A note on variable exponent Hörmander spaces

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Abstract. In this paper we introduce the variable exponent Hörmander spaces and we study some of their properties. In particular, it is shown that $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'$ is isomorphic to $\mathscr{B}^{\text{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$ (Ω open set in $\mathbb{R}^n$, $p_- > 1$ and the Hardy-Littlewood maximal operator M is bounded in $L_{p(\cdot)}$) extending a Hörmander's result to our context. As a consequence, a number of results on sequence space representations of variable exponent Hörmander spaces are given.

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1. Introduction and notation

1.1. Introduction

The Lebesgue spaces $L_{p(\cdot)}$ with variable exponent and the corresponding Sobolev spaces $W^m_{p(\cdot)}$ have been intensively investigated during the last years (see the recent book of Diening et al. [3]). These spaces are of interest in their own right and also have applications to PDE of non-standard growth (see e.g. [3, Chapter 13]) and to modelling electrorheological fluids and to image restoration (see [3, Chapter 14]). Our paper lies in this field of variable exponent function spaces. We introduce the variable exponent Hörmander spaces $\mathscr{B}_{p(\cdot)}$, $\mathscr{B}^c_{p(\cdot)}(\Omega)$ and $\mathscr{B}^{\text{loc}}_{p(\cdot)}(\Omega)$ (it is well known that the Hörmander spaces $\mathscr{B}_{p,k}$, $\mathscr{B}^c_{p,k}(\Omega)$ and $\mathscr{B}^{\text{loc}}_{p,k}(\Omega)$ play a crucial role in the theory of linear partial differential operators (see e.g. [8], [24], [6], [15], [16], [17])) and we study some of their properties. We also give a number of results on sequence space representations of the introduced spaces.

The organization of the paper is as follows. Section 2 contains some basic facts about variable exponent Lebesgue spaces and the definition of variable exponent Lebesgue spaces of entire analytic functions. In Section 3 we introduce the variable exponent Hörmander spaces $\mathscr{B}_{p(\cdot)}$, $\mathscr{B}^c_{p(\cdot)}(\Omega)$ and $\mathscr{B}^{\text{loc}}_{p(\cdot)}(\Omega)$ and we study some of their properties when the Hardy-Littlewood maximal operator M is bounded in $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p_-$ (convolution, density, completeness, embedding theorems, multiplication operators) and, by using Fourier multipliers and a result of Diening [2], we obtain a sequence space representation of the space $\mathscr{B}^c_{p(\cdot)}(]a,b[)$ (see Theorem 3.5/5). We also extend a result of Hörmander [8, Chapter XV, 15.2] to the variable exponent Hörmander spaces $\mathscr{B}^c_{p(\cdot)}(\Omega)$ (see Remark 3.6/2). In Section 4 we show that $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'$ is isomorphic to $\mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$ (see also [8, Chapter XV] and [16]) and another result on sequence space representation is given. Finally, two questions on complex interpolation and on sequence space representation of variable exponent Hörmander spaces are proposed.

1.2. Notation

Let E and F be topological linear spaces over $\mathbb{C}$. If E and F are (topologically) isomorphic we put $E \simeq F$. The (topological) dual of E is denoted by E' and is given the strong topology (i.e. the topology of uniform convergence on all the bounded subsets of E). We put $E \hookrightarrow F$ if E is a linear subspace of F and the canonical injection is continuous (if E is also dense in F we replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$). If $\{E_n\}_{n=1}^{\infty}$ is a sequence of Banach spaces, $\prod_{n=1}^{\infty} E_n$ denotes the topological product of the spaces E_n and $\oplus_{n=1}^{\infty} E_n$ their locally convex direct sum. The Fréchet space defined by the projective sequence of Banach spaces E_n and linking maps A_n will be denoted by $\mathrm{proj}_n(E_n, A_n)$ (or $\mathrm{proj}_n E_n$, for short). If $\{E_n\}_{n=1}^{\infty}$ is a sequence of topological linear spaces such that $E_n \hookrightarrow E_{n+1}$ for each n, then their inductive limit is denoted by $\mathrm{ind}_n E_n$ (see [9], [10]).

If $f \in L_1(\mathbb{R}^n)$ the Fourier transform of f, $\hat{f}$ or $\mathscr{F} f$, is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi x} dx$. If f is a function on $\mathbb{R}^n$, then $\tilde{f}(x) = f(-x)$ for $x \in \mathbb{R}^n$. B_r is the closed Euclidean ball $\{x : |x| \le r\}$ in $\mathbb{R}^n$. $C_0^\infty(\mathbb{R}^n)$ $(= \mathscr{D}(\mathbb{R}^n))$, $C_0^\infty(\Omega)$ $(= \mathscr{D}(\Omega))$ and $S(\mathbb{R}^n)$ are the usual Schwartz spaces (in the last space the norms $\max_{|\alpha| \le m} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^m |\partial^\alpha \varphi(x)|$, $m = 0, 1, 2, \ldots$, are denoted by $|\varphi|_m$). $\mathscr{D}'(\mathbb{R}^n)$, $\mathscr{D}'(\Omega)$ and $S'(\mathbb{R}^n)$ are their corresponding duals. $\mathscr{E}'(K)$ (K compact in $\mathbb{R}^n$) is the set of distributions on $\mathbb{R}^n$ with supports contained in K. O_M is the space of all functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi\psi \in S(\mathbb{R}^n)$ for all $\psi \in S(\mathbb{R}^n)$. The Fourier transform in $S'(\mathbb{R}^n)$ is also denoted by $\hat{}$ (or $\mathscr{F}$). If $u \in S'(\mathbb{R}^n)$, $\tilde{u}$ is defined by $\langle \varphi, \tilde{u} \rangle = \langle \tilde{\varphi}, u \rangle$ for all $\varphi \in S(\mathbb{R}^n)$; thus $\sim$ coincides with the operator $(2\pi)^{-n} \mathscr{F}^2$. In general, we will consider function spaces defined on the whole Euclidean space $\mathbb{R}^n$. So, in what follows, we shall omit the "$\mathbb{R}^n$" of their notation. The letter C will always denote a positive constant, not necessarily the same at each ocurrence.

2. Preliminaries

In this section we collect some basic facts about variable exponent spaces and we give the definition of variable Lebesgue spaces of entire analytic functions.

2.1. Variable exponent spaces

If $p(\cdot)$ is a measurable function on $\mathbb{R}^n$ with range in $[1, \infty]$, we put $\mathbb{R}^n_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}$, $p_- = \mathrm{ess\,inf}_{x \in \mathbb{R}^n} p(x)$, $p_+ = \mathrm{ess\,sup}_{x \in \mathbb{R}^n} p(x)$, and we define the modular functional

$$\rho(f) := \int_{\mathbb{R}^n \setminus \mathbb{R}^n_\infty} |f(x)|^{p(x)} dx + \|f\|_{L_\infty(\mathbb{R}^n_\infty)} .$$

$L_{p(\cdot)}$ denotes the set of all complex-valued measurable functions on $\mathbb{R}^n$ such that for some $\lambda > 0$, $\rho(f/\lambda) < \infty$. This set becomes a Banach space when equipped with the Luxemburg norm

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : \ \rho(f/\lambda) \le 1\} .$$

These spaces are referred to as variable exponent Lebesgue spaces since they generalize the standard Lebesgue spaces. In this paper, we shall also consider analogous spaces to classical Lebesgue spaces L_p, $0 < p < 1$, with variable exponents. Define $\mathscr{P}^0$ to be the set of all measurable functions on $\mathbb{R}^n$ with range in $(0,\infty)$ such that $p_- = \text{ess inf}_{x\in\mathbb{R}^n} p(x) > 0$ and $p_+ = \text{ess sup}_{x\in\mathbb{R}^n} p(x) < \infty$. Given $p(\cdot) \in \mathscr{P}^0$, we also define the space $L_{p(\cdot)}$ as above. This is equivalent to defining it to be the set of all measurable functions f such that $|f|^{p_0} \in L_{q(\cdot)}$, where $0 < p_0 \le p_-$ and $q(x) = \frac{p(x)}{p_0}$. We can define a quasi-norm on this space by

$$\|f\|_{p(\cdot)} := \| \, |f|^{p_0} \|^{1/p_0}_{q(\cdot)} \left(= \inf\left\{ \lambda > 0 : \ \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\} \right)$$

(see [1]). With this quasi-norm $L_{p(\cdot)}$ becomes a quasi-Banach space.

