ON WEIGHTED L_p–SPACES OF VECTOR–VALUED ENTIRE ANALYTIC FUNCTIONS

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Dedicated to Prof. Manuel Valdivia

Abstract. The weighted L_p–spaces of entire analytic functions are generalized to the vector–valued setting. In particular, it is shown that the dual of the space $L^K_{p,\rho}(E)$ is isomorphic to $L^{-K}_{p',\rho^{-1}}(E')$ when the function χ_K is an $L_{p,\rho}(E)$–Fourier multiplier. This result allows us to give some new characterizations of the so–called UMD–property, and to represent several ultradistribution spaces by means of spaces of vector sequences.

1. Introduction

It is well-known that the spaces $L^K_p = \{f \in S' : \operatorname{supp} \hat{f} \subset K\, , \|f\|_p < \infty\}$ ($0 < p \leq \infty$, K compact subset of $\mathbb{R}^n$) play a crucial role in the theory of function spaces (cf. [27] and [40]). If $0 < p \leq q \leq \infty$, K is a compact set in $\mathbb{R}^n$, and α is an arbitrary multi–index, then there is a constant $c > 0$ such that $\|\partial^\alpha f\|_q \leq c\,\|f\|_p$ for all $f \in L^K_p$. These are the Plancherel–Polya–Nikol'skij inequalities (cf. [27] for $p \geq 1$, and [40] for $0 < p \leq \infty$). In [38] and [33] these inequalities are extended to the weighted case by using Beurling's ultradistributions (for some exponential weights, e.g. $e^{\pm|x|^\beta}$, $0 < \beta < 1$, the theory of the usual tempered distributions $S'(\mathbb{R}^n)$ is inadequate), and the theory of the weighted L_p–spaces of entire analytic functions is developed.

In this paper the weighted L_p–spaces of entire analytic functions are generalized to the vector–valued setting and several applications to the geometry of Banach spaces and to the representation of function spaces are given (cf. also [2], [25], [32] and [41]). The organization of the paper is as follows. Section 2 contains some basic facts about vector–valued (Beurling) ultradistributions. In Section 3 we introduce the weighted L_p–spaces of vector–valued entire analytic functions $L^K_{p,\rho}(E)$ (see Def. 3.1) and we study their basic properties: E–valued maximal inequalities and Plancherel–Polya–Nikol'skij inequalities, completeness, approximation and density. Section 4 contains a discussion of the dual of the space $L^K_{p,\rho}(E)$. Here we prove that the natural mapping $N : L^{-K}_{p',\rho^{-1}}(E') \to \left(L^K_{p,\rho}(E)\right)' : g \to \langle f, Ng\rangle = \int_{\mathbb{R}^n} \langle f, g\rangle dx$ becomes an isomorphism when $p \in (1, \infty)$ and χ_K is an $L_{p,\rho}(E)$–Fourier multiplier (Th. 4.6 and Cor. 4.8). As a consequence we give some new characterizations of the so–called UMD–property (e.g. $E \in$ UMD if and only if E is reflexive and

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$L_p^Q(E)$ is a complemented subspace of $L_p(E)$, c.f. Cor. 4.7). By using a vector version of the Shannon sampling theorem (see also [12] and [38]), the inequalities (3.4) of Theorem 3.2 and the duality studied in Theorem 4.6, we represent weighted L_p–spaces of vector–valued entire analytic functions by means of spaces of vector sequences in Section 5. Finally, some other distribution spaces (Hörmander, Besov) are represented by using sequence spaces also.

Notation. The linear spaces we use are defined over $\mathbb{C}$. Let E and F be locally convex spaces. Then $L_b(E,F)$ is the locally convex space of all continuous linear operators equipped with the bounded convergence topology. The dual of E is denoted by E' and is given the strong topology so that $E' = L_b(E,\mathbb{C})$. $E\widehat{\otimes}_\varepsilon F$ (resp. $E\widehat{\otimes}_\pi F$) is the completion of the injective (resp. projective) tensor product of E and F. E and F are (topologically) isomorphic if there exists a one–to–one linear operator Φ mapping E onto F and such that Φ and Φ^{-1} are continuous operators. We write $E \hookrightarrow F$ if E is a linear subspace of F and the canonical injection is continuous. We replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$ if E is also dense in F. If $(E_n)_{n=1}^\infty$ is a sequence of locally convex spaces, $E_1 \oplus E_2 \oplus E_3 \oplus \cdots$ ($E^{(\mathbb{N})}$ if $E_n = E$ for all n) is the locally convex direct sum of the spaces E_n. C^∞, D, S, D' and S' have the usual meaning. A is the space of entire analytic functions in $\mathbb{C}^n$. In the E–valued case we write $C^\infty(E)$, $D(E)$, $S(E)$, $D'(E)$, $S'(E)$ and $A(E)$ (see [14] and [35]). Let $0 < p \le \infty$, $\rho : \mathbb{R}^n \to (0,\infty)$ a locally integrable function, and E a Banach space. Then $L_p(E)$ is the set of all Bochner measurable functions $f : \mathbb{R}^n \to E$ for which $\|f\|_p = \left(\int_{\mathbb{R}^n} \|f(x)\|_E^p\,dx\right)^{1/p}$ is finite (with the usual modification when $p = \infty$). $L_\infty^c(E)$ stands for all functions $f \in L_\infty(E)$ with compact support. $L_{p,\rho}(E)$ denotes the set of all Bochner measurable functions $f : \mathbb{R}^n \to E$ such that $\rho f \in L_p(E)$. Putting $\|f\|_{p,\rho} = \|\rho f\|_p$ for $f \in L_{p,\rho}(E)$, $L_{p,\rho}(E)$ becomes a quasi–Banach space (Banach space if $p \ge 1$) isomorphic to $L_p(E)$. When E is the field $\mathbb{C}$, we simply write L_p and $L_{p,\rho}$. If $f \in L_1(E)$ the Fourier transform of f, $\hat{f}$ or $\mathcal{F}f$, is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i\xi x}dx$. If f is a function on $\mathbb{R}^n$ then $\tilde{f}(x) = f(-x)$, $(\tau_h f)(x) = f(x-h)$ for $x, h \in \mathbb{R}^n$.

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2. Spaces of vector–valued (Beurling) ultradistributions

In this section we collect some basic facts about vector–valued (Beurling) ultradistributions. The results are "elementary" in the sense that the usual "scalar proofs" carry over to the vector–valued setting by using obvious modifications only. Comprehensive treatments of the theory of (scalar or vector–valued) ultradistributions can be found in [3], [15], [19] and [20]. Our notations are based on [3] and [33, pp. 14–19].

Let $\mathcal{M}$ be the set of all functions $\omega(x)$ on $\mathbb{R}^n$ such that $\omega(x) = \sigma(|x|)$ where $\sigma(t)$ is an increasing continuous concave function on $[0,\infty[$ with the following properties:

(i) $\sigma(0) = 0$,

(ii) $\int_0^\infty \frac{\sigma(t)}{1+t^2}dt < \infty$ (Beurling's condition),

(iii) there exist a real number a and a positive number b such that $\sigma(t) \geq a + b\log(1+t)$ for $t \geq 0$.

The assumption (ii) is essentially the Denjoy–Carleman non–quasi–analyticity condition (cf. [3, Sect. 1.5]). If $\omega \in \mathcal{M}$ and E is a Banach space, we denote by $D_\omega(E)$ the set of all functions $f \in L_1(E)$ with compact support such that $\|f\|_\lambda = \int_{\mathbb{R}^n} \|\hat{f}(x)\|_E \, e^{\lambda\omega(x)}dx < \infty$ for all $\lambda > 0$. For each compact subset K of $\mathbb{R}^n$, $D_\omega(K, E) = \{f \in D_\omega(E) : \operatorname{supp} f \subset K\}$, equipped with the topology induced by the family of norms $\{\|\cdot\|_\lambda : \lambda > 0\}$, is a Fréchet space and $D_\omega(E) = \operatorname{ind}_{\underset{K}{\to}} D_\omega(K, E)$ becomes a strict (LF)–space. Let $S_\omega(E)$ be the set of all functions $f \in L_1(E)$ such that both f and $\hat{f}$ are infinitely differentiable functions on $\mathbb{R}^n$ with $\vec{p}_{\alpha,\lambda}(f) = \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)}\|\partial^\alpha f(x)\|_E < \infty$ and $\vec{\pi}_{\alpha,\lambda}(f) = \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)}\|\partial^\alpha(\mathcal{F}f)(x)\|_E < \infty$ for all multi–indices α and all positive numbers λ. $S_\omega(E)$ with the topology induced by the family of seminorms $\{\vec{p}_{\alpha,\lambda}, \vec{\pi}_{\alpha,\lambda}\}$ is a Fréchet space and the Fourier transformation $\mathcal{F}$ is an automorphism of $S_\omega(E)$. If $E = \mathbb{C}$ then $D_\omega(E)$ and $S_\omega(E)$ coincide with the spaces D_ω and S_ω (cf. [3]). In this case we write $p_{\alpha,\lambda}$ and $\pi_{\alpha,\lambda}$ instead of $\vec{p}_{\alpha,\lambda}$ and $\vec{\pi}_{\alpha,\lambda}$. Let us recall that, by Beurling's condition, the space D_ω is non-trivial and the usual procedure of the partition of unity can be established with D_ω–functions (cf. [3, Th. 1.3.7]). Furthermore, $D_\omega \overset{d}{\hookrightarrow} D$ (cf. [3, Th. 1.3.18]) and D_ω is nuclear (cf. [42, Cor. 7.5]). On the other hand, $D_\omega = D \cap S_\omega$, $D_\omega \overset{d}{\hookrightarrow} S_\omega \overset{d}{\hookrightarrow} S$ (cf. [3, Prop. 1.8.6, Th. 1.8.7]) and S_ω is a nuclear space (cf. [15, p. 320]). Using the above results and [20, Th. 1.12] we can identify $D_\omega(E)$ with $D_\omega\widehat{\otimes}_\varepsilon E$ and $S_\omega(E)$ with $S_\omega\widehat{\otimes}_\varepsilon E$. A continuous linear operator from D_ω into E is said to be a (Beurling) ultradistribution with values in E. We write $D'_\omega(E)$ for the space of all E–valued (Beurling) ultradistributions endowed with the bounded convergence topology, thus $D'_\omega(E) = L_b(D_\omega, E)$ is isomorphic to $D'_\omega\widehat{\otimes}_\varepsilon E$. A continuous linear operator from S_ω into E is said to be an E–valued tempered ultradistribution. $S'_\omega(E)$ is the space of all E–valued (tempered) ultradistributions equipped with the bounded convergence topology. Also, $S'_\omega(E) = L_b(S_\omega, E)$ is isomorphic to $S'_\omega\widehat{\otimes}_\varepsilon E$ and the Fourier transformation $\mathcal{F}$ is an automorphism of $S'_\omega(E)$.

Next we recall the definition of $R(\omega)$ given in [38, Def. 1.3.1]. If $\omega \in \mathcal{M}$, then $R(\omega)$ denotes the collection of all Borel–measurable real functions $\rho(x)$ on $\mathbb{R}^n$ such that there exists a positive constant c with $0 < \rho(x) \leq c\, e^{\omega(x-y)}\rho(y)$ for all $x, y \in \mathbb{R}^n$. If $\rho \in R(\omega)$, then $c_1\, e^{-\omega(x)} \leq \rho(x) \leq c_2\, e^{\omega(x)}$ for all $x \in \mathbb{R}^n$ (here c_1 and c_2 are appropriate positive numbers). Two very interesting examples are $\rho(x) = (1+|x|)^d \in R(\log(1+|x|)^d)$, $d > 0$, and $\rho(x) = e^{d|x|^\beta} \in R(|x|^\beta)$, $d \in \mathbb{R} \setminus \{0\}$, $0 < \beta < 1$. If $u \in L_1^{\mathrm{loc}}$ and $\int_{\mathbb{R}^n} \varphi(x)u(x)dx = 0$ for all $\varphi \in D_\omega$, then $u = 0$ a.e. (see [3]). This result, the Hahn–Banach theorem and [9, Cor. II.27] prove that if $\rho \in R(\omega)$ and $p \in [1, \infty]$ we can identify $f \in L_{p,\rho}(E)$ with the E–valued tempered ultradistribution $\varphi \to \langle \varphi, f\rangle = \int_{\mathbb{R}^n} \varphi(x)f(x)dx$, $\varphi \in S_\omega$. Summarizing, we have the embeddings

$$\begin{tikzcd}
D_\omega(E) \arrow[r, hook, "d"] \arrow[d, hook, "d"] & S_\omega(E) \arrow[r, hook, "d"] \arrow[d, hook, "d"] & S'_\omega(E) \arrow[r, hook, "d"] & D'_\omega(E) \\
D(E) \arrow[r, hook, "d"] & S(E) \arrow[r, hook, "d"] & S'(E) \arrow[r, hook, "d"] \arrow[u, hook, "d"] & D'(E) \arrow[u, hook, "d"]
\end{tikzcd}$$

