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Additional Information

A note on Mackey topologies on Banach spaces

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Abstract

There is a maybe unexpected connection between three apparently unrelated notions concerning a given w^* -dense subspace Y of the dual X^* of a Banach space X : (i) The norming character of Y , (ii) the fact that (Y, w^*) has the Mazur property, and (iii) the completeness of the Mackey topology $\mu(X, Y)$, i.e., the topology on X of the uniform convergence on the family of all absolutely convex w^* -compact subsets of Y . To clarify these connections is the purpose of this note. The starting point was a question raised by M. Kunze and W. Arendt and the answer provided by J. Bonet and B. Cascales. We fully characterize $\mu(X, Y)$ -completeness or its failure in the case of Banach spaces X with a w^* -angelic dual unit ball—in particular, separable Banach spaces or, more generally, weakly compactly generated ones—by using the norming or, alternatively, the Mazur character of Y . We characterize the class of spaces where the original Kunze–Arendt question has always a positive answer. Some other applications are also provided.

1 Introduction

In [BC10], J. Bonet and B. Cascales answered in the negative a question of M. Kunze and W. Arendt—see [Kun08]—by showing that in the dual of $\ell_1[0, 1]$ there exists a closed and norming—in fact, 1-norming—subspace Y such that $(\ell_1[0, 1], \mu(\ell_1[0, 1], Y))$ is not complete. Here, $\mu(X, Y)$ stands for the *Mackey topology on X associated to a dual pair $\langle X, Y \rangle$* , i.e., the topology on X of the uniform convergence on the family of all absolutely convex and $w(Y, X)$ -compact subsets of Y . In [GM15] we gave some criteria for deciding whether, for a general Banach space X and a $w(X^*, X)$ -dense subspace Y of its dual, the

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space $(X, \mu(X, Y))$ is —or not— complete. We focussed on subspaces Y that contain a predual of X if available, in contrast with the situation in the original example of Bonnet and Cascales. We discussed also the case of the space c_0 , that fails to have a predual. It is worth to remind the reader that if X is a Banach space, $\mu(X, X^*)$ is just the topology defined by the norm —and so $(X, \mu(X, X^*))$ is complete—, and that, in the case that X has a predual $P \subset X^*$, the space $(X, \mu(X, P))$ is also complete —a consequence of the Krein–Šmulyan theorem—. Even for subspaces Y of X^* such that $P \subset Y$, the completeness of $(X, \mu(X, Y))$ is not guaranteed.

In this note, that extends and completes [BC10] and [GM15] (and provides some applications), we observe first that a result on Mazur spaces due to A. Wilansky in [Wi81] gives a unified approach to several situations treated there. The simplicity of the technique used adds extra insight into previous discussions. Then we characterize —by using the concept of norming subspace— the completeness of $(X, \mu(X, Y))$ when the Banach space X has an angelic dual ball, a situation that includes the separable Banach spaces X and, more generally, the weakly compactly generated ones —so in particular the reflexive Banach spaces, although for this last instance our results become irrelevant—. Finally, we apply our results, among other things, to characterize those Banach spaces where the answer to the Kunze–Arendt question mentioned above is always positive.

We adopt here the terminology of Banach spaces—as, e.g., in [FHHMZ11]—, even when dealing with locally convex spaces. For example, if X is a locally convex space, then X^* denotes its topological dual. If S is a subset of X^* , the topology on X of the pointwise convergence on points of S will be denoted by $w(X, S)$, and $w(S, X)$ will denote the topology on S of the pointwise convergence on points of X . Quite often, the topology $w(X^*, X)$ will be denoted, as it is usual, by w^* . We shall try to use the same name for a topology on a topological space and its restriction to any subset if no ambiguity is expected. The dual norm of a norm $\|\cdot\|$ will be denoted by $\|\cdot\|^*$, although we shall use $\|\cdot\|$ instead if there is no risk of misunderstanding.