Next we give some basic results about variable Lebesgue spaces. Their proofs can be found in [3] (see also [11]).

Lemma 2.1. *If $p(\cdot) \in \mathscr{P}^0$, then $\|f_k - f\|_{p(\cdot)} \to 0$ if and only if $\int_{\mathbb{R}^n} |f_k(x) - f(x)|^{p(x)} dx \to 0$. If $\|f_k - f\|_{p(\cdot)} \to 0$ then there exists a subsequence of (f_k) which converges a.e. to f.*

Lemma 2.2 (generalized Hölder inequality). *Let $p(\cdot) : \mathbb{R}^n \to [1,\infty]$. If $f \in L_{p(\cdot)}$ and $g \in L_{p'(\cdot)}$, then*

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \le C \, \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}$$

where $p'(\cdot)$ is the conjugate exponent function defined by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, $x \in \mathbb{R}^n$.

Lemma 2.3. *Let $p(\cdot) : \mathbb{R}^n \to [1,\infty]$. The natural mapping*

$$I : L_{p'(\cdot)} \to \left(L_{p(\cdot)}\right)' : g \to \langle f, I(g) \rangle = \int_{\mathbb{R}^n} f(x)g(x) \, dx$$

is an isomorphism if and only if $p_+ < \infty$. The space $L_{p(\cdot)}$ is reflexive if and only if $1 < p_- \le p_+ < \infty$.

Lemma 2.4.

1. *Let $p(\cdot) : \mathbb{R}^n \to [1,\infty]$. Then $S \hookrightarrow L_{p(\cdot)} \hookrightarrow L_1^{\text{loc}}, S'$. Furthermore, if $p_+ < \infty$, C_0^∞ and S are dense in $L_{p(\cdot)}$ and the mapping $S \times L_{p(\cdot)} \to L_{p(\cdot)} : (\varphi, f) \to \varphi f$ is continuous.*
2. *If $p(\cdot) \in \mathscr{P}^0$, $S \hookrightarrow L_{p(\cdot)}$ and the mapping $S \times L_{p(\cdot)} \to L_{p(\cdot)} : (\varphi, f) \to \varphi f$ is continuous.*

Many classical operators in harmonic analysis (maximal operators, Calderón-Zygmund operators, fractional integrals, ...) are bounded in $L_{p(\cdot)}$ whenever the Hardy-Littlewood maximal operator M is bounded in $L_{p(\cdot)}$ (c.f., e.g. [1], [2] and [3]). The next lemma provides a sufficient condition on $p(\cdot)$ for M to be bounded in $L_{p(\cdot)}$ and one important consequence of the Diening's characterization of variable Lebesgue spaces on which the maximal operator M is bounded:

Lemma 2.5.

1. *Let* $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ *be such that* $1 < p_- \le p_+ < \infty$. *Suppose that* $p(\cdot)$ *satisfies*
$$|p(x) - p(y)| \le \frac{C}{-\log|x-y|}, \qquad |x-y| \le \frac{1}{2},$$
and
$$|p(x) - p(y)| \le \frac{C}{\log(e+|x|)}, \qquad |x| \le |y| .$$
Then the Hardy-Littlewood maximal operator M *is bounded in* $L_{p(\cdot)}$.
2. *If* $1 < p_- \le p_+ < \infty$, *then the following conditions are equivalent:*
   (a) M *is bounded in* $L_{p(\cdot)}$,
   (b) M *is bounded in* $L_{p'(\cdot)}$,
   (c) M *is bounded in* $L_{p(\cdot)/q}$ *for some* $1 < q < p_-$,
   (d) M *is bounded in* $L_{(p(\cdot)/q)'}$ *for some* $1 < q < p_-$.

Remark 2.6. In spite of the previous results, the variable Lebesgue spaces have a number of undesired properties. For example, these spaces are not translation invariant and so the Young's inequality cannot be generalized to the spaces $L_{p(\cdot)}$ for non-constant $p(\cdot)$ (see [3] and [11]).

2.2. Variable exponent Lebesgue spaces of entire analytic functions

If K is a compact subset of $\mathbb{R}^n$, μ is a positive Borel measure on $\mathbb{R}^n$ and $0 < p \le \infty$, then
$$L_p^K(\mu) := \{f \in S' : \operatorname{supp} \hat{f} \subset K,\ f \in L_p(\mu)\}$$
($L_p^K := L_p^K(\mu)$ if μ is the Lebesgue measure). $(L_p^K(\mu), \|\cdot\|_p)$ is a quasi-Banach (Banach if $p \ge 1$) space (see [23], [19]; see also [20]).

If K is a compact subset of $\mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}^0$, then
$$L_{p(\cdot)}^K := \{f \in S' : \operatorname{supp} \hat{f} \subset K,\ \|f\|_{p(\cdot)} < \infty\} .$$
$(L_{p(\cdot)}^K, \|\cdot\|_{p(\cdot)})$ is a quasinormed (normed if $p_- \ge 1$) linear space. From the Paley-Wiener-Schwartz theorem it follows that the elements of $L_{p(\cdot)}^K$ are entire analytic functions of exponential type. When $p(\cdot) \equiv p$, a constant, then $L_{p(\cdot)}^K = L_p^K$ with equality of quasi-norms (resp. norms). We shall denote by S^K the collection of all $f \in S$ such that $\operatorname{supp} \hat{f} \subset K$; obviously $S^K \subset L_{p(\cdot)}^K$. The spaces $L_{p(\cdot)}^K$ have been introduced in [18].

3. Variable exponent Hörmander spaces

In this section we introduce the variable exponent Hörmander spaces $\mathscr{B}_{p(\cdot)}$, $\mathscr{B}^c_{p(\cdot)}(\Omega)$ and $\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ (see Definition 3.1) and we study some of their properties when the Hardy-Littlewood maximal operator M is bounded in $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p_-$: convolution, density, completeness, embedding theorems, multiplication operators, sequence space representations, ...

We begin with the variable exponent (and weight $k \equiv 1$) counterpart of [8, Definition 10.1.6] (see also [8, Sections 10 and 15] and [16], [17], [24]).

