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An algebraic approach to the structural properties of positive state control systems

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In this paper we deal with discrete-time linear control systems in which the state is constrained to lie in the positive orthant R^n_+ independently of the inputs involved, that is, the inputs can take negative values. Such (positive state) systems appear, for example, in ecology models where the removal of individuals from a population is described. Controllability and reachability are fundamental properties of a system that show its ability to move in space, which are analyzed from an algebraic point of view throughout the text, paying special attention to the single-input case. Copyright © 2017 John Wiley & Sons, Ltd.

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1. Introduction

In many biological processes, for instance, metabolism and drug ingestion, the state and input variables must be nonnegative values. The need to understand the properties of these kinds of systems induced the development of positive systems theory, which is well documented in the bibliography both from a control point of view and from an algebraic point of view [1, 6, 8, 15].

Nevertheless, there are many applications where it is not necessary to be so restrictive, and only the states must be nonnegative. For example, in several economic models [2, 14], or in population ecology where to describe the removal of individuals from a population, it is required that the control can take negative values ([4, 9, 13]). These processes can be represented by the (n-dimensional) discrete-time linear system with n states and m inputs

$$x(k+1) = Ax(k) + Bu(k), \quad k \in \mathbb{N}_0, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are nonnegative matrices, the state $x(\cdot) \in \mathbb{R}^n$ is a nonnegative vector and without restrictions for the control vector $u(\cdot) \in \mathbb{R}^m$, which is said to be *positive state system* according to reference [10]. From now on, let us denote them by $(A, B)_{ps} \geq 0$.

Differently from positive systems, only few authors have contributed to this emerging topic. It is worth mentioning Guiver *et al.* [10] who dealt with the controllability property of these systems, which was termed *positive state controllability*. It is understood that $(A, B)_{ps} \geq 0$ is positive state controllable in finite time if every nonnegative initial state x_0 can be transferred to any other nonnegative final state x_f in a finite number of steps and additionally the nonnegativity of the states involved is maintained. This property is equivalent to reachability ($x_0 = 0$) and null-controllability ($x_f = 0$) with positive state, which we briefly call *positive state reachability* and *positive state null-controllability*, respectively (or simply, reachability and null-controllability in this context).

Guiver *et al.* have demonstrated that under certain specific assumptions the problem of positive state controllability is equivalent to positive input controllability of a related positive system. Moreover, the authors give a characterization of positive state reachability in finite time in terms of the reachability matrix of the corresponding closed-loop system. However, in order to avoid the calculations of the reachability matrix, canonical forms representing all the pairs of matrices guaranteeing a certain structural property should be found. This algebraical approach would provides us with an effective way to check such a property avoiding both the aforementioned assumptions and the need to construct a related system, that is, directly into the open-loop system. In this work, after some preliminary results for a general positive state system where reachability and monomial reachability are connected, the general case for single-input positive state systems is characterized, and some examples from

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economy and population ecology models are presented. Finally, a multiple-input system showing the difficulty to extend the obtained results to these more complex systems is introduced.

The authors want to emphasize that apart from the analysis of these properties and of others as observability and stability of positive state systems, some interesting problems could be addressed in the future such as the study of reachability algorithms, of a canonical form by means of an appropriated positive similarity as well as the analysis of some property under feedback, among others.

This manuscript has been organized as follows: Section II examines some notations and basic definitions. In Section III, the preliminary results on positive state reachability are presented. Canonical forms for positive state controllability and reachability are given in the single-input case in Section IV. Later on, several applications are introduced both for single-input (Section V) and for multi-input model (Section 6) showing the great differences between both cases. Finally, in Section 7, the main contributions of this article are commented.

2. Notions and basic definitions

From now on, let us write $x > 0$ if $x \geq 0$ but it is not equal to 0. We recall that a **positive i-monomial vector** (or simply, i-monomial vector) is a positive multiple of the canonical vector e_i of R^n. Similarly, a **monomial matrix** is a nonsingular matrix having a unique positive entry in each row and column. Besides that, a monomial matrix whose nonzero entries are all unitary is called **permutation matrix**.

For any given positive state system $(A, B)_{ps} \geq 0$ described by equation (1), its solution is given by:

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-1-i} B u(i) = A^k x(0) + [B\ AB\ A^2B \cdots A^{k-1}B] \begin{bmatrix} u(k-1) \\ u(k-2) \\ u(k-3) \\ \vdots \\ u(0) \end{bmatrix}, \tag{2}$$

where the matrix

$$\mathcal{R}_k(A, B) = [B\ AB\ A^2B \cdots A^{k-1}B] \quad k = 1, 2 \ldots \tag{3}$$

is called **the reachability matrix in** k **steps**, $k = 1, 2, \ldots$. This matrix allows us to study the system states $x(k)$ that can be reached in a finite number of steps. Concretely, we are interested in the controllability property in order to be able to transfer any given dynamic system under control input from its initial state to any final desirable state, which is defined as follows.

