Mean square solution of Bessel differential equation with uncertainties

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Abstract

This paper deals with the study of a Bessel-type differential equation where input parameters (coefficient and initial conditions) are assumed to be random variables. Using the so-called L_p-random calculus and assuming moment conditions on the random variables in the equation, a mean square convergent generalized power series solution is constructed. As a result of this convergence, the sequences of the mean and standard deviation obtained from the truncated power series solution are convergent as well. The results obtained in the random framework extend their deterministic counterpart. The theory is illustrated in two examples in which several distributions on the random inputs are assumed. Finally, we show through examples that the proposed method is computationally faster than Monte Carlo method.

Keywords: Random differential equation, L_p–random calculus, Bessel differential equation

1. Introduction

Deterministic differential equations have demonstrated to be powerful tools to model a number of problems in physics, chemistry, epidemiology, engineering, etc. When they are put in practice, their inputs (coefficients, forcing term, initial/boundary conditions) need to be set from sampled data, which usually contain uncertainty. The main source of randomness come from measurement errors and complexity of the phenomenon under analysis. This leads to two main approaches in dealing with differential equations with randomness, namely, stochastic differential equations and random differential equations. On the one hand, stochastic differential equations consider uncertainty through an irregular Gaussian stochastic process termed as white noise, i.e., the derivative of the Wiener process. Their analytic and numerical study requires the so-called Itô calculus [1, 2]. On the other hand, random differential equations constitute natural extensions of their deterministic counterpart since the involved input parameters are considered directly

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random variables and/or stochastic process having a more regular behaviour. The advantage of considering random differential equations against stochastic differential equations is the wide range of well-known probability distributions that can be assigned to their input parameters such as beta, gamma, lognormal, gaussian, etc [3, 4, 5, 6, 7]. The analysis of random differential equations is based on the so-called L_p-random calculus, being mean square and mean fourth calculus specializations corresponding to $p = 2$ and $p = 4$, respectively, that have demonstrated to be very useful for this purpose [8, 9].

The goal of this paper is to construct a mean square solution for the Bessel random differential equation (r.d.e.)

$$t^2\ddot{X}(t) + t\dot{X}(t) + (t^2 - A^2)X(t) = 0, \qquad t > 0, \tag{1}$$

where A is assumed to be a random variable defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout the paper, we will assume that A is a non-negative random variable with probability 1 (w.p. 1), i.e.,

$$\mathbb{P}\left[\{\omega \in \Omega : A(\omega) \geq 0\}\right] = 1. \tag{2}$$

The construction of such solution will be performed by random generalized power series whose mean square convergence will be justified taking advantage of L_p-random calculus. From an applied point of view, it is important to point out that the computation of the rigorous solution of (1) in the mean square sense guarantees that the approximations generated by truncating the exact random power series solution of (1) will converge to the corresponding exact mean and variance. These two statistical moments are often the most relevant information required in applications. This advantage makes L_p-random calculus, and hence mean square convergence, the convenient framework to study random differential equation (1) instead of using alternative stochastic convergences such as almost surely, in probability and distribution. Furthermore we shall show later, through several numerical examples, that random generalized power series solution approach is faster than Monte Carlo sampling. This latter approach is the most widely used method to deal with random differential equations in applications.

The consideration of randomness in the A parameter that appears in the Bessel differential equation (1) can be motivated from physical considerations. The wave propagation generated by a electric field and its variations in the medium can be considered as being randomly varying due to unhomogeneous physical properties of the medium. As it is shown in [10], the governing equation for the electric field in a specific direction is given by a Bessel equation of the form (1), where A coefficient depends upon random medium parameters. From a mathematical point of view the Bessel differential equation is encountered when solving boundary value problems, such as separable solutions to Helmholtz equation in cylindrical or spherical coordinates. The A parameter determines the order of the Bessel functions found in the solution of equation (1). In the deterministic framework A parameter can take any real value. A natural generalization of this equation to the random context consists of assuming that A parameter together with the corresponding initial conditions are random variables rather than deterministic numbers. The extension to the random scenario of another classical second-order linear differential equations that appear in physics can be found in [11] and in the references therein. In [11], the study is conducted taking advantage of L_p-calculus. Another contributions solving random differential equations in the mean square sense include [12, 13, 14].

The paper is organized as follows. In Section 2 the main results regarding the so-called L_p-random calculus that will be required throughout the paper are summarized and/or established. Section 3 is devoted to construct two mean square convergent random generalized power series of the Bessel differential equation under mild conditions. Section 4 is addressed to apply the theoretical results established in Section 3 to construct a mean square solution of the random Bessel differential equation with two random initial conditions. Several illustrative examples as well as conclusions are presented in Section 5.

2. Preliminaries on L_p–random calculus

Hereinafter, the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a complete probability space. For the sake of clarity, first we will summarize the main definitions and results that will be used throughout this paper. Further details about them can be found in [1, 8, 9, 15]. We will also establish new technical results related to the so-called L_p–random calculus that will be required later.

Let $p \geq 1$ be a real number. A real random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called of order p (in short, p-r.v.), if

$$\mathrm{E}[|X|^p] < \infty ,$$

where E[ ] denotes the expectation operator. The set $\mathrm{L}_p(\Omega)$ of all the p-r.v.'s endowed with the norm

$$\|X\|_p = \left(\mathrm{E}[|X|^p]\right)^{1/p} ,$$

is a Banach space, [16, p.9]. Let $\{X_n : n \geq 0\}$ be a sequence in $\mathrm{L}_p(\Omega)$. We say that it is convergent in the p-th mean to $X \in \mathrm{L}_p(\Omega)$, if

$$\lim_{n\to\infty} \|X_n - X\|_p = 0.$$

This convergence is denoted by $X_n \xrightarrow[n\to+\infty]{p\text{−th mean}} X$. For $p = 2$, this 2-th mean convergence is usually referred to as mean square convergence.