A subspace Y of the dual X^* of a Banach space X is said to be *norming* (*1-norming*) whenever $\|x\|_Y := \sup\{\langle x, y^* \rangle : y^* \in B_Y\}$, $x \in X$, is an equivalent norm (respectively, is the original norm) on X . Observe that every norming subspace of X^* is w^* -dense. If S is a subset of a vector space E , then $[S]$ will denote the linear span of S , and if $x \in E$, then $[x]$ will denote the linear span of the set $\{x\}$. For other non-defined concepts we refer, e.g., to [FHHMZ11]. Our Banach spaces are always assumed to be real.

A. Grothendieck gave a characterization of the completion of a locally convex space —see, e.g., [Ko69, §21.9]— that, when applied to a dual pair $\langle X, Y \rangle$ and the associated Mackey topology $\mu(X, Y)$ on X , reads:

(G) *The completion of the locally convex space $(X, \mu(X, Y))$ can be identified to the set of all linear functionals $L: Y \rightarrow \mathbb{R}$ whose restriction $L|_K$ to any absolutely convex and w^* -compact subset K of Y is w^* -continuous.*

In particular, $(X, \mu(X, Y))$ is complete if, and only if, given a linear func-

tional $L: Y \rightarrow \mathbb{R}$ whose restriction to any absolutely convex and w^* -compact subset of Y is w^* -continuous, there is $x \in X$ such that $\langle x, y \rangle = L(y)$ for all $y \in Y$ —we will simply say that L belongs to X .

2 The main results

A topological space T is said to be *Fréchet–Urysohn* if the sequential closure and the closure of any subset of T agree. It is said to be *angelic*—a concept due to H. D. Fremlin—whenever every relatively countably compact K of T is relatively compact, and the sequential closure of K coincides with its closure. Of course, in the setting of compact topological spaces, both notions coincide. Any Banach space is angelic in its weak topology. If X is a Banach space, the closed unit ball B_{X^*} of its dual X^* , when endowed with the w^* -topology, is angelic if X is *weakly compactly generated*—WCG, in short, meaning that there exists a weakly compact and linearly dense subset of X —in particular, if X is separable. This happens, more generally, if X is weakly Lindelöf determined—see, e.g., [FHHMZ11, Chapter 14].

Along this paper, Y will denote a w^* -dense subspace of the dual X^* of a Banach space X . We shall provide criteria to decide whether $(X, \mu(X, Y))$ is or not complete.

The first set of criteria—Proposition 1 and Theorem 2—checks completeness by looking at the quality of (Y, w^*) being a Mazur space—a locally convex space E is said to be *Mazur*, or, in other terms, that has the *Mazur property*, if every sequentially continuous linear form defined on E is continuous—. The setting includes the separable or, more generally, the WCG Banach spaces. As we shall see later, Proposition 1 is essentially a reformulation of a result of A. Wilansky in the context of Banach spaces.

Proposition 1 *Let X be a Banach space, and Y be a w^* -dense subspace of X^* . Assume that every absolutely convex and compact subset of (B_Y, w^*) is Fréchet–Urysohn. If the space $(X, \mu(X, Y))$ is complete, then (Y, w^*) is Mazur.*

Surprisingly, for w^* -dense and $\|\cdot\|$ -closed subspaces Y , the converse holds without assumptions on the Banach space (Proposition 10): $(X, \mu(X, Y))$ is complete whenever (Y, w^*) is Mazur. This is a consequence of the fact that (Y, w^*) is locally complete—see Proposition 9 below—. As a byproduct, we have the following characterization.

Theorem 2 *Let X be a Banach space, and Y a w^* -dense and $\|\cdot\|$ -closed subspace of X^* . Assume that every absolutely convex and compact subset of (B_Y, w^*) is Fréchet–Urysohn. Then the two following statements are equivalent.*

1. *The space (Y, w^*) is Mazur.*
2. *The space $(X, \mu(X, Y))$ is complete.*

If Y is just w^* -dense, the implication (1) \Rightarrow (2) in Theorem 2 above may fail —see Example 12 below.

The second set of criteria —Proposition 3 and Theorem 4— checks completeness now in terms of the norming quality of the subspace Y . Again, the setting includes the separable and, more generally, the WCG Banach spaces.

