Lagrangian relaxation of the Generic Materials and Operations Planning model

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Abstract. The supply chain management requires increasingly proposals for the production programming planning that brings together its special singularities. Solving coexisting products and alternative processes or by-products must be allowed by the mathematical programming models. The Generic Materials and Operations Planning (GMOP) formulation allows operating with different materials and process lists. The paper presents a procedure to solve the versatile GMOP model by the Lagrange Relaxation. The subgradient update method of the lagrangian multiplier is compared with a linear update method. Obtaining lower bound faster compared to the linear method is allowed by the subgradient method, but the linear method provides better solutions after certain iterations.

Keywords: GMOP; Lagrangian Relaxation; Subgradient

1. Introduction

In the age of digital transformation, Supply Chains become increasingly complex, requiring more versatile and powerful mathematical programming models. At the same time, this situation forces us to work on the simplification of these models to facilitate their resolution in feasible time without losing their capacity of representation (Kong & Rönnqvist, 2014).

A global view of the Supply Chain includes supplier selection, location selection, product, process and transport (Stadtler & Kilger, 2008). It can be found with the Generic Materials and Operations Planning (GMOP) formulation (Garcia-Sabater, Maheut, & Marin-Garcia, 2013). A stroke modelling allows managing jointly the bill of material (BOM) and the bill of process (BOP). It allows the representation of a parallel process model, alternative packaging management, the decomposition of products and other possibilities inherent to stroke use. It is more versatile than the Gozinto structure (Maheut, 2013). Therefore, it is a MLCLSP (Multi-level, capacitated, lot-sizing problem) where the product structure and the process are incorporated.

Regarding its flexibility as a methodology, we must emphasize that it allows to detail alternative processes, the product location with packaging or the packaging type used (Maheut, Garcia-Sabater, & Mula, 2012). It allows to evaluate alternative production, product structures (Coronado-Hernández, Simancas-Mateus, Avila-Martinez, & Garcia-Sabater, 2017; Maheut, 2013), co-products (Coronado-Hernández, 2016; Vidal-Carreras, Garcia-Sabater, & Coronado-Hernandez, 2012) or alternative resources (Coronado-Hernández, Garcia-Sabater, Maheut, & Garcia-Sabater, 2010). But, as far as we have found, a Lagrangian relaxation has not been applied in order to be able to solve GMOP problems. In other equivalent models, Lagrangian relaxation has been used satisfactorily (Attanasio, Ghiani, Grandinetti, & Guerriero, 2006; Jeong & Yim, 2009; Kong & Rönnqvist, 2014; Lau, Zhao, Ge, & Lee, 2011; Lu, Lau, & Yiu, 2012; Pukkala, Heinonen, & Kurttila, 2009; Walther, Schmid, & Spengler, 2008).

As Kelly & Zyngier (2008) proposed decomposition as a natural way of dealing with large problems. The idea behind decomposition is to break the overall problem into a number of smaller sub-problems, which are easier to solve, and coordinate these sub-problems through a master problem (Kong & Rönnqvist, 2014). And, a well studied method of decomposition is through the lagrangian multiplier, Lagrangian

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Relaxation (LR) (Fisher, 1985). Some recent analysis of the reduction of calculation time with the decomposition can be found in the literature (Harb et al., 2015; Quddus, Ibne Hossain, Mohammad, Jaradat, & Roni, 2017; Sokoler et al., 2014) but the analysis of the behavior of the multipliers is missing, as a factor of coordination of the decentralized elements.

The main contribution of our work is to present the Lagrangian relaxation application methodology in the GMOP modeling method and a comparative analysis of two proposals for the recalculation of the Lagrangian multiplier updater gradient, such as the subgradient (Fisher, 1985) and the Conejo et al. (2006) proposal.

The rest of the paper is structured as follows: first, a short description of Lagragian Relaxation is introduced; second, the Subgradient Method and a linear update method for the GMOP are presented, third, experimental results are given and finally, the paper ends with a conclusion and future works.