Definition 1 *A positive state system* $(A, B)_{ps} \geq 0$ *is said to be* **positively state controllable** *if for any pair of states, initial* $x_0 = x(0)$ *and final* x_f*, there exists* $k \in \mathbb{N}$ *and a control sequence* $u(0), u(1), \ldots, u(k-1)$*, that steer the system states from the initial state* x_0 *to* x_f *at time* k*, i.e* $x(k) = x_f$*, through nonnegative states* $x(0), x(1), \ldots, x(k-1)$*.*

Generally, two more definitions related to the previous one can be found: controllability from the origin or reachability and controllability to the origin or null-controllability.

Definition 2 *A positive state system* $(A, B)_{ps} \geq 0$ *is said to be* **positively state reachable** *if for any final state* x_f *there exists* $k \in \mathbb{N}$ *and a control sequence* $u(0), u(1), \ldots, u(k-1)$*, such that steer the system state from the origin at zero instant, that is* $x(0) = 0$*, to* x_f *at time* k *through nonnegative states* $x(0), x(1), \ldots, x(k-1)$*. In this case,* x_f *is said to be* **reachable at finite time** k*.*

Additionally, it is said that a system $(A, B)_{ps} \geq 0$ is **positively monomial-state reachable** (see for example [17]) if every monomial vector of R^n_+ (equivalently, every canonical vector e_i, $i = 1, 2, \ldots, n$ of $\mathbb{R}^n_+$) is reachable for some finite time. Finally,

Definition 3 *A positive state system* $(A, B)_{ps} \geq 0$ *is said to be* **positively state null-controllable** *if for any initial state* x_0 *there exists* $k \in \mathbb{N}$ *and a control sequence* $u(0), u(1), \ldots, u(k-1)$*, such that steer the system states from* x_0*, at zero instant, to the final state* $x_f = 0$*, at time* k *through nonnegative states* $x(0), x(1), \ldots, x(k-1)$*. In this case,* x_0 *is said to be* **null-controllable at finite time** k*.*

For standard discrete-time systems, definitions 1 and 2 are equivalent, but not definition 3 (see [6]). Similarly to the positive case, these concepts are no longer equivalent in the positive state case as commented before, but there are many more positively state reachable systems than under positive constrains. All these differences are analyzed throughout the paper.

3. Preliminary results

Although from a theoretical point of view, any timing, finite or infinite, is possible to reach a state, the next results are always under conditions of finiteness to be useful in real-life practice. The next theorem states that it is enough to study if the canonical vectors of $\mathbb{R}^n_+$ are reachable at some finite time (positive monomial-state reachability) to achieve reachability.

Theorem 1 *Considering a positive state system $(A, B)_{ps} \geq 0$, then the pair (A, B) is positively monomial-state reachable in finite time N if and only if the same pair is positively state reachable in finite time N.*

Proof $\Rightarrow$) To simplify notation, let us define the following vectors:

$$v_1 = B, \; v_2 = AB, \; v_3 = A^2B, \; \ldots, \; v_N = A^{N-1}B$$
$$u_i(k) = u_{i,N-k}, \; k = 0, 1, \ldots, N-1, N \in \mathbb{N}.$$

By the definition of a positively monomial-state reachable system, every canonical vector e_i, $i = 1, 2, \ldots, n$ is positively state reachable at most in finite time N, then taking into account (2) and (3) (remind that $x(0) = 0$),

$$e_i = \mathcal{R}_N(A, B) \begin{bmatrix} u_i(N-1) \\ u_i(N-2) \\ \vdots \\ u_i(1) \\ u_i(0) \end{bmatrix} = \sum_{j=1}^{N} u_{ij} v_j \tag{4}$$

and the different intermediate states $x_i(0), x_i(1), \ldots, x_i(k-1)$ to reach e_i are also nonnegative. Consequently,

$$x_i(1) = Bu_i(0) > 0 \Rightarrow v_1 u_{iN} > 0 \Rightarrow \boxed{u_{iN} > 0} \tag{5}$$

$$x_i(2) = Bu_i(1) + ABu_i(0) \geq 0 \Rightarrow \boxed{v_1 u_{i,N-1} + v_2 u_{iN} \geq 0} \tag{6}$$

$$\vdots$$

$$x_i(N-1) = Bu_i(N-2) + ABu_i(N-3) + \cdots + A^{N-3}Bu_i(1) + A^{N-2}Bu_i(0) \geq 0$$

$$\Rightarrow \boxed{\sum_{j=1}^{N-1} u_{i,j+1} v_j \geq 0.} \tag{7}$$

Moreover, any vector $x_f \in \mathbb{R}^n_+$ is a positive combination of e_i, i.e., if $\alpha_i \geq 0$ then

$$x_f = \sum_{i=1}^{n} \alpha_i e_i = \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{N} u_{ij} v_j \right) = \sum_{j=1}^{N} \left(\sum_{i=1}^{n} \alpha_i u_{ij} \right) v_j.$$