Definition 3.1. Let $p(\cdot) \in \mathscr{P}^0$ be and let Ω be an open set in $\mathbb{R}^n$. Then

$$\mathscr{B}_{p(\cdot)} := \{u \in S' : \hat{u} \in L_{p(\cdot)}\} .$$

If $u \in \mathscr{B}_{p(\cdot)}$ we put $\|u\|_{\mathscr{B}_{p(\cdot)}} := \|\hat{u}\|_{p(\cdot)}$. $(\mathscr{B}_{p(\cdot)}, \|\cdot\|_{\mathscr{B}_{p(\cdot)}})$ is a quasi-normed space isomorphic to $(L_{p(\cdot)} \cap S', \|\cdot\|_{p(\cdot)})$ (a Banach space isomorphic to $L_{p(\cdot)}$ if $p_- \geq 1$). Now we consider the space

$$\mathscr{B}^c_{p(\cdot)}(\Omega) := \cup\{\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K) : K \Subset \Omega\} .$$

If every $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ is equipped with the topology induced by $\mathscr{B}_{p(\cdot)}$, then $\mathscr{B}^c_{p(\cdot)}(\Omega)$ (endowed with the corresponding inductive linear topology) becomes a strict inductive limit

$$\mathscr{B}^c_{p(\cdot)}(\Omega) := \mathrm{ind}_K[\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)] .$$

Finally,

$$\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega) := \{u \in \mathscr{D}'(\Omega) : \varphi u \in \mathscr{B}_{p(\cdot)} \text{ for all } \varphi \in C_0^\infty(\Omega)\} .$$

The topology of this space is generated by the seminorms (semiquasi-norms when $p_- < 1$) $u \to \|u\|_{p(\cdot),\varphi} := \|\varphi u\|_{\mathscr{B}_{p(\cdot)}}$, $\varphi \in C_0^\infty(\Omega)$.

Remark 3.2.

1. $\mathscr{B}_{p(\cdot)}$, $\mathscr{B}^c_{p(\cdot)}(\Omega)$ and $\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ are called variable exponent Hörmander spaces. If $p(\cdot) \equiv p$ and $p \geq 1$, these spaces coincide with the Hörmander spaces $\mathscr{B}_{p,1}$, $\mathscr{B}^c_{p,1}(\Omega)$ and $\mathscr{B}^{\mathrm{loc}}_{p,1}(\Omega)$ respectively (see [8]).
2. In general, the space $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ (K any compact in $\mathbb{R}^n$) is not a complemented subspace of $\mathscr{B}_{p(\cdot)}$. For example, if $n > 1$, $p(\cdot) \equiv p$ (a constant), $1 < p \neq 2 < \infty$ and $K = B_1$, then $\mathscr{B}_{p,1} \cap \mathscr{E}'(B_1)$ is not a complemented subspace of $\mathscr{B}_{p,1}$ since $\mathscr{B}_{p,1} \cap \mathscr{E}'(B_1)$ is isomorphic (via the Fourier transform) to $L_p^{B_1}$ and this space is not a complemented subspace of L_p by the Fefferman theorem (see [4] and [14]).

The following elementary fact will be used in the next theorem: "Let $F = \mathrm{ind}_j F_j$ be the strict inductive limit of a properly increasing sequence $F_1 \subset F_2 \subset \cdots$ of Banach spaces. Assume that every F_j is a complemented subspace of F_{j+1} and that G_j is a topological complement of F_j in F_{j+1}. Then the mapping $F_1 \oplus G_1 \oplus G_2 \oplus \cdots \to F : (f_1, g_1, g_2, \ldots) \to f_1 + g_1 + g_2 + \cdots$ is an isomorphism". We will also need the following lemmata.

Lemma 3.3. *Let $p(\cdot) \in \mathscr{P}^0$. Then the bilinear mapping $\Phi : S \times \mathscr{B}_{p(\cdot)} \to \mathscr{B}_{p(\cdot)}$, defined by $\Phi(\varphi, u) = \varphi * u$, is continuous. Furthermore, for every $u \in \mathscr{B}_{p(\cdot)}$, $\theta_\varepsilon * u \to u$ in $\mathscr{B}_{p(\cdot)}$ when $\varepsilon \to 0+$ (here $\theta_\varepsilon(x) = \frac{1}{\varepsilon^n}\theta(\frac{x}{\varepsilon})$ being θ a C_0^∞ function such that $\theta \geq 0$ and $\int_{\mathbb{R}^n} \theta(x)\,dx = 1$).*

Proof. If $\varphi \in S$ and $u \in \mathscr{B}_{p(\cdot)}$ then $\varphi * u \in O_M$ and $\widehat{\varphi * u} = \hat{\varphi}\hat{u}$. Hence and from Lemma 2.4/2 if follows that $\varphi * u \in \mathscr{B}_{p(\cdot)}$. Using again Lemma 2.4/2 and the continuity of the Fourier transform, we obtain

$$\|\varphi * u\|_{\mathscr{B}_{p(\cdot)}} = \|\widehat{\varphi * u}\|_{p(\cdot)} = \|\hat{\varphi}\hat{u}\|_{p(\cdot)} \leq C\,|\hat{\varphi}|_r\,\|\hat{u}\|_{p(\cdot)} \leq C\,|\varphi|_m\,\|u\|_{\mathscr{B}_{p(\cdot)}}$$

(here $r, m \in \mathbb{N}_0$ are independent both of φ and u). This proves the continuity of Φ. By Lemma 2.1

$$\|\theta_\varepsilon * u - u\|_{\mathscr{B}_{p(\cdot)}} = \|(\hat{\theta}_\varepsilon - 1)\hat{u}\|_{p(\cdot)}$$

goes to zero if and only if $\int_{\mathbb{R}^n}(|\hat{\theta}_\varepsilon(x) - 1|\,|\hat{u}(x)|)^{p(x)}dx \to 0$ but the latter is a consequence of the Lebesgue dominated convergence theorem in view of the estimate $|\hat{\theta}_\varepsilon(x) - 1|^{p(x)} \leq 2^{p_+}$, the fact that $\hat{\theta}_\varepsilon \to 1$ pointwise when $\varepsilon \to 0+$, and the integrability of the function $|\hat{u}(x)|^{p(x)}$. □

Lemma 3.4. *Let K be a compact subset of $\mathbb{R}^n$ and let $p(\cdot) \in \mathscr{P}^0$ be such that M is bounded in $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p_-$.*

1. *The convolution $S \times L^K_{p(\cdot)} \to L^K_{p(\cdot)} : (\varphi, f) \to \varphi * f$ is well defined and is bilinear and continuous.*
2. *(Inequalities of Plancherel-Polya-Nikol'skij type) Let α be a multiindex. Then there exists a constant C such that*

$$\|\partial^\alpha f\|_{p(\cdot)} \leq C\,\|f\|_{p(\cdot)}$$

holds for all $f \in L^K_{p(\cdot)}$.