2. Lagrangian Relaxation.

The Lagrange multiplier method is used in many different complex mathematical problems, since it can transform them into simpler problems by eliminating some “difficult" constraints. Fisher (2004) identifies Lorie-Savage's work on budgets as the first work, published for the first time in 1949 (Lorie & Savage, 1955). Goffin (1977) suggests that the main idea of the Lagrange relaxation method was from Agmon (1954). Further work on the Lagrange multiplier and its mathematical properties can be found (Barker, 1945; Gould, 1945). Galvao et al. (2011) propose that the first use of the Lagrange relaxation ideas should be assigned to Bilde & Krarup (1967). At the same time, equivalent works on the subgradient and its convergence can be found in the Russian literature of the time (Polyak, 1969).

But the turning point occurred in 1970 when Held & Karp (1970; 1971) successfully applied the Lagrange multiplier with the subgradient (SM) algorithm to update the Lagrange multiplier in the resolution of the travelling salesman problem. They narrowed the search trees to a minuscule level. Later in 1974, Held et al. (1974) presented the computational performance of the subgradient method and the cases of theoretical convergence; starting from the assumptions or conditions, the gradients “s^j" should tend to zero and the sum of all the gradients tend to infinity in successive iterations as Equation (1). Therefore, in the convexity assumptions and the Equation (1) conditions, the maximum solution of the successive solutions of the relaxed primal problem “$Z_D(u^j)$" as Equation (7) for each value of the lagrange multiplier "u^j" (in minimization problems) as Equation (10) provides the solution to the main problem “Z” as Equation (3). And in the cases of non-convexity problems, the different solutions of the relaxed primal problem provide lower bounds (or upper bounds on maximization problems) to the primal function target value. This difference is called duality gap (Conejo et al., 2006) as defined on Equation (13).

$$s^j \to 0 \text{ and } \sum_{j=0}^{q} s^j \to \infty \text{ then } Z_D(u^j) \to Z_D \leq Z \tag{1}$$

In 1974, Geoffrion (1974) established the name of “Lagrangean[2] Relaxation” (LR) to the method. The LR has been successfully applied to different difficult problems like Scheduling, General IP, Locations, Generalized assignments, Travelling salesman, etc, extensive review can be found in Fisher (2004) work.

[2] “Lagrangean” is by the French name of Joseph Louis Lagrange, and “Lagrangian” is by his original Italian name Giuseppe Lodovico Lagrangia.

Other extensions of the gradient methodology have been developed (Dimitri P Bertsekas, 1975; Wolfe, 1974) as the Conjugate Subgradient Optimization (Goffin, 1977), where memory is added to the algorithm, as a filter or a smoothing, Equation (2). Thus, g^j is the gradient proposed by the subgradient method. Therefore, when we choose β equal to zero value, we have the subgradient method (Boyd, Mutapcic, Xiao, & Mutapcic, 2008).

$$s^j = (1-\beta)g^j + \beta s^{j-1}\ ;\ 0 \le \beta \le 1 \qquad (2)$$

Other lines of work suggest modifying the gradient direction, within the SM, to avoid the oscillation problems of the method. Proposals such as Camerini (1975) or Bertsekas (2000), who propose the Stochastic Subgradient Method, that achieves better computational results than SM.

In general, the Lagrange multiplier can be updated by the methods of SM, Cutting Plane (Sáez, 2000), Bundle (Gaudioso, Giallombardo, & Miglionico, 2009), Trust Region (Conejo et al., 2006), Adaptive Gradient Algorithm (Duchi, Hazan, & Singer, 2011), Accelerated Gradient (Giselsson, Doan, Keviczky, Schutter, & Rantzer, 2013), Improved Subgradient Level Algorithm (Mao, Pan, Pang, & Chai, 2014), etc. For example, the Bundle method can give better direction than SM but requires more computational effort (Zhao, Luh, & Wang, 1999). Or, for example, Surrogate Relaxation updates the multiplier only with the calculation of part of the function (Chang, 2008; Fisher, Lageweg, Lenstra, & Kan, 1983). Or the Incremental Subgradient method (Narciso & Lorena, 1999) as a mixture between Surrogate and subgradient to avoid oscillations on the first iterations of the subgradient method. The Modified Surrogate Subgradient Method (Zhao et al., 1999) improves the Surrogate Relaxation direction. However, SM is the easiest to implement.