Therefore, the sequence of controls used to reach $x_f \in \mathbb{R}^n_+$ is $u(k) = \sum_{i=1}^{n} \alpha_i u_{i,N-k}$ with $k = 0, 1, \ldots, N-1$ with $u(0) \geq 0$ by (5).

Additionally, to verify that $x_f = x(N)$ is positively state reachable at finite time N we have to prove that every state involved in the previous steps needed to attain x_f is also nonnegative. The first state, $x(1) = Bu(0)$, is obviously nonnegative. Let us check the other ones:

$$x(2) = Bu(1) + ABu(0) = v_1 \sum_{i=1}^{n} \alpha_i u_{i,N-1} + v_2 \sum_{i=1}^{n} \alpha_i u_{i,N} = \sum_{i=1}^{n} \alpha_i (u_{i,N-1} v_1 + u_{iN} v_2)$$

is nonnegative by (6), and so on, until the last step

$$\begin{aligned} x(N-1) \quad &= Bu(N-2) + ABu(N-3) + \cdots + A^{N-3}Bu(1) + A^{N-2}Bu(0) = \\ &= \sum_{j=1}^{N-1} v_j \left(\sum_{i=1}^{n} \alpha_i u_{i,j+1} \right) = \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{N-1} u_{i,j+1} v_j \right) \end{aligned}$$

that is nonnegative by (7) too, which concludes the proof.

$\Leftarrow$) The inverse proof immediately follows by the definition of positive state reachability. ∎

The next example illustrates as this characterization facilitate the analysis of the reachability property of a system.

Example 1 *([11], Example 4.3) Given* $(A, B)_{ps} \geq 0$ *with*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is simple that the canonical vectors e_1 *and* e_2 *are reachable in one step using the sequence of controls* $u_1(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *and* $u_1(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, *respectively. Therefore, this system is positively state reachable in finite time* N.

From this theorem we can derive a necessary condition to be a reachable positive state system.

Lemma 1 *Let* $(A, B)_{ps} \geq 0$ *be a reachable n-dimensional positive state system in finite time* N, *then* $n \leq Nm$, *and the rank of the reachability matrix in N-steps* $\mathcal{R}_N(A, B)$ *associated is n, that is,*

$$Rank([B\ AB\ A^2B\ \cdots\ A^{N-1}B]) = n.$$

Proof If $(A, B)_{ps} \geq 0$ is positively state reachable then every canonical vector can also be reached, hence using equation (4) for each canonical vector, it is derived that

$$I_n = \mathcal{R}_N(A, B)U \quad \text{with} \quad U = \begin{bmatrix} u_1(N-1) & u_2(N-1) & \cdots & u_n(N-1) \\ u_1(N-2) & u_2(N-2) & \cdots & u_n(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(1) & u_2(1) & \cdots & u_n(1) \\ u_1(0) & u_2(0) & \cdots & u_n(0) \end{bmatrix}$$

where I_n is de $n \times n$ identity matrix. Therefore, by a basic property of the rank of a product of matrices, $\text{Rank}(I_n) \leq \min(\text{Rank}(\mathcal{R}_N(A, B)), \text{Rank}(U))$. Consequently, $n \leq \text{Rank}(\mathcal{R}_N(A, B))$ and $n \leq \text{Rank}(U)$.

In addition, $\mathcal{R}_N(A, B) \in \mathbb{R}^{n \times mN}$ and $U \in \mathbb{R}^{mN \times n}$ hence $\text{Rank}(\mathcal{R}_N(A, B)) \leq \min(n, mN) \leq n$, and $\text{Rank}(U) \leq \min(n, mN) \leq n$. Thus, combining both inequalites we obtain that $\text{Rank}(U) = n$ and $\text{Rank}(\mathcal{R}_N(A, B)) = n$, which is what had to be proven. Besides that, observe that $n \leq mN$ since $\text{Rank}(\mathcal{R}_N(A, B)) = n \leq \min(n, mN) \leq mN$. ∎

From the aforementioned lemma it is deduced that a system may be reachable in time 1 only if the matrix B has at least as many columns as the matrix A, that is, $n \leq m$. Similarly, a single-input positive state system ($m = 1$) may be reachable only if $n \leq N$. Furthermore, it is clear that, in the same way that for standard reachability and positive reachability (see [7]), any state that can be reached in a finite number of steps can also be reached within n steps with an appropriate choice of inputs. Thus, let us sum up these conclusions with the following proposition.