(commutative diagrams) and, when $1 \le p < \infty$,

$$S_\omega(E) \overset{d}{\hookrightarrow} L_{p,\rho}(E) \overset{d}{\hookrightarrow} S'_\omega(E) .$$

For $\varphi \in S_\omega$, $T \in S'_\omega(E)$ and $\psi \in S_\omega$, we define $\langle \psi, \varphi T\rangle = \langle \psi\varphi, T\rangle$. The "point–wise multiplication" $S_\omega \times S'_\omega(E) \to S'_\omega(E) : (\varphi, T) \mapsto \varphi T$ is an hypocontinuous bilinear mapping (by [15, p. 320] and [34, p. 424]). If $\varphi \in S_\omega$ and $T \in S'_\omega(E)$, we define $\varphi * T(x) = \langle \tau_x\tilde{\varphi}, T\rangle$, $x \in \mathbb{R}^n$. The function $\varphi * T : \mathbb{R}^n \to E$ is called the convolution of φ and T. $\varphi * T \in C^\infty(E)$ and, for every multi–index α, there exist positive constants C_α and Λ_α such that $\|\partial^\alpha(\varphi * T)(x)\|_E = \|(\partial^\alpha\varphi) * T(x)\|_E \le C_\alpha\, e^{\Lambda_\alpha \omega(x)}$ for all $x \in \mathbb{R}^n$. Thus, we can identify $\varphi * T$ with the E–valued tempered ultradistribution $\psi \to \langle \psi, \varphi * T\rangle = \int_{\mathbb{R}^n} \psi(x)(\varphi * T)(x)\, dx$, $\psi \in S_\omega$. The bilinear mapping $S_\omega \times S'_\omega(E) \to S'_\omega(E) : (\varphi, T) \mapsto \varphi * T$ is hypocontinuous also ([15, p. 320] and [34, p. 424]). One easily checks that

$$\langle \psi, \varphi * T\rangle = \langle \tilde{\varphi} * \psi, T\rangle \ , \quad (\varphi * T)^\wedge = \hat{\varphi}\,\hat{T} \ , \quad (\varphi T)^\wedge = (2\pi)^{-n}(\hat{\varphi} * \hat{T}) \ ,$$

for all $\varphi, \psi \in S_\omega$ and all $T \in S'_\omega(E)$.

We now state the vector–valued version of the Paley–Wiener–Schwartz theorem (cf. [3, Th. 1.8.14], [19, Th. 1.1] and [33, pp. 18–19] for the scalar case) that we shall use: If $T \in S'_\omega(E)$ and $\operatorname{supp}\hat{T} \subset \bar{B}_b$ then there exist an E–valued entire analytic function $U(\zeta)$ and a real number λ such that for any $\varepsilon > 0$

$$\|U(\xi + i\eta)\|_E \le C_\varepsilon\, e^{(b+\varepsilon)|\eta| + \lambda\omega(\xi)}$$

holds for all $\zeta = \xi + i\eta$ where C_ε depends on ε but not on ζ ($U(\zeta)$ is called an E–valued entire function of exponential type) and such that U represents T, i.e., such that $\langle \varphi, T\rangle = \int_{\mathbb{R}^n} \varphi(x)U(x)dx$ for all $\varphi \in S_\omega$.

3. Weighted L_p–spaces of vector–valued entire analytic functions. Basic properties

In this section we introduce the weighted L_p–spaces of vector–valued entire analytic functions $L^K_{p,\rho}(E)$ (see Def. 3.1) and we study some of their properties: E–valued maximal inequalities and Plancherel–Polya–Nikol'skij inequalities, completeness, approximation, density, ... In order to extend scalar assertions to vector–valued ones we follow [38, Ch. I], [33, Ch. I] and [41, Sect. 15, Ch. III].

We begin with the vector–valued counterpart of [38, Def. 1.4.1] and [33, Def. 1.5.1,p. 35].

Definition 3.1. Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$, $0 < p \le \infty$. Let K be a compact set in $\mathbb{R}^n$. Let E be a Banach space. Then

$$L^K_{p,\rho}(E) = \{ f \mid f \in S'_\omega(E) \,,\ \operatorname{supp}\mathcal{F}f \subset K \,,\ \|f\|_{L^K_{p,\rho}(E)} = \|f\|_{p,\rho} < \infty \} \ .$$

$(L^K_{p,\rho}(E)\,, \|\cdot\|_{L^K_{p,\rho}(E)})$ is a quasi–normed (normed if $p \ge 1$) linear space.

Remark. We shall write $L^K_{p,\rho}$ instead of $L^K_{p,\rho}(\mathbb{C})$. It is immediate to verify that if $f \in L^K_{p,\rho}(E)$ and $e' \in E'$ then $e' \circ f \in L^K_{p,\rho}$. If $\rho(x) \equiv 1$, then we put $L^K_{p,1}(E) = L^K_p(E)$. We shall denote by $S^K_\omega(E)$ (S^K_ω if $E = \mathbb{C}$) the collection of all $f \in S_\omega(E)$ such that $\operatorname{supp}\hat{f} \subset K$.

Theorem 3.2. *Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$ and $0 < p \le \infty$. Let K be a compact set in $\mathbb{R}^n$. Let E be a Banach space.*

(i) *Let $0 < r < \infty$. Then there exist two positive numbers c_1 and c_2 such that for all $f \in L^K_{p,\rho}(E)$ and for all $x \in \mathbb{R}^n$*

$$\sup_{z\in\mathbb{R}^n} \rho(x-z)\frac{\|\nabla f(x-z)\|_E}{1+|z|^{n/r}} \le c_1 \sup_{z\in\mathbb{R}^n} \rho(x-z)\frac{\|f(x-z)\|_E}{1+|z|^{n/r}} \le \le c_2 \left[\left(M\|\rho f\|^r_E\right)(x)\right]^{1/r} . \tag{3.1}$$

(ii) *Let $0 < r < p$. Then there exists a positive number c such that*

$$\left\|\sup_{z\in\mathbb{R}^n} \rho(\cdot-z)\frac{\|f(\cdot-z)\|_E}{1+|z|^{n/r}}\right\|_p \le c\,\|f\|_{p,\rho} \tag{3.2}$$

holds for all $f \in L^K_{p,\rho}(E)$.

(iii) *(Plancherel–Polya–Nikol'skij inequalities). Let $p \le q \le \infty$ and let α be a multi–index. Then there exists a positive number c such that*

$$\|\partial^\alpha f\|_{q,\rho} \le c\,\|f\|_{p,\rho} \tag{3.3}$$

holds for all $f \in L^K_{p,\rho}(E)$.

(iv) *There exist three positive numbers h_0, c_1, c_2 such that*

$$c_1 \left\|(\rho(x^k)f(x^k))\right\|_{l_p(\mathbb{Z}^n,E)} \le h^{-n/p}\|f\|_{p,\rho} \le \le c_2 \left\|(\rho(x^k)f(x^k))\right\|_{l_p(\mathbb{Z}^n,E)} \tag{3.4}$$

holds for all h with $0 < h \le h_0$, all sets $\{x^k\}_{k\in\mathbb{Z}^n}$ with $x^k \in Q^h_k = \prod_{j=1}^n [hk_j, h(k_j+1)[$ and all $f \in L^K_{p,\rho}(E)$.

(v) *If $p \le q \le \infty$ we have the topological embeddings $S^K_\omega \hookrightarrow L^K_{p,\rho}(E) \hookrightarrow L^K_{q,\rho}(E) \hookrightarrow S'_\omega(E)$.*

(vi) *$L^K_{p,\rho}(E)$ is a quasi–Banach (Banach if $p \ge 1$) space.*

(vii) *Translations and differentiations generate continuous linear operators in $L^K_{p,\rho}(E)$.*

(viii) *The mapping $S_\omega \times L^K_{p,\rho}(E) \longrightarrow L^K_{p,\rho}(E)$: $(\varphi, f) \to \varphi * f$ is well–defined and is bilinear and continuous.*

Proof. (i) Let $\varphi \in S_\omega$ with $\varphi(0) = 1$ and $\operatorname{supp}\hat{\varphi} \subset \bar{B}_1$. Given $f \in L^K_{p,\rho}(E)$, we consider the functions $f_\varepsilon(x) = \varphi(\varepsilon x)f(x)$ for $0 < \varepsilon \le 1$. Obviously for $\varepsilon \to 0+$, $\|f_\varepsilon(x) - f(x)\|_E \to 0$ for every x. Moreover, for every $e' \in E'$, $\widehat{e'\circ f} = e' \circ \hat{f}$ has compact support, so $e' \circ f_\varepsilon = \varphi(\varepsilon\cdot)(e' \circ f) \in S_\omega$ ([33, p. 17]). Since $\operatorname{supp}\widehat{e'\circ f_\varepsilon} \subset \operatorname{supp}\widehat{\varphi(\varepsilon\cdot)} + \operatorname{supp}\widehat{e'\circ f} \subset \bar{B}_\varepsilon + K = K_\varepsilon$ it follows that $e' \circ f_\varepsilon \subset S^{K_\varepsilon}_\omega$ (thus $f_\varepsilon \in S^{K_\varepsilon}_\omega(E)$). On the other hand, there exists a constant $c > 0$ such that

$$\sup_{z\in\mathbb{R}^n} \rho(x-z)\frac{|\phi(x-z)|}{1+|z|^{n/r}} \le c\left[\left(M|\rho\phi|^r\right)(x)\right]^{1/r}$$

for all $\phi \in S^{K_1}_\omega$ and for all $x \in \mathbb{R}^n$ (see [33, Th. 1.4.2]). By using this maximal inequality and the Hahn–Banach theorem, we get for $x \in \mathbb{R}^n$ and $0 < \varepsilon \le 1$

$$\sup_{z\in\mathbb{R}^n} \rho(x-z)\frac{\|f_\varepsilon(x-z)\|_E}{1+|z|^{n/r}} = \sup_{\|e'\|\le 1}\left(\sup_{z\in\mathbb{R}^n} \rho(x-z)\frac{|(e'\circ f_\varepsilon)(x-z)|}{1+|z|^{n/r}}\right) \le$$

$$\le c \sup_{\|e'\|\le 1} \left[\left(M\left|\rho\left(e'\circ f_\varepsilon\right)\right|^r\right)(x)\right]^{1/r} \le c\left[\left(M\|\rho f_\varepsilon\|^r_E\right)(x)\right]^{1/r} \le$$

$$\le c\left[\left(M\|\rho f\|^r_E\right)(x)\right]^{1/r} .$$

Finally, passing to the limit as $\varepsilon \to 0+$ we obtain the right–side of (3.1). The first inequality of (3.1) is shown in a similar way by using the corresponding scalar inequality ([33, Th. 1.4.2]).

(ii) (3.2) is a consequence of the right–side of (3.1) and the Hardy–Littlewood maximal inequality (since $p/r > 1$).

(iii) Following the scalar case (see [33, Prop. 1.4.3]) and using the maximal inequalities (3.1) one can show the inequalities (3.3) for arbitrary functions of $S_\omega^K(E)$. Then, using the approximation procedure in (i) and Fatou's lemma one obtains (3.3) for all $f \in L_{p,\rho}^K(E)$.

(iv) Let Λ be a bounded open set with $\Lambda \supset K$. Reasoning as in the scalar case (see [33, Prop. 1.4.4]) and using (3.1) and (3.3) one can find constants $h_0, c_1, c_2 > 0$ such that (3.4) holds for all $h \in]0, h_0]$, all sets $\{x^k\}$ with $x^k \in Q_k^h$ and all $f \in S_\omega^{\bar{\Lambda}}(E)$. Then, using the approximation procedure in (i) (if $f \in L_{p,\rho}^K(E)$ then $f_\varepsilon \in S_\omega^{\bar{\Lambda}}(E)$ when $\varepsilon \to 0+$) one obtains (3.4) for all $f \in L_{p,\rho}^K(E)$.

(v) is an immediate consequence of the Plancherel–Polya–Nikol'skij inequalities (3.3) and of the topological embeddings $S_\omega(E) \hookrightarrow L_{p,\rho}(E)$ $(0 < p \le \infty)$ and $L_{p,\rho}(E) \hookrightarrow S'_\omega(E)$ $(1 \le p \le \infty)$.