Other research uses the Lagrange multiplier to relax constraints and separate the model into independent sub-problems. These sub-problems can be solved independently. We have the Dantzig-Wolfe (1960) decomposition, or the Benders (1962) decomposition. Benders proposes to dualize the function and then relax the constraints to generate separable sub-problems, where a centralized problem must add the solutions of each sub-problem with the constraints and function that cannot be decomposed. Dantzig-Wolfe method uses the Lagrange multiplier to coordinate by raising prices for shared resources when sub-problems request more resources. And Benders method requests the prices proposed by sub-problems to reduce or increase shared resources. The Lagrangian Decomposition (LD), on the other hand, proposes to duplicate a set of variables and relax the constraints that correlate these duplications. Therefore, LD allows to divide the problems with the connection variables duplication (Lidestam & Rönnqvist, 2011). The problems divided by LD allow parallel computing (Jeet & Kutanoglu, 2007). Guignard et al (1987) demonstrated that LD creates better bounds than LR.

Other research lines propose, for non-differential and non-convex problems, to provide convex properties by implementing quadratic terms when the Lagrange multiplier is updated. The Aumented Lagrangian "convexifies" the problems (D. P. Bertsekas, 1979). In 1977, Goffin (1977) reviewing the convergence of SM for some non-differential problems recommended a second order function. In general, the Augmented Lagragian Relaxation is slower but provides viable solutions compared to the LR, which normally requires meta-heuristics to obtain a viable solution to the main function (Beltran & Heredia, 2002). Boudin et al. (2005) recommend starting with LR to set a good limit and then an Augmented Lagrangian Relaxation to get a valid solution for the main function. However, to find precise solutions, high weights are required in the quadratic function and it requires longer computing times (Tosserams, Etman, Papalambros, & Rooda, 2006). Another alternative is to linearize or decompose the Augmented Lagrangian Relaxation in order to be able to separate the sub-problems (Li & Ierapetritou, 2012). An Accelerated Distributed Augmented Lagrangian method (Fu & Diabat, 2015) has also been proposed.

The LR allows to be an initial method that can be combined with other methods, for example the Branch and Cut (Karuppiah & Grossmann, 2008; Nishi, Hiranaka, & Inuiguchi, 2010) or the Branch and Bound where the LR accelerates the process (Kuno & Utsunomiya, 2000).

The SM has several disadvantages. First, generally, solutions to the primal function are not generated, at least feasible solutions close to the optimal solution. The relaxed function solutions are generally not valid in the main function, since the relaxed condition usually fails (Sherali & Choi, 1996). The reason is that marginal changes in Lagrange multipliers can lead to different integer values and then changes in the objective functions of sub-problems (Gunnerud & Foss, 2010). On the other hand, there are procedures that with small modifications of the Lagrange multipliers allow us to find valid solutions to the main function (Barahona & Anbil, 2000; Conejo et al., 2006). Secondly, usually at the beginning, there is a zigzagging and later a slow convergence, more noticeable with high values of σ_k, Equation (11). Thirdly, convergence depends on the values of σ_k (Boyd et al., 2008). Finally, a valid initial solution is necessary, so a heuristic has to be used to obtain the initial value. But SM has the advantage that its mathematical formulation allows us to control convergence all the times, and has the goodness of the solution found (Araúzo, Del-Olmo-Martínez, Laviós, & De-Benito-Martín, 2015). In addition, it is simple and works on most problems (Pukkala et al., 2009).