Proposition 3 *Let X be a Banach space and let Y be a w^* -dense subspace of X^* . If $(X, \mu(X, Y))$ is complete, then Y is norming.*

In case that Y is norm closed we get an equivalence when the closed dual unit ball is angelic in the w^* -topology.

Theorem 4 *Let X be a Banach space, and Y a w^* -dense and $\|\cdot\|$ -closed subspace of X^* . Assume that (B_{X^*}, w^*) is angelic. Then, the two following conditions are equivalent:*

1. *The space Y is norming.*
2. *$(X, \mu(X, Y))$ is complete.*

If Y is just w^* -dense, the implication (1) \Rightarrow (2) in Theorem 4 above may fail —see again Example 12.

Let us collect in a single result the two main contributions —Theorems 2 and 4— of this note. It is formulated in a somehow less than optimal —albeit more useful— context. As mentioned above, Theorem 5 below applies to all weakly compactly generated —in particular, all separable— Banach spaces.

Theorem 5 *Let X be a Banach space such that (B_{X^*}, w^*) is angelic. Let Y be a w^* -dense and $\|\cdot\|$ -closed subspace of X^* . Then the three following statements are equivalent.*

1. *$(X, \mu(X, Y))$ is complete.*
2. *Y is norming.*
3. *(Y, w^*) is Mazur.*

The rest of the paper includes the proofs of these statements, examples to test their scope, intermediate results, applications to former and new examples, and some consequences. Here is a short summary.

Proposition 1 applies to the example given by J. Bonnet and B. Cascales in [BC10] consisting of a w^* -dense and $\|\cdot\|$ -closed subspace $Y (= C[0, 1])$ of $\ell_\infty[0, 1]$ such that $(\ell_1[0, 1], \mu(\ell_1[0, 1], Y))$ is not complete —see Subsection 5.1. It also includes the result in [GM15] showing that $(\ell_1(\Gamma), \mu(\ell_1(\Gamma), Y))$ is not complete whenever $Y := c_0(\Gamma) \oplus [x^{**}] (\subset \ell_\infty(\Gamma))$ and x^{**} has uncountable support.

Theorem 2 applies, in particular, to the result in [GM15] showing that $(\ell_1(\Gamma), \mu(\ell_1(\Gamma), Y))$ is complete whenever $Y := c_0(\Gamma) \oplus [x^{**}] (\subset \ell_\infty(\Gamma))$ and x^{**} has countable support.

Proposition 3 gives *in all Banach spaces X having an infinite-dimensional quotient X^{**}/X* (e.g., in the space $X := c_0$) w^* -dense and closed subspaces Y of X^* such that $(X, \mu(X, Y))$ is not complete. This solves, in particular, a question left open for c_0 in [GM15]. This fact, together with the result that quasi-reflexive Banach spaces are WCG spaces, leads us to a characterization of those Banach spaces where the Kunze–Arendt question —see [Kun08]— cannot be solved in the negative —see Subsection 5.2.

Theorem 4 includes and improves several results in [GM15], for example Corollary 3 there —the assumption used that $\mu(X, P)$ should be complete for a separable Banach space X and a $\|\cdot\|$ -closed and norming subspace P of X^* was superfluous— and Corollary 13 there —in fact showing that this result is true for any $\|\cdot\|_1$ -closed and norming subspace H of ℓ_1 , not necessarily a hyperplane.

3 Proofs

The following result is [Wi81, Theorem 3.3].

Theorem 6 (A. Wilansky) *Let X be a complete locally convex space such that every equicontinuous subset of X^* is Fréchet–Urysohn in the w^* -topology. Then (X^*, w^*) is a Mazur space.*

Proposition 1 above is a straightforward consequence of the previous theorem, formulated in the setting of Banach spaces. We provide the proof of the proposition for the sake of completeness. As we shall subsequently see, Proposition 1 solves several questions on the subject of this note.

We shall need the following simple result (see, e.g., [Fl80, Exercise 3.26]).