(vi) Let (f_j) be a Cauchy sequence in $L_{p,\rho}^K(E)$. Since $L_{p,\rho}(E)$ is complete there exists an $f \in L_{p,\rho}(E)$ such that $f_j \to f$ in $L_{p,\rho}(E)$. Passing to a subsequence, if necessary, we can suppose that $f_j \to f$ a.e. By (3.3) and the estimate $1/\rho(x) \le ce^{\omega(x)}$ for all $x \in \mathbb{R}^n$, we have $\sup_{x\in\mathbb{R}^n} \{e^{-\omega(x)}\|f_j(x)\|_E : j = 1, 2, \dots\} < \infty$. Then, using Fatou's lemma and the E–valued dominated convergence theorem, we obtain that $f \in S'_\omega(E)$ and $f_j \to f$ in $S'_\omega(E)$. Thus, $\hat{f}_j \to \hat{f}$ in $S'_\omega(E)$, $\operatorname{supp}\hat{f} \subset K$, $f \in L_{p,\rho}^K(E)$ and $f_j \to f$ in $L_{p,\rho}^K(E)$.

(vii) Let $h \in \mathbb{R}^n$. Then τ_h is a continuous linear operator in $L_{p,\rho}^K(E)$ by virtue of the estimate $\rho(x+h) \le c\, e^{\omega(h)}\rho(x)$, $x \in \mathbb{R}^n$, and of the formula $\widehat{\tau_h f} = e^{-ih(\cdot)}\hat{f}$, $f \in S'_\omega(E)$. By the Plancherel–Polya–Nikol'skij inequalities ∂^α is a continuous linear operator in $L_{p,\rho}^K(E)$ for all multi–indices α.

(viii) Let $\varphi \in S_\omega$ and let $f \in L_{p,\rho}^K(E)$. Then $\varphi * f \in S'_\omega(E)$ and $\widehat{\varphi * f} = \hat{\varphi}\,\hat{f}$. Thus $\operatorname{supp}\widehat{\varphi * f} \subset K$ and so, by the Paley–Wiener–Schwartz theorem for E–valued ultradistributions, $\varphi * f(x) = \langle \tau_x\tilde{\varphi}, f\rangle = \int_{\mathbb{R}^n} \varphi(x-y)f(y)\,dy$ becomes the restriction to $\mathbb{R}^n$ of an E–valued entire function of exponential type. On the other hand, it follows from the proof of the Proposition in [38, p. 40], that there exist positive constants c and Λ such that for any $\phi \in S_\omega$ and $g \in L_{p,\rho}^K$

$$\rho(x-z)\frac{|\phi * g(x-z)|}{1+|z|^{n/r}} \le c\,p_{0,\Lambda}(\phi)\sup_{\xi\in\mathbb{R}^n}\rho(x-\xi)\frac{|g(x-\xi)|}{1+|\xi|^{n/r}}$$

with $0 < r < p$. Then, by using the Hahn–Banach theorem, we get

$$\rho(x-z)\frac{\|\varphi * f(x-z)\|_E}{1+|z|^{n/r}} \le c\,p_{0,\Lambda}(\varphi)\sup_{\xi\in\mathbb{R}^n}\rho(x-\xi)\frac{\|f(x-\xi)\|_E}{1+|z|^{n/r}}$$

for all $x, z \in \mathbb{R}^n$. Finally, the estimate (3.2) yields

$$\|\varphi * f\|_{p,\rho} \le c\,p_{0,\Lambda}(\varphi)\|f\|_{p,\rho}$$

which completes the proof.

□

Remark 3.3. 1. The constants which appear in the inequalities (3.1), (3.2), (3.3) and (3.4) and the constant h_0 in (iv) are independent of the Banach space E.

2. Observe that the study of the spaces $L^K_{p,\rho}(E)$ is not reduced to the study of the spaces $L^K_{p,\rho}\widehat{\otimes}_\varepsilon E$ (resp. $L^K_{p,\rho}\widehat{\otimes}_\pi E$): Let us assume $1 < p < \infty$, $\rho(x) \equiv 1$, $\mathring{K} \neq \emptyset$ and that E is reflexive and contains a copy of l_r (resp. has a quotient isomorphic to l_r) with $p' \le r < \infty$ (resp. with $p \le r' < \infty$). Let $Q \subset K$ be a cube with sides parallel to the axes. As is well known, χ_Q is a Fourier multiplier in L_p (see [39, Lemma 2.2.4]), thus L^Q_p is a complemented subspace of L^K_p and $L^Q_p \simeq l_p$ (see [39, Th. 2.11.2]). By using these results and properties of the tensor products of Banach spaces (see [9, Chapter VIII]), we have that $L^K_p\widehat{\otimes}_\varepsilon E$ (resp. $L^K_p\widehat{\otimes}_\pi E$) contains a copy (resp. has a quotient isomorphic to) of $l_p\widehat{\otimes}_\varepsilon l_r$ (resp. $l_p\widehat{\otimes}_\pi l_r$). Since $l_p\widehat{\otimes}_\varepsilon l_r$ ($p' \le r < \infty$) and $l_p\widehat{\otimes}_\pi l_r$ ($p \le r' < \infty$) are not reflexive (see [16]), it follows that neither $L^K_p\widehat{\otimes}_\varepsilon E$ nor $L^K_p\widehat{\otimes}_\pi E$ are reflexive (see [43, p. 31]). However, E being reflexive, $L^K_p(E)$ is reflexive. Consequently, $L^K_p(E)$ is not isomorphic to $L^K_p\widehat{\otimes}_\varepsilon E$ (resp. to $L^K_p\widehat{\otimes}_\pi E$).

However, we should point out that the topology that $L^K_{p,\rho}(E)$ induces on $L^K_{p,\rho}\otimes E$ is always finer than the ε–topology and coarser than the π–topology (for any $\rho \in R(\omega)$, $1 \le p \le \infty$, K compact and E Banach space).

Now we shall prove that $S^K_\omega \otimes E$ is dense in $L^K_{p,\rho}(E)$. In general this is not the case. For example, if $0 < p \le \infty$ and $0 < \beta < 1$, the space $L^{\{0\}}_{p,e^{-|x|^\beta}}$ is infinite–dimensional but $S^{\{0\}}_{|x|^\beta}$ contains only the function $\varphi(x) \equiv 0$ (see [38, Remark 1.4.3]); on the other hand, p must be $< \infty$ since, e.g., if K is uncountable then L^K_∞ is not separable ($\{e^{ik(\cdot)} : k \in K\} \subset L^K_\infty$ and $\|e^{ik(\cdot)} - e^{ik'(\cdot)}\|_\infty = 2$ when $k \neq k'$) but S^K (as subspace of S) is separable, thus S^K is not dense in L^K_∞.

Let us recall that a bounded open Ω in $\mathbb{R}^n$ has the segment property if there exist open balls V_j and vectors $y^j \in \mathbb{R}^n$, $j = 1, \dots, N$, such that $\bar{\Omega} \subset \cup_{j=1}^N V_j$ and $(\bar{\Omega} \cap V_j) + ty^j \subset \Omega$ for $0 < t < 1$ and $j = 1, \dots, N$. For instance, if Ω is convex or if $\partial\Omega \in C^{0,1}$ then Ω has the segment property.

Theorem 3.4. *Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$ and let K be the closure of a bounded open Ω in $\mathbb{R}^n$. Let E be a Banach space. If $0 < p < \infty$ and Ω has the segment property, then $S^K_\omega(E)$ and $S^K_\omega \otimes E$ are dense in $L^K_{p,\rho}(E)$.*

Proof. Let $\varepsilon_0 > 0$ such that $K + \bar{B}_{\varepsilon_0} \subset \cup_{j=1}^N V_j$. Then we can find $\psi_j \in D_\omega(V_j)$ so that $\psi_j \ge 0$ and $\sum_{j=1}^N \psi_j = 1$ in $K + \bar{B}_{\varepsilon_0}$ (cf. [3]). Put $\varphi_j = \mathcal{F}^{-1}\psi_j \in S^{V_j}_\omega$. Then by Theorem 3.2 (viii) the convolution operators $\Phi_j f = \varphi_j * f$ are bounded in $L^K_{p,\rho}(E)$. Besides, $\sum_{j=1}^N \Phi_j f = f$ for all $f \in L^K_{p,\rho}(E)$. Next, reasoning as in the scalar case (see [38, Prop. 1.4.4]) and using the approximation procedure in Theorem 3.2 (i), it is possible to approximate every $\Phi_j f$ by functions of $S^K_\omega(E)$. Consequently, $S^K_\omega(E)$ is dense in $L^K_{p,\rho}(E)$. Finally, since $S^K_\omega(E) \hookrightarrow L^K_{p,\rho}(E)$ and $S^K_\omega \otimes E$ is dense in $S^K_\omega(E)$ (see the next lemma) the proof is complete. $\square$

Lemma 3.5. *Let $\omega \in \mathcal{M}$ and let K be a compact set in $\mathbb{R}^n$. Let E be a Banach space. Then $S^K_\omega\widehat{\otimes}_\varepsilon E = S^K_\omega(E)$.*

Proof. Firstly, reasoning as in the classical case (i.e., when $\omega(x) = \log(1+|x|)$; see, e.g., [34]), one proves that $S_\omega(E)$ coincides with the set of all $f : \mathbb{R}^n \to E$ such that $e' \circ f \in S_\omega$ for any $e' \in E'$. Therefore, $S_\omega^K(E)$ coincides with the collection of all $f : \mathbb{R}^n \to E$ such that $e' \circ f \in S_\omega^K$. Next, since S_ω is a nuclear Fréchet space (see [15]), the subspace S_ω^K also is a nuclear Fréchet space (see, e.g., [36, p. 514]). Then, by using [20, Th. 1.12, p. 666], we see that the mapping

$$\begin{array}{cccc} \Phi : & L_b\left(\left(S_\omega^K\right)', E\right) & \to & S_\omega^K(E) \\ & T & \to & f(x) = T(\delta_x) \end{array}$$

is an algebraic isomorphism. Finally, since the graph of Φ is closed and $S_\omega^K(E)$ and $L_b\left(\left(S_\omega^K\right)', E\right)$ $(= S_\omega^K \widehat{\otimes}_\varepsilon E$, see, e.g., [36, p. 525]) are Fréchet spaces, the closed graph theorem shows that Φ becomes a topological isomorphism. Consequently, $S_\omega^K \widehat{\otimes}_\varepsilon E$ coincides algebraic and topologically with $S_\omega^K(E)$. □

Remark 3.6. 1. Theorem 3.4 generalizes the Proposition in [38, p. 40] tho the E–valued case.

2. Let $0 < p < \infty$, K a compact set in $\mathbb{R}^n$ and let E be a Banach space. Then the canonical injection $j_E : L_p^K(E) \to A(E)$ is continuous (we suppose $A(E)$ equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}^n$). In fact, let $1 \le p < \infty$ and $K = \{x : |x_j| \le b_j\ , \ j = 1, \dots, n\}$. Let $L_p^K \otimes_p E$ be the space $L_p^K \otimes E$ equipped with the topology induced by $L_p^K(E)$. By Remark 3.3/2 the identity mapping $id : L_p^K \otimes_p E \to L_p^K \otimes_\varepsilon E$ is continuous, thus it may be extended to a continuous linear mapping $\widehat{id} : L_p^K(E) \to L_p^K \widehat{\otimes}_\varepsilon E$ since $L_p^K(E)$ is the completion of $L_p^K \otimes_p E$ by Theorem 3.2/(vi) and Theorem 3.4. On the other hand, as a consequence of a compactness theorem by Nikol'skij [27, p. 127], the canonical injection $j : L_p^K \to A$ is also continuous. Furthermore, it is well–known that $A(E) = A \widehat{\otimes}_\varepsilon E$ (see, e.g., [14, Ch. II, p. 81]). Finally, since the diagram

$$\begin{tikzcd} L_p^K(E) \arrow[rr, "j_E"] \arrow[dr, "\widehat{id}"'] & & A(E) = A\widehat{\otimes}_\varepsilon E \\ & L_p^K \widehat{\otimes}_\varepsilon E \arrow[ur, "j\widehat{\otimes}_\varepsilon id_E"'] & \end{tikzcd}$$

is commutative, it follows that j_E is continuous. In the general case, i.e., when $0 < p < \infty$ and K is any compact set in $\mathbb{R}^n$, we use this result and the Plancherel–Polya–Nikol'skij inequalities.

4. $L_{p,\rho}(E)$–Fourier multipliers. Duality

In this section we shall calculate the dual of the space $L_{p,\rho}^K(E)$. In fact, we shall show that the natural mapping $N : L_{p',\rho^{-1}}^{-K}(E') \to \left(L_{p,\rho}^K(E)\right)' : g \to \langle f, Ng\rangle = \int_{\mathbb{R}^n} \langle f(x), g(x)\rangle dx$ becomes an isomorphism when $p \in (1,\infty)$ and χ_K is an $L_{p,\rho}(E)$–Fourier multiplier (see Theorem 4.6 and Corollary 4.8). Some new characterizations of the so–called UMD–property will also be given. These results will be used in the next section in order to represent several distribution spaces by means of spaces of vector sequences.