Also, despite the time elapsed since the first works with the Lagrange multiplier, it is still used in different research lines, growing day by day. Some recent works that we can find are a LR for a lot size problem (Zhang, Jiang, & Pan, 2012), un Augmented Lagrangian Relaxation for a cluster coordination problem (Qu et al., 2015), an LR with a localization problem (Diabat, Battaïa, & Nazzal, 2015) or LD in a planning problem in a supply chain (Lidestam & Rönnqvist, 2011), but we have not been able to find a case applying SM to the GMOP formulation in a peer reviewed work.

For a general integer function, called primal, with the constraints as Equations (4) to(6), we have , Equation (3).

$$Z = \min \sum_{\mathrm{i=1}}^{\mathrm{m}} \sum_{\mathrm{k=1}}^{\mathrm{n}} \mathrm{C_{ik}x_{ik}} \tag{3}$$

$$s.t. \sum_{i=1}^{m} A_i x_{ik} \leq b_k; \mathrm{k} = 1,..,n \tag{4}$$

$$\sum_{k=1}^{n} B_k x_{ik} \leq c_i; i = 1,..,m \tag{5}$$

$$x_{ik} \geq 0 \; and \; integral \tag{6}$$

Where the constraints Equation (4) are the complicated condition. Then, if this constraint is removed we should have an easier problem to solve (Lemaréchal, 2001). So, we can transfer the constraint to the objective function with a penalty of its non-compliance, such that it penalizes the breach of the aforementioned eliminated constraint (Fisher, 1985). It is also interpreted as the cost that the objective function will pay for non-compliance with the constraint. Therefore, we will have the function LR Equation (7), called dual, which will give us a lower limit to the main function Equation (3) and its greater value will be on Equation (8).

$$Z_D(u^j) = \min \sum_{k=1}^{n}\sum_{i=1}^{m} C_{ik}x_{ik} + \sum_{k=1}^{n} u_k^j \left(\sum_{i=1}^{m} A_i x_{ik} - b_k\right) \quad (7)$$

$$Z_D = \max_j Z_D(u^j) = \max_j \min \sum_{k=1}^{n}\sum_{i=1}^{m} C_{ik}x_{ik} + \sum_{k=1}^{n} u_k^j \left(\sum_{i=1}^{m} A_i x_{ik} - b_k\right) \quad (8)$$

$$u_k^j \geq 0;\ \forall k, j \quad (9)$$

The new function on Equation (8) holds the constraints, Equations (5), (6) and adds the constraint Equation (9) for the multiplier. The vector u^j is the Lagrange multipliers for the k constraints we have relaxed, where j is the number of iterations we will perform on the Lagrange multiplier. The Lagrange multipliers will be updated with the SM algorithm Equation (10), where h_k is the non-fulfilment of each constraint k. u_k^j is greater than or equal to zero, Equation (9). Therefore, a negative term is added to the main function which includes the non-compliance of the removed constraint.

$$u_k^{j+1} = \max\{0, u_k^j + s^j h_k\} = \max\left\{0, u_k^j + s^j\left(\sum_{i=1}^{m} A_i x_{ik} - b_k\right)\right\} \forall\ k \quad (10)$$

The process starts with $u_k^0 = 0\ \forall k$ that allows calculating the first value of the relaxed function $Z_D(u^0)$.

Being s^j the gradient of the SM, Equation (11).

$$s^j = \frac{\sigma_j\left(Z^* - Z_D(u^j)\right)}{\sum_{k=1}^{n}\left\|\left(\sum_{i=1}^{m} A_i x_{ik} - b_k\right)\right\|^2} \quad (11)$$

Where σ_j is a scalar that must satisfy $0 < \sigma_j \leq 2\ \forall j$. Z^* is the minimum value we have of the principal function (3) and $Z_D(u^j)$ is the value of the relaxed function, Equation (7), for a given multiplier vector u^j.