Lemma 7 *Let A be a Fréchet–Urysohn topological space. Let B be a topological space. Then, every sequentially continuous function $f: A \rightarrow B$ is continuous.*

Proof of Proposition 1. Let $L: Y \rightarrow \mathbb{R}$ be a sequentially continuous linear form on (Y, w^*) . Let K be any absolutely convex and w^* -compact subset of Y . It is $\|\cdot\|$ -bounded —a consequence of the Banach–Mackey theorem— so we may assume, without loss of generality, that $K \subset B_Y$. It follows that (K, w^*) is Fréchet–Urysohn. According to Lemma 7, $L|_K$ is continuous on (K, w^*) , hence an element of the completion of $(X, \mu(X, Y))$ in view of Grothendieck’s characterization (G) above. There exists then an element $x \in X$ such that $L(y) = \langle x, y \rangle$ for all $y \in Y$, and so L is w^* -continuous. \square

A locally convex space X is said to be *locally complete* if the normed space generated by any bounded absolutely convex and closed subset of X is complete. The following result —see, e.g., [BP87, Theorem 5.1.11]— provides a characterization of this property.

Proposition 8 *Let X be a locally convex space. Then, the following are equivalent:*

1. X is locally complete.

2. The closed absolutely convex hull of any null sequence in X is compact.

Probably the following proposition is well known, although we did not find any reference in the literature. This is why we provide a proof. By the way, it shows that the hypothesis in [GM15, Lemma 8] on the local completeness of the space $(H, w(H, c_0))$, where H was a $\|\cdot\|$ -closed and w^* -dense hyperplane of ℓ_1 , was superfluous.

Proposition 9 *Let X be a Banach space and Y a $\|\cdot\|$ -closed and w^* -dense subspace of X^* . Then the space (Y, w^*) is locally complete.*

Proof. We shall prove that the space (Y, w^*) satisfies (2) in Proposition 8. Let $(y_n^*)_{n \in \mathbb{N}}$ be a null sequence in (Y, w^*) . It is, then, norm-bounded and the operator $T: X \rightarrow c_0$ defined by $Tx = (y_n^*(x))_{n \in \mathbb{N}}$ for every $x \in X$ is well defined and norm-to-norm continuous. The adjoint operator $T^*: \ell_1 \rightarrow X^*$ is $w(\ell_1, c_0)$ - w^* -continuous —also norm to norm continuous—. Since B_{ℓ_1} is a $w(\ell_1, c_0)$ -compact subset of ℓ_1 , $T^*B_{\ell_1}$ is a w^* -compact subset of X^* . Note that $T^*e_n^* = y_n^*$ for all $n \in \mathbb{N}$ (where e_n^* is the n -th vector of the canonical basis of ℓ_1), so $\Gamma\{y_n^*: n \in \mathbb{N}\} = T^*\Gamma\{e_n^*: n \in \mathbb{N}\} \subset T^*B_{\ell_1}$. It follows easily that $T^*B_{\ell_1} = \overline{\Gamma\{y_n^*: n \in \mathbb{N}\}}^{w^*}$. Due to the fact that $B_{\ell_1} = \overline{\Gamma\{e_n^*: n \in \mathbb{N}\}}^{\|\cdot\|_1}$ and that $Y (\subset X^*)$ is $\|\cdot\|$ -closed, we get that $T^*B_{\ell_1} \subset Y$. \square

Proposition 10 *Let X be a Banach space, and let Y be a $\|\cdot\|$ -closed and w^* -dense subspace of X^* . If (Y, w^*) is Mazur then the space $(X, \mu(X, Y))$ is complete.*

Proof Let $L: Y \rightarrow \mathbb{R}$ be an element of the completion of $(X, \mu(X, Y))$ and let $(y_n^*)_{n \in \mathbb{N}}$ be a w^* -null sequence in Y . By Proposition 9, the set $K = \overline{\Gamma\{y_n^*: n \in \mathbb{N}\}}^{w^*}$ is an absolutely convex and w^* -compact subset of Y . Therefore, by Grothendieck's completeness criterion —see (G) above— L restricted to K is w^* -continuous. In particular, $L(y_n^*) \rightarrow 0$. This shows that L is w^* -sequentially continuous, hence an element in X . \square

Proof of Theorem 2. It follows from Propositions 1 and 10. \square

Let X be a Banach space. A useful fact —to be used several times below— is that a w^* -dense subspace Y of X^* is norming if, and only if, $X + Y^\perp$ is $\|\cdot\|$ -closed in X^{**} . This is formulated and proved in, e.g., [FHHMZ11, Exercise 5.3] for Y w^* -dense and $\|\cdot\|$ -closed. Certainly, it also holds if we only ask Y to be w^* -dense.