Proposition 1 *Let the pair* $(A, B)_{ps} \geq 0$ *be reachable in finite time, then every nonnegative state* $x(k) \in \mathbb{R}^n_+$ *can be reached in at most n steps. Besides that, if* $m = 1$, *the number of steps needed to reach every nonnegative state is exactly n.*

Corollary 1 *Let* $(A, B)_{ps} \geq 0$ *be a reachable n-dimensional positive state system in finite time, then the rank of the reachability matrix in n-steps* $\mathcal{R}_n(A, B)$ *is n.*

Example 2 *Let* $(A, b)_{ps} \geq 0$ *with*

$$A = \begin{bmatrix} * & 1 & 0 \\ * & 0 & 1 \\ * & 0 & 1 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \mathcal{R}_3(A, b) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

with $*$ *a nonnegative entry.*

Clearly, canonical vector e_3 *can be reached in one step. In the same way, canonical vector* e_2 (e_3) *in two steps (three steps) using the sequence of controls* $u(0) = 1$, *and* $u(1) = -1$ $(u(0) = 1,\ u(1) = -1,\ \text{and}\, u(2) = 0)$, *which maintains the nonnegativity of every state in each previous step. Therefore,* $(A, b)_{ps} \geq 0$ *is reachable. Observe that in this case the rank of the reachability matrix* $\mathcal{R}_3(A, b)$ *is 3 and that this same system is not positively reachable (using only nonnegative inputs).*

The next example illustrates that a condition on the rank of the reachability matrix in n-steps is not a sufficient condition to guarantee the reachability property of the system.

Example 3 *Let* $(A, b)_{ps} \geq 0$ *be a system given by the pair*

$$A = \begin{bmatrix} * & 1 & 0 \\ * & 1 & 1 \\ * & 1 & 1 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \mathcal{R}_3(A, b) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

with $*$ *a nonnegative entry.*

Its reachability matrix $\mathcal{R}_3(A, b)$ *has rank equal to 3 but it is not positively state reachable because the sequence of controls to attain* e_1 *is* $u(0) = 1$, $u(1) = -2$ *and* $u(2) = 0$, *which has previously steered the initial state to the non-positive state* $x(2) = [0\ 0\ -1]^T$ *in two steps.*

4. Single-Input Positive State Systems

A generalization of the structure given by the matrices involved in example 2 provides us with a class of reachable positive state systems. We have proved that a single-input positive state system may be reachable only if the input vector b have a specific structure.

Theorem 2 *Let* $(A, B)_{ps} \geq 0$ *be a positive state system. If it is positively state reachable then the control matrix* b *is a monomial vector.*

Proof We prove this result by reduction ad absurdum. Let us suppose that $b = b_1 e_{i_1} + b_2 e_{i_2} + \ldots + b_r e_{i_r}$, with $r \geq 2$, $b_j > 0$, for $j = 1, 2, \ldots, r$. Since $(A, B)_{ps} \geq 0$ is positively state reachable hence all canonical vectors $e_{i_1}, e_{i_2}, \ldots, e_{i_r}$ are positively state reachable. Let k_j, $j \in \{1, 2, \ldots, r\}$ the number of steps needed to reach the corresponding canonical vector e_{i_j}, then for $j = 1, 2, \ldots, r$:

$$e_{i_j} = bu_{i_j}(k_j - 1) + Abu_{i_j}(k_j - 2) + \cdots + A^{k_j - 2}bu_{i_j}(1) + A^{k_j - 1}bu_{i_j}(0)$$

with $u_{i_j}(0) > 0$. Note that, since b is not a monomial vector, $k_j > 1$, for all $j = 1, 2, \ldots, r$.