Held et al. (1974) recommended as a general rule, but not always, to establish $\sigma_j = 2$ for 2p iterations, where p is a quantification of the size of the problem, and then successively reduce by half σ_j and the number of iterations, until the number of iterations reaches a threshold. Then, σ_j is reduced by half each iteration time until s^j reaches a small value as stop criterion. Held et al. (1974) commented that this protocol does not fulfil the second condition of Equation (1) $\sum_{j=0}^{q} s_j \to \infty$ and that almost never converges to the optimum, but generates good limits to the main function. Guignard (2003) says that the convergence of the SM is unpredictable and for some problems converge quickly, with enough reliability, whereas in other problems it has an erratic behaviour.

Boyd et al (2008) studied different effects of the σ_j between 0.05, 0.01, 0.005 and reported that for higher values, the method has a faster convergence but greater initial oscillations and converges to a greater dual gap, Equation (12), than with smaller factors. They also point out the good results with 1/j as multiplier refresher method.

$$\text{GAP} = \frac{\|Z_d - Z^*\|}{Z^*} \quad (12)$$

Conejo et al. (2006) propose the updated gradient method Equation (13) as a proposal that fulfils Equation (1).

$$s^j = \frac{1}{a + bj} where\ a \text{ and } b\ are\ scalar\ constants \quad (13)$$

And give as values to the parameters a=1 and b=0.1

Fisher (2004) collects as an alternative algorithm for the SM, to start with $\sigma_0 = 2$ and halve σ_j as long as $Z_D(u^j)$ has not been able to improve the lower limit on a fixed number of iterations. This rule also does not guarantee to satisfy the requirements of Equation (1) but it is said that it works well empirically.

Therefore, to apply the SM we have to answer some previous questions. First we have to select the difficult constraints. Secondly, we have to define a feasible first solution to the main function Z*, knowing that Held et al. (1974) observed that the final result did not seem to depend on this value, but Boyd et al. (2008) states that to reduce the initial uncertainty by a factor of 10^3, at least 10^6 iterations were required. Thirdly, we have to select update rules for σ_j. Finally, we have to set the stopping criteria, such as the number of iterations, the minimum dual gap and the smallest step size for the constant σ_j which implies small variations in the Lagrange multipliers.

3. The Lagrangian Relaxation approach to solve GMOP

The GMOP formulation is a representation of the problem of batch size, multi-level, multi-sub-process and multi-post-process, multi-structure, multi-period, with limited capacity (Garcia-Sabater et al., 2013). It overcomes the limitation of the Gozinto structure with the use of Strokes (Coronado-Hernández et al., 2010; Maheut, 2013; Vidal-Carreras et al., 2012). And since lot size problems with limited capacity have been proved to be NP-Hard even for the case of a single product (Bitran & Yanasse, 1982), the GMOP model is an NP-Hard problem. In large-scale problems, it is computationally difficult to obtain an optimal or near-optimal solution in a reasonable computational time.

The list of indexes, parameters and variables can be seen in Table 1. The main function is a balance between inventory maintenance costs, penalties for delays in service, batch preparation and the production, as shown in Equation (14). It has the constraints Equations (15) - (18). The constraint of Equation (15) is the stock equilibrium and connects the logistic part (stocks, delays and demand) with production (consumption of components and creation of new products). The constraint of Equation (16) defines the capacity limitation of resources. The constraint of Equation (17) establishes the batch preparation requirement when manufactured in period t with a stroke k. Finally, the set of constraints Equation (18) establishes the requirements for variables.

Table 1 The indexes, parameters and variables used in GMOP.

Index	
i	Index set of products (includes products, packaging and site)
t	Index set of planning periods
r	Index set of resources
k	Index set of strokes
Parameters	
$D_{i,t}$	Demand of product i for period t
$H_{i,t}$	Cost of storing a unit of product i in period t
$CO_{k,t}$	Cost of stroke k in period t
$CS_{k,t}$	Cost of the setup of stroke k in period t
$CB_{i,t}$	Cost of purchasing product i in period t
$SO_{i,k}$	Number of units i that generates a stroke k
$SI_{i,k}$	Number of units i that stroke k consumes