If $q > p \geq 1$, and $\{X_n : n \geq 0\}$ is a convergent sequence in $\mathrm{L}_q(\Omega)$, that is, q-th mean convergent to $X \in \mathrm{L}_q(\Omega)$, then $\{X_n : n \geq 0\}$ is in $\mathrm{L}_p(\Omega)$ and it is p-th mean convergent to $X \in \mathrm{L}_p(\Omega)$. In general, $\mathrm{L}_q(\Omega) \subset \mathrm{L}_p(\Omega)$ for $q > p \geq 1$, [16, p.13]. Moreover, using the Cauchy-Schwarz inequality one can demonstrate that [17, p. 415]

$$\|XY\|_q \leq \|X\|_{2q}\|Y\|_{2q}, \quad X, Y \in L_{2q}(\Omega),\ q \geq 1. \tag{3}$$

From these facts it is easy to establish the following

Proposition 1. *Let $\{X_n : n \geq 0\}$ be a sequence in $\mathrm{L}_{2q}(\Omega)$, $q \geq 1$. If $Y \in \mathrm{L}_{2q}(\Omega)$ and $X_n \xrightarrow[n\to+\infty]{2q\text{−th mean}} X$ then, $YX_n \xrightarrow[n\to+\infty]{q\text{−th mean}} YX$.*

Let $\mathcal{T} \subset \mathbb{R}$ be an interval. If $\mathrm{E}[|X(t)|^p] < +\infty$ for all $t \in \mathcal{T}$, then $\{X(t) : t \in \mathcal{T}\}$ is called a stochastic process of order p (in short, p-s.p.). The stochastic process $\{X(t) : t \in \mathcal{T}\}$ in $L_p(\Omega)$ is said to be p-th mean continuous at $t \in \mathcal{T}$ if

$$\|X(t+h) - X(t)\|_p \xrightarrow[h\to 0]{} 0, \quad t, t+h \in \mathcal{T}. \tag{4}$$

If there exists a stochastic process $\frac{dX(t)}{dt} \in \mathrm{L}_p(\Omega)$ such that

$$\left\| \frac{X(t+h) - X(t)}{h} - \frac{dX(t)}{dt} \right\|_p \xrightarrow[h\to 0]{} 0, \quad t, t+h \in \mathcal{T}, \tag{5}$$

then we say that the stochastic process $X(t)$ is p-th mean differentiable at $t \in \mathcal{T}$ and its p-th mean derivative at t is given by $\frac{dX(t)}{dt}$. The notation $\dot{X}(t)$ is also used for the p-th mean derivative of the stochastic process $X(t)$ at the point t.

Example 1. *Let* $Z \in \mathrm{L}_{2q}(\Omega)$, $q \geq 1$. *Clearly the stochastic process* $\{X(t) = Z \ln t : t > 0\} \in \mathrm{L}_{2q}(\Omega)$ *and its* $2\,q$*-th mean derivative is given by* $\{\dot{X}(t) = \frac{Z}{t} : t > 0\}$*:*

$$\lim_{h \to 0} \left\| \frac{Z \ln(t+h) - Z \ln(t)}{h} - \frac{Z}{t} \right\|_{2q} = \|Z\|_{2q} \lim_{h \to 0} \left| \frac{\ln(t+h) - \ln(t)}{h} - \frac{1}{t} \right| = 0,$$

since the deterministic function $\ln(t)$ *is differentiable for each* $t > 0$ *and* $\|Z\|_{2q} < \infty$ *because* $Z \in L_{2q}(\Omega)$.

The proof of the two following propositions are easily adapted for $p \geq 2$ with the corresponding results for the case of $p = 4$ given in [9].

Proposition 2. *Let* $\{X(t) : t \in \mathcal{T} \subset \mathbb{R}\}$ *be a stochastic process in* $\mathrm{L}_p(\Omega)$. *If it is p-th mean differentiable at t, then it is p-th mean continuous at t.*

Remark 1. Since q-th mean convergence entails p-th mean convergence when $q \geq p \geq 1$, then if a stochastic process is q-th mean differentiable (continuous) then it is also p-th mean differentiable (continuous).

Example 2. *In the context of Example 1 by Proposition 2, the stochastic process* $X(t)$ *is* $2q$*-th mean continuos. Moreover, it is also p-th mean differentiable, and hence p-th mean continuous for* $1 \leq p \leq 2q$.

Proposition 3 (product q-th mean derivative rule). *Let* $\{W(t) : t \in \mathcal{T}\}$ *and* $\{X(t) : t \in \mathcal{T}\}$, $\mathcal{T} \subset \mathbb{R}$ *be* $2q$*-th mean differentiable stochastic processes in* $\mathrm{L}_{2q}(\Omega)$. *Let* $\frac{dW(t)}{dt}$ *and* $\frac{dX(t)}{dt}$ *denote their* $2q$*-th mean derivatives, respectively. Then* $W(t)X(t)$ *is q-th mean differentiable at t and its q-th mean derivative is given by*

$$\frac{d\left(W(t)X(t)\right)}{dt} = \frac{dW(t)}{dt} X(t) + W(t) \frac{dX(t)}{dt}.$$

Next, we state a result to legitimate the 4-th mean differentiation of 4-th mean convergent series. Its proof can be found in [18].

Proposition 4. *Assume that for* $n \geq 1$, *the process* $\{X_n(t) : t \in \mathcal{T} \subset \mathbb{R}\}$, *satisfies*

1. $X_n(t)$ *is* 4*-th mean differentiable and* $\dot{X}_n(t)$ *is* 4*-th mean continuous,*
2. $X(t) = \sum_{n \geq 1} X_n(t)$ *is* 4*-th mean convergent,*
3. $\sum_{n \geq 1} \dot{X}_n(t)$ *is uniform* 4*-th mean convergent in a neighborhood of each* $t \in \mathcal{T}$.

Then, for each $t \in \mathcal{T}$, $X(t)$ *is* 4*-th mean differentiable and* $\dot{X}(t) = \sum_{n \geq 1} \dot{X}_n(t)$.

$$X(t) = \sum_{n=0}^{\infty} \frac{X^{(n)}(0)}{n!} t^n, \tag{6}$$

where $X^{(n)}(0)$ denotes the derivative of order n of the s.p. $X(t)$ evaluated at the point $t = 0$, in the p-th mean sense. The p-th mean derivative for the composition of two stochastic processes will be needed later. Some conditions under which the random *chain rule* can be applied were given in [14]. In order to state that result, we first remember the concept of almost surely sample path continuous stochastic process.

Definition 1. *([15, p.55]) We say that a stochastic process* $\{X(t): t \in \mathcal{T}\}$ *defined on an interval* $\mathcal{T}$ *is almost surely sample path continuous or that* $\{X(t) : t \in \mathcal{T}\}$ *has continuous paths with probability one (w.p. 1) if*

$$\mathbb{P}\left[\bigcup_{t\in\mathcal{T}}\left\{\omega \in \Omega : \lim_{h\to 0} |X(t+h)(\omega) - X(t)(\omega)| \neq 0\right\}\right] = 0\,.$$

A very useful result to check that a stochastic process is almost surely continuous is the Kolmogorov's criterion.