Let us consider, without loss of generality, that $1 < k_1 \leq k_2 \leq \ldots \leq k_r \leq n$. In this case, every canonical vector can be rewritten as:

$$e_{i_j} = \alpha_{j_0} b + \alpha_{j_1} Ab + \cdots + \alpha_{j_{k_j - 1}} A^{k_j - 1} b + \alpha_{j_{k_j}} A^{k_j} b + \cdots + \alpha_{j_{n-1}} A^{n-1} b, \quad \text{with}$$

$$\alpha_{j_s} = \begin{cases} u_{i_j}(k_j - 1 - s) & \text{if} \quad 0 \leq s \leq k_j - 1 \\ 0 & \text{if} \quad k_j \leq s \leq n - 1 \end{cases}, \text{ being } \alpha_{j_{k_j - 1}} = u_{i_j}(0) > 0. \tag{8}$$

Therefore,

$$\begin{aligned} b &= \textstyle\sum_{j=1}^{r} b_j(\alpha_{j_0} b + \alpha_{j_1} A + \cdots + \alpha_{j_{k_j - 1}} A^{k_j - 1} + \alpha_{j_{k_j}} A^{k_j} b + \cdots + \alpha_{j_{n-1}} A^{n-1} b) = \\ &= b(b_1\alpha_{1_0} + b_2\alpha_{2_0} + \cdots + b_r\alpha_{r_0}) + Ab(b_1\alpha_{1_1} + b_2\alpha_{2_1} + \cdots + b_r\alpha_{r_1}) + \cdots + A^{n-1}b(b_1\alpha_{1_{n-1}} + b_2\alpha_{2_{n-1}} + \cdots + b_r\alpha_{r_{n-1}}). \end{aligned}$$

If, for $s = 0, 1, \ldots, n - 1$, we define

$$\beta_s = b_1\alpha_{1_s} + b_2\alpha_{2_s} + \cdots + b_r\alpha_{r_s} \tag{9}$$

then it is easy to see that

$$(\beta_0 - 1)b + \beta_1 Ab + \cdots + \beta_{n-1} A^{n-1} b = 0.$$

Moreover, observe that by construction (see equations (8) and (9)), at least $\beta_s > 0$ for $s = k_1 - 1, k_2 - 1, \ldots, k_r - 1$. But by Corollary 1, the set $\{b, Ab, \ldots, A^{n-1}b\}$ is linearly independent, hence $\beta_0 = 1$, and $\beta_1 = \ldots = \beta_{n-1} = 0$, which is a contradiction. Consequently, vector b must be monomial. ∎

Theorem 3 *Let* $(A, b)_{ps} \geq 0$ *be a single-input positive state system. If* $(A, b)_{ps} \geq 0$ *is reachable, then there exists a permutation matrix* P *such that*

$$P^T AP = \begin{bmatrix} * & + & 0 & \cdots & 0 \\ * & 0 & + & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \cdots & + \\ * & * & * & \cdots & * \end{bmatrix}, \text{ and } P^T b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ + \end{bmatrix} \tag{10}$$

with $+$ *a positive entry and* $*$ *a nonnegative entry.*

Proof Applying Theorem 2, we derive that b is a monomial vector, that is, $b = \alpha e_{j_1}$, $\alpha > 0$, for some $j_1 \in \{1, \ldots, n\}$. Let us check that for every canonical vector $e_{j_r} \in \mathbb{R}^n_+$ with $j_r \in \{1, 2, \ldots, n\}$, and $j_r \neq j_1$, there exists a column s_r of A such that $col_{s_r}(A) = \alpha_r e_{j_r} + \beta_r e_{j_1}$, for $\alpha_r > 0, \beta_r \geq 0$.

Assuming reachability, every canonical vector e_{j_r} is reachable. Since $x(1)$ is a j_1-monomial vector and $j_r \neq j_1$ then there exists a $k \in \mathbb{N}$ such that the canonical vector e_{j_r} (a j_r-monomial vector) is reached in $k+1$-steps, that is:

$$e_{j_r} = x(k+1) = bu(k) + Ax(k) \tag{11}$$

where $x(k) \geq 0, \forall k$.

Note that, on the one hand, $x(k)$ should be nonzero ($x(k) > 0$) since it is assumed that $j_r \neq j_1$ and, on the other hand, $u(k) \leq 0$ since otherwise the above equality is impossible. Moreover, if $u(k) = 0$ then $e_{j_r} = Ax(k)$, and it is only possible if A has at least one column that is an j_r-monomial vector, which proves the previous affirmation for $\beta_r = 0$.

Assuming that $u(k) < 0$, then

$$Ax(k) = e_{j_r} - \alpha u(k) e_{j_1} \neq 0$$

with $Ax(k) > 0$ being a nonnegative linear combination of some columns of A (combining those columns corresponding to the nonnegative entries of $x(k)$). It forces that there exists at least a column s_r of A such that $col_{s_r}(A) = \alpha_r e_{j_r} + \beta_r e_{j_1}$, for $\alpha_r > 0, \beta_r \geq 0$.