LT_k	Lead time of stroke k
$KAP_{r,t}$	Capacity availability of resource r in period t (in time units)
M	A sufficiently large number
$TO_{k,r}$	Capacity of the resource r required for performing one unit of stroke k(in time units)
$TS_{k,r}$	Capacity required of resource r for setup of stroke k (in time units)
σ_m	Constant used to calculate Lagrangian multiplier
Variables	
$z_{k,t}$	Amount of strokes k to be performed in period t
$\delta_{k,t}$	=1 if stroke k is performed in period t (0 otherwise)
$w_{i,t}$	Purchase quantity for product i in period t
$x_{k,t}$	Stock level of product i on hand at the end of period t
$u_{r,t}^{j}$	Lagrangian multipliers

$$Z_P: min \sum_t \sum_i \left(H_{i,t} x_{i,t}\right) + \sum_t \sum_k \left(CS_{k,t}\delta_{k,t} + CO_{k,t} z_{k,t}\right) + \sum_t \sum_i \left(CB_{i,t} w_{i,t}\right) \quad (14)$$

$$x_{i,t} = x_{i,t-1} - D_{i,t} + w_{i,t} - \sum_k \left(SI_{i,k} z_{k,t}\right) + \sum_k \left(SO_{i,k} z_{k,t-LT_k}\right), \forall i, t \quad (15)$$

$$\sum_k \left(TS_{k,r}\delta_{k,t}\right) + \sum_k \left(TO_{k,r} z_{k,t}\right) \leq KAP_r \ \forall r, t \quad (16)$$

$$z_{k,t} - M\delta_{k,t} \leq 0, \forall k, t \quad (17)$$

$$x_{i,t} \geq 0;\ w_{i,t} \geq 0, \forall i, t;\ z_{k,t} \in \mathbb{Z}^+;\ \delta_{k,t} \in \{0,1\}\ \forall k, t \quad (18)$$

Now, we have to select the relaxation constraint that allow us to generate a simpler model as the Lagrange Relaxation

The inventory balance Equation (15) can decompose the problem into sub-problems, some related to production and others to logistics management. The relaxation of the capacity limitation Equation (16) simplifies the problem by decoupling each resource and period, so that the different products do not have the joint constraint of sharing the resource, allowing us to see the problem as a set of sub-mono-product, mono-resource or mono-locality problems, therefore no more NP-hard if we follow Gupta & Maranas (1999) conclusion on their equivalent problem. The relaxation of constraint Equation (17) will disconnect the preparation requirement, so the preparation cost will not be taken into account when optimizing the function. The function will recommend the production requested by the demand without delay or storage. The relaxation of the constraint of Equation (18) would avoid the constraint of integers.

Gupta & Maranas (1999) argue that the limits obtained with the relaxation of available capacity constraints are better, closer to the optimum, than the relaxation of the other constraints in a McDonald and Karimi (1997) medium-term production planning model.

Therefore, the RL of the constraint of the available capacity of each resource Equation (16) of the GMOP problem Equation (14) is applied in formulation of Equation (17).

$$Z_d: \max_u min \sum_t \sum_i \left(H_{i,t} x_{i,t}\right) + \sum_t \sum_k \left(CS_{k,t}\delta_{k,t} + CO_{k,t} z_{k,t}\right) + \sum_t \sum_i \left(CB_{i,t} w_{i,t}\right) + \sum_r \sum_t u_{r,t}^j \left[\sum_k \left(TS_{k,r}\delta_{k,t}\right) + \sum_k \left(TO_{k,r} z_{k,t}\right) - KAP_{r,t}\right] \quad (17)$$

The updating of the Lagrange multiplier vector $u^j_{r,t}$ is done with the positive value of Equation (18). And s^j is the gradient in each j^{th} interaction and h^j_{rt} is the non-compliance of each relaxed constraint Equation (16) of each period and resource Equation (19).

$$u_{r,t}^{k+1} = u_{r,t}^j + s^j h^j_{rt} \quad (18)$$

$$h^j_{rt} = h\left(u_{r,t}^j\right) = \left[\sum_k \left(TS_{k,r}\delta_{k,t}\right) + \sum_k \left(TO_{k,r} z_{k,t}\right) - KAP_{r,t}\right]^j \quad \forall r, t \quad (19)$$