Proof. See [18, Theorems 3.5/2, 3.5/5]. □

Theorem 3.5. *Let Ω be an open set in $\mathbb{R}^n$ and let $p(\cdot) \in \mathscr{P}^0$.*

1. *$C_0^\infty(\Omega) \hookrightarrow \mathscr{B}^c_{p(\cdot)}(\Omega)$ and $C_0^\infty(\Omega)$ is sequentially dense in $\mathscr{B}^c_{p(\cdot)}(\Omega)$.*
2. *If M is bounded in $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p_-$, then $\mathscr{B}^c_{p(\cdot)}(\Omega) \hookrightarrow S'$. (If $p_- \geq 1$, the hypothesis on M is not necessary.)*
3. *Let M be as in 2. Then the inductive limit $\mathscr{B}^c_{p(\cdot)}(\Omega)$ is regular (i.e. every bounded set in $\mathscr{B}^c_{p(\cdot)}(\Omega)$ is contained and bounded in some step) and complete. (If $p_- \geq 1$, the hypothesis on M is not necessary.)*
4. *Let M as in 2. Let $\varphi \in S$ (resp. P a polynomial in $\mathbb{R}^n$). Then the multiplication operator M_φ (resp. M_P) is continuous from $\mathscr{B}^c_{p(\cdot)}(\Omega)$ into $\mathscr{B}^c_{p(\cdot)}(\Omega)$.*
5. *Assume $n = 1$. Let $p(\cdot)$ be such that $1 < p_- \leq p_+ < \infty$ and M is bounded in $L_{p(\cdot)}(\mathbb{R})$. Let $\Omega =]a, b[$ $(-\infty \leq a < b \leq \infty)$. Let $a_j \searrow a$, $b_j \nearrow b$, and we put $K_j = [a_j, b_j]$ for $j = 1, 2, \ldots$ Then $\mathscr{B}^c_{p(\cdot)}(\Omega) \simeq L^{-K_1}_{p(\cdot)} \oplus G_1 \oplus G_2 \oplus \cdots$ being each G_j an infinite dimensional complemented subspace of $L^{-K_{j+1}}_{p(\cdot)}$. In the case $p(\cdot) \equiv p$, the space $\mathscr{B}^c_{p(\cdot)}(\Omega)$ is isomorphic to $(l_p(\mathbb{Z}))^{(\mathbb{N})}$. Finally, if $p(\cdot) \equiv$*

p and $0 < p < 1$ then the Banach envelope of each step $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$ is isomorphic to l_1.

Proof.

1. It is easily seen that the natural mapping $C_0^\infty(\Omega) \to \mathscr{B}^c_{p(\cdot)}(\Omega) : \varphi \to \langle \theta, \varphi \rangle = \int_{\mathbb{R}^n} \theta \varphi \, dx \, (\theta \in S)$ is well defined and is linear and injective. Next we prove that it is continuous. If K is any compact subset of Ω then $\langle \cdot, \varphi \rangle \in \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ for all $\varphi \in C_0^\infty(K)$. Then if $\varphi_j \to 0$ in $C_0^\infty(K)$ it results that $\varphi_j \to 0$ in S, i.e. $\hat{\varphi}_j \to 0$ in S, thus $\hat{\varphi}_j \to 0$ in $L_{p(\cdot)}$ which implies that $\langle \cdot, \varphi_j \rangle \to 0$ in $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$. Since $C_0^\infty(\Omega)$ and $\mathscr{B}^c_{p(\cdot)}(\Omega)$ are inductive limits the required continuity is shown. In order to prove the density we take any u in $\mathscr{B}^c_{p(\cdot)}(\Omega)$ and we apply Lemma 3.3. Then, $\theta_\varepsilon * u \to u$ in $\mathscr{B}_{p(\cdot)}$ (without loss of generality, we can assume that $\operatorname{supp} \theta \subset B_1$). If $u \in \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ and $K + B_{\varepsilon_0} \subset \Omega$ then $(\theta_\varepsilon * u)_{\varepsilon \le \varepsilon_0} \subset C_0^\infty(\Omega)$ and $\theta_\varepsilon * u \to u$ in $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K + B_{\varepsilon_0})$, therefore $\theta_\varepsilon * u \to u$ in $\mathscr{B}^c_{p(\cdot)}(\Omega)$.
2. Let K be any compact subset of Ω and consider the following diagram
$$\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K) \xrightarrow{\mathscr{F}} L^{-K}_{p(\cdot)} \xrightarrow{j} S' \xrightarrow{\mathscr{F}^{-1}} S'$$
where $\mathscr{F}$ is the Fourier transform, j is the canonical injection and $\mathscr{F}^{-1}$ is the inverse Fourier transform. Since $\mathscr{F}$ is an isomorphism, j is continuous (use the hypothesis on M and [18, Theorem 3.5/4]) and $\mathscr{F}^{-1}$ is an automorphism, it results that the canonical injection $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K) \to S'$ is also continuous. In consequence, $\mathscr{B}^c_{p(\cdot)}(\Omega) \hookrightarrow S'$.
3. It is easy to see that $\mathscr{B}^c_{p(\cdot)}(\Omega)$ coincides with the inductive limit $\operatorname{ind}_j[\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)]$ where $\{K_j\}_{j=1}^\infty$ is any fundamental sequence of compact subsets of Ω. Then, since M is bounded in $L_{p(\cdot)/p_0}$, it follows from [18, Theorem 3.5/4] that each $L^{-K_j}_{p(\cdot)}$ is complete (i.e. a quasi-Banach space). Thus each step $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$ is also complete and so $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$ is closed in $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_{j+1})$ for all j. In view of [9, Theorem 2 p. 84, Theorem 4 p. 86], the inductive limit $\mathscr{B}^c_{p(\cdot)}(\Omega)$ is regular and complete.
4. If K is any compact subset of Ω, it is sufficient to show that M_φ and M_P are bounded operators from $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ into $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$. But this is a consequence of the following commutative diagrams and Lemma 3.4:
$$\begin{array}{ccc} \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K) & \xrightarrow{M_\varphi} & \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K) \\ \mathscr{F} \big\downarrow & & \big\uparrow \mathscr{F}^{-1} \\ L^{-K}_{p(\cdot)} & \longrightarrow & L^{-K}_{p(\cdot)} \\ f & \longrightarrow & (2\pi)^{-n} \hat{\varphi} * f \end{array} \qquad \begin{array}{ccc} \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K) & \xrightarrow{M_{(-ix)^\alpha}} & \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K) \\ \mathscr{F} \big\downarrow & & \big\uparrow \mathscr{F}^{-1} \\ L^{-K}_{p(\cdot)} & \xrightarrow{\partial^\alpha} & L^{-K}_{p(\cdot)} \end{array}$$
5. We have $\mathscr{B}^c_{p(\cdot)}(\Omega) = \operatorname{ind}_j[\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)]$ and here each step $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$ is isomorphic (via Fourier transform) to $L^{-K_j}_{p(\cdot)}$. On the other hand, by [2, Theorem 8.14], the Hilbert transform is bounded in $L_{p(\cdot)}(\mathbb{R})$. Hence it follows,

reasoning as in the classical case, that every χ_{-K_j} is an $L_{p(\cdot)}(\mathbb{R})$-Fourier multiplier (i.e. the operator associated with χ_{-K_j}, $S_{-K_j}(f) = \mathscr{F}^{-1}(\chi_{-K_j}\hat{f})$, is bounded in $L_{p(\cdot)}(\mathbb{R})$). Then, since $L_{p(\cdot)}(\mathbb{R}) = L_{p(\cdot)}^{-K_j} \oplus \ker S_{-K_j}$, we get $L_{p(\cdot)}^{-K_{j+1}} = L_{p(\cdot)}^{-K_j} \oplus \left(\ker S_{-K_j} \cap L_{p(\cdot)}^{-K_{j+1}}\right)$ which shows that $G_j = \ker S_{-K_j} \cap L_{p(\cdot)}^{-K_{j+1}}$ is an infinite-dimensional (if Q is a compact interval such that $Q \subset -K_{j+1} \setminus (-K_j + [-\varepsilon,\varepsilon])$, for a sufficiently small $\varepsilon > 0$, then $G_j \supset S^Q$) topological complement of $L_{p(\cdot)}^{-K_j}$ in $L_{p(\cdot)}^{-K_{j+1}}$. Next, using the fact previous to Lemma 3.3, we obtain the required isomorphism: $\mathscr{B}^c_{p(\cdot)}(\Omega) \simeq L_{p(\cdot)}^{-K_1} \oplus G_1 \oplus G_2 \oplus \cdots$. If $p(\cdot) \equiv p$ we know (see e.g. [22, pp. 239,240]) that every $L_p^{-K_j}$ is isomorphic to $l_p(\mathbb{Z})$, thus G_j is isomorphic to an infinite-dimensional complemented subspace of $l_p(\mathbb{Z})$ and so, since the space $l_p(\mathbb{Z})$ is prime [12, Theorem 2.a.3], G_j becomes isomorphic to $l_p(\mathbb{Z})$. We conclude that $\mathscr{B}^c_p(\Omega) \simeq \left(l_p(\mathbb{Z})\right)^{(\mathbb{N})}$ (see also [6], [16]). If $0 < p < 1$ Hoffmann [7] proved that the Banach envelope of the quasi-Banach space $L_p^{[-\pi,\pi]}$ is isomorphic to l_1, thus our claim is an immediate consequence of this result.