Is is obvious that we have determined $n-1$ different columns of A, one column for each canonical vector. Note that the remaining column of A can also be in the same way that those columns already determined, that is the addition of a j_s-monomial vector with a j_1-monomial vector, or alternatively one of the following types, $col(A) = 0$, $col(A) = \beta e_{j_1}$ with $\beta > 0$, or a column with at least two different positive entries of the position j_1.

Now, let us observe that the system is reachable depending on the result of the product of A with b. Since b is j_1-monomial then $A \cdot b = col_{j_1} A$. If $col_{j_1} A = 0$ or $col_{j_1}(A) = \beta e_{j_1}$ with $\beta > 0$ then clearly,

$$rank \left[b \mid Ab \mid \cdots \mid A^{n-1} b \right] \leq 1$$

and therefore the system cannot be reachable.

Now, let us see that if $col_{j_1} A$ has at least two positive entries two different positive entries of the position j_1, that is, $col_{j_1} A = \alpha_1 e_{j_{r_1}} + \alpha_2 e_{j_{r_2}} + w$ with $j_{r_1}, j_{r_2} \neq j_1$, $\alpha_{r_1} > 0$, $\alpha_{r_2} > 0$ and $w \geq 0$ then the system is not reachable. Let us calculate $x(2)$:

$$x(2) = Abu(0) + bu(1) = col_{j_1} A u(0) + bu(1) = (\alpha_{r_1} e_{j_{r_1}} + \alpha_{r_2} e_{j_{r_2}}) u(0) + A \cdot w + bu(1).$$

We recall that $u(0) > 0$ thus, although we take $u(1) < 0$ satisfying $A \cdot w + bu(1) \geq 0$, $x(2)$ has at least two positive entries two different positive entries of the position j_1, too. Thus, in 2-steps no monomial vectors can be reached.

Moreover, by the structure of the columns of A, the vector $A\alpha_{r_1} e_{j_{r_1}} + A\alpha_{r_2} e_{j_{r_2}}$ has at least two different positive entries of the position j_1, and in the same way $x(3)$ since

$$x(3) = Ax(2) + bu(2) = (\alpha_{r_1} A e_{j_{r_1}} + \alpha_{r_2} A e_{j_{r_2}}) u(0) + A^2 \cdot w + Abu(1) + bu(2)$$

with $A^2 \cdot w + Abu(1) \geq 0$. Therefore, no monomial vectors can be reached in 3-steps and so consecutively for $k = 4, \ldots, n$-steps. In this case, the system is not reachable again.

As a consequence, if the system is reachable then $col_{j_1} A = \alpha_2 e_{j_2} + \beta_2 e_{j_1}$, for $\alpha_2 > 0, \beta_2 \geq 0$. Hence, the j_2-monomial vector can be reached in 2-steps since

$$x(2) = Abu(0) + bu(1) = \alpha_2 e_{j_2} + \beta_2 e_{j_1} + bu(1)$$

is a j_2-monomial vector for a suitable choice of $u(1) \leq 0$.

Reasoning as in the case of $x(1)$ (a j_1-monomial vector) but for $x(2)$ (a j_2-monomial vector), we can reach in k-steps a j_k-monomial vector for $k = 3, 4, \ldots, n$ satisfying that $A \cdot e_{j_k} = \alpha_{k+1} e_{j_{k+1}} + \beta_{k+1} e_{j_1}$, $\alpha_{k+1} > 0$, and $\beta_{k+1} \geq 0$. Thus, constructing the permutation matrix Q such that j_k is assigned to $n+1-k$, for $k = 1, \ldots, n$, we obtain the similar system (12) by taking P the transpose of Q. ∎

Moreover, the inverse implication is true too.

Lemma 2 *The positive state system* $(A, b)_{ps} \geq 0$ *where*

$$A = \begin{bmatrix} * & + & 0 & \cdots & 0 \\ * & 0 & + & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \cdots & + \\ * & * & * & \cdots & * \end{bmatrix}, \text{ and } b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ + \end{bmatrix}$$

with $+$ *a positive entry and* $*$ *a nonnegative entry is positively state reachable.*