Theorem 1. *([1, p. 12]) Assume that the stochastic process* $\{X(t): t \in [0,T]\}$ *satisfies that, for all* $T > 0$*, there exist positive constants* α, β, D *such that*

$$\mathrm{E}\left[|X(t) - X(s)|^{\alpha}\right] \leq D|t-s|^{1+\beta} \quad 0 \leq s, t \leq T.$$

Then the s.p. $\{X(t): t \in [0,T]\}$ *is almost surely sample path continuous.*

Example 3. *Let* $Z \in \mathrm{L}_p(\Omega)$ *with* $p \geq 2$*. Then the stochastic process* $\{X(t) = Z \ln t : t \in [r_1, r_2],\ 0 < r_1 < r_2\}$*, that belongs to* $\mathrm{L}_p(\Omega)$*, is almost surely path continuous (w.p. 1). In fact, using the mean value theorem for the deterministic function* $\ln y$ *on* $y \in [t, s]$ *one gets*

$$\mathrm{E}\left[|X(t) - X(s)|^p\right] = \mathrm{E}\left[|Z|^p\right] |\ln(t) - \ln(s)|^p = \mathrm{E}\left[|Z|^p\right] \frac{1}{\xi} |t-s|^p, \quad \xi \in (t, s).$$

Since $Z \in L_p(\Omega)$*, then* $\mathrm{E}\left[|Z|^p\right] < \infty$*. Thus, taking* $\alpha = p > 0$*,* $D = \frac{1}{\xi}\mathrm{E}[|Z|^p] > 0$ *and* $\beta = p - 1 > 0$ *in Theorem 1, it follows that* $X(t)$ *is almost surely path continuous on* $[r_1, r_2]$*.*

Theorem 2 (chain rule). *([14]) Let* $f(x)$ *be a deterministic real function with continuous derivative* $f'(x)$ *and the stochastic process* $\{X(t) : t > 0\} \in \mathrm{L}_{2q}(\Omega)$*, with* $q \geq 1$*, satisfying*

1. $X(t)$ *is* $2\,q$*-th mean differentiable.*
2. $X(t)$ *is almost surely path continuous w.p. 1.*
3. *There exist* $r > 2\,q$ *and* $\delta > 0$ *such that* $\sup_{s\in[-\delta,\delta]} \mathrm{E}\left[\left|f'(x)|_{x=X(t+s)}\right|^r\right] < +\infty$.

Then, the stochastic process $f(X(t)) \in \mathrm{L}_q(\Omega)$ *is* q*-th mean differentiable and its* q*-th mean derivative is given by*

$$\frac{df(X(t))}{dt} = f'(x)\Big|_{x=X(t)} \frac{dX(t)}{dt}\,.$$

The following result is a consequence of Theorem 2.

Proposition 5. *If* $Z \in L_{2q}(\Omega)$*,* $q \geq 1$*, and there exist positive numbers* $r > 2q$ *and* $\delta > 0$ *such that*

$$\sup_{s\in[-\delta,\delta]} \mathrm{E}\left[e^{Zr\ln(t+s)}\right] < \infty, \tag{7}$$

then $e^{Z\ln t}$ *is* q*-th mean differentiable at* t *and*

$$\frac{d}{dt}\left(e^{Z\ln t}\right) = \frac{d}{dt}\left(t^Z\right) = \frac{Z}{t} e^{Z\ln t}. \tag{8}$$

Remark 2. The condition (7) guarantees that $t^Z \in L_r(\Omega)$ for $r > 2q$.

Proof 1. *Let us take $f(x) = e^x$, $X(t) = Z \ln t$, then from Example 1, Example 3 (with $p = 2q \geq 2$) and Theorem 2, it follows that $f(X(t)) = e^{Z \ln t}$ is q-th mean differentiable at t and its q-th mean derivative is given by* (8). □

Next, we show that the set of random variables which satisfy the conditions of Proposition 5 is not empty.

Example 4. *Let Z be a random variable with Beta distribution: $Z \sim Be(\alpha,\beta)$, $\alpha > 0, \beta > 0$. By taking $t > 0$, $\delta = \delta(t) = \frac{t}{2}$, $r > 2q$, $q \geq 1$ in Proposition 5, it follows that $t+s \in [t-\delta, t+\delta] \subset]0, \infty[$ and*

$$\mathrm{E}\left[e^{Zr\ln(t+s)}\right] = \int_0^1 (t+s)^{zr} f_Z(z)\,\mathrm{d}z \leq K(t,s),$$

where $f_Z(z)$ is the probability density function of r.v. Z and

$$K(t,s) = \begin{cases} 1 & \text{if } (t+s) \leq 1, \\ (t+s)^r & \text{if } (t+s) > 1. \end{cases}$$

Therefore $\sup_{s\in[-\delta,\delta]} \mathrm{E}\left[e^{Zr\ln(t+s)}\right] < \infty$. Moreover,

$$\mathrm{E}\left[Z^{2q}\right] = \frac{\Gamma(\beta)\Gamma(\alpha+\beta)\Gamma(\alpha+2q)}{\Gamma(\alpha+\beta+2q)}.$$

Thus, by Proposition 5 the q-th mean derivative of the s.p. t^Z is given by Zt^{Z-1}.

3. Constructing a solution of the Bessel random differential equation

This section is devoted to construct a 2-th mean convergent solution for the Bessel random differential equation (1). Inspirated by the classical Fröbenius method, we seek solutions in form of generalized series

$$X(t) = t^Z \sum_{n=0}^{\infty} X_n t^n, \tag{9}$$

where Z and X_n, $n \geq 0$, are random variables and t^Z is a stochastic process defined as $t^Z := e^{Z \ln t}$. In order to impose that series (9) satisfies random differential equation (1), its two first 2-th mean derivatives must be computed. The following results provides sufficient conditions to this end.

Lemma 3. *Let $\{X_n,\ n \geq 0\}$ be a sequence in $L_4(\Omega)$ and let $\sum_{n=0}^{\infty} X_n t^n$ be 4-th mean convergent for $0 < r_1 \leq t \leq r_2$. If Z satisfies the following conditions*

i) $Z \in L_{16}(\Omega)$.

ii) There exist $r > 16$ and $\delta > 0$ such that $\sup_{s\in[-\delta,\delta]} \mathrm{E}\left[e^{Zr\ln(t+s)}\right] < \infty$,

then $X(t) := t^Z \sum_{n=0}^{\infty} X_n t^n$ belongs to $L_2(\Omega)$ and the first and second 2-th mean derivatives, $\dot{X}(t)$ and $\ddot{X}(t)$, are given by

$$\dot{X}(t) = \sum_{n=0}^{\infty} (n+Z) X_n t^{n+Z-1}, \quad \ddot{X}(t) = \sum_{n=0}^{\infty} X_n (n+Z)(n+Z-1) t^{n+Z-2}. \tag{10}$$

Proof 2. *By hypothesis ii) and Remark 2 it follows*

$$t^Z \in L_r(\Omega) \quad with \quad r > 16. \tag{11}$$

Therefore $t^Z \in L_4(\Omega)$. *In addition, we also have* $\sum_{n=0}^{\infty} X_n t^n \in L_4(\Omega)$. *Then* $X(t) := t^Z \sum_{n=0}^{\infty} X_n t^n \in L_2(\Omega)$.

Since $\sum_{n=0}^{\infty} X_n t^n$ *is 4-th mean convergent on* $0 < r_1 \leq t \leq r_2$, *Proposition 4 implies that for each* $t \in [r_1, r_2]$, *the stochastic process* $Y(t) := \sum_{n=0}^{\infty} X_n t^n$ *is 4-th mean differentiable and its* 4*-th mean derivative is given by* $\dot{Y}(t) = \sum_{n=1}^{\infty} nX_n t^{n-1}$.