□

Remark 3.6.

1. Let us recall that a bounded open set Ω in $\mathbb{R}^n$ has the segment property if there exist open balls V_j and vectors $y^j \in \mathbb{R}^n$, $j = 1,\ldots,N$, such that $\bar{\Omega} \subset \cup_{j=1}^N V_j$ and $(\bar{\Omega} \cap V_j) + ty^j \subset \Omega$ for $0 < t < 1$ and $j = 1,\ldots,N$. In [18, Theorem 3.5/3] it is shown that if M is bounded in $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p_-$ and $K = \bar{O}$, being O a bounded open set with the segment property, then S^K is dense in $L^K_{p(\cdot)}$. By using this result, it is immediate to check that $C_0^\infty(K)$ is also dense in $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ ($p(\cdot)$ and K as before). This improves Theorem 3.5/1 (versus the additional hypothesis that M is bounded in $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p_-$) since all open set in $\mathbb{R}^n$ has a fundamental sequence of compact subsets which are the closures of open sets with the segment property.
2. In [8, Chapter XV, 15.2] Hörmander obtains a family of seminorms defining the inductive limit topology of $\mathscr{B}^c_{2,k}(\Omega)$ when k is a Hörmander weight. In this note we extend this result to the variable exponent Hörmander spaces $\mathscr{B}^c_{p(\cdot)}(\Omega)$ when $p(\cdot)$ is as in Lemma 3.4 and $p_- \geq 1$: If $(\theta_j)_{j=1}^\infty$ is a $C_0^\infty(\Omega)$-partition of unity on Ω, then the inductive limit topology of $\mathscr{B}^c_{p(\cdot)}(\Omega)$ is generated by the system of seminorms
$$\|u\|_{(C_i)} := \sum_{i=1}^\infty C_i \|\theta_i u\|_{\mathscr{B}_{p(\cdot)}}, \qquad u \in \mathscr{B}^c_{p(\cdot)}(\Omega)\ ,\ (C_i)_{i=1}^\infty \in (\mathbb{R}_+)^{\mathbb{N}}\ .$$
Let $\{K_j\}_{j=1}^\infty$ be a fundamental sequence of compact subsets of Ω. For each $u \in \mathscr{B}^c_{p(\cdot)}(\Omega)$, $\theta_i u = 0$ for all i large enough. Indeed, if $\operatorname{supp} u \subset K_j$ there exist a relatively compact open set W_j and a positive integer m_j such that $K_j \subset W_j \subset \bar{W}_j \subset \Omega$ and $\theta_1(x) + \cdots + \theta_{m_j}(x) = 1$ in W_j. Therefore $\theta_i = 0$ in W_j for $i > m_j$ and so $\theta_i u = 0$ for these indexes. Consequently, $u = \sum_{i=1}^{m_j} \theta_i u$. Hence

it follows that, for each $(C_i)_{i=1}^{\infty} \in (\mathbb{R}_+)^{\mathbb{N}}$, the mapping $\mathscr{B}^c_{p(\cdot)}(\Omega) \to [0,\infty[: u \to \|u\|_{(C_i)}$ is a seminorm on $\mathscr{B}^c_{p(\cdot)}(\Omega)$. Let $\mathscr{T}$ be the topology generated by these seminorms. If $(C_i)_{i=1}^{\infty} \in (\mathbb{R}_+)^{\mathbb{N}}$, the Lemma 3.4/1 and the continuity of the Fourier transform show that exist a positive integer k and a positive constant C such that

$$\|u\|_{(C_i)} = \sum_{i=1}^{m_j} C_i \,\|\theta_i u\|_{\mathscr{B}_{p(\cdot)}} = \sum_{i=1}^{m_j} C_i \,\|\widehat{\theta_i u}\|_{p(\cdot)} = (2\pi)^{-n} \sum_{i=1}^{m_j} C_i \,\|\hat{\theta}_i * \hat{u}\|_{p(\cdot)} \le C \Big(\sum_{i=1}^{m_j} C_i \,|\theta_i|_k\Big) \|u\|_{\mathscr{B}_{p(\cdot)}}$$

holds for all $u \in \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$. This proves the continuity of the canonical injection $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j) \to \mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]$. Since this is valid for all j, the topology $\mathscr{T}$ is coarser than the inductive limit topology. Let us see the reverse inclusion: Let $\{K_{j_l}\}_{l=1}^{\infty}$ be a subsequence of $\{K_j\}_{j=1}^{\infty}$ such that $\operatorname{supp}\theta_l \subset K_{j_l}$ for all l. If $\|\cdot\|$ is a continuous seminorm in the space $\mathscr{B}^c_{p(\cdot)}(\Omega)$ equipped with the inductive limit topology, its restriction to each step is continuous. So there are constants $C_l > 0$ such that $\|u\| \le C_l \,\|u\|_{\mathscr{B}_{p(\cdot)}}$ for all $u \in \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_{j_l})$ $l = 1,2,\ldots$ Let $u \in \mathscr{B}^c_{p(\cdot)}(\Omega)$. If $\operatorname{supp} u \subset K_j$, we know that $u = \sum_{i=1}^{m_j} \theta_i u$ for some positive integer m_j. Since each $\theta_i u$ is in $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_{j_i})$, we obtain

$$\|u\| \le \sum_{i=1}^{m_j} \|\theta_i u\| \le \sum_{i=1}^{m_j} C_i \,\|\theta_i u\|_{\mathscr{B}_{p(\cdot)}} = \sum_{i=1}^{\infty} C_i \,\|\theta_i u\|_{\mathscr{B}_{p(\cdot)}} = \|u\|_{(C_i)} \,.$$

In consequence the two topologies coincide.

In the next theorem we show a number of basic properties of the spaces $\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ that we shall need to study the duality $\langle \mathscr{B}^c_{p(\cdot)}(\Omega), \mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)\rangle$.

Theorem 3.7. *Let Ω be an open set in $\mathbb{R}^n$ and let $p(\cdot) \in \mathscr{P}^0$ be such that M is bounded in $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p_-$. Then:*

1. *$\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ is a metrizable topological linear space,*
2. *We have the natural embeddings $C^\infty(\Omega) \hookrightarrow \mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$, $C_0^\infty(\Omega) \overset{d}{\hookrightarrow} \mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$,*
3. *$\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ is complete (i.e. an F-space; a Fréchet space if $p_- \ge 1$).*

Proof.