Proof of Proposition 3. Assume that Y is not norming. Then $X + Y^\perp$ is not $\|\cdot\|$ -closed in X^{**} . Find $x^{**} \in \overline{X + Y^\perp}^{\|\cdot\|} \setminus (X + Y^\perp)$. Let $(x_n + y_n^{**})_{n=1}^\infty$ be a sequence $\|\cdot\|$ -converging to x^{**} , where $x_n \in X$ and $y_n^{**} \in Y^\perp$ for all $n \in \mathbb{N}$. Given an absolutely convex and w^* -compact subset K of Y ,

note that $x^{**}|_K$ is w^* -continuous, since it is the uniform limit of the sequence $((x_n + y_n^{**})|_K)_{n=1}^\infty = (x_n|_K)_{n=1}^\infty$. By Grothendieck's completeness criterion — see (G) above — we deduce that $x^{**}|_Y$ belongs to the completion of $(X, \mu(X, Y))$. Therefore, $(X, \mu(X, Y))$ is not complete. \square

In the proof of Theorem 4 the following lemma will be needed.

Lemma 11 *Let X be a Banach space, and let Y be a norming subspace of X^* . If $x^{**} \in X^{**} \setminus (X + Y^\perp)$ then the space $Z = Y \cap \ker(x^{**})$ is a norming subspace of X^* .*

Proof. We claim that $Z^\perp = Y^\perp + [x^{**}]$. To see this, observe first that $Y^\perp + [x^{**}] \subset Z^\perp$. Take now $z^{**} \in Z^\perp$. The linear form $z^{**}|_Y: Y \rightarrow \mathbb{R}$ vanishes on Z . It follows that $z^{**}|_Y = \lambda x^{**}|_Y$ for some $\lambda \in \mathbb{R}$. Thus $z^{**} - \lambda x^{**} \in Y^\perp$, and so $z^{**} \in Y^\perp + \lambda x^{**} \subset Y^\perp + [x^{**}]$. Since $X + Y^\perp$ is $\|\cdot\|$ -closed in X^{**} , so it is $X + Z^\perp$ — see, e.g., [FHHMZ11, Exercise 5.11] — and Z is then norming. \square

Proof of Theorem 4. In view of Proposition 3, only the implication (1) \Rightarrow (2) needs a proof. Assume that Y is norming. By Theorem 2 it is enough to show that (Y, w^*) is Mazur. Let $L: Y \rightarrow \mathbb{R}$ be a w^* -sequentially continuous linear mapping. Clearly, L is $\|\cdot\|$ -continuous. Let $x^{**} \in X^{**}$ be a Hahn–Banach extension of L to X^* . Assume that $x^{**} \notin X + Y^\perp$. By Lemma 11 we know that $Z = Y \cap \ker(x^{**})$ is a (proper) $\|\cdot\|$ -closed norming subspace of X^* . Therefore, there exists a positive scalar α such that

$$B_Y \subset B_{X^*} \subset \overline{\alpha B_Z}^{w^*} \subset \alpha B_{X^*}.$$

Fix $y^* \in B_Y$ such that $y^* \notin Z$. Since $y^* \in B_{X^*}$ we can find, by assumption, a sequence $(\alpha z_n^*)_{n \in \mathbb{N}}$, where $z_n^* \in B_Z$ for all $n \in \mathbb{N}$, such that $\alpha z_n^* \rightarrow y^*$ in the topology w^* . But then

$$L(\alpha z_n^*) = \langle x^{**}, \alpha z_n^* \rangle = 0 \not\rightarrow \langle x^{**}, y^* \rangle = L(y^*) (\neq 0),$$

and so $L (= x^{**}|_Y)$ is not w^* -sequentially continuous. This is a contradiction, hence $x^{**} \in X + Y^\perp$, and so $L: Y \rightarrow \mathbb{R}$ is w^* -continuous. This is what we wanted to show. \square