By Proposition 5, assumptions i) and ii) guarantee that $t^Z = e^{Z \ln t}$ *has 8-th mean derivative and its 8-th mean derivative is given by* Zt^{Z-1}, *which coincides with the 4-th mean derivative. Thus applying the product q-th mean derivative rule (Proposition 3) to* t^Z *and* $Y(t)$ *with* $q = 2$, *we have the 2-th mean derivative of* $X(t)$:

$$\dot{X}(t) = t^Z \sum_{n=0}^{\infty} nX_n t^{n-1} + Zt^{Z-1} \sum_{n=0}^{\infty} X_n t^n = \sum_{n=0}^{\infty} (n+Z) X_n t^{n+Z-1}.$$

Let us justify the commutation of the terms t^Z *and* Zt^{Z-1} *with the infinite sums implicitly used in the last step above to compute* $\dot{X}(t)$. *By (11),* $t^{Z-1} \in L_{16}(\Omega)$. *Moreover by i),* $Z \in L_{16}(\Omega)$ *and hence, by applying inequality* (3) *for* $q = 8$, *one gets* $Zt^{Z-1} \in L_8(\Omega) \subset L_4(\Omega)$. *Therefore, the commutation is justified by Proposition 1.*

By similar reasoning to the one used to justify the existence and computation of the 2-th mean derivative $\dot{X}(t)$, *one can legitimate the following representation for the second order 2-th mean derivative of* $X(t)$

$$\begin{aligned} \ddot{X}(t) &= t^Z \sum_{n=0}^{\infty} n(n-1) X_n t^{n-2} + Zt^{Z-1} \sum_{n=0}^{\infty} nX_n t^{n-1} \\ &+ Zt^{Z-1} \sum_{n=0}^{\infty} nX_n t^{n-1} + Z(Z-1) t^{Z-2} \sum_{n=0}^{\infty} X_n t^n \\ &= \sum_{n=0}^{\infty} (n+Z)(n+Z-1) X_n t^{n+Z-2}. \end{aligned} \tag{12}$$

Here the hypothesis made on Z also justify the commutations implicitly used in (12). □

Remark 3. Regarding our goal, which is to construct a rigorous solution random series, $X(t)$, of the form $X(t) = t^Z Y(t)$ to the Bessel r.d.e. given by (1). Lemma 3 tell us knowledge of the 4-th mean convergence of $Y(t) = \sum_{n=0}^{\infty} X_n t^n$ guarantees $X(t)$ is a 2-th mean solution if Z satisfies hypotheses i) and ii) of Lemma 3.

Keeping this in mind, we continue by inserting expressions (9) and (10) into the random Bessel differential equation (1)

$$\begin{aligned} 0 &= t^2 \ddot{X}(t) + t\dot{X}(t) + (t^2 - A^2) X(t) \\ &= \sum_{n=0}^{\infty} X_n (n+Z)(n+Z-1) t^{n+Z} + \sum_{n=0}^{\infty} (n+Z) X_n t^{n+Z} + (t^2 - A^2) \sum_{n=0}^{\infty} X_n t^{n+Z} \\ &= t^Z \left\{ (Z^2 - A^2) X_0 + \left[(1+Z)^2 - A^2 \right] X_1 t + \sum_{n=2}^{\infty} \left[\left\{ (n+Z)^2 - A^2 \right\} X_n + X_{n-2} \right] t^n \right\}. \end{aligned}$$

Since $t^{Z(\omega)} = e^{Z(\omega)\ln t} \neq 0$, $\forall \omega \in \Omega$, w.p. 1, above relation yields

$$(Z^2 - A^2)X_0 + \left[(1+Z)^2 - A^2\right] X_1 t + \sum_{n=2}^{\infty} \left[\left\{(n+Z)^2 - A^2\right\} X_n + X_{n-2}\right] t^n = 0. \tag{13}$$

In order for this relation to be satisfied for all t, let us take $Z = A$ and assume

$$\mathbb{P}\left[\{\omega \in \Omega : \ X_0(\omega) \neq 0\}\right] = 1, \tag{14}$$

i.e., X_0 is a non-zero random variable w.p. 1. Since A satisfies hypothesis (2), if

$$X_1 = 0, \quad \text{w.p. } 1, \tag{15}$$

and

$$X_n = -\frac{X_{n-2}}{(n+A)^2 - A^2} = -\frac{X_{n-2}}{n(n+2A)}, \quad n \geq 2, \quad \text{w.p. } 1, \tag{16}$$

then relation (13) holds for all t.

From (15) and (16), one deduces

$$X_{2n+1} = 0, \quad n \geq 0, \quad \text{w.p. } 1, \tag{17}$$

$$X_{2n} = \frac{(-1)^n X_0}{4^n n! \prod_{i=1}^{n}(A+i)}, \quad n \geq 1, \quad \text{w.p. } 1. \tag{18}$$

Therefore, taking into account Lemma 3 with $Z = A$ and Remark 3, a rigorous solution to (1) is given by

$$X_1(t) = t^A Y_1(t), \quad \text{where} \quad Y_1(t) = X_0 + \sum_{n=1}^{\infty} \frac{(-1)^n X_0}{4^n n! \prod_{i=1}^{n}(A+i)} t^{2n}, \tag{19}$$

provided that the 4-th mean convergence of $Y_1(t)$ can be justified. Under the assumption that $X_0 \in L_4(\Omega)$, if we show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n X_0}{4^n n! \prod_{i=1}^{n}(A+i)} t^{2n} \tag{20}$$

is 4-th mean convergent, then the series defining the s.p. $Y_1(t)$ will be also 4-th mean convergent. By hypothesis (2) one gets

$$A(\omega) + i \geq i \geq 1, \qquad i = 1, \ldots, n, \quad \text{w.p. } 1,$$

which implies

$$0 < \frac{1}{A(\omega) + i} \leq 1, \qquad i = 1, \ldots, n, \quad \text{w.p. } 1.$$

Hence, one obtains

$$\left(\frac{1}{\prod_{i=1}^{n}(A(\omega)+i)}\right)^4 \leq 1, \quad \text{w.p. } 1.$$

By multiplying both sides of the above inequality by $(X_0(\omega))^4$, which is non-negative w.p. 1, and taking the expectation operator, then by definition of the $\| \, \|_4$–norm, one gets

$$\left\| \frac{X_0}{\prod_{i=1}^{n}(A+i)} \right\|_4 \leq \|X_0\|_4.$$

As a consequence,

$$\sum_{n=1}^{\infty}\left\|\frac{(-1)^n X_0}{4^n n!\prod_{i=1}^{n}(A+i)}t^{2n}\right\|_4 \le \sum_{n=1}^{\infty}\left\|\frac{X_0}{\prod_{i=1}^{n}(A+i)}\right\|_4 \frac{|t|^{2n}}{4^n n!} \le \sum_{n=1}^{\infty}\frac{\|X_0\|_4}{4^n n!}|t|^{2n}. \tag{21}$$

