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On Reich type $\lambda - \alpha$-nonexpansive mapping in Banach spaces with applications to $L_1([0,1])$

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Abstract

In this manuscript we introduce a new class of monotone generalized nonexpansive mappings and establish some weak and strong convergence theorems for Krasnoselskii iteration in the setting of a Banach space with partial order. We consider also an application to the space $L_1([0,1])$. Our results generalize and unify the several related results in the literature.

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1. Introduction and preliminaries

The study of the existence of fixed point of nonexpansive mappings, initiated in 1965 independently by Browder [5], Göhde [11] and [16], is one of dynamic research subject in nonlinear functional analysis. In [16], Kirk proved that a self-mapping on a nonempty bounded closed and convex subset of a reflexive Banach space possesses a fixed point if it is nonexpansive and the corresponding subset has a normal structure. In 1992, Veeramani obtained a more general result in this direction by introducing the notion of $T-$regular set [23].

On the other hand, in 1967, Opial introduced in [18] a class of spaces for which the asymptotic center of a weakly convergent sequence coincides with

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the weak limit point of the sequence. A Banach space X is said to have the Opial property, if for each weakly convergent sequence $\{x_n\}$ in X with limit z, $\liminf \|x_n - z\| < \liminf \|x_n - y\|$ for all $y \in X$ with $y \neq z$. In 1972, Gossey and Lami Dozo noticed in [12] that all the spaces of this class have normal structure. It is well known that Hilbert spaces, finite dimensional Banach spaces and l^p-spaces, $(1 < p < \infty)$, have the Opial property [8]. In 2008, Suzuki introduced in [21] a new class of mappings satisfying the so-called (C)-condition which also includes nonexpansive mappings and proved that such mappings on a nonempty weakly compact convex set in a Banach space which satisfies Opial's condition have a fixed point. In 2011, Falset *et al.* proposed in [8] mappings satisfying (C_λ)-condition, $\lambda \in (0,1)$, respectively. In [1] Aoyama and Kohsaka introduced a new class of nonexpansive mappings, and obtained a fixed point result for such mappings. Finally, in 2017, in [19] Shukla et al proposed a new generalization and introduce the deneralized $\alpha-$nonexpansive mapping and obtained a fixed theorem for such mappings. All the results cited above were obtained, in the weak case, with Opial's condition.

In this report, we propose a generalization of the results of Shukla *et al.* [19] by introducing a class of $\lambda-\alpha$- generalized nonexpansive mapping. In addition, we establish some weak and strong convergence theorems for Krasnoselskii iteration in an ordered Banach space with partial order $\leq$. We also consider an application in the context of $L_1([0,1])$. The presented results in this report, extend, generalize and unify a number of existing results on the the topic in the literature.

Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ the set of the real numbers. For a non-empty K of a real Banach space X, a mapping $T : K \to K$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in K$. Moreover, a selfmapping T is called quasinonexpansive [7] if $\|T(x) - y\| \leq \|x - y\|$ for all $x \in K$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T .

Definition 1.1 ([12, 22])**.** The norm of a Banach space X is called uniformly convex in every direction, in short, we say that X is UCED, if for $\varepsilon \in (0,2]$ and $z \in X$ with $\|z\| = 1$, there exists $\delta(\varepsilon, z) > 0$ such that for all $x, y \in X$ with where $\|x\| \leq 1,\ \|y\| \leq 1$ and $x - y \in \{tz : t \in [-2, -\varepsilon] \cup [\varepsilon, 2]\}$

$$\|x + y\| \leq 2(1 - \delta(\varepsilon, z)).$$

Lemma 1.2 ([21])**.** *For a Banach space X, the following are equivalent:*

- (i) *X is UCED.*
- (ii) *If $\{x_n\}$ is a bounded sequence in X , then the function f on X defined by $f(x) = \limsup \|x_n - x\|$ is strictly quasiconvex , that is, $f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}$ for all $\lambda \in (0,1)$ and $x, y \in X$ with $x \neq y$.*

Lemma 1.3 ([9])**.** *Let (z_n) and (w_n) be bounded sequences in a Banach space X and let λ belongs to $(0,1)$. Suppose that $z_{n+1} = \lambda w_n + (1-\lambda) z_n$ and $\|w_{n+1} - w_n\| \leq \|z_{n+1} - z_n\|$ for all $n \in \mathbb{N}$. Then $\lim \|w_n - z_n\| = 0$.*

Definition 1.4 ([21]). Let K be a nonempty subset of a Banach space X. We say that a mapping $T : K \to K$ satisfies (C)-condition on K if for $x, y \in K$ we have

$$\frac{1}{2}\|x - T(x)\| \leq \|x - y\| \Rightarrow \|T(x) - T(y)\| \leq \|x - y\|.$$

It is clear that each nonexpansive mapping satisfies the condition (C) but the converse is not true. For details and counterexamples see e.g. [10].

Definition 1.5 ([8]). Let K be a nonempty subset of Banach space X and $\lambda \in (0, 1)$. We say that a mapping $T : K \to K$ satisfies (C_λ) -condition on if for all $x, y \in K$, we have

$$\lambda\|x - T(x)\| \leq \|x - y\| \Rightarrow \|T(x) - T(y)\| \leq \|x - y\|.$$

Note that if $\lambda = \frac{1}{2}$, then (C_λ)-condition implies (C)-condition. For more details and examples, see e.g. Falset *et al.* [8].