Proof From Theorem 1, it suffices to prove that every canonical vector is positively state reachable to verify that the positive state system $(A, B)_{ps} \geq 0$ is positively state reachable. Note that b is proportional to the last canonical vector, i.e. $b = \alpha e_n$ with $\alpha > 0$ hence it is straightforward to obtain e_n in one step since $x(1) = bu(0) = e_n$. The following canonical vectors can be reached in successive steps. From (1), in the second step we get:

$$x(2) = Ax(1) + bu(1) = Ae_n + bu(1)$$

where $Ax(1)$ is the n-th column of A, with pattern $[0, \dots, 0, +, *]^T$, and $bu(1)$ is a vector proportional to the last canonical vector, with pattern $[0, \dots, 0, +]^T$. Therefore, there exists a control $u(1) \leq 0$ such that it is possible to attain the canonical vector e_{n-1} and thus $x(2) = e_{n-1}$. Working in that way, we obtain the remaining canonical vectors until getting the fist canonical vector at step n because

$$x(n) = Ax(n-1) + bu(n-1) = Ae_2 + bu(n-1)$$

where $Ax(n-1)$ is the second column of A, with structure $[+, 0, \dots, 0, *]^T$, and $bu(1)$ is a vector proportional to e_n. Then, there exists a control $u(n-1) \leq 0$ such that it is possible to reach the canonical vector e_1, which finishes the proof. ∎

Therefore, we can sum up the preceding results as follows.

Theorem 4 *Let $(A, b)_{ps} \geq 0$ be a single-input positive state system, then $(A, b)_{ps} \geq 0$ is reachable if and only if there exists a permutation matrix P such that*

$$P^T AP = \begin{bmatrix} * & + & 0 & \cdots & 0 \\ * & 0 & + & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \cdots & + \\ * & * & * & \cdots & * \end{bmatrix}, \text{ and } P^T b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ + \end{bmatrix} \qquad (12)$$

with + a positive entry and $*$ a nonnegative entry.

This canonical form allows us to deduce a new canonical form for the positively state controllable single-input case.

Proposition 2 *Considering a positive state single-input system $(A, B)_{ps} \geq 0$, this system is positively state completely controllable if, and only if, there exists a permutation matrix P such that*

$$P^T AP = \begin{bmatrix} 0 & + & 0 & \cdots & 0 \\ 0 & 0 & + & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & + \\ * & * & * & \cdots & * \end{bmatrix}, \text{ and } P^T b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ + \end{bmatrix} \qquad (13)$$

with + a positive entry and $*$ a nonnegative entry.

Proof The positive state controllability is equivalent to positive state reachability and null-controllability. Thus, from reachability and Theorem 4, there exists a permutation matrix P such that the structure of matrices $P^T AP$ and $P^T B$ are fully determined by canonical form (12). This implies that positive state null-controllability is only possible if the zero vector is reached from a initial vector $x_0 > 0$ in exactly n-steps, but furthermore it is simply to prove that it is only possible if and only if $P^T AP$ and $P^T B$ has the structure given in equation (12) under permutation similarity. ∎

It is important to recall that null-controllability under positive constrains implies that A is nilpotent (see [16]). However, clearly by its structure, the state matrix of a positively state null-controllable system has not to necessarily be nilpotent.

5. Applications of single-input positive state systems

5.1. Management of a production chain

Typical examples of positive state system in economy are the production chains. Let us see an example ([2]) where we consider the industrial system formed by the following three compartments:

- $x_1(k)$: amount of raw material stored at instant k and used by the primary industry to produce simple objects.
- $x_2(k)$: amount of raw material contained in the objects of the primary production at instant k. A fraction p_i of it is used for a secondary production of more sophisticated objects and another fraction $r_i + c_i$ is bought by consumers who can recycle (c_i) these kinds of objects as new raw material after its usage or not (so that it leaves the production chain (r_i)).
- $x_3(k)$: amount of raw material contained in the more sophisticated objects of the secondary production at instant k, which again a fraction of it is recycled after usage and another one is lost.

This system can be represented by a discrete-time system given by the equations:

$$\begin{cases} x_1(k+1) = (1-p_1)x_1(k) + c_2x_2(k) + c_3x_3 \\ x_2(k+1) = (1-p_2-r_2-c_2)x_2(k) + p_1x_1(k) \\ x_3(k+1) = (1-r_3-c_3)x_3(k) + p_2x_2(k). \end{cases}$$

Let us suppose that we are interesting in selling a fraction of the raw material to another production chain, but ensuring that the original production chain has the supply it needs. That situation implies that the system has a negative control but the state must be nonnegative. That is, we have a positive state system (1) with

$$A = \begin{bmatrix} 1-p_1 & c_2 & c_3 \\ p_1 & 1-p_2-r_2-c_2 & 0 \\ 0 & p_2 & 1-r_3-c_3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and similar to

$$A = \begin{bmatrix} 1-r_3-c_3 & p_2 & 0 \\ 0 & 1-p_2-r_2-c_2 & p_1 \\ c_3 & c_2 & 1-p_1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

using the permutation matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Note that, by Proposition 2, this production chain is a positively state controllable system only if $1-p_2-r_2-c_2 = 0$ and $1-r_3-c_3 = 0$, which implies that the material in step i for $i = 2, 3$ only depends on the material in the previous step $i-1$, $i = 2, 3$.