Therefore, the random series (20) has been majorized, in $\|\,\|_4$–norm, by the following scalar power series

$$\sum_{n=1}^{\infty}\alpha_n(t), \quad \alpha_n(t)=\frac{\|X_0\|_4}{4^n n!}|t|^{2n}, \quad n\ge 1. \tag{22}$$

Now, we check by the ratio or D'Alembert test, that this series is convergent on the whole real line

$$\lim_{n\to\infty}\frac{\alpha_{n+1}(t)}{\alpha_n(t)}=\lim_{n\to\infty}\frac{|t|^2}{4(n+1)}=0, \quad \forall t\in\mathbb{R}. \tag{23}$$

Summarizing, taking into account Lemma 3, the following result has been established:

Theorem 4. *Let A and X_0 be random variables such that*

i) $A\in L_{16}(\Omega)$ satisfies condition (2).

ii) There exist positive numbers $r>16$ and $\delta>0$ such that

$$\sup_{s\in[-\delta,\delta]} \mathrm{E}\left[e^{Ar\ln(t+s)}\right]<\infty.$$

iii) $X_0\in L_4(\Omega)$ satisfies condition (14).

Then, the stochastic process $X_1(t)$ defined by (19) is a mean square solution of the Bessel random differential equation (1) on the interval $0<r_1\le t\le r_2<\infty$.

Remark 4. Assuming that A satisfies hypotheses i) and ii) of Theorem 4. From (21), if $X_0\in L_{16}(\Omega)$, then $Y_1(t)\in L_{16}(\Omega)$. Moreover, as $X_1(t)=t^A Y_1(t)$ and $t^A\in L_r(\Omega)$ with $r>16$ (see Remark 2), if $X_0\in L_{16}(\Omega)$, then $X_1(t)\in L_8(\Omega)\subset L_2(\Omega)$.

Now, we seek a second mean square solution to the Bessel random differential equation (1). For this end, we keep the assumption (14), take $Z=-A$ in (13) and assume that Z satisfies hypotheses of Lemma 3. Then, assuming

$$A(\omega)\in\bigcup_{m=0}^{\infty}[a_m,b_m], \;\; m<a_m<b_m<m+1, \;\; m\ge 0 \;\text{ integer}, \quad \text{w.p. } 1, \tag{24}$$

one obtains a second rigorous solution to (1) on $[r_1,r_2]$ given by

$$X_2(t)=t^{-A}Y_2(t), \quad \text{where} \quad Y_2(t)=X_0+\sum_{n=1}^{\infty}\frac{(-1)^n X_0}{4^n n!\prod_{i=1}^{n}(-A+i)}t^{2n}, \tag{25}$$

if $Y_2(t)$ is 4-th mean convergent on $[r_1,r_2]$. Indeed, if

$$d_i:=\min\{i-b_{i-1},|i-a_i|\} \quad \text{for each } i=1,2,\ldots$$

Then,

$$0<d_i<|i-A(\omega)|, \qquad \forall i=1,2,\ldots, \quad \text{w.p. } 1,$$

which implies that

$$0 < \left(\frac{X_0(\omega)}{\prod_{i=1}^{n} |i - A(\omega)|} \right)^4 < \left(\frac{X_0(\omega)}{\prod_{i=1}^{n} d_i} \right)^4, \quad \forall n \geq 1, \text{ integer}, \quad \text{w.p. } 1.$$

Therefore, by definition of $\| \|_4$–norm one gets

$$\left\| \frac{X_0}{\prod_{i=1}^{n}(i - A)} \right\|_4 < \frac{1}{\prod_{i=1}^{n} d_i} \|X_0\|_4, \quad \forall n \geq 1 \text{ integer}.$$

Let $\epsilon > 0$ such that $0 < \epsilon < d_n$ for all $n = 1, 2, \ldots$, then

$$\left\| \frac{X_0}{\prod_{i=1}^{n}(i - A)} \right\|_4 < \frac{1}{\prod_{i=1}^{n} d_i} \|X_0\|_4 < \frac{1}{\epsilon^k} \|X_0\|_4.$$

Therefore, the random series given in (25) has been majorized in $\| \|_4$-norm as follows

$$\sum_{n=1}^{\infty} \left\| \frac{(-1)^n X_0}{4^n n! \prod_{i=1}^{n}(-A + i)} t^{2n} \right\|_4 < \sum_{n=1}^{\infty} \frac{\|X_0\|_4}{4^n n! \epsilon^n} |t|^{2n}. \tag{26}$$

Using the ratio or D'Alembert test we check that the majorant series is convergent on the whole real line

$$\sum_{n=1}^{\infty} \beta_n(t), \quad \beta_n(t) = \frac{\|X_0\|_4}{4^n n! \epsilon^n} |t|^{2n}, \ n \geq 1 \ ;$$

$$\lim_{n \to \infty} \frac{\beta_{n+1}(t)}{\beta_n(t)} = \lim_{n \to \infty} \frac{|t|^2}{4(n+1)\epsilon} = 0, \ \forall t \in \mathbb{R}.$$

This proves the random series $Y_2(t)$ is 4-th mean convergent for $t > 0$. Summarizing, the following result has been established:

Theorem 5. *Let A and X_0 be random variables satisfying*

i) $A \in L_{16}(\Omega)$.

ii) $A(\omega) \in \bigcup_{n=0}^{\infty} [a_n, b_n]$ *w.p. 1, where* $n < a_n < b_n < n + 1$ *for all* $n = 0, 1, 2, \ldots$

iii) There exist positive numbers $r > 16$ *and* $\delta > 0$ *such that*

$$\sup_{s \in [-\delta, \delta]} \mathrm{E}\left[e^{-Ar \ln(t+s)} \right] < \infty,$$

iv) $X_0 \in L_4(\Omega)$ *satisfies condition* (14).

If $d_n = \min\{n - b_{n-1}, |n - a_n|\}$ *and there exists* $\epsilon > 0$ *such that* $0 < \epsilon < d_n$ *for all* $n = 1, 2, \ldots$, *then the stochastic process* $X_2(t)$ *given by (25) is a second mean square solution of the Bessel random differential equation (1) on the interval* $[r_1, r_2]$, $0 < r_1 < r_2$.

Remark 5. From (26), if $X_0 \in L_{16}(\Omega)$, then $Y_2(t) \in L_{16}(\Omega)$. As $t^{-A} \in L_{16}(\Omega)$ (see Remark 2), $X_2(t) = t^{-A} Y_2(t) \in L_8(\Omega)$ if $X_0 \in L_{16}(\Omega)$.