Throughout the paper, the pair $(X, \leq)$ will denote an ordered Banach space where X is a Banach space endowed with a partial order " $\leq$ ".

Definition 1.6. A self-mapping T defined on an ordered Banach space $(X, \leq)$ is said to be monotone if for all $x, y \in X$,

$$x \leq y \Rightarrow T(x) \leq T(y).$$

Definition 1.7 ([1]). Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be α-nonexpansive if for all $x, y \in K$ and $\alpha < 1$,

$$\|T(x) - T(y)\|^2 \leq \alpha\|T(x) - y\|^2 + \alpha\|x - T(y)\|^2 + (1 - 2\alpha)\|x - y\|^2$$

Definition 1.8 ([19]). Let K be a nonempty subset of an ordered Banach space $(X, \leq)$. A mapping $T : K \to K$ will be called a generalized α-nonexpansive mapping if there exists $\alpha \in (0, 1)$ such that

$$\begin{cases} \frac{1}{2}\|x - T(x)\| \leq \|x - y\| \textit{ implies} \\ \|T(x) - T(y)\| \leq \alpha\|T(x) - y\| + \alpha\|T(y) - x\| + (1 - 2\alpha)\|x - y\| \end{cases}$$

for all $x, y \in K$ with $x \leq y$.

Remark 1.9. When $\alpha = 0$, a generalized-nonexpansive mapping is reduced to a mapping satisfying (C)-condition. The converse is false. For more details and counterexamples see e.g. [19] and [14, 13].

2. Reich type $(\lambda - \alpha)$-nonexpansive mappings

Definition 2.1. Let K be a nonempty subset of an ordered Banach space $(X, \leq)$. A mapping $T : K \to K$ will be called Reich type $(\lambda - \alpha)$-nonexpansive mappings if there exists $\lambda \in (0, 1)$ and $\alpha \in [0, 1)$ such that

$$\lambda\|x - T(x)\| \leq \|x - y\| \Rightarrow \|T(x) - T(y)\| \leq R^\alpha_T(x, y), \tag{2.1}$$

where

$$R^\alpha_T(x, y) := \alpha(\|T(x) - y\| + \|T(y) - x\|) + (1 - 2\alpha)\|x - y\|$$

for all $x, y \in K$ with $x \leq y$. In addition, if the mapping T is monotone, we say that monotone Reich type $(\lambda - \alpha)$-nonexpansive mapping.

Remark 2.2. We point the following special cases:

(1) When $\alpha = 0$, a Reich type $\lambda - \alpha$-nonexpansive mapping reduced to a mapping satisfying condition (C_λ), see e.g. [8].

(2) If $\lambda = \frac{1}{2}$, it becomes a generalized α-nonexpansive condition.

Proposition 2.3. *Let K be a nonempty subset of an ordered Banach space $(X, \leq)$ and $T : K \to K$ be a Reich type $(\lambda - \alpha)$-nonexpansive mapping with a fixed point $z \in K$ with $x \leq z$. Then T is quasinonexpansive.*

Proof. Since $z \in K$ fixed point, $0 = \lambda\|z - T(z)\| \leq \|z - x\|$, we have

$$\|z - T(x)\| \leq \alpha\|z - T(x)\| + \alpha\|T(z) - x\| + (1 - 2\alpha)\|z - x\| \leq \|z - x\|.$$

□

Definition 2.4. Let T be a monotone self-mapping on a nonempty convex subset of an ordered Banach space $(X, \leq)$. For a fix $\lambda \in (0, 1)$ and for an initial point $x_1 \in K$, the Krasnoselskii iteration sequence $\{x_n\} \subset K$ is defined by

$$x_{n+1} = \lambda T(x_n) + (1 - \lambda)x_n \ , \ n \geq 1. \tag{2.2}$$

In the sequel we need the following lemmas.

Lemma 2.5 ([17])**.** *Let $x, y, z \in X$ and $\lambda \in (0, 1)$. Suppose p is the point of segment $[x, y]$ which satisfies $\|x - p\| = \lambda\|x - y\|$, then,*

$$\|z - p\| \leq \lambda\|z - y\| + (1 - \lambda)\|z - x\| \tag{2.3}$$

Lemma 2.6 ([15])**.** *Let K be convex and $T : K \to K$ be monotone. Assume that $x_1 \in K$, $x_1 \leq T(x_1)$. Then the sequence $\{x_n\}$ defined by (2.2) satisfies:*

$$x_n \leq x_{n+1} \leq T(x_n) \leq T(x_{n+1}),$$

for $n \geq 1$. Moreover, if $\{x_n\}$ has two subsequences which converge to y and z, then we must have $y = z$.

It is easy to see that by the mimic of the idea used in Lemma 2.6, we get that

$$T(x_{n+1}) \leq T(x_n) \leq x_{n+1} \leq x_n,$$

by assuming the initial condition as $T(x_1) \leq x_1$.