Definition 4.1. Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$, $1 \le p < \infty$ and E be a Banach space. A function $m \in L_\infty$ is said to be an $L_{p,\rho}(E)$–Fourier multiplier if there is a constant C such that for all $f \in S_\omega(E)$ we have

$$\|\mathcal{F}^{-1}(m\mathcal{F}f)\|_{p,\rho} \le C\|f\|_{p,\rho}\ . \tag{4.1}$$

The set of all $L_{p,\rho}(E)$–Fourier multipliers will be denoted by $M_{p,\rho}(E)$ and the smallest constant C such that (4.1) holds by $\|m\|_{M_{p,\rho}(E)}$. For an $L_{p,\rho}(E)$–Fourier multiplier m the operator $f \to \mathcal{F}^{-1}(m\mathcal{F}f)$ extends uniquely to a bounded operator on $L_{p,\rho}(E)$ which will be denoted by T_m.

Remark. If $\rho(x) \equiv 1$ then we put $M_{p,1}(E) = M_p(E)$ (M_p if $E = \mathbb{C}$). We shall write $M_{p,\rho}$ instead of $M_{p,\rho}(\mathbb{C})$. If $m \in M_{p,\rho}$ the corresponding operator on $L_{p,\rho}$ will also be denoted by T_m. If $m \in M_{p,\rho}(E)$ then $m \in M_{p,\rho}$ but, in general, the converse does not hold. For example, if Q is a cube with sides parallel to the axes, $p \in (1,\infty)$ and $E \notin$ UMD, then $\chi_Q \in M_p$ but $\chi_Q \notin M_p(E)$ (see Corollary 4.7).

Lemma 4.2. *Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$, $p \in (1,\infty)$ and E be a Banach space. If $m \in M_{p,\rho}(E)$, then $\widetilde{m} \in M_{p',\rho^{-1}}(E')$.*

Proof. Since $m \in M_{p,\rho}$ and the identity

$$\int_{\mathbb{R}^n} \varphi(x)\mathcal{F}^{-1}(m\hat{\psi})(x)\,dx = \int_{\mathbb{R}^n} \psi(x)\mathcal{F}^{-1}(\widetilde{m}\hat{\varphi})(x)\,dx \tag{4.2}$$

holds for all $\varphi, \psi \in L_2$, a duality argument proves that $\widetilde{m} \in M_{p',\rho^{-1}}$. Let us now consider the following diagram

$$\begin{tikzcd}
(L_{p,\rho}(E))' \arrow[rr, "T'_m"] & & (L_{p,\rho}(E))' \\
L_{p',\rho^{-1}}(E') \arrow[u, "I"] & & L_{p',\rho^{-1}}(E') \arrow[u, "I"] \\
& S_\omega \otimes_{p',\rho^{-1}} E' \arrow[ul, "i"] \arrow[ur, dashed, "Z"'] &
\end{tikzcd}$$

where T'_m is the adjoint of the operator T_m associated with the $L_{p,\rho}(E)$–Fourier multiplier m, I is the isometric embedding $\langle f, I(g)\rangle = \int_{\mathbb{R}^n}\langle f, g\rangle dx$ for $f \in L_{p,\rho}(E)$ and $g \in L_{p',\rho^{-1}}(E')$, $S_\omega \otimes_{p',\rho^{-1}} E'$ is the space $S_\omega \otimes E'$ equipped with the topology induced by $L_{p',\rho^{-1}}(E')$, i is the natural injection and Z is the map defined by $Z\left(\sum \varphi_j \otimes e'_j\right) = \sum (T_{\widetilde{m}}\varphi_j) \otimes e'_j$ for $\sum \varphi_j \otimes e'_j \in S_\omega \otimes E'$. By virtue of (4.2) we get

$$\begin{aligned}
\langle \psi \otimes e, I\left(Z(\varphi \otimes e')\right)\rangle &= \int_{\mathbb{R}^n} \langle \psi(x)e, T_{\widetilde{m}}\varphi(x)e'\rangle dx = \\
= \int_{\mathbb{R}^n} \psi(x)\mathcal{F}^{-1}(\widetilde{m}\hat{\varphi})(x)\,dx\,\langle e, e'\rangle &= \int_{\mathbb{R}^n} \varphi(x)\mathcal{F}^{-1}(m\hat{\psi})(x)\,dx\ \langle e, e'\rangle = \\
= \int_{\mathbb{R}^n} \langle \mathcal{F}^{-1}(m\hat{\psi})(x)e, \varphi(x)e'\rangle dx &= \langle T_m\left(\psi \otimes e\right), I\left(\varphi \otimes e'\right)\rangle = \\
&= \langle \psi \otimes e, T'_m\left(I\left(i(\varphi \otimes e')\right)\right)\rangle
\end{aligned}$$

for all $\varphi, \psi \in S_\omega$, $e \in E$ and $e' \in E'$. Since $S_\omega \otimes E$ is dense in $L_{p,\rho}(E)$ we conclude that the diagram is commutative. Therefore $I \circ Z$ is bounded and, since I is isometric, Z is also bounded. Consequently, $\widetilde{m} \in M_{p',\rho^{-1}}(E')$. $\square$

We shall omit the proof of the following simple result.

Lemma 4.3. *Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$, $1 \le p < \infty$ and E a Banach space. If $f \in L_{p,\rho}(E)$ has compact support, then there exists a sequence $(h_j)_1^\infty \subset D_\omega(E)$ such that $h_j \to f$ in $L_{p,\rho}(E)$ as $j \to \infty$ and, for all j, $\operatorname{supp} h_j \subset K$ where K is some fixed compact neighborhood of* $\operatorname{supp} f$.

The next lemma is a simple consequence of some results in [10, Ch. II]. We shall give the proof for the sake of completeness. We shall employ the following notation (see [10, Ch. II]): Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$, $1 < p < \infty$ and let E be a Banach space. Let Σ be the ring of measurable subsets $A \subset \mathbb{R}^n$ such that $\int_A \rho^{-p'}(x)\,dx < \infty$. A Σ–partition π of $\mathbb{R}^n$ is any finite disjoint collection $\{A_j\} \subset \Sigma$. Then

$$V_{p',\rho^{-1}}(E') = \{m \,|\, m : \Sigma \to E'\,,\ m \text{ finitely additive,} \\ m(A) = 0 \text{ if } \operatorname{Vol}_n(A) = 0\,,\ |m|_{p',\rho^{-1}} < \infty\}$$

where $|m|_{p',\rho^{-1}} = \sup\Big\{\Big(\sum_\pi \frac{\|m(A)\|_{E'}^{p'}}{\left(\int_A \rho^{-p'}dx\right)^{p'-1}}\Big)^{1/p'} : \pi = \Sigma\text{–partition of } \mathbb{R}^n\Big\}$. With the norm $|\cdot|_{p',\rho^{-1}}$, $V_{p',\rho^{-1}}(E')$ is a Banach space, and the mapping $V_{p',\rho^{-1}}(E') \to (L_{p,\rho}(E))' : m \to \int_{\mathbb{R}^n} (\cdot)\rho^{p'}(x)\,dm(x)$ is an isomorphism (isometric).

Lemma 4.4. *Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$, $1 < p < \infty$ and E be a Banach space. Let U be a continuous map from $\mathbb{R}^n$ into E' such that* $\sup\{|\int_{\mathbb{R}^n}\langle f, U\rangle dx| : \|f\|_{p,\rho} \le 1\,,\ f \in L^c_\infty(E)\} = C < \infty$. *Then $U \in L_{p',\rho^{-1}}(E')$ and $\|U\|_{p',\rho^{-1}} = C$.*

Proof. Putting $I(f) = \int_{\mathbb{R}^n}\langle f(x), U(x)\rangle dx$, $f \in L^c_\infty(E)$, and using the hypothesis we see that $I(f)$ becomes a continuous linear form on $L^c_\infty(E)$ equipped with the topology induced by $L_{p,\rho}(E)$. Let $\bar I$ be the continuous extension of I to $L_{p,\rho}(E)$ and let $m \in V_{p',\rho^{-1}}(E')$ be such that $\bar I(f) = \int_{\mathbb{R}^n} \rho^{p'}(x) f(x)\,dm(x)$, $f \in L_{p,\rho}(E)$, and $\|\bar I\| = \|I\| = C = |m|_{p',\rho^{-1}}$ [10, Th. 1, p. 259]. Then $\int_{\mathbb{R}^n}\langle f(x), U(x)\rangle dx = \int_{\mathbb{R}^n} \rho^{p'}(x) f(x)\,dm(x)$ for all $f \in L^c_\infty(E)$ and therefore, taking $f = \rho^{-p'}\chi_A \otimes e$ with $e \in E$ and A measurable and bounded in $\mathbb{R}^n$, we obtain $\langle e, \int_A \rho^{-p'}(x)U(x)\,dx\rangle = \langle e, m(A)\rangle$. Hence it follows that $m(A) = \int_A \rho^{-p'}(x)U(x)\,dx$ for all measurable and bounded A. On the other hand, for any compact K in $\mathbb{R}^n$, we have

$$\left(\int_K \|U(x)\|^{p'}\rho^{-p'}(x)\,dx\right)^{1/p'} = \\ = \sup\Big\{\Big(\sum_\pi \frac{\|m_U^K(B)\|_{E'}^{p'}}{\left(\int_B \rho^{-p'}dx\right)^{p'-1}}\Big)^{1/p'} : \pi = \mathcal{B}(K)\text{–partition of } K\Big\}$$

where $m_U^K(B) = \int_B U(x)\rho^{-p'}(x)\,dx$ for all $B \in \mathcal{B}(K)$ (see [10, Ch. II]). Since $m_U^K(B) = m(B)$ for $B \in \mathcal{B}(K)$, it results that

$$\left(\int_K \|U(x)\|^{p'}\rho^{-p'}(x)\,dx\right)^{1/p'} \le |m|_{p',\rho^{-1}} = C\ .$$

Varying K we get $U \in L_{p',\rho^{-1}}(E')$ and $\|U\|_{p',\rho^{-1}} \le C$. By using now the isometric embedding $L_{p',\rho^{-1}}(E') \hookrightarrow V_{p',\rho^{-1}}(E') : g \to m_g(A) = \int_A g\,\rho^{-p'}\,dx$ for all $A \subset \Sigma$, it follows that $\|U\|_{p',\rho^{-1}} = |m_U|_{p',\rho^{-1}}$ (see again [10, Ch. II]). Finally, since $U \in L_{p',\rho^{-1}}(E')$, it is easy to see that $\bar I(f) = \int\langle f(x), U(x)\rangle dx$ for all $f \in L_{p,\rho}(E)$; hence it follows that $m = m_U$ which completes the proof. $\square$

Lemma 4.5. *Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$, $p \in (1,\infty)$ and K be a compact set in $\mathbb{R}^n$ such that $\mathring{K} \neq \emptyset$ and $\operatorname{Vol}_n(\partial K) = 0$. Assume that $\rho = \tilde{\rho}$ and that $L^K_{p,\rho}$ is an invariantly complemented subspace of $L_{p,\rho}$. Then $\chi_K \in M_{p,\rho}$.*

Proof. Let P be a translation invariant bounded projection in $L_{p,\rho}$ such that $\operatorname{Im} P = L^K_{p,\rho}$. By [33, Lemma 5.1.3] we may assume, without loss of generality, that $\rho(0) = 1$ and $\rho \in C^\infty$. Then, we can find an $m \in L_\infty$ such that $\widehat{P\varphi} = m\hat{\varphi}$ for all $\varphi \in D$ (see [23]). Since D is dense in the space $L_2 \cap L_{p,\rho}$ equipped with the norm $\|\cdot\|_\cap = \max(\|\cdot\|_2, \|\cdot\|_{p,\rho})$, we also have that $\widehat{Pf} = m\hat{f}$ for any $f \in L_2 \cap L_{p,\rho}$. In fact, if $f \in L_2 \cap L_{p,\rho}$ and the sequence $(\varphi_j) \subset D$ satisfies $\|\varphi_j - f\|_\cap \to 0$, then by Plancherel's theorem $\|m\hat{\varphi}_j - m\hat{f}\|_2 \to 0$ and thus $m\hat{\varphi}_j \to m\hat{f}$ in S'_ω; also $\|P\varphi_j - Pf\|_{p,\rho} \to 0$ and therefore $P\varphi_j \to Pf$ in S'_ω and $\widehat{P\varphi_j} \to \widehat{Pf}$ in S'_ω. Since $\widehat{P\varphi_j} = m\hat{\varphi}_j$ it results that $\widehat{Pf} = m\hat{f}$ as we required. Applying this property we see that

$$m\hat{f} = \widehat{Pf} = \widehat{P^2 f} = \widehat{P(Pf)} = m\widehat{Pf} = m^2\hat{f}$$

for any $f \in L_2 \cap L_{p,\rho}$. Hence it follows that $m^2 = m$ a.e. and so $m = \chi_A$ a.e. where $A = \{x : m(x) = 1\}$. Since $\operatorname{supp} \chi_A \hat{f} \subset K$ for any $f \in L_2 \cap L_{p,\rho}$, we get $\operatorname{Vol}_n(A \setminus K) = 0$. On the other hand, $\chi_A = 1$ in $\mathring{K}$ a.e. (for any $x \in \mathring{K}$ there exists $\varphi_x \in D_\omega(\mathring{K})$ such that $\varphi_x = 1$ in a neighborhood of x, then if $f_x = \mathcal{F}^{-1}\varphi_x$ we see that $\varphi_x = \hat{f}_x = \widehat{Pf_x} = \chi_A \hat{f}_x = \chi_A \varphi_x$ and so $\chi_A = 1$ in a neighborhood of x), that is, $\operatorname{Vol}_n(\mathring{K} \setminus A) = 0$. In consequence $\chi_A = \chi_K$ a.e. and so $\chi_K \in M_{p,\rho}$. □