4. Computing the mean square solution of the initial value problem of Bessel random differential equation

Now, we search a solution stochastic process to the Bessel random differential equation (1) satisfying the following initial conditions

$$X(t_0) = \eta_1, \qquad \dot{X}(t_0) = \eta_2, \tag{27}$$

being η_1, η_2 random variables. Let us consider

$$X(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t), \tag{28}$$

where $X_1(t)$ and $X_2(t)$ are the solutions of (1) defined by (19) and (25), respectively, and α_i, $i = 1, 2$, are random variables to be determined in such a way that (28) satisfies initial conditions (27). In order to assure that $X(t)$ is a mean square solution to the initial value problem (1) and (27), we must prove that $X(t) \in L_2(\Omega)$. For that it is sufficient to show that α_i, $X_i(t) \in L_4(\Omega)$, $i = 1, 2$, for each $t \in [r_1, r_2]$, $0 < r_1 < r_2$.

An algebraic computation shows that

$$\alpha_1 = \frac{\eta_1 \dot{X}_2(t_0) - \eta_2 X_2(t_0)}{W(X_1, X_2)(t_0)}, \qquad \alpha_2 = \frac{\eta_2 X_1(t_0) - \eta_1 \dot{X}_1(t_0)}{W(X_1, X_2)(t_0)}, \tag{29}$$

being $W(X_1, X_2)(t_0) = X_1(t_0)\dot{X}_2(t_0) - \dot{X}_1(t_0)X_2(t_0)$ the wronskian of the solutions $\{X_1(t), X_2(t)\}$ at $t = t_0 \in [r_1, r_2]$.

Let us take $X_0 = 1$ in (19) and (25). If A satisfies condition ii) of Theorem 5 and $\mathcal{K} := \{\omega \in \Omega : A(\omega) \in \bigcup_{n=1}^{\infty}[a_n, b_n],\ n < a_n < b_n < n+1,\ n = 0, 1, 2, \ldots\}$ then $P(\mathcal{K}) = 1$ and for each $\omega \in \mathcal{K}$ the wronskian of the following functions

$$\begin{aligned} J_A(t)(\omega) &= D_0^1(\omega) X_1(t)(\omega), & D_0^1(\omega) :&= \frac{1}{2^{A(\omega)}\Gamma(A(\omega)+1)}, \\ J_{-A}(t)(\omega) &= D_0^2(\omega) X_2(t)(\omega), & D_0^2(\omega) :&= \frac{1}{2^{-A(\omega)}\Gamma(-A(\omega)+1)}, \end{aligned} \tag{30}$$

is given by

$$W(J_A, J_{-A})(t)(\omega) = -\frac{2\sin(A(\omega)\pi)}{\pi t}, \quad t > 0.$$

Taking into account the reflection formula $\Gamma(A(\omega))\Gamma(1 - A(\omega)) = \frac{\pi}{\sin(\pi A(\omega))}$ and

$$W(J_A, J_{-A})(t)(\omega) = D_0^1(\omega) D_0^2(\omega) W(X_1, X_2)(t)(\omega), \quad t > 0,\ \omega \in \mathcal{K},$$

it follows that

$$\begin{aligned} W(X_1(t), X_2(t))(\omega) &= -\frac{2\sin(A(\omega)\pi)\Gamma(A(\omega)+1)\Gamma(-A(\omega)+1)}{\pi t} \\ &= -\frac{2\sin(A(\omega)\pi)A(\omega)\Gamma(A(\omega))\Gamma(1-A(\omega))}{\pi t} \\ &= -\frac{2\sin(A(\omega)\pi)A(\omega)\left(\frac{\pi}{\sin(\pi A(\omega))}\right)}{\pi t} = -\frac{2A(\omega)}{t}. \end{aligned} \tag{31}$$

These properties used of the Bessel functions as well as Gamma function can be found in [19].

Let us prove that $X_i(t) \in L_4(\Omega)$, $i = 1, 2$, for each t in $[r_1, r_2]$, $0 < r_1 < r_2$. As $X_i(t)$ has the general form (9) is sufficient to prove that $t^Z \in L_8(\Omega)$ and $\sum_{n=0}^{\infty} X_n t^n \in L_8(\Omega)$. Since $X_0 = 1$ w.p. 1, by Theorems 4 and 5, and Remarks 4 and 5 it follows that $X_i(t) \in L_8(\Omega) \subset L_4(\Omega)$, $i = 1, 2$. We shall now prove that, $\alpha_i \in L_4(\Omega)$, $i = 1, 2$. For that, first notice that, by hypothesis (24), for each $\omega \in \mathcal{K}$, $0 < a_0 < A(\omega)$. Then applying (31) for $t = t_0$ and (29), one follows

$$\begin{aligned} |\alpha_1(\omega)|^4 &= \left| \frac{\eta_1(\omega)\dot{X}_2(t_0)(\omega) - \eta_2(\omega)X_2(t_0)(\omega)}{W(X_1, X_2)(t_0)(\omega)} \right|^4 \\ &< \left| \frac{t_0}{2a_0} \right|^4 \left| \eta_1(\omega)\dot{X}_2(t_0)(\omega) - \eta_2(\omega)X_2(t_0)(\omega) \right|^4 . \end{aligned}$$

Now, we prove that $\dot{X}_2(t_0) \in L_4(\Omega)$. Indeed, in the proof of Lemma 3 we showed that

$$\dot{X}_2(t_0) = (t_0)^{-A}\dot{Y}_2(t_0) + (-A)(t_0)^{-A-1}Y_2(t_0).$$

Moreover, from (25) one gets

$$\dot{Y}_2(t_0) = 2\sum_{n=1}^{\infty} \frac{(-1)^n X_0}{4^n (n-1)! \prod_{i=1}^{n}(-A+i)} (t_0)^{2n-1}.$$

By replacing $\|\cdot\|_4$ for $\|\cdot\|_8$ in the inequality given by (26), it follows that $\dot{Y}_2(t_0) \in L_8(\Omega)$. In addition, $-A$, $(t_0)^{-A-1}$, $(t_0)^{-A}$ are in $L_{16}(\Omega)$, therefore $\dot{X}_2(t_0) \in L_4(\Omega)$. By assuming that η_i, $i = 1, 2$, and A are independent random variables, one follows that

$$\|\alpha_1\|_4 < \frac{t_0}{2a_0} \left(\|\eta_1\|_4 \|\dot{X}_2(t_0)\|_4 + \|\eta_2\|_4 \|X_2(t_0)\|_4 \right) < \infty,$$

if $\eta_i \in L_4(\Omega)$, $i = 1, 2$. Therefore $\alpha_1 \in L_4(\Omega)$. Similar arguments show that $\alpha_2 \in L_4(\Omega)$. Finally, by expressing

$$X(t) = \eta_1 \left[\frac{\dot{X}_2(t_0)t^A Y_1(t) - \dot{X}_1(t_0)t^{-A}Y_2(t)}{W(X_1, X_2)(t_0)} \right] + \eta_2 \left[\frac{X_1(t_0)t^{-A}Y_2(t) - X_2(t_0)t^A Y_1(t)}{W(X_1, X_2)(t_0)} \right]$$

it is also shown that $X(t)$ is a 2-th mean solution of (1) and (27). Summarizing the following result has been established:

Theorem 6. *Let $\eta_i \in L_4(\Omega)$, $i = 1, 2$, and let $X_0 = 1$ in (19) and (25). Let A be a random variable satisfying conditions i), ii) of Theorem 4, and conditions i)-iii) of Theorem 5. Assume that A is independent of random variables η_i, $i = 1, 2$. If there exists $\epsilon > 0$ as in Theorem 5, then the initial value problem*

$$t^2\ddot{X}(t) + t\dot{X}(t) + (t^2 - A^2)X(t) = 0, \qquad X(t_0) = \eta_1, \quad \dot{X}(t_0) = \eta_2, \tag{32}$$

$t_0, t \in [r_1, r_2]$, $\quad 0 < r_1 < r_2 < \infty$, *has a solution stochastic process $X(t) \in L_2(\Omega)$ given by*

$$X(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t), \tag{33}$$

with

$$\alpha_1 = \frac{\eta_1\dot{X}_2(t_0) - \eta_2 X_2(t_0)}{W(X_1, X_2)(t_0)}, \qquad \alpha_2 = \frac{\eta_2 X_1(t_0) - \eta_1\dot{X}_1(t_0)}{W(X_1, X_2)(t_0)}, \tag{34}$$

being

$$W(X_1, X_2)(t_0) = X_1(t_0)\dot{X}_2(t_0) - \dot{X}_1(t_0)X_2(t_0), \tag{35}$$

for each $t \in [r_1, r_2]$.

4.1. Determining statistical information associated to the solution of the Bessel random differential equation

So far sufficient conditions under which $X(t)$ given by (33)–(34) defines a mean square solution to the initial value problem (32) have been provided. Since $X(t)$ is a stochastic process, it is also important to give its main statistical functions in order to describe it from a probabilistic standpoint. In general, it is done by means of the expectation and variance (or equivalently, standard deviation) functions. Because the solution is represented through a random infinite series, truncation is required to keep computationally feasible. The following result will play a key role to legitimate the approximations of the expectation and variance of the solution $X(t)$ by truncating of its infinite series representation.

Proposition 6. *Let $\{H_n : n \geq 0\}$ be a 2-th mean convergent sequence of random variables in $L_2(\Omega)$ and let us denote its limit by $H \in L_2(\Omega)$. Then,*

$$\mathrm{E}[H_n] \xrightarrow[n\to+\infty]{} \mathrm{E}[H_n], \quad \mathrm{E}\left[(H_n)^2\right] \xrightarrow[n\to+\infty]{} \mathrm{E}\left[H^2\right]. \tag{36}$$

And, as a consequence,

$$\mathrm{Var}[H_n] \xrightarrow[n\to+\infty]{} \mathrm{Var}[H]. \tag{37}$$

At this point, notice that we can take advantage of this result because mean square convergence of the infinite series defining $X(t)$ for $t > 0$ has been rigorously established in Theorem 6.

To deal with the approximations of the mean and variance, first it is convenient to introduce the following notation:

$$S(n) := \frac{(-1)^n}{4^n n!}, \quad U(n; A) := \prod_{i=1}^{n} (A + i). \tag{38}$$

Hence, according to (33)–(35) the truncated series of order N of $X(t)$, can be expressed as follows

$$X_N(t) = \eta_1 K(t; t_0, A, N) + \eta_2 F(t; t_0, A, N), \tag{39}$$

where

$$K(t; t_0, A, N) := \frac{\dot{X}_2^N(t_0) X_1^N(t) - \dot{X}_1^N(t_0) X_2^N(t)}{W(X_1^N, X_2^N)(t_0)}, \tag{40}$$

$$F(t; t_0, A, N) := \frac{X_2^N(t) X_1^N(t_0) - X_1^N(t) X_2^N(t_0)}{W(X_1^N, X_2^N)(t_0)}, \tag{41}$$

being

$$X_1^N(t) = t^A \left[1 + \sum_{n=1}^{N} \frac{S(n)}{U(n; A)} t^{2n}\right], \quad X_2^N(t) = t^{-A} \left[1 + \sum_{n=1}^{N} \frac{S(n)}{U(n; -A)} t^{2n}\right], \tag{42}$$

$$\dot{X}_1^N(t) = t^A \left[\sum_{n=1}^{N} \frac{2nS(n)}{U(n; A)} t^{2n-1}\right] + A t^{A-1} \left[1 + \sum_{n=1}^{N} \frac{S(n)}{U(n; A)} t^{2n}\right], \tag{43}$$

and

$$\dot{X}_2^N(t) = t^{-A} \left[\sum_{n=1}^{N} \frac{2nS(n)}{U(n; -A)} t^{2n-1}\right] - A t^{-A-1} \left[1 + \sum_{n=1}^{N} \frac{S(n)}{U(n; -A)} t^{2n}\right]. \tag{44}$$

By assuming pairwise independence among random model inputs A, η_1 and η_2, from (39), one gets the following approximations to the expectation and variance for the truncated solution stochastic process, $X_N(t)$, of $X(t)$

$$\mathrm{E}[X_N(t)] = \mathrm{E}[\eta_1]\mathrm{E}[K(t;t_0,A,N)] + \mathrm{E}[\eta_2]\mathrm{E}[F(t;t_0,A,N)] \tag{45}$$

and

$$\begin{aligned} \mathrm{E}[(X_N(t))^2] &= \mathrm{E}[(\eta_1)^2]\mathrm{E}[(K(t;t_0,A,N))^2] + \mathrm{E}[(\eta_2)^2]\mathrm{E}[(F(t;t_0;,A,N))^2] \\ &+ 2\mathrm{E}[\eta_1]\mathrm{E}[\eta_2]\mathrm{E}[K(t;t_0,A,N)F(t;t_0,A,N)], \end{aligned} \tag{46}$$

where the variance is approximated using the above expressions taking into account the well-known relationship

$$\mathrm{Var}[X_N(t)] = \mathrm{E}[(X_N(t))^2] - (\mathrm{E}[X_N(t)])^2 . \tag{47}$$

5. Examples and conclusions

This section is devoted to illustrate the theoretical results previously established. We will show two examples where the input parameters A, η_1 and η_2 are assumed to be random variables with probabilistic distributions such as uniform, beta, Gaussian, etc. Approximations for the expectation and the standard deviation to the solution of the Bessel random differential equation (32) will be computed using different truncation order from expressions (45)–(47) and (38)–(44). The obtained numerical results will be compared against the ones computed by Monte Carlo simulations.

