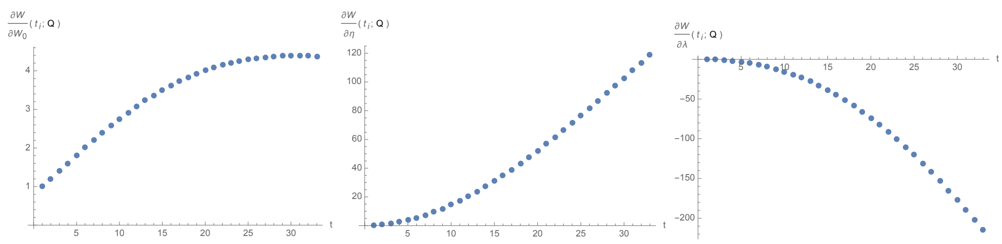

Figure 5: Analytic sensibility values of matrix (22). Left: $\frac{\partial W}{\partial W_0}(t_i;\mathbf{Q})$. Center: $\frac{\partial W}{\partial \eta}(t_i;\mathbf{Q})$. Right: $\frac{\partial W}{\partial \lambda}(t_i;\mathbf{Q})$. $t_i = i$, $1 \le i \le 33 = N$.

Then, from (21) and (22), one obtains the covariance matrix of model parameters

$$\Sigma_{\mathbf{Q}} = \sigma^2 \left((\chi(\mathbf{Q}))^{\mathrm{T}} \chi(\mathbf{Q})\right)^{-1} = \begin{bmatrix} 0.0029288 & -0.000812275 & -0.000400288 \\ -0.00081227 & 0.000268075 & 0.000136915 \\ -0.000400288 & 0.000136915 & 0.0000705259 \end{bmatrix}. \quad (23)$$

It is worth nothing that λ parameter has the smallest variability (0.0000705259). This is in agreement with the fact that it is the most critical parameter regarding the sensibility of the model.

Summarizing, based on inverse frequentist technique for parameter estimation approach the following probabilistic distribution has been assigned to model parameters

$$\mathbf{Q} = (W_0, \eta, \lambda) \sim \mathrm{N}_3(\mu_{\mathbf{Q}}; \Sigma_{\mathbf{Q}}) \quad (24)$$

where $\mu_{\mathbf{Q}}$ and $\Sigma_{\mathbf{Q}}$ are given by (19) and (23), respectively.

At this point, we are ready to take advantage of the theoretical results previously established about the 1-p.d.f. of the solution s.p. to the random Bertalanffy model. In Figure 6 (left), the 1-p.d.f. is plotted for every time of the whole sample. As the 1-p.d.f. is leptokurtic and it has little variance, we have plotted it only for values of w where it is greater than 10^{-12}. In Figure 6 (right), a more detailed plot for times values $t \in \{1.1, \ldots, 2\}$ is shown.

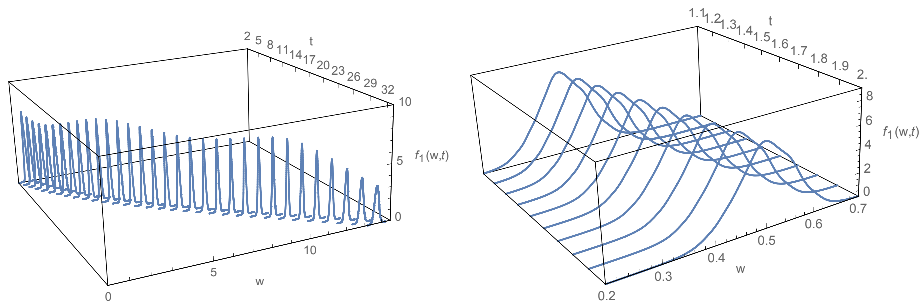

Figure 6: Left: 1-p.d.f. of the solution stochastic process to random Bertalanffy model (14) given by (17) for all the times of the sample, $t \in \{2, \ldots, 33 = N\}$. Model parameters are assumed to have the multinormal distribution defined by (24), (19) and (23). Right: Detailed representation of the 1-p.d.f. for the times $t \in \{1.1, 1.2, \ldots, 2\}$.