1. Let $u \in \mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega) \setminus \{0\}$ and let $\varphi \in C_0^\infty(\Omega)$ such that $\langle \varphi, u\rangle \neq 0$. Then $\|u\|_{p(\cdot),\theta} > 0$ for all $\theta \in C_0^\infty(\Omega)$ such that $\theta = 1$ in $\operatorname{supp}\varphi$. Thus the topology of $\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ is Hausdorff. We now consider a fundamental sequence $\{K_j\}_{j=1}^{\infty}$ of compact subsets of Ω and choose $\varphi_j \in C_0^\infty(\Omega)$ such that $\varphi_j \equiv 1$ on K_j and $\operatorname{supp}\varphi_j \subset \overset{\circ}{K}_{j+1}$, $j = 1,2,\ldots$ Then the topology of $\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ is also generated by the system of semiquasi-norms $\{\|\cdot\|_{p(\cdot),\varphi_j} : j = 1,2,\ldots\}$. In order to prove this, we

choose a function $\varphi \in C_0^\infty(\Omega)$ and an integer j such that $\operatorname{supp}\varphi \subset K_j$. Then, for each $u \in \mathscr{B}_{p(\cdot)}^{\mathrm{loc}}(\Omega)$, we have $\varphi u = \varphi(\varphi_j u)$ where $\varphi_j u \in \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_{j+1})$. Thus, by using Lemma 3.4/1 and the continuity in S of the Fourier transform, we get a positive constant C and a positive integer m such that

$$\|u\|_{p(\cdot),\varphi} = \|\varphi u\|_{\mathscr{B}_{p(\cdot)}} = \|\varphi(\varphi_j u)\|_{\mathscr{B}_{p(\cdot)}} = \left\|\widehat{\varphi(\varphi_j u)}\right\|_{p(\cdot)} = (2\pi)^{-n}\|\hat{\varphi} * \widehat{\varphi_j u}\|_{p(\cdot)}$$
$$\leq C\,|\varphi|_m\,\|\varphi_j u\|_{\mathscr{B}_{p(\cdot)}} = C\,|\varphi|_m\,\|u\|_{p(\cdot),\varphi_j}$$

holds for all $u \in \mathscr{B}_{p(\cdot)}^{\mathrm{loc}}(\Omega)$. This proves 1.

2. We will only show the density of $C_0^\infty(\Omega)$ in $\mathscr{B}_{p(\cdot)}^{\mathrm{loc}}(\Omega)$. Let $u \in \mathscr{B}_{p(\cdot)}^{\mathrm{loc}}(\Omega)$. Given $\varphi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$ we have to prove that there exists a function $\phi \in C_0^\infty(\Omega)$ such that $\|u-\phi\|_{p(\cdot),\varphi} < \varepsilon$. Let $\psi \in C_0^\infty(\Omega)$ so that $\psi = 1$ in $\operatorname{supp}\varphi$. Then $\psi u \in \mathscr{B}_{p(\cdot)}^{c}(\Omega)$, and there exists a sequence (χ_ν) in $C_0^\infty(\Omega)$ such that $\chi_\nu \to \psi u$ in $\mathscr{B}_{p(\cdot)}^{c}(\Omega)$ when $\nu \to \infty$ (apply Theorem 3.5/1). Hence and from Theorem 3.5/4 if follows that $\varphi\chi_\nu \to \varphi(\psi u) = \varphi u$ in $\mathscr{B}_{p(\cdot)}^{c}(\Omega)$ and, a fortiori, in $\mathscr{B}_{p(\cdot)}$. Therefore, putting $\phi = \chi_\nu$ with ν sufficiently large, we have $\|u-\phi\|_{p(\cdot),\varphi} < \varepsilon$.
3. Let $(u_j)_{j=1}^\infty$ be a Cauchy sequence in $\mathscr{B}_{p(\cdot)}^{\mathrm{loc}}(\Omega)$ (only consider sequences in virtue of 1). By 2 and the completeness of $\mathscr{D}'(\Omega)$, u_j has a limit u in $\mathscr{D}'(\Omega)$. Let us see that $u \in \mathscr{B}_{p(\cdot)}^{\mathrm{loc}}(\Omega)$. Let $\varphi \in C_0^\infty(\Omega)$. Obviously, $\varphi u_j \to \varphi u$ in $\mathscr{D}'(\Omega)$. Furthermore, $(\varphi u_j)_{j=1}^\infty$ is a Cauchy sequence in the quasi-Banach space $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(\operatorname{supp}\varphi)$. Let v be the limit of φu_j in this space. From Theorem 3.5/2 we conclude that $\varphi u = v$. Hence it follows that $u \in \mathscr{B}_{p(\cdot)}^{\mathrm{loc}}(\Omega)$ and that $u_j \to u$ in $\mathscr{B}_{p(\cdot)}^{\mathrm{loc}}(\Omega)$.

□

4. The dual of $\mathscr{B}_{p(\cdot)}^{c}(\Omega)$

In [8, Chapter XV], Hörmander studies the behaviour of the Fourier-Laplace transform in the space $\mathscr{B}_{2,k}^{c}(\Omega) = \operatorname{ind}_K[\mathscr{B}_{2,k} \cap \mathscr{E}'(K)]$ when Ω is an open convex set in $\mathbb{R}^n$ and k satisfies the estimate $k(x+y) \leq (1+C|x|)^N k(y)$, $x, y \in \mathbb{R}^n$ (C and N positive constants). For this he analyses the inductive topology in $\mathscr{B}_{2,k}^{c}(\Omega)$, proves the isomorphism $\left(\mathscr{B}_{2,k}^{c}(\Omega)\right)' \simeq \mathscr{B}_{2,1/\tilde{k}}^{\mathrm{loc}}(\Omega)$ and shows that every continuous seminorm in $\mathscr{B}_{2,k}^{c}(\Omega)$ is bounded by a seminorm of the form $u \to \left(\int_{\mathbb{C}^n} |\hat{u}(\zeta)|^2 e^{-2\phi(\zeta)} d\lambda(\zeta)\right)^{1/2}$, where $\hat{u}$ is the Fourier-Laplace transform of u and ϕ is plurisubharmonic (see [8, Section 15.2]). In [16, Section 3] the former isomorphism is extended to Hörmander spaces in the sense of Beurling and Björck. A number of applications of this duality (to sequence space representations of several ultradistributions spaces and to linear partial differential operators) are also given in [16] and [17]. In this section we extend the former isomorphism to variable exponent Hörmander spaces. As a consequence, some results on sequence space representations of variable exponent

Hörmander spaces are given. Finally, we propose a question on interpolation of the spaces $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$.

Lemma 4.1. *Let $p \in \mathscr{P}^0$ be with $p_- > 1$ and let f be a measurable function on $\mathbb{R}^n$ such that $f\varphi \in L_1$ for all $\varphi \in S$ and such that $B := \sup\{ |\int_{\mathbb{R}^n} f\varphi\, dx| : \varphi \in S,\ \|\varphi\|_{p'(\cdot)} \le 1 \} < \infty$. Then $f \in L_{p(\cdot)}$.*

Proof. We define the functional $u(\varphi) := \int_{\mathbb{R}^n} f\varphi\, dx$, $\varphi \in S$. Since $|u(\varphi)| \le B\, \|\varphi\|_{p'(\cdot)}$ for all $\varphi \in S$, it follows that $u \in (S, \|\cdot\|_{p'(\cdot)})'$ and, since S is dense in $L_{p'(\cdot)}$ (use Lemma 2.4/1), u has a unique continuous linear extension U to $L_{p'(\cdot)}$. Next, by duality (Lemma 2.3), we can find a function $g \in L_{p(\cdot)}$ such that $\int_{\mathbb{R}^n} g\varphi\, dx = U(\varphi) = \int_{\mathbb{R}^n} f\varphi\, dx$ for all $\varphi \in S$. In consequence, $f = g$ and $f \in L_{p(\cdot)}$. □