Example 5. *Let us consider the random initial value problem* (32) *where A has a uniform distribution on the interval $[\frac{1}{10},\frac{9}{10}]$, i.e., $A \sim U\left([\frac{1}{10},\frac{9}{10}]\right)$, and assume that random initial conditions η_i, $i = 1,2$, have Beta distributions, $\eta_i \sim Be(a_i;b_i)$, $i = 1,2$, where $a_1 = 1$, $b_1 = 3$, $a_2 = 2$ and $b_2 = 5$. Following the arguments exhibited in Example 4, it is straightforward to check that A satisfies conditions of the Theorem 6. Example 4 also justifies that each η_i, $i = 1,2$ has finite moments. Then, by Theorem 6, there exists a solution stochastic process, $X(t)$, given by (33)–*(35). *Let us compute reliable numerical approximations of the mean,* $\mathrm{E}[X_N(t)]$, *and standard deviation, $\sigma_N(t) = +\sqrt{\mathrm{Var}[X_N(t)]}$, of the solution process $X(t)$ from its truncated expression $X_N(t)$ of order N. For this purpose, expressions (45)–(47) and* (38)–(44) *are used assuming that random variables η_1, η_2 and A are pairwise independent. The obtained results for the mean and the standard deviation are shown in Tables 1-2, respectively. Approximations using Monte Carlo sampling with m simulations for the mean, $\tilde{\mu}_X^m(t)$, and the standard deviation, $\tilde{\sigma}_X^m(t)$, are also collected in these tables. From these data we observe that both methods agree.*

Table 1: Approximations of the mean by the proposed truncated series method ($\mathrm{E}[X_N(t)]$) and Monte Carlo sampling ($\tilde{\mu}_X^m(t)$) using different orders of truncation N and number m of simulations, respectively, at some selected time points t in the context of Example 5.

t	$\mathrm{E}[X_N(t)]$; $N = 10$	$\mathrm{E}[X_N(t)]$; $N = 20$	$\tilde{\mu}_X^m(t)$; $m = 50000$	$\tilde{\mu}_X^m(t)$; $m = 100000$
1.0	0.250000	0.250000	0.247345	0.249427
2.0	0.343619	0.343619	0.341780	0.343579
2.5	0.276392	0.276392	0.275382	0.276553
3.5	0.031725	0.031725	0.032386	0.032022
4.0	-0.088935	-0.088935	-0.087747	-0.088672

Table 2: Approximations of the standard deviation by the proposed truncated series method ($\sigma_N(t)$) and Monte Carlo sampling ($\tilde{\sigma}_X^m(t)$) using different orders of truncation N and number m of simulations, respectively, at some selected time points t in the context of Example 5.

t	$\sigma_N(t)$; $N = 10$	$\sigma_N(t)$; $N = 20$	$\tilde{\sigma}_X^m(t)$; $m = 50000$	$\tilde{\sigma}_X^m(t)$; $m = 100000$
1.0	0.193646	0.193649	0.192932	0.193486
2.0	0.165496	0.165132	0.193486	0.165337
2.5	0.128705	0.128705	0.128607	0.128744
3.5	0.078538	0.078538	0.078478	0.078603
4.0	0.090635	0.090635	0.090362	0.090572

Example 6. *In this second example, we consider the random initial value problem* (32) *and we assume that A has a truncated beta distribution on [d,1-d] , $d = 1 \times 10^{-7}$, with parameters $\alpha = 1$ and $\beta = 3$; η_1 has a standard Gaussian distribution, $\eta_1 \sim N(0;1)$ and η_2 has with uniform distribution on $[0,1]$, $\eta_2 \sim U([0,1])$. It is straightforward to check that hypotheses of Theorem 6 hold true and, therefore a solution of the form* (33)–(35) *can be constructed. In Tables 3-4 approximations of the mean and the standard deviation of the solution process to initial value problem* (32) *using the proposed truncated series method and Monte Carlo simulations are shown. From the obtained tables we observe a high agreement between both approximations.*

Table 3: Approximations of the mean by the proposed truncated series method ($\mathrm{E}[X_N(t)]$) and Monte Carlo sampling ($\tilde{\mu}_X^m(t)$) using different orders of truncation N and number m of simulations, respectively, at some selected time points t in the context of Example 6.

t	$\mathrm{E}[X_N(t)]$; $N = 10$	$\mathrm{E}[X_N(t)]$; $N = 20$	$\tilde{\mu}_X^m(t)$; $m = 50000$	$\tilde{\mu}_X^m(t)$; $m = 100000$
1.0	0	0	0.005396	-0.005113
2.0	0.293729	0.293729	0.297004	0.289315
2.5	0.307795	0.307795	0.309329	0.304980
3.5	0.147978	0.147978	0.146352	0.148902
4.0	0.024559	0.024559	0.022073	0.026854

Table 4: Approximations of the standard deviation by the proposed truncated series method ($\sigma_N(t)$) and Monte Carlo sampling ($\tilde{\sigma}_X^m(t)$) using different orders of truncation N and number m of simulations, respectively, at some selected time points t in the context of Example 6.

t	$\sigma_N(t)$; $N = 10$	$\sigma_N(t)$; $N = 20$	$\tilde{\sigma}_X^m(t)$; $m = 50000$	$\tilde{\sigma}_X^m(t)$; $m = 100000$
1.0	1.000000	1.000000	1.004220	1.002310
2.0	0.670807	0.670807	0.674305	0.672227
2.5	0.364257	0.364257	0.366328	0.364783
3.5	0.322779	0.322779	0.323623	0.323559
4.0	0.484010	0.484010	0.485911	0.485238

Finally, from Tables 1-4 it is observed that the absolute error of the numerical results with the truncated series method for $N = 10$ and $N = 20$ is less than 1×10^{-6}. A comparison of the CPU time used in Mathematica® 7.0 to compute some numerical results presented in Tables 1-4 is shown in Table 5. These data show that the proposed truncated series method is faster than the Monte Carlo Method.

In this paper mean square convergent generalized power series solution of the random Bessel differential equation (32) have been constructed taking advantage of L_p-random calculus together with random Fröbenius method. The results obtained extend their deterministic counterpart under mild conditions. In addition, general expressions to approximate both the mean and the variance of the solution have been determined. An important feature of our analysis is that these

Table 5: Execution time for computing the mean and variance for Examples 5 and 6 implemented on Intel® Core™ 2 Duo, 4GB, 2.4GHz.

Methods	Monte Carlo 10×10^4 simulations CPU(seconds)	Truncated series method truncation order N=20 CPU(seconds)	% Increase
Example 5	94.30	31.46	300.2
Example 6	94.32	3.49	2702.6

approximations are guaranteed to converge to their respective exact values. To illustrate the reliability of the results, two examples have been provided. Finally, we want to point out that our approach can be very useful to continue studying, from a probabilistic standpoint, other kind of Bessel differential equations (Weber, Kelvin, Neumann, etc) as well as another important second-order linear differential equations usually encountered in physics such as Jacobi, hypergeometric, etc.

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Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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