Theorem 4.6. *Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$ and $p \in (1,\infty)$. Let K be the closure of a bounded open in $\mathbb{R}^n$ with the segment property and let E be a Banach space. If $\chi_K \in M_{p,\rho}(E)$, then the mapping $N : L^{-K}_{p',\rho^{-1}}(E') \to \left(L^K_{p,\rho}(E)\right)' : g \to \langle f, Ng\rangle = \int_{\mathbb{R}^n} \langle f(x), g(x)\rangle dx$ becomes an isomorphism. Conversely, if the former mapping N is an isomorphism, $\rho = \tilde{\rho}$ and $\operatorname{Vol}_n(\partial K) = 0$, then $\chi_K \in M_{p,\rho}(E)$.*

Proof. ($\Longrightarrow$) Denote by S_K the operator associated with the $L_{p,\rho}(E)$–Fourier multiplier χ_K. Since $L^K_{p,\rho}(E)$ is complete (see Theorem 3.2/(vi)) and $S^K_\omega(E)$ is dense in $L^K_{p,\rho}(E)$ (Theorem 3.4) it is easy to check that $\operatorname{Im} S_K = L^K_{p,\rho}(E)$, S_K is a projection and $L_{p,\rho}(E) = L^K_{p,\rho}(E) \oplus \ker S_K$. Analogously, we get $L_{p',\rho^{-1}}(E') = L^{-K}_{p',\rho^{-1}}(E') \oplus \ker S_{-K}$ where S_{-K} is the operator associated with the $L_{p',\rho^{-1}}(E')$–Fourier multiplier χ_{-K} (see Lemma 4.2). Furthermore, from the proof of Lemma 4.2, it results that the identity

$$\int_{\mathbb{R}^n} \langle S_K f(x), g(x)\rangle dx = \int_{\mathbb{R}^n} \langle f(x), S_{-K} g(x)\rangle dx \tag{4.3}$$

holds for all $f \in L_{p,\rho}(E)$ and for all $g \in L_{p',\rho^{-1}}(E')$.

Now study the properties of the mapping N. By Hölder's inequality N is well–defined and it is linear and continuous. Let us see that it is injective. Suppose $Ng = 0$, i.e., $\int_{\mathbb{R}^n} \langle f(x), g(x)\rangle dx = 0$ for all $f \in L^K_{p,\rho}(E)$. Then, if $f \in L_{p,\rho}(E)$ and

$f = f_1 + f_2$ with $f_1 \in L^K_{p,\rho}(E)$ and $f_2 \in \ker S_K$, we obtain, from the identity (4.3),

$$\int_{\mathbb{R}^n} \langle f(x), g(x)\rangle dx = \int_{\mathbb{R}^n} \langle f_1(x), g(x)\rangle dx + \int_{\mathbb{R}^n} \langle f_2(x), g(x)\rangle dx =$$
$$= \int_{\mathbb{R}^n} \langle f_2(x), S_{-K} g(x)\rangle dx = \int_{\mathbb{R}^n} \langle S_K f_2(x), g(x)\rangle dx = 0 \ .$$

But since the mapping $L_{p',\rho^{-1}}(E') \to (L_{p,\rho}(E))' : h \to \int_{\mathbb{R}^n} \langle \cdot, h(x)\rangle dx$ is an isometric embedding, it follows that $g \equiv 0$ as we required.

Next we prove that $\operatorname{Im} N = \left(L^K_{p,\rho}(E)\right)'$. For this we consider the diagram

$$\left(L^K_{p,\rho}(E)\right)' \xrightarrow{S'_K} (L_{p,\rho}(E))' \xrightarrow{j'} (S_\omega(E))' \xrightarrow{\Phi} S'_\omega(E')$$

where S'_K is the adjoint of S_K, j' is the adjoint of the natural injection $S_\omega \overset{j}{\hookrightarrow} L_{p,\rho}(E)$ and Φ is the operator defined by $\langle e, \langle \varphi, \Phi(v)\rangle\rangle = \langle \varphi \otimes e, v\rangle$ for all $v \in (S_\omega(E))'$, $\varphi \in S_\omega$ and $e \in E$ (as is well–known, see [36, p. 524], Φ is a topological isomorphism since S_ω is nuclear [15]). Put $\Lambda = \Phi \circ j' \circ S'_K$. Let $u \in \left(L^K_{p,\rho}(E)\right)'$. If $\varphi \in D_\omega(\complement(-K))$ and $e \in E$, we have

$$\langle e, \langle \varphi, \widehat{\Lambda u}\rangle\rangle = \langle e, \langle \hat{\varphi}, \Lambda u\rangle\rangle = \langle e, \langle \hat{\varphi}, \Phi(j'(S'_K(u)))\rangle\rangle = \langle S_K(\hat{\varphi} \otimes e), u\rangle =$$
$$= \langle \mathcal{F}^{-1}(\chi_K \hat{\hat{\varphi}}) \otimes e, u\rangle = (2\pi)^n \langle \mathcal{F}^{-1}(\chi_K \tilde{\varphi}) \otimes e, u\rangle = 0$$

since $\chi_K \tilde{\varphi} = 0$. Hence it follows that $\operatorname{supp} \widehat{\Lambda u} \subset -K$. Then the Paley–Wiener–Schwartz theorem for E'–valued ultradistributions shows that there exists an E'–valued entire function of exponential type U such that $\langle \varphi, \Lambda u\rangle = \int_{\mathbb{R}^n} \varphi(x) U(x)\, dx$ for all $\varphi \in S_\omega$. This implies that $\langle S_K f, u\rangle = \int_{\mathbb{R}^n} \langle f, U\rangle dx$ for all $f \in S_\omega \otimes E$ and since S_K is a bounded operator, $S_\omega \otimes E$ is dense in $S_\omega(E)$ and there exist constants $C > 0$, $\lambda \in \mathbb{R}$ such that $\|U(x)\|_{E'} \le C e^{\lambda \omega(x)}$ for any $x \in \mathbb{R}^n$, it is clear that this identity also holds for all $f \in S_\omega(E)$. By using Lemma 4.3 we also get

$$\langle S_K f, u\rangle = \int_{\mathbb{R}^n} \langle f(x), U(x)\rangle dx$$

for all $f \in L^c_\infty(E)$, therefore

$$\sup\left\{\left|\int_{\mathbb{R}^n} \langle f(x), U(x)\rangle dx\right| : \|f\|_{p,\rho} \le 1\,,\ f \in L^c_\infty(E)\right\} \le$$
$$\le \|m\|_{M_{p,\rho(E)}} \|u\|_{\left(L^K_{p,\rho}(E)\right)'} < \infty \ .$$

Hence an application of Lemma 4.4 gives $U \in L_{p',\rho^{-1}}(E')$ and thus $\Lambda u \in L^{-K}_{p',\rho^{-1}}(E')$. Furthermore, since $S^K_\omega(E)$ is dense in $L^K_{p,\rho}(E)$ and for all $f \in S^K_\omega(E)$ we have

$$\langle f, N(\Lambda u)\rangle = \int_{\mathbb{R}^n} \langle f(x), U(x)\rangle dx = \langle S_K f, u\rangle = \langle f, u\rangle \ ,$$

it follows that $N(\Lambda u) = u$. To complete the proof we apply the open mapping theorem.

($\Longleftarrow$) Consider the diagram

$$L_{p',\rho^{-1}}(E') \xrightarrow{I} (L_{p,\rho}(E))' \xrightarrow{R} \left(L^K_{p,\rho}(E)\right)' \xrightarrow{N} L^{-K}_{p',\rho^{-1}}(E')$$

where I is the natural isometric embedding, R is the restriction operator and N is the given topological isomorphism. Putting $P = N^{-1} \circ R \circ I$, it is easy to see that P is a translation invariant bounded projection in $L_{p',\rho^{-1}}(E')$ with $\operatorname{Im} P = L^{-K}_{p',\rho^{-1}}(E')$. Hence it follows that the mapping $P_{ee'} : L_{p',\rho^{-1}} \to L^{-K}_{p',\rho^{-1}} : g \to e \circ P(g \otimes e')/\langle e, e' \rangle$ is a translation invariant bounded projection in $L_{p',\rho^{-1}}$ such that $\operatorname{Im} P_{ee'} = L^{-K}_{p',\rho^{-1}}$ provided $\langle e, e' \rangle \neq 0$. By Lemma 4.5 it results that $\chi_{-K} \in M_{p',\rho^{-1}}$ (and $P_{ee'} = S_{-K}$ = the operator associated with the $L_{p',\rho^{-1}}$–Fourier multiplier χ_{-K}). Therefore, the mapping $S_\omega \otimes_{p',\rho^{-1}} E' \to L_{p',\rho^{-1}}(E') : \sum \varphi_j \otimes e'_j \to \sum (S_{-K}\varphi_j) \otimes e'_j$ is well–defined and it is bounded (since it coincides with $P|_{S_\omega \otimes E'}$), that is, $\chi_{-K} \in M_{p',\rho^{-1}}(E')$. Then, by Lemma 4.2, $\chi_K \in M_{p,\rho}(E'')$ and so $\chi_K \in M_{p,\rho}(E)$. □

As a consequence of this theorem we can give some characterizations of the so–called UMD–property (cf. [30], [8]). Let us recall that a Banach space E is UMD provided that for $1 < p < \infty$ martingale difference sequences $d = (d_1, d_2, \dots)$ in $L_p([0,1], E)$ are unconditional, i.e. $\|\varepsilon_1 d_1 + \varepsilon_2 d_2 + \cdots\|_p \leq C_p(E)\|d_1 + d_2 + \cdots\|_p$ whenever $\varepsilon_1, \varepsilon_2, \dots$ are numbers in $\{-1, 1\}$. This property is also equivalent to the boundedness of the Hilbert transform on $L_p(\mathbb{R}, E)$ (see [4], [5]).

Corollary 4.7. *Let E be a Banach space and Q the cube $[-1,1]^n$. Then for all $p \in (1, \infty)$ the following statements are equivalent:*

(i) *$E \in$ UMD.*
(ii) *$\chi_Q \in M_p(E)$.*
(iii) *$L^Q_{p'}(E')$ and $\big(L^Q_p(E)\big)'$ are isomorphic via the natural mapping.*
(iv) *$L^Q_p(E)$ is an invariantly complemented subspace of $L_p(E)$.*
(v) *E is reflexive and $L^Q_p(E)$ is a complemented subspace of $L_p(E)$.*

Proof. By the results in [4] and [5], (i) is equivalent to (ii). The equivalence between (ii) and (iii) is a consequence of Theorem 4.6. The operator S_Q associated with the $L_p(E)$–Fourier multiplier χ_Q is a translation invariant bounded projection in $L_p(E)$ and $\operatorname{Im} S_Q = L^Q_p(E)$ (see the proof of Theorem 4.6), thus (ii) implies (iv). Conversely, if P is a translation invariant bounded projection in $L_p(E)$ such that $\operatorname{Im} P = L^Q_p(E)$ then reasoning as we did in the part ($\Longleftarrow$) of Theorem 4.6 we get $\chi_Q \in M_p(E)$, so (iv) implies (ii). From [1] (cf. also [30]), we know that UMD implies reflexivity (actually, super–reflexivity), therefore (iv) ($\Leftrightarrow$ (i)) implies (v). We now show that (v) implies (iv). Since E is reflexive, $L_p(E)$ becomes a reflexive space (cf., e.g., [9]) and then we can apply [29, A6, p. 80] and argue exactly as in [29, Lemma 3.1, p. 59]. Thus $L^Q_p(E)$ becomes an invariantly complemented subspace of $L_p(E)$. □

Let us now recall the definition of A_p functions. A positive, locally integrable function ω on $\mathbb{R}^n$ is in A^*_p provided, for $1 < p < \infty$,

$$A^*_p(\omega) = \sup_R \left(\frac{1}{|R|}\int_R \omega\, dx\right)\left(\frac{1}{|R|}\int_R \omega^{-p'/p} dx\right)^{p/p'} < \infty ,$$

where R runs over all bounded n–dimensional intervals. If R runs over all cubes in $\mathbb{R}^n$ then ω is in A_p and the corresponding supremum is denoted by $A_p(\omega)$. A_p is the class of Muckenhoupt. The basic properties of these functions can be found in [26], [7] and [13, Ch. IV].