Lemma 4.2. *Let $p(\cdot) \in \mathscr{P}^0$ be such that $p_- > 1$ and such that M is bounded in $L_{p(\cdot)}$. Let K be a locally integrable function in $\mathbb{R}^n \setminus \{0\}$ such that $\widehat{K} \in L_\infty$, $|K(x)| \le C\,|x|^{-n}$ and $|\nabla K(x)| \le C\,|x|^{-(n+1)}$ for all $x \neq 0$. Then the singular integral operator T, defined by $Tf(x) = K * f(x)$, is bounded in $L_{p(\cdot)}$.*

Proof. See [1, p. 247]. □

Theorem 4.3. *Let Ω be an open set in $\mathbb{R}^n$ and let $p(\cdot) \in \mathscr{P}^0$ be such that $p_- > 1$ and M is bounded in $L_{p(\cdot)}$. Then $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'$ is isomorphic to $\mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$.*

Proof. Let J be the natural embedding

$$C_0^\infty(\Omega) \to \mathscr{B}^c_{p(\cdot)}(\Omega) : \varphi \to \langle \theta, J(\varphi) \rangle = \int_{\mathbb{R}^n} \theta\varphi\, dx \qquad (\theta \in S)$$

(see Theorem 3.5/1) and consider its adjoint operator $J' : \left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)' \to \mathscr{D}'(\Omega)$. Let us see that $\mathrm{Im}\, J' \subset \mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$. Let $l \in \left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'$. We have to show that $\varphi J'(l) \in \mathscr{B}_{\widetilde{p'(\cdot)}}$, i.e. $\mathscr{F}(\varphi J'(l)) \in L_{\widetilde{p'(\cdot)}}$, for all $\varphi \in C_0^\infty(\Omega)$. Let us fix such a φ and set $K = \mathrm{supp}\, \varphi$. Then $\mathscr{F}^2(\varphi J'(l)) = (2\pi)^n \widetilde{\varphi J'(l)} = (2\pi)^n \hat{\varphi} \widetilde{J'(l)}$ and so $\mathrm{supp}\, \mathscr{F}^2(\varphi J'(l)) \subset -K$. This implies, by the Paley-Wiener-Schwartz theorem, that $\mathscr{F}(\varphi J'(l))$ is an entire analytic function of n complex variables such that for any $\varepsilon > 0$

$$|\mathscr{F}(\varphi J'(l))(z)| \le A_\varepsilon (1+|x|)^\lambda e^{(\sigma+\varepsilon)|y|}$$

holds for all $z = x + iy$ with $x, y \in \mathbb{R}^n$ ($\lambda \in \mathbb{R}$ is a constant and A_ε depends on ε but not on z) (see e.g. [21, p. 272]). We now prove that $\sup\{ |\int_{\mathbb{R}^n} \mathscr{F}(\varphi J'(l))\theta\, dx| : \theta \in S,\ \|\theta\|_{\widetilde{p(\cdot)}} \le 1 \} < \infty$ since, once this is established, Lemma 4.1 yields $\mathscr{F}(\varphi J'(l)) \in L_{\widetilde{p'(\cdot)}}$. Fix an element $\theta \in S$ such that $\|\theta\|_{\widetilde{p(\cdot)}} \le 1$. Then, from the continuity of l on the step $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ and from the continuity in $L_{p(\cdot)}$ of the operator $\theta \to \theta * \hat{\varphi}$ (Lemma 4.2), we get

$$\left| \int_{\mathbb{R}^n} \mathscr{F}(\varphi J'(l))\theta\, dx \right| = \left| \langle \theta, \mathscr{F}(\varphi J'(l)) \rangle \right| = \left| \langle J(\hat{\theta}\varphi), l \rangle \right| \le C\, \|J(\hat{\theta}\varphi)\|_{\mathscr{B}_{p(\cdot)}}$$
$$= C\, \|\mathscr{F}(\hat{\theta}\varphi)\|_{p(\cdot)} = C\, \|\tilde{\theta} * \hat{\varphi}\|_{p(\cdot)} \le C\, \|\tilde{\theta}\|_{p(\cdot)} = C\, \|\theta\|_{\widetilde{p(\cdot)}} \le C\,.$$

Therefore, we have shown that $\operatorname{Im} J' \subset \mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$. Next we prove that the mapping

$$J' : \left(\mathscr{B}^{c}_{p(\cdot)}(\Omega)\right)' \to \mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$$

is onto: Let $\{K_j\}_{j=1}^{\infty}$ be a fundamental sequence of compact subsets of Ω such that every K_j is the closure of an open set with the segment property; moreover, let $\chi_j \in C_0^\infty(\Omega)$ be such that $\chi_j(x) = 1$ whenever $x \in K_j$, $j = 1, 2, \dots$ Let $u \in \mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$. For each j we define l_j by $\langle J(\varphi), l_j\rangle := \langle \varphi, u\rangle$ for all $\varphi \in C_0^\infty(K_j)$. Then, taking into account the generalized Hölder inequality, we get

$$\begin{aligned} |\langle J(\varphi), l_j\rangle| = |\langle \varphi, u\rangle| = |\langle \varphi, \chi_j u\rangle| = C\left|\int_{\mathbb{R}^n} \widehat{\chi_j u}\tilde{\hat{\varphi}}\,dx\right| \\ \le C\,\|\tilde{\hat{\varphi}}\|_{\widetilde{p(\cdot)}}\|\widehat{\chi_j u}\|_{(\widetilde{p(\cdot)})'} = C\,\|\hat{\varphi}\|_{p(\cdot)} = C\,\|J(\varphi)\|_{\mathscr{B}_{p(\cdot)}} \end{aligned}$$

for all $\varphi \in C_0^\infty(K_j)$. Hence and from the density of $J(C_0^\infty(K_j))$ in the step $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$ (use Lemma 2.4/1 and Remark 3.6/1) it follows that l_j extends to a unique continuous linear form $\bar{l}_j$ on $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$. Finally, since $\bar{l}_{j+1}$ and $\bar{l}_j$ coincide on $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$, we easily obtain an l in $\left(\mathscr{B}^{c}_{p(\cdot)}(\Omega)\right)'$ with $J'(l) = u$. To sum up, J' is an algebraic isomorphism from $\left(\mathscr{B}^{c}_{p(\cdot)}(\Omega)\right)'$ onto $\mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$. To see that J' is a (topological) isomorphism it suffices to prove that it is continuous since that those spaces are Fréchet spaces ($\mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$ is a Fréchet space by Lemma 2.4/1 and Theorem 3.7/3; $\mathscr{B}^{c}_{p(\cdot)}(\Omega)$ is a (DF)-space by Theorem 3.5/3 and [10, (4) p. 402] and so its strong dual is also a Fréchet space (see [10, (1) p. 397])). Suppose that $l_\nu \to 0$ in $\left(\mathscr{B}^{c}_{p(\cdot)}(\Omega)\right)'$. Fix φ in $C_0^\infty(\Omega)$. Then, taking into account Lemma 2.3 and Lemma 2.5/2, we get

$$\begin{aligned} \|\varphi J'(l_\nu)\|_{\mathscr{B}_{\widetilde{p'(\cdot)}}} = \|\mathscr{F}(\varphi J'(l_\nu))\|_{\widetilde{p'(\cdot)}} \\ \le C \sup\left\{\left|\int_{\mathbb{R}^n} \mathscr{F}(\varphi J'(l_\nu))\theta\,dx\right| : \theta \in S,\ \|\theta\|_{\widetilde{p(\cdot)}} \le 1\right\} \\ = C \sup\left\{\left|\langle J(\hat{\theta}\varphi), l_\nu\rangle\right| : \theta \in S,\ \|\theta\|_{\widetilde{p(\cdot)}} \le 1\right\}. \end{aligned}$$