Lemma 2.7. *Let K be a nonempty convex subset of an ordered Banach space $(X, \leq)$ and $\{x_n\}$ is the iteration sequence defined by (2.2) in K. Let $T : K \to K$ be a monotone Reich type $\lambda - \alpha$-nonexpansive mapping with $\lambda \in (\frac{1}{3}, 1)$ and $\alpha \in [0, 1)$. Suppose also that $y_n = T(x_n)$, $n \geq 1$. If, for $x_1 \in K$ with $x_1 \leq y_1 = T(x_1)$ we have*

$$\|y_n - x_{n+1}\| \leq (3\lambda - 1)\|y_n - x_n\|, \text{ for all } n \in \mathbb{N}, \tag{2.4}$$

then the sequence $\{\|y_n - x_n\|\}$ is decreasing.

Proof. On account of the definition of Krasnoselskii iteration we have

$$x_{n+1} = \lambda y_n + (1-\lambda)x_n, \text{ with } y_n = T(x_n). \tag{2.5}$$

It means that x_{n+1} belongs to the segment $]x_n, y_n[$, and hence we have

$$\|x_n - y_n\| = \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|. \tag{2.6}$$

Furthermore, (2.7) yields that

$$\|x_n - x_{n+1}\| = \|x_n - [\lambda T(x_n) + (1-\lambda)x_n]\| = \lambda\|x_n - T(x_n)]\|. \tag{2.7}$$

On account of the triangle inequality together with the fact that T is monotone Reich type $(\lambda-\alpha)$-nonexpansive mapping, we derive that

$$\begin{aligned} \|y_{n+1} - x_{n+1}\| &\leq \|y_{n+1} - y_n\| + \|y_n - x_{n+1}\| \\ &= \|T(x_{n+1}) - T(x_n)\| + \|y_n - x_{n+1}\| \\ &\leq \alpha(\|T(x_n) - x_{n+1}\| + \|T(x_{n+1}) - x_n\|) \\ &\qquad +(1-2\alpha)\|x_n - x_{n+1}\| + \|y_n - x_{n+1}\| \\ &= (1+\alpha)\|y_n - x_{n+1}\| + \alpha\|y_{n+1} - x_n\| + (1-2\alpha)\|x_n - x_{n+1}\| \\ &= (1+\alpha)\|y_n - x_{n+1}\| + \alpha\|y_{n+1} - x_n\| \\ &\qquad +(1+\alpha)\|x_n - x_{n+1}\| - 3\alpha\|x_n - x_{n+1}\|. \end{aligned} \tag{2.8}$$

On account of (2.6) the left hand side of the inequality of (2.8) turns into

$$= (1+\alpha)\|x_n - y_n\| + \alpha\|y_{n+1} - x_n\| - 3\alpha\|x_n - x_{n+1}\|, \tag{2.9}$$

Taking the inequality (2.7) into account, the expression (2.9) turns into

$$\begin{aligned} &\leq (1+\alpha)\|x_n - y_n\| + \alpha\|y_{n+1} - x_n\| - 3\lambda\alpha\|x_n - y_n\| \\ &= (1+\alpha-3\lambda\alpha)\|x_n - y_n\| + \alpha\|y_{n+1} - x_n\| \end{aligned} \tag{2.10}$$

Employing the assumption (2.4) of the lemma, we estimate the expression (2.10) from above as

$$\begin{aligned} &= (1+\alpha-3\lambda\alpha)\|x_n - y_n\| + \alpha\,(3\lambda-1)\,\|y_n - x_n\| \\ &= \|x_n - y_n\|. \end{aligned} \tag{2.11}$$

By combining (2.8)- (2.11), for each n, we deduce that

$$\|y_{n+1} - x_{n+1}\| \leq \|x_n - y_n\|,$$

which complete the proof. □

In the following proposition, we extend the Goebel-Kirk inequality [9] from the class of nonexpansive mappings into the class of monotone generalized$(\lambda-\alpha)$-nonexpansive mapping.

Proposition 2.8. *Let K be a nonempty convex subset of an ordered Banach space $(X, \leq)$. Let $T : K \to K$ be a monotone Reich type $(\lambda - \alpha)$-nonexpansive mapping with $\lambda \in (\frac{1}{3}, 1)$ and $\alpha \in (0, 1)$. For $x_1 \in K$ with $x_1 \leq T(x_1)$, we set $y_n = T(x_n)$ where $\{x_n\}$ is the iteration sequence defined by (2.2) in K satisfies the assumption (2.4). Then, we have*

$$\|y_{i+n} - x_i\| \geq (1-\lambda)^{-n}[\|y_{i+n} - x_{i+n}\| - \|y_i - x_i\|] + (1+n\lambda)\|y_i - x_i\|, \tag{2.12}$$

for all $i, n \in \mathbb{N}$.