Corollary 4.8. *Let $\omega \in \mathcal{M}$, $1 < p < \infty$, $\rho \in R(\omega)$, $\rho^p \in A^*_p$ and let E be a Banach space with the UMD–property. Then $\big(L^I_{p,\rho}(E)\big)'$ and $L^{-I}_{p',\rho^{-1}}(E')$ are isomorphic, via the natural mapping, for all compact n–dimensional intervals I.*

Proof. By the next theorem, $\chi_I \in M_{p,\rho}(E)$ for any compact n–dimensional interval I. The corollary now follows from Theorem 4.6. $\square$

Remark. The former corollary extends the theorem in [38, p. 43] (see also [38, p. 24] and [38, p. 40]).

If ω is a positive, locally integrable function on $\mathbb{R}^n$ and $1 < p < \infty$, then the partial sum operators S_I are uniformly bounded (for all n–dimensional intervals I) in $L_p(\omega\, dx)$ if and only if $\omega \in A_p^*$ (see [13, Th. 6.2, p. 453]). In the next theorem, this result is partially extended to the vector–valued setting. The extension is essentially a consequence of Burkholder's theorem [5] and Theorem 1.3 in [31].

Theorem 4.9. *If ω is in A_p^* ($1 < p < \infty$) and the Banach space E is in UMD, then the partial sum operators S_I ($S_I = \mathcal{F}^{-1}(\chi_I \hat{f})$ for $f \in S(E)$) are uniformly bounded (for all n–dimensional intervals I) in $L_p(\omega\, dx, E)$.*

Proof. Case $n = 1$. By [13, Th. 2.6, p. 399], there is $\varepsilon > 0$ such that $\omega \in A_{p-\varepsilon}$. Let $\beta = \frac{p}{p-\varepsilon}$ and $q = r' = \beta$ (r' is the conjugate exponent of r). Then, $p > \beta$, the $L(E)$–valued Hilbert kernel K ($K(x,y)(e) = \frac{e}{\pi(x-y)}$, $x, y \in \mathbb{R}$, $x \neq y$, $e \in E$) satisfies (D_1') and (D_r') of [31, Def. 1.1, p. 30], and the Hilbert transform H is a bounded linear operator on $L_q(E)$ (cf. [5]). Therefore, H satisfies the conditions of Theorem 1.3 of [31] and so H becomes a bounded linear operator on $L_p(\omega\, dx, E)$. Finally, by using the relationship between S_I and H (e.g., $S_{(a,b)}f = \frac{i}{2}\left[e^{ia(\cdot)}H(e^{-ia(\cdot)}f) - e^{ib(\cdot)}H(e^{-ib(\cdot)}f)\right]$ for $f \in S \otimes E$, $-\infty < a < b < \infty$) and the denseness of $S \otimes E$ in $L_p(\omega\, dx, E)$ (ω is in A_p), it results that

$$\sup_I \|S_I\|_{L(L_p(\omega\, dx, E))} \leq 1.5 + \|H\|_{L(L_p(\omega\, dx, E))} < \infty \ .$$

Case $n > 1$. We shall assume $n = 2$ since this case contains all the essential difficulties of the general situation. By [13, p. 464] there is $\varepsilon > 0$ such that $\omega \in A_{p-\varepsilon}^*$, and by [13, Th. 6.2, p. 453] there exist measurable null sets $N_1, N_2 \subset \mathbb{R}$ such that $\omega(x_1, \cdot) \in A_{p-\varepsilon}$ for all $x_1 \in \mathbb{R} \setminus N_1$, $\omega(\cdot, x_2) \in A_{p-\varepsilon}$ for all $x_2 \in \mathbb{R} \setminus N_2$ and

$$\sup_{x_1 \in \mathbb{R} \setminus N_1} A_{p-\varepsilon}(\omega(x_1, \cdot)) \ , \ \sup_{x_2 \in \mathbb{R} \setminus N_2} A_{p-\varepsilon}(\omega(\cdot, x_2)) \leq A_{p-\varepsilon}^*(\omega) \ . \tag{4.4}$$

Then, reasoning as we did in the case $n = 1$, analyzing in detail the constants which appear throughout the proof of Theorem 1.3 of [31], and using (4.4) we obtain a constant C, independent of $x_1 \in \mathbb{R} \setminus N_1$ and of $x_2 \in \mathbb{R} \setminus N_2$, such that

$$\int_{-\infty}^{\infty} \|Hf(x_1)\|_E^p \omega(x_1, x_2) dx_1 \leq C^p \int_{-\infty}^{\infty} \|f(x_1)\|_E^p \omega(x_1, x_2) dx_1$$

for all $f \in L_p(\omega(x_1, x_2)dx_1, E)$ and for all $x_2 \in \mathbb{R} \setminus N_2$, and such that

$$\int_{-\infty}^{\infty} \|Hf(x_2)\|_E^p \omega(x_1, x_2) dx_2 \leq C^p \int_{-\infty}^{\infty} \|f(x_2)\|_E^p \omega(x_1, x_2) dx_2$$

for any $f \in L_p(\omega(x_1, x_2)dx_2, E)$ and for any $x_1 \in \mathbb{R} \setminus N_1$. Hence it follows that

$$\|S_{I_1}\|_{L(L_p(\omega(x_1,x_2)dx_1, E))} \ , \|S_{I_2}\|_{L(L_p(\omega(x_1,x_2)dx_2, E))} \leq 1.5 + C \tag{4.5}$$

for all intervals $I_1, I_2 \subset \mathbb{R}$, for all $x_2 \in \mathbb{R} \setminus N_2$ and for all $x_1 \in \mathbb{R} \setminus N_1$.

Next, let I_1 be an interval of the x_1–axis and let $S_{I_1}^1$ be the mapping

$$\begin{array}{llll} S_{I_1}^1 : & S \otimes S \otimes E[L_p(\omega\, dx_1 dx_2, E)] & \to & L_p(\omega\, dx_1 dx_2, E) \\ & f & \to & S_{I_1}^1 f(x_1, x_2) = S_{I_1} f(\cdot, x_2)(x_1) \ . \end{array}$$

Then, by Fubini's theorem and (4.5) we get

$$\int_{\mathbb{R}^2} \|S^1_{I_1} f(x_1,x_2)\|^p_E\, \omega(x_1,x_2)dx_1dx_2 =$$

$$= \int_{\mathbb{R}\smallsetminus N_2} \Big[\int_{-\infty}^{\infty} \|S_{I_1} f(\cdot,x_2)(x_1)\|^p_E\, \omega(x_1,x_2)dx_1\Big] dx_2 \le$$

$$\le k^p \int_{\mathbb{R}\smallsetminus N_2} \|f(\cdot,x_2)\|^p_{L_p(\omega(x_1,x_2)dx_1,E)} dx_2 =$$

$$= k^p \int_{\mathbb{R}\smallsetminus N_2} \Big[\int_{-\infty}^{\infty} \|f(x_1,x_2)\|^p_E\, \omega(x_1,x_2)dx_1\Big] dx_2 = k^p \|f\|^p_{L_p(\omega\, dx_1dx_2,E)}$$

for any $f \in S\otimes S\otimes E$ (being $k = 1.5{+}C$). Since $S\otimes S\otimes E$ is dense in $L_p(\omega\, dx_1dx_2, E)$ it follows that $S^1_{I_1}$ becomes a bounded linear operator on $L_p(\omega\, dx_1dx_2, E)$ with norm independent of I_1. Analogously, it is shown that $S^2_{I_2}$ becomes a bounded linear operator on $L_p(\omega\, dx_1dx_2, E)$ with norm independent of I_2. Finally, if $I = I_1 \times I_2$ is a 2–dimensional interval, we have $S_{I_1\times I_2} = S^1_{I_1} \circ S^2_{I_2}$. This remark completes the proof of the theorem. □

Remark 4.10. 1. The examples 1 and 2 and the theorem in [38, 1.4.5, pp. 41–46] led us to Theorem 4.9.

2. In [11], C. Fefferman showed that the characteristic function of a euclidean ball in $\mathbb{R}^n$ is not an L_p–Fourier multiplier when $p \in (1,\infty) \smallsetminus \{2\}$ and $n > 1$. Mitiagin in [24] extended this result to compact sets K in $\mathbb{R}^n$ which have at least one point of strict convexity ($x \in \partial K$ is a point of strict convexity of K if for some $\varepsilon > 0$ the set $K \cap B_\varepsilon(x)$ is convex and at each point of $\partial K \cap B_{\varepsilon/2}(x)$ there exists only one hyperplane supporting $\partial K \cap B_{\varepsilon/2}(x)$). By using this result it is easily seen that $L^{-K}_{p'}(E')$ and $\big(L^K_p(E)\big)'$ are not topologically isomorphic (via the natural mapping) if K is a compact in $\mathbb{R}^n$ with any point of strict convexity, $K = \bar{\Omega}$ (Ω open set with segment property), $\mathrm{Vol}_n(\partial K) = 0$, $p \in (1,\infty) \smallsetminus \{2\}$, $n > 1$ and E is any Banach space (cf. [38, pp. 45–46] and Theorem 4.6).

3. Taking into account that any translation invariant bounded projection on L_∞ comes from a Borel measure on $\mathbb{R}^n$ [17], it is easy to check that L^{-K}_∞ and $\big(L^K_1\big)'$ are not topologically isomorphic (again via the natural mapping) for any compact set K in $\mathbb{R}^n$.

5. Isomorphism properties

In this section we represent weighted L_p–spaces of vector-valued entire analytic functions by means of spaces of vector sequences. Some other distribution spaces are represented by using sequence spaces also. The basic tools used are a vector version of the Shannon sampling theorem, the inequalities (3.4) of Theorem 3.2 and the duality studied in Theorem 4.6.

We begin with an extension of the Shannon theorem (see also [12, pp. 55–56], [38, p. 30]):

Theorem 5.1. *Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$, $p \in [1,\infty)$ and Q_b the cube $[-b,b]^n$. Let E be a Banach space. Suppose $f \in S^{Q_b}_\omega(E)$, $g \in S'_\omega$ and $\operatorname{supp}\hat{g} \subset Q_b$. Then, for all $x \in \mathbb{R}^n$, we have*

$$f * g(x) = \sum_{k\in\mathbb{Z}^n} \Big(\frac{\pi}{b}\Big)^n f(\frac{\pi}{b}k) g(x - \frac{\pi}{b}k) \tag{5.1}$$

(absolute convergence) and if $g \in L^{Q_b}_{p,\rho}$ *then* (5.1) *also holds in the norm of* $L^{Q_b}_{p,\rho}(E)$. *In particular, if* $g(x) = \left(\frac{b}{\pi}\right)^n \prod_{j=1}^n \frac{\sin bx_j}{bx_j}$, *we get the representation*

$$f(x) = \sum_{k\in\mathbb{Z}^n} f(\frac{\pi}{b}k) \prod_{j=1}^n \frac{\sin(bx_j - k_j\pi)}{bx_j - k_j\pi} \tag{5.2}$$

for all $x \in \mathbb{R}^n$ *(absolute convergence) and if* $\chi_{Q_b} \in M_{p,\rho}$ *(*$\Rightarrow g \in L^{Q_b}_{p,\rho}$*) then* (5.2) *also holds in the norm of* $L^{Q_b}_{p,\rho}(E)$.

Proof. Case $E = \mathbb{C}$. Suppose first that $g \in S^{Q_b}_\omega$. Then, from the classical case (see [12, p. 55]) we obtain (5.1). Suppose now that $g \in S'_\omega$ and $\operatorname{supp}\hat{g} \subset \overset{\circ}{Q}_b$. For $\varepsilon > 0$ let $g_\varepsilon(x) = \varphi(\varepsilon x)g(x)$, where $\varphi \in S_\omega$ satisfies $\varphi(0) = 1$ and $\operatorname{supp}\hat{\varphi} \subset \bar{B}_1$. By [33, Prop. 2, p. 17], $g_\varepsilon \in S^{\overset{\circ}{Q}_b}_\omega$ for each sufficiently small $\varepsilon > 0$. Thus $f * g_\varepsilon(x) = \sum_{k\in\mathbb{Z}^n} \left(\frac{\pi}{b}\right)^n f(\frac{\pi}{b}k)g_\varepsilon(x - \frac{\pi}{b}k)$. Furthermore, since there exist positive constants c, Λ such that $|g(x)| \leq ce^{\Lambda\omega(x)}$ for all $x \in \mathbb{R}^n$ (g is an entire function of exponential type by the Paley–Wiener–Schwartz theorem), the sum $\sum_{k\in\mathbb{Z}^n}|f(\frac{\pi}{b}k)||g(x-\frac{\pi}{b}k)|$ is finite. Then, taking the limit as ε tends to 0, and using the dominated convergence theorem we get that (5.1) holds for each $x \in \mathbb{R}^n$.