5.2. Epidemiology

In [8], a system is introduced describing the process of ascertaining in a medical center whether an individual is affected by a certain disease or not by using a test consisting of blood drawn followed by an analysis (positive, negative or ambiguous). Namely, the state equations of the systems are:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 0 & r & 0 \\ 1 & 0 & a \\ 0 & p & c \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(k)$$

where $u(k)$ are individuals presenting some symptoms of such a disease, supervised by a general practitioner, going to perform the test during the week k, $x_1(k)$, $x_2(k)$ and $x_3(k)$ represent the number of individuals waiting for a test, the number of individuals performing the test and the number of individuals under treatment (ill individuals), respectively. In addition, p is the probability that the test is positive, r the probability that the test has to be repeated (because initial test is ambiguous), a certain percentage a of individuals under treatment need to perform the test once more, and a certain percentage c of individuals remain under treatment.

Although initially this system was considered as a positive system, note that in that case we may find an individual having to consecutively perform the test if it was always ambiguous. However, this same model can be considered as a positive state system if those individuals who have no symptoms can be remove from the list of patients waiting for a test by the general practitioner.

Note that this positive state system is reachable since the control action can quit the individuals having to unnecessarily repeat the test.

6. Introduction to multiple-input positive state systems

In [12] a discrete-time system (1) with

$$A = \begin{bmatrix} 0.0995 & 0 & 0 & 0 & 0 \\ 0 & 0.968 & 0 & 0 & 0 \\ 0.005 & 0.032 & 0.92 & 0 & 0 \\ 0 & 0 & 0.08 & 0.267 & 0 \\ 0 & 0 & 0 & 0.733 & 0.748 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 15 & 0 & 0 & 0 & 0 \\ 0 & 38 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 0 & 14 \end{bmatrix}$$

is used to control the amount of pollution in the five Great Lakes of North America. Let us suppose now that the incoming pollutant flows stop and that we implement measures to reduce pollution in each water reservoir. That is, we apply negative controls instead of the previous positive controls. To guarantee the usefulness of model, we need to ensure that states do not become negative over time. Otherwise, estimation of pollutant would be incorrect, specifically, it would be underestimated. Therefore, we can consider this model as a multi-input positive state system. But, note that, although the structure of matrix A does not hold Theorem 2, it is a monomial positively state reachable system because B is a monomial matrix and hence, by Theorem 1, it is a positively state reachable system, which allows us to control the amount of pollution in the five Great Lakes.

We can also find the opposite situation. In the multiple-input case, a positive state system $(A, B)_{ps} \geq 0$ can be reachable having some nonmonomial columns in the input matrix B, which are necessary to guarantee such property.

Example 4 *Let $(A, B)_{ps} \geq 0$ be a system given by*

$$A = \begin{bmatrix} * & 1 & 0 \\ * & 1 & 0 \\ * & 1 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

with $*$ a nonnegative entry.

It is clear that canonical vectors e_3 and e_2 can be reached in one step. In the same way, canonical vector e_1 in two steps using the sequence of controls $u(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $u(1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, which maintains the nonnegativity of the states in each previous step. Therefore, $(A, B)_{ps} \geq 0$ is reachable but it is not positively reachable (using only nonnegative inputs).

The above example reveals the great differences between the single-input case and the multi-input case. This last open-problem would deserve a special attention in the future. This question may be tackled using digraph theory, more specifically, using the monomial subdigraphs belonging to a digraph of a reachable pair $(A, B)_{ps} \geq 0$ (see [3, 5, 16]).

7. Conclusions

In this article, we have dealt with positive state systems, that is, discrete-time linear control systems in which the state is constrained to lie in the positive orthant R^n_+ independently of the inputs involved. Namely, we have focused on the study of certain fundamental properties of the system, mainly controllability and reachability, both positive state and positive monomial-state. In fact, we have verified that if every monomial vector of R^n_+ is reachable at some finite time for any given positive state system then reachability is ensured for such a system. Therefore, positive state reachability is equivalent to positive monomial-state reachability. Moreover, several examples and results have illustrated the great differences with respect to the positive/standard case. For the single-input case, the structure of the state and control matrices of the system has been completely characterized. These results have been applied to the management of a production chain and to an epidemiology problem. Finally, the multi-input case have been also analyzed through an application to the control of the amount of pollution in the five Great Lakes of North America.

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