4 (Counter)examples

Example 12 Neither (1) \Rightarrow (2) in Theorem 2 nor (1) \Rightarrow (2) in Theorem 4 are true in general if we remove the condition of $\|\cdot\|$ -closedness of Y . A counterexample that works for both (taken from [BC10]) consists of the space $X := c_0$ and the subspace φ of X^* ($= \ell_1$) of all its finitely-supported vectors. (B_{ℓ_1}, w^*) is metrizable, hence angelic. The space φ is clearly (1-) norming, while $(c_0, \mu(c_0, \varphi))$ is not complete. The reason for this last assertion is that every absolutely convex and $w(\varphi, c_0)$ -compact subset of φ must be finite-dimensional (due to the Baire Category Theorem), and so $\mu(c_0, \varphi)$ is the topology on c_0 of the pointwise convergence. In this case, this topology is not even sequentially complete. Moreover, the space $(\varphi, w(\varphi, c_0))$ is easily seen to be Mazur.

Example 13 Observe that Theorem 4 does not hold under the more general assumption —used in Proposition 1 and Theorem 2, for example— that the absolutely convex and w^* -compact subsets of Y , endowed with the topology w^* , are Fréchet–Urysohn. In fact, the example of Bonet and Cascales —see Proposition 15— is a counterexample. Indeed, absolutely convex and w^* -compact subsets of $Y := C[0, 1]$ are w^* -angelic —by the way, in the proof of Proposition 15 below it is proved that (Y, w^*) is angelic—, and Y is (1-) norming. However, the space $(\ell^1[0, 1], \mu(\ell_1[0, 1], C[0, 1]))$ was shown not to be complete. Thus, Theorem 4 cannot be applied to this example, although Proposition 1 does, as we shall show below —see Propositions 15.

Example 14 Let Y be a w^* -dense subspace of the dual of a Banach space X . From the proof of Theorem 2 we deduce that $(X, \mu(X, Y))$ is complete whenever Y is norm-closed and (Y, w^*) is a Mazur space. The converse is not true, for example $X = C([0, \omega_1])$ and $Y = X^*$. Indeed, $(X^*, \mu(X^*, X))$ is complete but (X^*, w^*) is not Mazur —see [Kap86]—. On the other hand if $(X, \mu(X, Y))$ is complete, Proposition 3 implies that Y is necessarily a norming subspace. The converse of this fact is not true as shows J. Bonet and B. Cascales example —see subsection 5.1—. This suggests that the equivalence between $(X, \mu(X, Y))$ being complete and Y being norming is satisfied in a much bigger class of Banach spaces than the one given in Theorem 5.

5 Applications

5.1 An application to the example of J. Bonet and B. Cascales in [BC10].

Note that $C[0, 1]$, i.e., the space of continuous functions on the interval $[0, 1]$, is a (closed) subspace of $(\ell_\infty[0, 1], \|\cdot\|_\infty)$, and it is 1-norming with respect to the pair $(\ell_1[0, 1], \ell_\infty[0, 1])$, in particular, w^* -dense in $\ell_\infty[0, 1]$.

Note, too, that the Riemann integral $R: (C[0, 1], w(C[0, 1], \ell_1[0, 1])) \rightarrow \mathbb{R}$ is a sequentially continuous non-continuous linear form. Indeed, if $(f_n)_{n \in \mathbb{N}}$ is a $w(C[0, 1], \ell_1[0, 1])$ -null sequence in $C[0, 1]$, it is $\|\cdot\|_\infty$ -bounded and pointwise null, and so the sequential continuity is a consequence of the Lebesgue Dominated Convergence Theorem. Note that R is not an element in $\ell_1[0, 1]$, as elements there have countable support.