Next, suppose that $g \in S'_\omega$ and $\operatorname{supp}\hat{g} \subset Q_b$. We first show that there exist a family $\{g_t : 0 < t < 1\} \subset S'_\omega$ and positive numbers c, λ such that $\operatorname{supp}\hat{g}_t \subset \overset{\circ}{Q}_b$ for $t \in (0, 1)$, $|g_t(x)| \leq ce^{\lambda\omega(x)}$ for $x \in \mathbb{R}^n$ and $t \in (0, 1)$, and $g_t(x) \to g(x)$, as t tends to 0, for each $x \in \mathbb{R}^n$. For this we argue as in [38, p. 40]: Since Q_b has the segment property there exist open balls V_j and vectors y^j, $j = 1, \dots, N$, such that $Q_b \subset \cup_{j=1}^N V_j$ and $(Q_b \cap V_j) + ty^j \subset \overset{\circ}{Q}_b$ for $0 < t < 1$ and $j = 1, \dots, N$. Let $\varepsilon_0 > 0$ such that $Q_b + \bar{B}_{\varepsilon_0} \subset \cup_1^N V_j$ and let $\psi_j \in D(V_j)$ so that $\psi_j \geq 0$ and $\sum_1^N \psi_j = 1$ in $Q_b + \bar{B}_{\varepsilon_0}$. Put $\varphi_j = \mathcal{F}^{-1}\psi_j \in S^{V_j}_\omega$. Then it is easily seen that the functions $g_t = \sum_{j=1}^N e^{ity^j(\cdot)}(\varphi_j * g)$ satisfy the required conditions. Consequently, we get

$$f * g_t(x) = \sum_{k\in\mathbb{Z}^n} \left(\frac{\pi}{b}\right)^n f(\frac{\pi}{b}k)g_t(x - \frac{\pi}{b}k)\ .$$

Taking the limit as t tends to 0 and by using the dominated convergence theorem again we obtain (5.1). If E is any Banach space, it suffices to notice that the sum $\sum_{k\in\mathbb{Z}^n}\|f(\frac{\pi}{b}k)\|_E|g(x - \frac{\pi}{b}k)|$ is finite and then to make use of the Hahn–Banach theorem.

Finally, if $g \in L^{Q_b}_{p,\rho}$ we get

$$\sum_{k\in\mathbb{Z}^n} \|g(\cdot - \frac{\pi}{b}k) \otimes f(\frac{\pi}{b}k)\|_{p,\rho} \leq c\|g\|_{p,\rho} \sum_{k\in\mathbb{Z}^n} e^{\omega(\frac{\pi}{b}k)}\|f(\frac{\pi}{b}k)\|_E < \infty$$

c being the constant of the estimate $\rho(x + y) \leq c\,e^{\omega(x)}\rho(y)$. Hence it follows, taking into account the completeness of $L^{Q_b}_{p,\rho}(E)$ and the topological embedding $L^{Q_b}_{p,\rho}(E) \hookrightarrow L^{Q_b}_{\infty,\rho}(E)$ (Theorem 3.2), that (5.1) also holds in the norm of $L^{Q_b}_{p,\rho}(E)$ and the proof of the theorem is complete. □

Theorem 5.2. *Let* $\omega \in \mathcal{M}$, $\rho \in R(\omega)$, $p \in (1, \infty)$, $Q_b = [-b, b]^n$ *and let* E *be a Banach space. If* $\chi_{Q_b} \in M_{p,\rho}(E)$, *then the mapping* $\Phi : L^{Q_b}_{p,\rho}(E) \to l_p(\mathbb{Z}^n, E) : f \to \left(\rho(\frac{\pi}{b}k)f(\frac{\pi}{b}k)\right)_{k\in\mathbb{Z}^n}$ *is an isomorphism.*

Proof. By virtue of Theorem 3.2/(iv), we can find positive numbers $h(\leq \pi/b)$, c_1 such that

$$c_1\|(\rho(x^m)f(x^m))\|_{l_p(\mathbb{Z}^n,E)} \leq h^{-n/p}\|f\|_{p,\rho} \tag{5.3}$$

(resp. $c_1\|\big(\rho^{-1}(x^m)g(x^m)\big)\|_{l_{p'}(\mathbb{Z}^n,E')} \leq h^{-n/p'}\|g\|_{p',\rho^{-1}}$) holds for all sets $\{x^m : m \in \mathbb{Z}^n\}$ with $x^m \in Q^h_m = \prod_{j=1}^n [hm_j, h(m_j+1)[$ and for all $f \in L^{Q_b}_{p,\rho}(E)$ (resp. $g \in L^{Q_b}_{p',\rho^{-1}}(E')$). Since for each $k \in \mathbb{Z}^n$ there exists a unique $m \in \mathbb{Z}^n$ such that $\frac{\pi}{b}k \in Q^h_m$, it follows from (5.3) that $\|\Phi(f)\|_{l_p(\mathbb{Z}^n,E)} \leq \frac{h^{-n/p}}{c_1}\|f\|_{p,\rho}$ for all $f \in L^{Q_b}_{p,\rho}(E)$. So Φ becomes a bounded linear operator. Now put $g_k(x) = \prod_{j=1}^n \frac{sin(bx_j - k_j\pi)}{bx_j - k_j\pi}$ and observe that $\chi_{Q_b} \in M_{p,\rho} \cap M_{p',\rho^{-1}}$. Then, by Theorem 3.4 and Theorem 5.1, the closed linear span of the set $\{g_k \otimes e : k \in \mathbb{Z}^n, e \in E\}$ (resp. $\{g_k \otimes e' : k \in \mathbb{Z}^n, e' \in E'\}$) is $L^{Q_b}_{p,\rho}(E)$ (resp. $L^{Q_b}_{p',\rho^{-1}}(E')$). Consequently, in order to show that the estimate $\|f\|_{p,\rho} \leq C\,\|\Phi(f)\|_{l_p(\mathbb{Z}^n,E)}$, where C is a constant, holds in $L^{Q_b}_{p,\rho}(E)$, it will be enough to consider functions f in the span $\{g_k \otimes e\}$. Let f be such a function and let c_2 such that $\|g\|_{p',\rho^{-1}} \leq c_2\|Ng\|$ for all $g \in L^{Q_b}_{p',\rho^{-1}}(E')$ (here $N : L^{Q_b}_{p',\rho^{-1}}(E') \to \big(L^{Q_b}_{p,\rho}(E)\big)'$ is the natural isomorphism, see Theorem 4.6). Then, we have

$$\begin{aligned}
\|f\|_{p,\rho} &= \sup\big\{|Tf| : T \in \big(L^{Q_b}_{p,\rho}(E)\big)'\ ,\ \|T\| \leq 1\big\} \leq \\
&\leq \sup\Big\{\Big|\int_{\mathbb{R}^n}\langle f,g\rangle dx\Big| : g \in L^{Q_b}_{p',\rho^{-1}}(E')\ ,\ \|g\|_{p',\rho^{-1}} \leq c_2\Big\} = \\
&= \sup\Big\{\Big|\int_{\mathbb{R}^n}\langle f,g\rangle dx\Big| : g \in \mathrm{span}\{g_k \otimes e'\}\ ,\ \|g\|_{p',\rho^{-1}} \leq c_2\Big\}\ .
\end{aligned}$$

Fix now $g \in \mathrm{span}\{g_k \otimes e'\}$ such that $\|g\|_{p',\rho^{-1}} \leq c_2$. We can suppose, without loss of generality, that $f = \sum_{|k|\leq N} g_k \otimes e_k$ and $g = \sum_{|k|\leq N} g_k \otimes e'_k$ for any positive integer N. Then, taking into account (5.3) and that $\{g_k : k \in \mathbb{Z}^n\}$ is an orthogonal system in L_2, we get

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n}\langle f,g\rangle dx\Big| &= \Big|\sum_{|k|,|l|\leq N}\int_{\mathbb{R}^n} g_k g_l\, dx\ \langle e_k, e'_l\rangle\Big| = \\
&= (2b)^n\Big|\sum_{|k|\leq N}\langle e_k, e'_k\rangle\Big| \leq (2b)^n \sum_{|k|\leq N}\|e_k\|_E\,\|e'_k\|_{E'} = \\
&= (2b)^n \sum_{|k|\leq N}\|\rho(\frac{\pi}{b}k)f(\frac{\pi}{b}k)\|_E\,\|\rho^{-1}(\frac{\pi}{b}k)g(\frac{\pi}{b}k)\|_{E'} \leq \\
&\leq (2b)^n\Big\|\Big(\rho(\frac{\pi}{b}k)f(\frac{\pi}{b}k)\Big)\Big\|_{l_p(\mathbb{Z}^n,E)}\Big\|\Big(\rho^{-1}(\frac{\pi}{b}k)g(\frac{\pi}{b}k)\Big)\Big\|_{l_{p'}(\mathbb{Z}^n,E')} \leq \\
&\leq (2b)^n\frac{h^{-n/p'}}{c_1}\|g\|_{p',\rho^{-1}}\|\Phi(f)\|_{l_p(\mathbb{Z}^n,E)} \leq \\
&\leq (2b)^n\frac{h^{-n/p'}}{c_1}c_2\|\Phi(f)\|_{l_p(\mathbb{Z}^n,E)} = c_3\|\Phi(f)\|_{l_p(\mathbb{Z}^n,E)}\ .
\end{aligned}$$

In consequence, $\|f\|_{p,\rho} \leq c_3\|\Phi(f)\|_{l_p(\mathbb{Z}^n,E)}$ as was required. We complete the proof by showing that Φ is surjective. Let $(v_k)_{k\in\mathbb{Z}^n} \in l_p(\mathbb{Z}^n,E)$. Let $\mathcal{F}(\mathbb{Z}^n)$ be the family of all finite subsets of $\mathbb{Z}^n$ and for each J in $\mathcal{F}(Z^n)$ let $s_J = \sum_{k\in J}\rho^{-1}(\frac{\pi}{b}k)g_k \otimes v_k$.

Then the net $\{s_J : J \in \mathcal{F}(\mathbb{Z}^n), \supset\}$ is a Cauchy net in $L^{Q_b}_{p,\rho}(E)$ since for $J, K \in \mathcal{F}(\mathbb{Z}^n)$ we have

$$\|s_J - s_K\|_{p,\rho} = \Big\| \sum_{k\in J\smallsetminus K} (\cdot) - \sum_{k\in K\smallsetminus J} (\cdot)\Big\|_{p,\rho} \leq$$

$$\leq c_3 \Big\| \sum_{k\in J\smallsetminus K} \Phi(\cdot) - \sum_{k\in K\smallsetminus J} \Phi(\cdot)\Big\|_{l_p(Z^n,E)} = c_3 \Big(\sum_{k\in J\Delta K} \|v_k\|^p_E \Big)^{1/p}$$

where $J\Delta K$ is the symmetric difference of J and K. Since $L^{Q_b}_{p,\rho}(E)$ is complete, that net converges and its limit f satisfies $f(\frac{\pi}{b}k) = \rho^{-1}(\frac{\pi}{b}k)v_k$ for all $k \in \mathbb{Z}^n$, thus $\Phi(f) = (v_k)_{k\in\mathbb{Z}^n}$. □

Remark 5.3. 1. Theorem 5.2 extends some results in [38, 1.4.6] (cf. also [33, p. 42]) to the E–valued case.

2. It is easy to check that if $\chi_{[-1,1]^n} \in M_{p,\rho}(E)$ then $\chi_Q \in M_{p,\rho}(E)$ for every cube Q in $\mathbb{R}^n$. Under these conditions, by using Theorem 5.2, we easily get that the spaces $L^Q_{p,\rho}(E)$ are isomorphic to $l_p(\mathbb{Z}^n, E)$.

In the next corollaries the following well-known isomorphisms will be used: $l_p(\mathbb{Z}^n, E) \simeq l_p(E)$ and $L_{p,\rho}(E) \simeq L_p([0, 1], E)$.