Proof. Inspired the techniques used in [9], we shall use the method of the induction to prove our assertion. It is evident that (2.12) is trivially true for all i if $n = 0$. We assume that the inequality (2.12) holds for a given n and for all i. By replacing i by $i + 1$ in (2.12), we get

$$\begin{aligned} \|y_{i+n+1} - x_{i+1}\| \geq (1-\lambda)^{-n}[&\|y_{i+n+1} - x_{i+n+1}\| \\ &- \|y_{i+1} - x_{i+1}\|] + (1+n\lambda)\|y_{i+1} - x_{i+1}\|. \end{aligned} \tag{2.13}$$

On the other hand, due to Krasnoselskii iteration, we have $x_{n+1} = \lambda y_n + (1 - \lambda)x_n$ with $y_n = T(x_n)$ and also

$$\|x_{i+1} - x_i\| = \|\lambda y_i + (1-\lambda)x_i - x_i\| = \lambda\|y_i - x_i\|. \tag{2.14}$$

The observation in (2.14) provide to apply Lemma 2.5 that yields

$$\|y_{i+n+1} - x_{i+1}\| \leq \lambda\|y_{i+n+1} - y_i\| + (1-\lambda)\|y_{i+n+1} - x_i\|.$$

Regarding that T is a monotone Reich type $(\lambda - \alpha)$-nonexpansive mapping, we have

$$\begin{aligned} \|y_{i+n+1} - x_{i+1}\| &\leq (1-\lambda)\|y_{i+n+1} - x_i\| + \lambda \textstyle\sum_{k=0}^{n} \|y_{i+k+1} - y_{i+k}\| \\ &\leq (1-\lambda)\|y_{i+n+1} - x_i\| \\ &\qquad + \lambda \textstyle\sum_{k=0}^{n}(\alpha\|x_{i+k+1} - y_{i+k}\| + \alpha\|y_{i+k+1} - x_{i+k}\| \\ &\qquad + (1-2\alpha)\|x_{i+k+1} - x_{i+k}\|) \end{aligned}$$

So, we derive that

$$\begin{aligned} \|y_{i+n+1} - x_i\| \geq (1-\lambda)^{-1}\|y_{i+n+1} - x_{i+1}\| - (1-\lambda)^{-1}\lambda\alpha B_{in} \\ - (1-\lambda)^{-1}\lambda(1-2\alpha)A_{in}, \end{aligned} \tag{2.15}$$

where

$$A_{in} = \sum_{k=0}^{n} \|x_{i+k+1} - x_{i+k}\|$$

and

$$B_{in} = \sum_{k=0}^{n}[\|x_{i+k+1} - y_{i+k}\| + \|y_{i+k+1} - x_{i+k}\|].$$

Taking the assumption (2.4) and (2.14) into account, we derive that

$$\begin{aligned} \|y_{i+k+1} - x_{i+k}\| + \|y_{i+k} - x_{i+k}\| &\leq (3\lambda - 1)\|y_{i+k} - x_{i+k}\| + \|y_{i+k} - x_{i+k}\| \\ &= 3\lambda\|y_{i+k} - x_{i+k}\| = 3\|x_{i+k} - x_{i+k+1}\|, \end{aligned} \tag{2.16}$$

for all $k \in 0, 1, ..., n$. Regarding the definition of Krasnoselskii iteration we have $x_{i+k+1} = \lambda y_{i+k} + (1-\lambda)x_{i+k}$, with $y_{i+k} = T(x_{i+k})$. In other words, $x_{i+k} \le x_{i+k+1} \le y_{i+k}$ and we have

$$\|x_{i+k} - y_{i+k}\| = \|x_{i+k+1} - x_{n+1}\| + \|x_{i+k+1} - y_{i+k}\|. \tag{2.17}$$

Now, by revisiting the inequality (2.16) by keeping the equality (2.17) in mind, we find

$$\begin{aligned} \|y_{i+k+1} - x_{i+k}\| + \|x_{i+k+1} - y_{i+k}\| &= \|y_{i+k+1} - x_{i+k}\| + \|y_{i+k} - x_{i+k}\| \\ &\quad - \|x_{i+k} - x_{i+k+1}\| \\ &\le 2\|x_{i+k} - x_{i+k+1}\| \end{aligned} \tag{2.18}$$

which implies $B_{in} \le 2A_{in}$. Accordingly, the inequality (2.15) becomes

$$\|y_{i+n+1} - x_i\| \ge (1-\lambda)^{-1}\|y_{i+n+1} - x_{i+1}\| - \lambda(1-\lambda)^{-1}A_{in}. \tag{2.19}$$

Employing the inequality (2.13) in (2.19), we find that

$$\begin{aligned} \|y_{i+n+1} - x_i\| \ge &(1-\lambda)^{-(n+1)}[\|y_{i+n+1} - x_{i+n+1}\| - \|y_{i+1} - x_{i+1}\|] \\ &+ (1-\lambda)^{-1}(1+n\lambda)\|y_{i+1} - x_{i+1}\| - \lambda(1-\lambda)^{-1}A_{in}. \end{aligned}$$