But the set $A = \{J(\hat{\theta}\varphi) : \theta \in S,\ \|\theta\|_{\widetilde{p(\cdot)}} \le 1\}$ is bounded in $\mathscr{B}^{c}_{p(\cdot)}(\Omega)$ (in fact, if $K = \operatorname{supp}\varphi$ then $A \subset \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ and so, reasoning as in the first part of the proof, it results that $\sup\left\{\|J(\hat{\theta}\varphi)\|_{\mathscr{B}_{p(\cdot)}} : \theta \in S,\ \|\theta\|_{\widetilde{p(\cdot)}} \le 1\right\} < \infty$, but this shows that A is bounded in that step and thus in $\mathscr{B}^{c}_{p(\cdot)}(\Omega)$) which implies, in virtue of the previous estimate and of the convergence to 0 in $\left(\mathscr{B}^{c}_{p(\cdot)}(\Omega)\right)'$ of (l_ν), that

$$\|\varphi J'(l_\nu)\|_{\mathscr{B}_{\widetilde{p'(\cdot)}}} \le C \sup_{u \in A} |\langle u, l_\nu\rangle| \to 0$$

when $\nu \to \infty$. Since φ is arbitrary, we have shown that $J'(l_\nu) \to 0$ in $\mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$. □

Remark 4.4. "If E is the inductive limit of an increasing sequence $E_1[\mathscr{T}_1] \subset E_2[\mathscr{T}_2] \subset \cdots$ of quasi-Banach spaces such that $\mathscr{T}_{n+1}$ induces on E_n the topology $\mathscr{T}_n$ and the dual E_n' of $E_n[\mathscr{T}_n]$ separates the points of E_n for each n, then the (strong) dual E'

is isomorphic to the projective limit of the Banach spaces E_n' via the natural mapping $E' \to \operatorname{proj}_n E_n' : u \to (u|_{E_n})_{n=1}^{\infty}$". (We shall omit the proof of this simple result.) Hence and from Theorem 3.5/3 it follows that if Ω is an open set in $\mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}^0$ is such that M is bounded in $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p_-$, then $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'$ is isomorphic to the projective limit of the Banach spaces $\left(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)\right)'$ (here $\{K_j\}_{j=1}^{\infty}$ is any fundamental sequence of compact subsets of Ω). In particular, if Ω is as in Theorem 3.5/5 and $p(\cdot) \equiv p$ with $0 < p < 1$ then $\left(\mathscr{B}^c_p(\Omega)\right)' \simeq \operatorname{proj}_j X_j$ where the Banach spaces X_j are isomorphic to l_∞ (use Theorem 3.5/5 and recall that a quasi-Banach space and its Banach envelope have isomorphic duals (see e.g. [13, Corollary 1])).

The sequence space representation $\mathscr{B}^{\mathrm{loc}}_{1,k}(\Omega) \simeq l_1^{\mathbb{N}}$ was established by Vogt in [24] (Ω an open set in $\mathbb{R}^n$ and k a temperate weight function on $\mathbb{R}^n$). In [6] and [16] (see also [15], [17]) the sequence space representations $\mathscr{B}^{\mathrm{loc}}_{p,k}(\Omega) \simeq l_p^{\mathbb{N}}$, $1 < p \leq \infty$ (Ω open set in $\mathbb{R}^n$, k a temperate weight function on $\mathbb{R}^n$ in [6] and k a temperate weight function on $\mathbb{R}^n$ with k^p in the generalized Muckenhoupt class A_p^* (see [5, p. 453]) when $p < \infty$ in [16]) were obtained. Next we give a result on function sequence space representation of variable exponent Hörmander spaces.

Corollary 4.5. *Let $p(\cdot)$ be such that $1 < p_- \leq p_+ < \infty$ and M is bounded in $L_{p(\cdot)}(\mathbb{R})$. Let $\Omega =]a,b[$ ($-\infty \leq a < b \leq \infty$). Let $a_j \searrow a$, $b_j \nearrow b$, and we put $K_j = [a_j, b_j]$ for $j = 1,2,\ldots$ Then $\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ is isomorphic to $L^{-K_1}_{p(\cdot)} \times \prod_{j=1}^{\infty} H_j$ where each H_j is isomorphic to an infinite dimensional complemented subspace of $L^{-K_{j+1}}_{p(\cdot)}$ (thus $\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ is isomorphic to a complemented subspace of $\left(L_{p(\cdot)}(\mathbb{R})\right)^{\mathbb{N}}$).*

Proof. From Lemma 2.5/2 if follows that M is also bounded in $L_{p'(\cdot)}(\mathbb{R})$ (and thus in $L_{\widetilde{p'(\cdot)}}(\mathbb{R})$), then using Theorem 4.3 and Theorem 3.5/5, and taking into account that the dual of a locally convex direct sum of Banach spaces is isomorphic to the product of their duals (see e.g. [10, p. 287]) and that the dual of $L^{-K_j}_{\widetilde{p'(\cdot)}}$ is isomorphic to $L^{K_j}_{\widetilde{p(\cdot)}}$ [18, Theorem 4.3], we get the isomorphisms

$$\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega) \simeq \left(\mathscr{B}^c_{\widetilde{p'(\cdot)}}(\Omega)\right)' \simeq \left(L^{-K_1}_{\widetilde{p'(\cdot)}} \oplus (\oplus_{j=1}^{\infty} G_j)\right)' \simeq L^{K_1}_{\widetilde{p(\cdot)}} \times \prod_{j=1}^{\infty} G_j' \simeq L^{-K_1}_{p(\cdot)} \times \prod_{j=1}^{\infty} H_j$$

where, for all j, H_j is an infinite dimensional complemented subspace of $L^{-K_{j+1}}_{p(\cdot)}$ (the last isomorphism is the operator $\sim$). □

Some questions

- It would also be interesting to obtain sequence space representations of the space $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(Q)$ (Q a cube in $\mathbb{R}^n$) and of the spaces $\mathscr{B}^c_{p(\cdot)}(\Omega)$ and $\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ (Ω open set in $\mathbb{R}^n$).
- Characterize the variable exponents $p_0(\cdot), p_1(\cdot)$ (with $1 \leq (p_j)_- \leq (p_j)_+ < \infty$, $j = 0,1$) and the compact subsets K of $\mathbb{R}^n$ such that $\big[\mathscr{B}_{p_0(\cdot)} \cap \mathscr{E}'(K), \mathscr{B}_{p_1(\cdot)} \cap$

$\mathscr{E}'(K)\big]_{[\theta]} \simeq \mathscr{B}_{p_\theta(\cdot)} \cap \mathscr{E}'(K)$ (complex interpolation) where $0 < \theta < 1$ and $\frac{1}{p_\theta(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{p_1(x)}$ (we know that $\left[\mathscr{B}_{p_0(\cdot)}, \mathscr{B}_{p_1(\cdot)}\right]_{[\theta]} \simeq \mathscr{B}_{p_\theta(\cdot)}$ (use the definition of $\mathscr{B}_{p(\cdot)}$ and [3, Theorem 7.1.2]); see also [23, pp. 66-78]).

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