Corollary 5.4. *Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$ and $p \in (1,\infty)$. Let K be the closure of a bounded open set in $\mathbb{R}^n$ with the segment property. Let E be a Banach space. If $\chi_{[-1,1]^n}, \chi_K \in M_{p,\rho}(E)$, then the space $L^K_{p,\rho}(E)$ is isomorphic to $l_p(E)$.*

Proof. Let $Q^{(1)}$ and $Q^{(2)}$ two cubes such that $Q^{(1)} \subset K \subset Q^{(2)}$. By the hypothesis, $L^{Q^{(i)}}_{p,\rho}(E)$ $(i = 1, 2)$ and $L^K_{p,\rho}(E)$ are complemented subspaces of $L_{p,\rho}(E)$ (see the proof of Theorem 4.6) and, by the previous remark, the spaces $L^{Q^{(i)}}_{p,\rho}(E)$ are isomorphic to $l_p(E)$. Therefore, $L^K_{p,\rho}(E)$ is isomorphic to a complemented subspace of $l_p(E)$ and $l_p(E)$ is isomorphic to a complemented subspace of $L^K_{p,\rho}(E)$. Since $l_p(l_p(E)) \simeq l_p(E)$, we are in a position to apply Pełczyński's decomposition method to conclude that $L^K_{p,\rho}(E) \simeq l_p(E)$. □

Corollary 5.5. *Let $\omega \in \mathcal{M}$, $\rho \in R(\omega)$ and $p \in (1,\infty)$. Let E be a Banach space with a symmetric basis and such that $l_p(E)$ is not isomorphic to $L_p(E)$. If $\chi_{[-1,1]^n} \in M_{p,\rho}(E)$, then the space $\ker S_Q$ (S_Q is the operator associated with the $L_{p,\rho}(E)$–Fourier multiplier χ_Q) is isomorphic to $L_p(E)$ for every cube Q in $\mathbb{R}^n$.*

Proof. By the hypothesis, we have $L_{p,\rho}(E) = L^Q_{p,\rho}(E) \oplus \ker S_Q$. Since $L_{p,\rho}(E) \simeq L_p([0, 1], E)$, it follows from [6] that $L_{p,\rho}(E)$ is a primary space and so either $L^Q_{p,\rho}(E)$ or $\ker S_Q$ is isomorphic to $L_{p,\rho}(E)$. But, by Remark 5.3/2, $L^Q_{p,\rho}(E) \simeq l_p(E)$ and since $L_p(E)$ and $l_p(E)$ are not isomorphic, we conclude that $\ker S_Q \simeq L_{p,\rho}(E) \simeq L_p(E)$. □

Remark. Let us mention some particular cases of Corollary 5.5: Let ω, ρ and p as in Corollary 4.8 and let Q be a cube. Assume $E = l_2$ and $p \neq 2$. By [21, p. 316], $L_p([0, 1], l_2) \simeq L_p$ and $l_p(l_2)$ is not isomorphic to L_p. Thus, by Corollary 5.5, the space $\ker S_Q$ is isomorphic to L_p. If $E = l_p$ and $p \neq 2$, $\ker S_Q$ is isomorphic to $L_2(l_p)$ since $l_2(l_p)$ is not an $\mathcal{L}_p$–space (cf. [21, p. 317]). Finally, if $E = l_p$ then $\ker S_Q$ is isomorphic to $L_p(l_p)$.

We consider now the spaces of Hörmander $B^c_{p,k}(\Omega,E)=\bigcup\{B_{p,k}(E)\cap\mathcal{E}'(K,E):$ K compact subset of $\Omega\}$. Here Ω is an open set in $\mathbb{R}^n$, $p\in[1,\infty]$, k is a temperate weight function on $\mathbb{R}^n$ [18, Def. 10.1.1], E is a Banach space, $B_{p,k}(E)=\{T\in S'(E):\widehat{T}\in L_{p,k}(E)\}$ and $\mathcal{E}'(K,E)=\{T\in D'(E):\operatorname{supp}T\subset K\}$. $B_{p,k}(E)$ becomes a Banach space with the norm $\|T\|_{B_{p,k}(E)}=\|\widehat{T}\|_{p,k}$ and $B^c_{p,k}(\Omega,E)$ is equipped with the inductive linear topology defined by the Banach spaces $\big(B_{p,k}(E)\cap\mathcal{E}'(K,E)\,,\,\|\cdot\|_{B_{p,k}(E)}\big)$, that is, $B^c_{p,k}(\Omega,E)=\operatorname{ind}_{\underset{K}{\to}}[B_{p,k}(E)\cap\mathcal{E}'(K,E)]$. For definitions, notation and elementary facts about these spaces see [18, Ch. X] (see also [25]). In [42] Vogt obtains the representation $B^c_{1,k}(\Omega)\simeq l_1^{(\mathbb{N})}$ (here $B^c_{1,k}(\Omega)=B^c_{1,k}(\Omega,\mathbb{C})$). We shall prove next that $B^c_{p,k}(\Omega,E)\simeq(l_p(E))^{(\mathbb{N})}$ for $p\in(1,\infty)$. The following elementary fact will be used: "Let $F=\operatorname{ind}_{\underset{j}{\to}}F_j$ be the strict inductive limit of a properly increasing sequence $F_1\subset F_2\subset\cdots$ of Banach spaces. Assume that every F_j is a complemented subspace of F_{j+1} and we put $F_{j+1}=F_j\oplus G_j$. Then, the mapping $F_1\oplus G_1\oplus G_2\oplus\cdots\to F=(f_1,g_1,g_2,\dots)\to f_1+g_1+g_2+\cdots$ is an isomorphism."

Corollary 5.6. *Let Ω be an open set in $\mathbb{R}^n$, $p\in(1,\infty)$ and k a temperate weight function on $\mathbb{R}^n$ with $k^p\in A^*_p$. Suppose that $\dim E<\infty$, $E=l_2$ or $E=l_p$. Then the space $B^c_{p,k}(\Omega,E)$ is isomorphic to $l_p^{(\mathbb{N})}$ if $\dim E<\infty$ or $E=l_p$, and to $(l_p(l_2))^{(\mathbb{N})}$ if $p\neq 2$ and $E=l_2$.*

Proof. Let (K_j) be a covering of Ω consisting of compact sets such that $K_j\subset\overset{\circ}{K}_{j+1}$, $K_j=\overline{\overset{\circ}{K}_j}$ and $\overset{\circ}{K}_j$ has the segment property (we may also assume, w.l.o.g., that each K_j is a finite union of n–dimensional compact intervals) and suppose that $E=l_2$ and $p\neq 2$. Then, $B^c_{p,k}(\Omega,l_2)=\operatorname{ind}_{\underset{j}{\to}}[B_{p,k}(l_2)\cap\mathcal{E}'(K_j,l_2)]$. In this inductive limit, the step $B_{p,k}(l_2)\cap\mathcal{E}'(K_j,l_2)$ is isomorphic (via the Fourier transform) to $L^{-K_j}_{p,k}(l_2)$ and this space is isomorphic, by Corollary 4.8 and Corollary 5.4, to $l_p(l_2)$. Furthermore, $L^{-K_j}_{p,k}(l_2)$ is a complemented subspace of $L^{-K_{j+1}}_{p,k}(l_2)$: $L^{-K_j}_{p,k}(l_2)\oplus\Big[\ker S_{-K_j}\cap L^{-K_{j+1}}_{p,k}(l_2)\Big]=L^{-K_{j+1}}_{p,k}(l_2)$. Thus, the space $G_j=\ker S_{-K_j}\cap L^{-K_{j+1}}_{p,k}(l_2)$ is isomorphic to an infinite–dimensional complemented subspace of $l_p(l_2)$. Then, by a result of Odell [28], G_j must be isomorphic to l_p, l_2, $l_p\oplus l_2$ or $l_p(l_2)$. But G_j contains a complemented copy of $l_p(l_2)$ (if Q is a cube such that $Q\subset K_{j+1}\setminus(K_j+\bar{B}_\varepsilon)$, for a sufficiently small $\varepsilon>0$, then $G_j\supset L^Q_{p,k}(l_2)$) and so G_j cannot be isomorphic to either l_p or l_2. If G_j were isomorphic to $l_p\oplus l_2$ then, since $l_1(l_p\oplus l_2)\simeq l_p\oplus l_2$, we could apply Pełczyński's decomposition method and conclude that $l_p(l_2)\simeq l_p\oplus l_2$, but this is false by [21, Ex. 8.2]. Therefore, necessarily $G_j\simeq l_p(l_2)$. In consequence, taking into account that $B^c_{p,k}(\Omega,l_2)\simeq L^{-K_1}_{p,k}(l_2)\oplus G_1\oplus G_2\oplus\cdots$, it results that $B^c_{p,k}(\Omega,l_2)\simeq(l_p(l_2))^{(\mathbb{N})}$. If $\dim E<\infty$ or $E=l_p$, one can reason in a similar way (recalling that the space l_p is prime [22, Th. 2.a.3]) and obtain the isomorphism $B^c_{p,k}(\Omega,E)\simeq l_p^{(\mathbb{N})}$. □

It is well-known that the Besov spaces $B^s_{p,q}$ $(=B^s_{p,q}(\mathbb{R}^n))$ are isomorphic to $l_q(l_p)$ (cf. [37] and [39]). Following Triebel's approach [39, Sect. 2.11.2], we shall show next the vector–valued counterpart of this result: $B^s_{p,q}(E)$ $(=B^s_{p,q}(\mathbb{R}^n,E))$ is isomorphic to $l_q(l_p(E))$. For definitions, notation and basic results about vector–valued Besov spaces see [32] and [2].

Corollary 5.7. *Let $1 < p < \infty$, $1 \le q \le \infty$, $-\infty < s < \infty$ and let E be a Banach space with the UMD–property. Then $B^s_{p,q}(E)$ is isomorphic to $l_q(l_p(E))$.*

Proof. By the "lifting theorem" for vector–valued Besov spaces (cf. [2, Th. 6.1]) we may assume that $s > 0$. Let $q_j = [-2^j, 2^j]^n$, $j = 0, 1, 2, \dots$ and $Q_0 = q_0$, $Q_j = q_j \setminus \mathring{q}_{j-1}$ for $j = 1, 2, \dots$ By [5] (see also Corollary 4.7), the characteristic function χ_j of Q_j is an $L_p(E)$–Fourier multiplier and so, if P_j denotes the operator associated with χ_j, a homogeneity argument shows that there exists a number c independent of $j = 0, 1, 2, \dots$ such that

$$\|P_j f\|_p \le c\|f\|_p \ , \qquad f \in L_p(E) \ . \tag{5.4}$$

Let $(\varphi_j)_{j=0}^{\infty}$ a dyadic resolution of unity in the sense of Definition p. 24 in [32]. Then, by [32, (36) p. 29], (5.4) is also valid for φ_j instead of χ_j, that is, also

$$\|\mathcal{F}^{-1}\varphi_j * f\|_p \le c\|f\|_p \ , \qquad f \in L_p(E) \ . \tag{5.5}$$

By using (5.4) and (5.5) we can proceed as in the scalar case (see [39]) and prove that

$$B^s_{p,q}(E) = \left\{ f \in L_p(E) : \|f\|^*_{B^s_{p,q}(E)} = \|(P_j f)\|_{l^s_q(L_p(E))} < \infty \right\} \tag{5.6}$$

and that $\|f\|_{B^s_{p,q}(E)} = \|(\mathcal{F}^{-1}\varphi_j * f)\|_{l^s_q(L_p(E))}$ and $\|f\|^*_{B^s_{p,q}(E)}$ are equivalent norms in the space $B^s_{p,q}(E)$ (we omit the details). Then, the mapping

$$\begin{array}{rccc} A: & B^s_{p,q}(E) & \to & \left(\sum_{j=0}^{\infty} \oplus L_p^{Q_j}(E)\right)_q \\ & f & \to & \left(2^{sj} P_j f\right)_{j=0}^{\infty} \end{array}$$

is well–defined and it is linear, injective and continuous. Furthermore, if $(f_j)_{j=0}^{\infty} \in \left(\sum_{j=0}^{\infty} \oplus L_p^{Q_j}(E)\right)_q$ then $f = \sum_{j=0}^{\infty} 2^{-sj} f_j \in B^s_{p,q}(E)$ and $Af = (f_j)_{j=0}^{\infty}$. In fact, obviously $\sum_{j=0}^{\infty} 2^{-sj} f_j$ converges in $L_p(E)$. Put $f = \sum_{j=0}^{\infty} 2^{-sj} f_j$. Since $P_k \circ P_j = 0$ if $j \ne k$ (if $f \in L_p(E)$ then $P_j f \in L_p^{Q_j}(E)$ and so, by Theorem 3.4, we can find a sequence $(g_\nu)_{\nu=0}^{\infty} \subset S^{Q_j}(E)$ such that $g_\nu \to P_j f$ in $L_p(E)$, thus $P_k g_\nu = \mathcal{F}^{-1}(\chi_k \mathcal{F} g_\nu) \underset{\nu}{\to} P_k(P_j f)$ and so $P_k(P_j f) = 0$) we see that $P_k f = \sum_{j=0}^{\infty} 2^{-sj} P_k f_j = \sum_{j=0}^{\infty} 2^{-sj} P_k(P_j f) = 2^{-sk} f_k$. Hence, it follows that $f \in B^s_{p,q}(E)$ and that $Af = (f_j)$. Therefore, A is an isomorphism. This completes the proof since by Corollary 5.4 each space $L_p^{Q_j}(E)$ is isomorphic to $l_p(E)$. □

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