On account of (2.14), the estimation above turns into

$$\begin{aligned} \|y_{i+n+1} - x_i\| \ge &(1-\lambda)^{-(n+1)}[\|y_{i+n+1} - x_{i+n+1}\| - \|y_{i+1} - x_{i+1}\|] \\ &+ (1-\lambda)^{-1}(1+n\lambda)\|y_{i+1} - x_{i+1}\| - \lambda^2(1-\lambda)^{-1}C_{in}, \end{aligned} \tag{2.20}$$

where $C_{in} := \sum_{k=0}^{n} \|y_{i+k} - x_{i+k}\|$. By bearing, Lemma 2.7, in mind, we find that

$$C_{in} := \sum_{k=0}^{n} \|y_{i+k} - x_{i+k}\| \le (n+1)\|y_i - x_i\|.$$

Consequently, (2.20) can be estimated above as

$$\begin{aligned} \|y_{i+n+1} - x_i\| \ge &(1-\lambda)^{-(n+1)}[\|y_{i+n+1} - x_{i+n+1}\| - \|y_{i+1} - x_{i+1}\|] \\ &+ (1-\lambda)^{-1}(1+n\lambda)\|y_{i+1} - x_{i+1}\| - \lambda^2(1-\lambda)^{-1}(n+1)\|y_i - x_i\| \\ &= (1-\lambda)^{-(n+1)}[\|y_{i+n+1} - x_{i+n+1}\| - \|y_i - x_i\|] \\ &+ [(1-\lambda)^{-1}(1+n\lambda) - (1-\lambda)^{-(n+1)}]\|y_{i+1} - x_{i+1}\| \\ &+ [(1-\lambda)^{-(n+1)} - \lambda^2(1-\lambda)^{-1}(n+1)]\|y_i - x_i\|, \end{aligned} \tag{2.21}$$

by adding and substraction the same term $(1-\lambda)^{-(n+1)}\|y_{i+1} - x_{i+1}\|$. Notice that $(1-\lambda)^{-1}(1+n\lambda) - (1-\lambda)^{-(n+1)} \le 0$. Thus, regarding this observation together with Lemma 2.7, the inequality (2.21) changed into

$$\begin{aligned}\|y_{i+n+1} - x_i\| \geq (1-\lambda)^{-(n+1)}[\|y_{i+n+1} - x_{i+n+1}\| - \|y_i - x_i\|] \\ + [(1-\lambda)^{-1}(1+n\lambda) - (1-\lambda)^{-(n+1)}]\|y_i - x_i\| \\ + [(1-\lambda)^{-(n+1)} - \lambda^2(1-\lambda)^{-1}(n+1)]\|y_i - x_i\| \\ \\ = (1-\lambda)^{-(n+1)}[\|y_{i+n+1} - x_{i+n+1}\| - \|y_i - x_i\|] \\ + (1+(n+1)\lambda)\|y_i - x_i\|\end{aligned}$$

which completes the proof of Proposition 2.8. □

Theorem 2.9. *Let K be a nonempty, convex and compact subset of an ordered Banach space $(X, \leq)$. Let $T : K \to K$ be a monotone Reich type $\lambda - \alpha$-nonexpansive mapping with $\lambda \in (\frac{1}{3}, 1)$. For $x_1 \in K$ with $x_1 \leq T(x_1)$, we set $y_n = T(x_n)$ where $\{x_n\}$ is the iteration sequence defined by (2.2) in K satisfies the assumption (2.4). Then $\{x_n\}$ converges to some $x \in K$ with $x_n \leq x$ and,*

$$\lim_n \|x_n - T(x_n)\| = 0 \tag{2.22}$$

Proof. We shall divide the proof in two cases: $\alpha = 0$ and $\alpha \in (0, 1)$. Suppose, first, that $\alpha = 0$. Due to the definition (2.2) of the sequence $\{x_n\}$, we have

$$\lambda\|x_n - y_n\| = \|x_n - x_{n+1}\|, \text{ for all } n \geq 1.$$

On account of Lemma 2.6, we have $x_n \leq x_{n+1}$, for all $n \geq 1$. Therefore condition (2.1) implies that,

$$\|T(x_n) - T(x_{n+1})\| = \|y_n - y_{n+1}\| \leq R_T^\alpha(x_n, x_{n+1}) = \|x_n - x_{n+1}\|,$$

since $\alpha = 0$. Employing Lemma 1.3, the inequality above yields that

$$\lim_n \|x_n - T(x_n)\| = 0.$$

In the following, we shall consider the second case $\alpha \in (0, 1)$. The proof of this case mainly adopted from the proof of Theorem 3.1 in [15]. Since K is compact, there exists a subsequence of $\{x_n\}$ which converges to $x \in K$. On account of Lemma 2.6, the sequence $\{x_n\}$ converges to x and $x_n \leq x$, for $n \geq 1$. To show our assertion (2.22), suppose, on the contrary, that

$$\lim_n \|x_n - T(x_n)\| = R > 0.$$

As $x_1 \leq x_n \leq x$, we then have

$$\|x_n - x_1\| \leq \|x - x_1\| \text{ for all } n \geq 1. \tag{2.23}$$

Due to triangle inequality we have

$$\begin{aligned}\|y_{i+n} - x_i\| = \|T(x_{i+n}) - x_i\| \leq \|T(x_{i+n}) - x_{i+n}\| + \|x_{i+n} - x_1\| + \|x_1 - x_i\| \\ \leq \|T(x_1) - x_1\| + 2\|x - x_1\|\end{aligned} \tag{2.24}$$

for any $i, n \geq 1$, due to (2.23) and Lemma 2.7. Since all conditions are satisfied in Proposition 2.8, we have (2.12). Letting $i \to \infty$ in the inequality (2.12), we derive that

$$\lim_{i\to\infty} \|y_{i+n} - x_i\| \geq (1 + n\lambda)R, \tag{2.25}$$

where we used that

$$\lim_{i\to\infty} (\|T(x_i) - x_i\| - \|T(x_{i+n}) - x_{i+n}\|) = R - R = 0,$$

for any $n \geq 1$. Combining (2.24) and (2.25), we find

$$(1 + n\lambda)R \leq \lim_{i\to\infty} \|y_{i+n} - x_i\| \leq \|T(x_1) - x_1\| + 2\|x - x_1\|$$