The following result was proved in [BC10] by using the Krein–Šmulyan Theorem. Here it is obtained as a consequence of Proposition 1 and the previous observation about the Riemann integral.

Proposition 15 (J. Bonet, B. Cascales, [BC10]) *The space $(\ell_1[0, 1], \mu(\ell_1[0, 1], C[0, 1]))$ is not complete.*

Proof The space $(C[0, 1], \mathcal{T}_p)$ is angelic, where \mathcal{T}_p denotes the topology of the pointwise convergence (see, e.g., [FHHMZ11, Theorem 3.58], or [F180, Theorem 3.7]), so it is the space $(C[0, 1], w(C[0, 1], \ell_1[0, 1]))$ (see [F180, Theorem 3.3.2

(ii)), and, in particular, the topological space $(B_{C[0,1]}, w(C[0,1], \ell_1[0,1]))$ (an alternative argument is that on every absolutely convex and $w(C[0,1], \ell_1[0,1])$ -compact subset of $C[0,1]$, the topologies $w(C[0,1], \ell_1[0,1])$ and \mathcal{T}_p agree). Moreover, the space $(C[0,1], w(C[0,1], \ell_1[0,1]))$ is not Mazur —see the note on the Riemann integral above—. The conclusion follows from Proposition 1. \square

Remark 16 Observe that $C[0,1]$ is $\|\cdot\|_\infty$ -closed in $\ell_\infty[0,1]$ and (1-) norming for $\ell_1[0,1]$. Theorem 4 and Proposition 15 together show then that $(B_{\ell_\infty[0,1]}, w^*)$ is not angelic, something that can be checked directly. This is to say that we cannot apply Theorem 4 to prove Proposition 15 above.

5.2 Answering Kunze–Arendt question

Theorem 4 allows us to show a simple characterization of those Banach spaces where the answer to the Kunze–Arendt question is always affirmative —see [BC10] for details on its origin—. Those are, precisely, the *quasi-reflexive* ones, i.e., those spaces X for which X^{**}/X is finite-dimensional.

Corollary 17 *A Banach space is quasi-reflexive if, and only if, $(X, \mu(X, Y))$ is complete for any w^* -dense subspace Y of X^* .*

Proof Every quasi-reflexive space is WCG [Val77]. Let Y be a w^* -dense subspace of X^* . A result of W. J. Davis and J. Lindenstrauss [DL72] shows that Y is norming. It is enough now to apply Theorem 4 to conclude that $(X, \mu(X, Y))$ is complete.

Conversely, assume that X is not quasi-reflexive. Then we can find a w^* -dense subspace Y of X^* which is not norming —see again [DL72]—. Then, by Theorem 4 —or just by Proposition 3— we deduce that $(X, \mu(X, Y))$ can not be complete. \square

This applies in particular to the space c_0 . We proved in [GM15] that for every $\|\cdot\|$ -closed and w^* -dense hyperplane H of ℓ_1 —i.e., the kernel of an element $x^{**} \in \ell_\infty \setminus c_0$ —, the space $(c_0, \mu(c_0, H))$ is complete. This is now a consequence of Theorem 4 above, since H is norming for c_0 . At that time we could not find w^* -dense and $\|\cdot\|$ -closed subspaces Y of ℓ_1 such that $(c_0, \mu(c_0, Y))$ is not complete. Now we may easily provide examples of this situation by using Theorem 4, either by relying on the result mentioned above in this subsection, or by giving a concrete example of a $\|\cdot\|_1$ -closed w^* -dense and not norming subspace of ℓ_1 . We quote [FHHMZ11, Exercise 3.92] for this: take a partition $\{I\} \cup \{I_k : k \in I\}$ of \mathbb{N} into disjoint infinite sets. Let $S := \{y \in \ell_1 : \text{for each } k \in I, y_k = \frac{1}{k} \sum_{n \in I_k} y_n\}$. The space S is w^* -dense in ℓ_1 but not norming. Observe, too, that S is $\|\cdot\|_1$ -closed in ℓ_1 .