The mean and the variance functions of the fish weight over the time can be determined as follows

$$\mu_W(t) = \mathbb{E}[W(t)] = \int_{-\infty}^{\infty} w f_1(w,t)\,\mathrm{d}w, \quad (\sigma_W(t))^2 = \mathbb{V}[W(t)] = \int_{-\infty}^{\infty} w^2 f_1(w,t)\,\mathrm{d}w - (\mu_W(t))^2, \quad (25)$$

where $f_1(w,t)$ is defined by (17). In order to construct confidence intervals, first let us fix a time value $\hat{t} \geq 1$ and $\alpha \in (0,1)$, and secondly determine $\hat{w}_1 = w_1(\hat{t})$ and $\hat{w}_2 = w_2(\hat{t})$ such that

$$\int_0^{\hat{w}_1} f_1(w,\hat{t})\,\mathrm{d}w = \frac{\alpha}{2} = \int_{\hat{w}_2}^1 f_1(w,\hat{t})\,\mathrm{d}w. \quad (26)$$

Then, $(1-\alpha)\times 100\%$-confidence interval is specified by

$$1-\alpha = \mathbb{P}\left(\{\omega \in \Omega : W(\hat{t};\omega) \in [\hat{w}_1, \hat{w}_2)]\}\right) = \int_{\hat{w}_1}^{\hat{w}_2} f_1(w,\hat{t})\,\mathrm{d}w\,. \tag{27}$$

In Figure 7, both the mean function (solid line), 99%–confidence intervals (dashed lines) and real data (blue points) are shown. From this graphical representation one observes that the proposed random Bertalanffy model captures satisfactorily the data uncertainty.

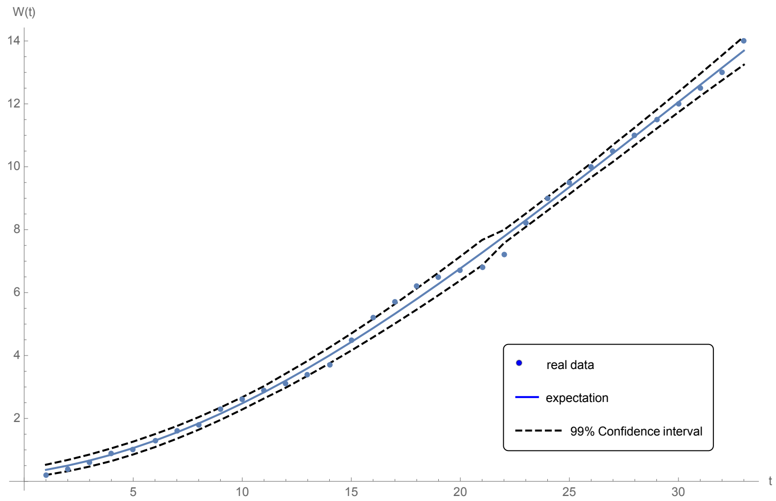

Figure 7: Expectation (solid line) and 99%–confidence intervals (dotted lines). Points represent fish weigh.

5. Conclusions

In this paper the Random Variable Transformation (RVT) method has been successfully used to determine a full probabilistic description of the Bernoulli random differential equation. This description has been made through the computation of the first probability density function of the solution stochastic process of that important equation. The study has been conducted in the general case that all input parameters and the initial condition are absolutely continuous random variables. Two important features are that neither probabilistic independence among random variables nor specific probabilistic distributions have been assumed throughout our analysis. These facts provide a generality to our study. Therefore, any joint probability density function can be considered for the model input parameters. Furthermore, it has been shown the usefulness of the theoretical results obtained to model satisfactorily a real problem. Finally, we want to point out that the proposed technique can be applied to compute the first probability density function of the solution stochastic process for other random (ordinary or partial) differential equations.

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Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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