Thus, the inequality can be fulfilled only if $R = 0$ which yields the inequality (2.22). □

Lemma 2.10. *Let K be a nonempty subset of an ordered Banach space $(X, \leq)$ and $T : K \to K$ be a monotone Reich type $(\lambda - \alpha)$-nonexpansive mapping with $\lambda \in]0, \frac{1}{2}]$. Then for each $x, y \in K$ with $x \leq y$:*

(i) $\|T(x) - T^2(x)\| \leq \|x - T(x)\|$

(ii) *either* $\lambda\|x - T(x)\| \leq \|x - y\|$ *or* $\lambda\|T(x) - T^2(x)\| \leq \|T(x) - y\|$

(iii) *either* $\|T(x) - T(y)\| \leq \alpha\|T(x) - y\| + \alpha\|x - T(y)\| + (1 - 2\alpha)\|x - y\|$

or $\|T^2(x) - T(y)\| \leq \alpha\|T(x) - T(y)\| + \alpha\|T^2(x) - y\| + (1 - 2\alpha)\|T(x) - y\|$

Proof. (i) Since we have $\lambda\|x - T(x)\| \leq \|x - T(x)\|$ for all $\lambda \in]0, \frac{1}{2}]$, by the definition of Reich type $(\lambda - \alpha)$-nonexpansive mapping we get the desired results. Indeed,

$$\|T(x) - T^2(x)\| \leq \alpha\|x - T^2(x)\| + (1 - 2\alpha)\|x - T(x)\|.$$

Thus (i) hold for $\alpha = 0$.

(ii) Suppose, on the contrary, that $\lambda\|x - T(x)\| > \|x - y\|$ and $\|T(x) - T^2(x)\| > \|T(x) - y\|$. Then, by triangle inequality together with the assumption (i), we find that

$$\begin{aligned}\|x - T(x)\| \leq \|x - y\| + \|T(x) - y\| &< \lambda\|x - T(x)\| + \lambda\|T(x) - T^2(x)\| \\ &\leq 2\lambda\|x - T(x)\|.\end{aligned}$$

Since $\lambda \leq \frac{1}{2}$ we obtain $\|x - T(x)\| < \|x - T(x)\|$ which is a contradiction. Thus, we obtain the desired result.

(iii) The proof of (iii) follows from (ii). We skip the details.

□

Lemma 2.11. *Let K be a nonempty subset of an ordered Banach space $(X, \leq)$ and $T : K \to K$ be a monotone Reich type $(\lambda-\alpha)$-nonexpansive mapping with $\lambda \in (0, \frac{1}{2}]$. Then for each $x, y \in K$ with $x \leq y$,*

$$\|x - T(y)\| \leq (\frac{3+\alpha}{1-\alpha})\|x - T(x)\| + \|x - y\|.$$

Proof. It is the mimic of the proof of Lemma 3.8 of [19]. So, we skip the details. □

Using the above two lemmas, we can prove the following.

Theorem 2.12. *Let K be a nonempty convex and a compact subset of an ordered Banach space $(X, \leq)$ and be $T : K \to K$ a monotone Reich type $(\lambda-\alpha)$-nonexpansive mapping with $\lambda \in (\frac{1}{3}, \frac{1}{2}]$. Select $x_1 \in K$ such that $x_1 \leq T(x_1)$, and for $n \geq 1$, denote $y_n = T(x_n)$ where $\{x_n\}$ is the iteration sequence defined by (2.2) in K satisfying, for all $n \in \mathbb{N}$, the assumption (2.4) . Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Theorem 2.9, we have

$$\lim_n \|x_n - T(x_n)\| = 0.$$

Since K is compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in K$ such that $\{x_{n_k}\}$ converges to z. Employing Lemma 2.11, we have,

$$\|x_{n_k} - T(z)\| \leq (\frac{3+\alpha}{1-\alpha})\|x_{n_k} - T(x_{n_k})\| + \|x_{n_k} - z\|$$

for all $k \in \mathbb{N}$. Thus, the sequence $\{x_{n_k}\}$ converges to $T(z)$ and hence $T(z) = z$. Since z is a fixed point of T, by Proposition 2.3, we find that

$$\|x_{n+1} - z\| \leq \lambda\|T(x_n) - z\| + (1-\lambda)\|x_n - z\| \leq \|x_n - z\|$$

for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ converges to z . □

We say that a Banach space X has the Opial property [18] if for every weakly convergent sequence $\{x_n\}$ in X with a limit z, fulfils

$$\liminf_{n\to\infty} \|x_n - z\| < \liminf_{n\to\infty} \|x_n - y\|,$$

for all $y \in X$ with $y \neq z$. It is a very rich class, for examples, all Hilbert spaces, sequence spaces $\ell_p, (1 < p < \infty)$, and finite dimensional Banach spaces have the Opial property. Unexpectedly, $L_p[0, 2\pi], (p \neq 2)$ do not have the Opial property [9],[10].