5.3 Comparing Proposition 1 and a result in [BC10]

The following result is proved in [BC10]:

Proposition 18 (J. Bonet, B. Cascales) *Let X be a Banach space, and let Y be a proper (i.e., $Y \neq X^*$) w^* -dense subspace of X^* . Assume that*

1. *The norm-bounded $w(X, Y)$ -compact subsets of X are weakly compact.*
2. *The space $(X, w(X, Y))$ is not Mazur.*

Then, $(X, \mu(X, Y))$ is not complete.

By the way, (1) in Proposition 18 above is the conclusion of a result of H. Pfitzner that solves in the positive the so-called *boundary problem* of G. Godefroy. Pfitzner's theorem says [Pfl10]: *Let X be a Banach space, B be a bounded subset of X and J a boundary for X . If B is $w(X, J)$ -compact then it is $w(X, X^*)$ -compact.* A (James) *boundary* for a Banach space X is a subset J of S_{X^*} such that for every $x \in X$ there exists $x^* \in J$ such that $\|x\| = \langle x, x^* \rangle$. An example of a boundary is the set of all extreme points of B_{X^*} . If J is a boundary then $\text{span}(J)$ is obviously a 1-norming subspace of X^* . Note that the conclusion of Pfitzner's Theorem holds trivially for a *strong boundary*, i.e., a boundary J for X such that $\overline{\text{conv}}^{\|\cdot\|}(J) = B_{X^*}$. There are boundaries that are not strong, as the example $J := \{\pm\delta_t : t \in [0, 1]\} \subset C[0, 1]^*$ shows.

As a consequence —and it was already remarked in [BC10]— any couple (X, Y) , where X is a Banach space that does not contain a copy of $\ell_1[0, 1]$ and $Y \subset X^*$ is a subspace containing a boundary for X , satisfies (1) and (2) in Proposition 18.

Note that if X is a Banach space such that (B_{X^*}, w^*) is angelic and $Y := \overline{[J]}^{\|\cdot\|}$ for J a James boundary for X , it follows from the completeness of $(X, \mu(X, Y))$, Pfitzner's theorem mentioned above, and the classical Eberlein–Šmul'yan theorem that, every bounded and $w(X, J)$ -relatively countably compact subset of X is w -relatively compact, and from the same Pfitzner's result and James' weak compactness theorem that if B is a $w(X, Y)$ -closed subset of X and every element in Y attains its supremum on B , then B is w -compact.

We note that Proposition 18 is independent of Proposition 1 in the following sense: A) there are couples (X, Y) that satisfy (1) and (2) in Proposition 18 although they do not satisfy the hypothesis in Proposition 1, and B) conversely, there are couples (X, Y) that satisfy the hypothesis of Proposition 1 and do not satisfy (1) and (2) in Proposition 18.

Indeed,

A) Put $X := C[0, 1]$ (clearly, this space does not contain a copy of $\ell_1[0, 1]$, since it is separable and $\ell_1[0, 1]$ is not), and let Y be the $\|\cdot\|$ -closed linear span of the boundary $\{\pm\delta_t : t \in [0, 1]\}$. Note that (Y, w^*) is Mazur. This is a consequence of Theorem 5: Since $C[0, 1]$ is separable, $(B_{C[0, 1]^*}, w^*)$ is angelic, and clearly Y is (1-) norming.

B) Put $Z := \ell_1[0, 1]$ and $W := C[0, 1]$. We showed above —see Subsection 5.1— that Proposition 1 can be applied to the couple (Z, W) . However, the sequence $(e_{1/n} - e_0)_{n \in \mathbb{N}}$ in $\ell_1[0, 1]$ is $w(\ell_1[0, 1], C[0, 1])$ -null, and

so $K := \{e_{1/n} - e_0 : n \in \mathbb{N}\} \cup \{0\}$ is a $w(\ell_1[0, 1], C[0, 1])$ -compact although not w -compact in $\ell_1[0, 1]$, so (1) in Proposition 18 fails. By the way, it is not difficult, by using Theorem 5, to show that (2) in Proposition 18 fails too, i.e., $(\ell_1[0, 1], w(\ell_1[0, 1], C[0, 1]))$ is Mazur.

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