Proposition 2.13. *Let K be a nonempty subset of an ordered Banach space $(X, \leq)$ with the Opial property and $T : K \to K$ be a monotone Reich type $(\lambda-\alpha)$-nonexpansive mapping with $\lambda \in (\frac{1}{3}, \frac{1}{2}]$. If $\{x_n\}$ converges weakly to z and*

$$\lim_n \|x_n - T(x_n)\| = 0,$$

then $T(z) = z$.

Proof. By Lemma 2.11, we have,

$$\|x_n - T(z)\| \le (\frac{3+\alpha}{1-\alpha})\|x_n - T(x_n)\| + \|x_n - z\|$$

for $n \in \mathbb{N}$ and hence,

$$\liminf_n \|x_n - T(z)\| \le \liminf_n \|x_n - z\|$$

We claim that $T(z) = z$. Indeed, if $T(z) \neq z$, the Opial property implies,

$$\liminf_n \|x_n - z\| < \liminf_n \|x_n - T(z)\|$$

which is a contradiction with inequality (2.22) . □

Theorem 2.14. *Let K be a nonempty convex and weakly compact subset of an ordered Banach space $(X, \le)$ with the Opial property and $T : K \to K$ be a monotone Reich type $(\lambda - \alpha)$-nonexpansive mapping with $\lambda \in (\frac{1}{3}, \frac{1}{2}]$. . Select $x_1 \in K$ such that $x_1 \le T(x_1)$, and for $n \ge 1$, denote $y_n = T(x_n)$ where $\{x_n\}$ is the iteration sequence defined by (2.2) in K satisfying, for all $n \in \mathbb{N}$, the assumption (2.4) . Then $\{x_n\}$ converges weakly to a fixed point of T .*

Proof. By Theorem 2.9, we have

$$\lim_n \|x_n - T(x_n)\| = 0.$$

Since K is weakly compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in K$ such that $\{x_{n_k}\}$ converges weakly to z . By Proposition 2.13, we deduce that z is a fixed point of T. As in the proof of Theorem 2.12, we can prove that $\{\|x_n - z\|\}$ is a nonincreasing sequence. We prove our assertion by *reductio de absurdum*. Suppose, on the contrary, that $\{x_n\}$ does not converge to z. Then there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to ω and $\omega \neq z$. We note that $T(\omega) = \omega$. From the Opial property,

$$\begin{aligned}\lim_n \|x_n - z\| &= \lim_k \|x_{n_k} - z\| < \lim_k \|x_{n_k} - \omega\| = \lim_n \|x_n - \omega\| \\ &= \lim_j \|x_{n_j} - \omega)\| < \lim_j \|x_{n_j} - z\| = \lim_n \|x_n - z\|,\end{aligned}$$

a contradiction that complete the proof. □

The following theorem directly follows from Theorems 2.12 and 2.14. So, to avoid the repetition, we skip the details.

Theorem 2.15. *Let K be a convex subset of an ordered Banach space $(X, \le)$, and $T : K \to K$ be a monotone Reich type $(\lambda - \alpha)$-nonexpansive mapping with $\lambda \in (\frac{1}{3}, \frac{1}{2}]$. Assume that either of the following holds:*

(i) *K is compact;*
(ii) *K is weakly compact and X has the Opial property.*

Then T has a fixed point.

Finally, we will give a generalization of a fixed point theorem due to Browder [5], Göhde [11] and Suzuki [21].

Theorem 2.16. *Let K be a convex and weakly compact subset of a UCED ordered Banach space $(X, \leq)$. Let $T : K \to K$ be a monotone Reich type $(\lambda - \alpha)$-nonexpansive mapping with $\lambda \in (\frac{1}{3}, \frac{1}{2}]$. Then T has a fixed point.*

Proof. We construct an iterative sequence $\{x_n\}$ in K by starting $x_1 \in K$ as

$$x_{n+1} = \frac{1}{2}(T(x_n) + x_n) \text{ with } \|T(x_{n+1}) - x_n\| \leq \|T(x_n) - x_n\|,$$

for all $n \in \mathbb{N}$. Then by Theorem 2.9, we have

$$\lim_n \|x_n - T(x_n)\| = 0$$

holds. Define a continuous convex function f from K to $[0, +\infty)$ by

$$f(x) = \limsup_n \|x_n - x\|$$

for all $x \in K$. Since K is weakly compact and f is weakly lower semicontinuous, there exists $z \in K$, such that

$$f(z) = \min\{f(x) : x \in K\}$$

Since, by Lemma 2.11:

$$\|x_n - T(z)\| \leq (\frac{3+\alpha}{1-\alpha})\|x_n - T(x_n)\| + \|x_n - z\|,$$

we then have, $f(T(z)) \leq f(z)$. Since $f(z)$ is the minimum, $f(T(z)) = f(z)$ holds. If $T(z) \neq z$, then since f is strictly quasiconvex (Lemma 1.2) we have,

$$f(z) \leq f(\frac{z + f(z)}{2}) < \max\{f(z), f(T(z))\} = f(z).$$

which is a contradiction. Hence $T(z) = z$. □

3. Application to $L_1([0, 1])$

As an application, we consider $L_1([0, 1])$ the Banach space of real valued functions defined on $[0, 1]$ with absolute value Lebesgue integrable, i.e., $\int_0^1 |f(x)|dx < \infty$.

We recall some definitions which can be found in e.g. [3]. As usual, $f = 0$ if and only if the set $\{x \in [0, 1] : f(x) = 0\}$ has Lebesgue measure 0, then, we say $f = 0$ almost everywhere. An element of $L_1([0, 1])$ is therefore seen as a class of functions. The norm of any $f \in L_1([0, 1])$ is given by

$$\|f\| = \int_0^1 |f(x)|dx$$

From now on, we will write L_1 instead of $L_1([0, 1])$. Recall that $f \leq g$ if and only if $f(x) \leq g(x)$ almost everywhere, for any $f, g \in L_1$. We adopt the convention $f \leq g$ if and only if $g \leq f$. We remark that order intervals are closed for convergence almost everywhere and convex. Recall that an order interval is a subset of the form

$$[f, \to) = \{g \in L_1 \ : \ f \leq g\} \text{ or } (\leftarrow, f] = \{g \in L_1 \ : \ g \leq f\},$$

for any $f \in L_1$.
As a direct consequence of this, the subset

$$[f,g] = \{h \in L_1 : f \leq h \leq g\} = [f,\rightarrow) \cap (\leftarrow, g]$$

is closed and convex, for any $f, g \in L_1$.

Let K be a nonempty subset of L_1 which is equipped with a vector order relation $\leq$. A map $T : K \to K$ is called monotone if for all $f \leq g$ we have $T(f) \leq T(g)$.

Remark 3.1. Since $L_1([0,1])$ fails to be uniformly convex, Theorem 2.12 can't not be used to get a fixed point result for monotone generalized $\lambda-\alpha$ nonexpansive mappings in $L_1([0,1])$. As an alternative, we will use an interesting property for the convergence almost everywhere contained in the following lemma.

Lemma 3.2 ([4])**.** *If (f_n) is a sequence of uniformly L^p -bounded functions on a measure space, and if $f_n \to f$ almost everywhere, then*

$$\liminf_n \|f_n\|_p^p = \liminf_n \|f_n - f\|_p^p + \|f\|_p^p$$

for all $0 < p < \infty$.

In particular, this result holds when $p = 1$.

On account of Lemma 2.11 and Lemma 3.2, we shall prove the following.

Theorem 3.3. *Let $K \subset L_1$ be nonempty, convex and compact for the convergence almost everywhere. Let $T : K \to K$ be a monotone Reich type $(\lambda - \alpha)$-nonexpansive mapping with $\alpha \in]\frac{1}{3}, \frac{1}{2}]$. Select $f_1 \in K$ such that $f_1 \leq T(f_1)$,, and for $n \geq 1$, denote $g_n = T(f_n)$ where (f_n) is the iteration sequence defined by (2.2) in K satisfying, for all $n \in \mathbb{N}$, the assumption (2.4) . Then the sequence (f_n) converges almost everywhere to some $f \in K$ which is a fixed point of T, i.e., $T(f) = f$. Moreover, $f_1 \leq f$.*

Proof. Theorem 2.9 implies that (f_n) converges almost everywhere to some $f \in K$ where $f_n \to f$, for any $n \geq 1$. Since (f_n) is uniformly bounded, lemma 3.2 [4] implies

$$\liminf_n \|f_n - T(f)\| = \liminf_n \|f_n - f\| + \|f - T(f)\|$$

Theorem 2.9 implies

$$\liminf_n \|f_n - T(f_n)\| = 0.$$

Therefore we get

$$\liminf_n \|f_n - T(f)\| = \liminf_n \|f_n - f\| + \|f - T(f)\|$$

On the other hand, we know that each $f_n \leq f$ for each $n \geq 1$, so, by assumption (2.1), we have,

$$\liminf_n \|f_n - f\| + \|f - T(f)\| \leq \liminf_n (\alpha\|f_n - T(f)\| + \alpha\|T(f_n) - f\| + (1-2\alpha)\|f_n - f\|)$$

And, by Lemma 2.11, we have,

$$\liminf_n \|f_n - f\| + \|f - T(f)\| \leq \liminf_n (\alpha \frac{3+\alpha}{1-\alpha} \|f_n - T(f)\| + \alpha \|T(f_n) - f\| + (1-2\alpha)\|f_n - f\|)$$

Again, by application of the Theorem 2.9, we obtain,

$$\liminf_n \|f_n - f\| + \|f - T(f)\| \leq \liminf_n (1-\alpha)\|f_n - T(f)\| + \alpha \|T(f_n) - f\|)$$

And like,

$$\liminf_n \|f_n - f\| = \|T(f_n) - f\|$$

we then have,

$$\liminf_n \|f_n - f\| + \|f - T(f)\| \leq \liminf_n \|f_n - T(f)\|,$$

that implies

$$\|f - T(f)\| = 0$$

or

$$T(f) = f.$$

□

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