The term s^j, the gradient, from Equation (18) is calculated by Equation (20) SM or Equation (13) (Conejo et al., 2006). Equation (20) is the slope function where it increases with the difference between the value of the best known solution of Equation (14) and the last value of the relaxed function of Equation (17), and divided by the square of non-compliance of all relaxed constraints and it is multiplied by the factor σ_m Equation (21).

$$s^j = \frac{\sigma_m \left(Z^* - Z_d\left(u_{r,t}^{j-1}\right)\right)}{\sum_{r,t}\left[\sum_k \left(TS_{k,r}\delta_{k,t}\right) + \sum_k \left(TO_{k,r} z_{k,t}\right) - KAP_{r,t}\right]^2} \quad (20)$$

$$0 \leq \sigma_m \leq 2, \forall m \quad (21)$$

In our case, we will start the parameter σ_m with the value of 2 and if after 10 iterations the result is not improved on Equation (17), we update the value of σ_m to half of it.

4 Numerical experiments

In order to evaluate the two methods, we have been implemented the algorithms in a C# programme that recollects random data generated in an EXCEL® sheet and calls to the Gurobi Optimizer 7.0® all this on a virtual machine with 16 cores. The results are sent back to the calculation sheet to compare between the optimal problem solution obtained by Gurobi® and the solution with the LR. We have randomly generated ten data sets to test both algorithms, Table 2 shows the parameter values which were used to generate test data and Table 3 lists the parameter values used for the Lagrangian procedures.

Table 2 Selected range of values for parameters in the test problems.

Parameters	Value
i	1-7
t	1-10
r	1-5
k	1-10
$D_{i,t}$	Random uniform from [100,1.000]
$h_{i,t}$	Random uniform from [10,20]
$CO_{k,t}$	Random uniform from [5,8]
$CS_{k,t}$	Random uniform from [5,10]

$CB_{i,t}$	Random uniform from [10,20]
$SO_{i,k}$	Random uniform from [35,50]
$SI_{i,k}$	Random uniform from [4,8]
LT_k	Random uniform from [1,2]
$KAP_{r,t}$	Random uniform from [400,500]
M	100.000
$TO_{k,r}$	Random uniform from [2,5]
$TS_{k,r}$	Random uniform from [5,10]

Table 3 Parameters for the Lagrangian relaxation procedure.

Parameters	**Value**
Maximum iteration count	1.000
σ_m (SM)	Initial value of 2.
Maximum number of iterations before halving is σ_m (SM)	10
Initial Primal value (SM)	300.000
Initial multiplier value (SM)	0
a (Conejo et al. 2006)	1
b (Conejo et al. 2006)	0.1

In order to evaluate the quality of the algorithms we used four criteria.

- The iteration required to narrow the gap, Equation (12), to less than 1% on both method, named as "It1%(SM)" and "It1%(Conejo)".

- The iteration when there is the maximum difference between each method gaps, named as "ItMax".

- The iteration when the Conejo et al. (2006) method has lower gap than the SM method, named as "ItConejo".

- Following Zhang et al. (2012) the distances between the upper bound (UB) and the optimal solution (OP) proposed by Gurobi®, being at least 24% faster than with the Lagrangian relaxation used, as UGAP and the distances between the lower bound (LB) and the optimal solution (OP), as LGAP. The distances are calculated as Equation (22) and we record the value at the iterations 10, 50, 100, 500, 1000.

$$\mathrm{UGAP} = \frac{2(\mathrm{UB} - \mathrm{OP})}{(\mathrm{UB} + \mathrm{OP})} ; \mathrm{LGAP} = \frac{2(\mathrm{OP} - \mathrm{LB})}{(\mathrm{OP} + \mathrm{LB})} \tag{22}$$

The results can be viewed in Table 4 and Table 5.

Table 4 Performance of the Lagrangian relaxation procedures.

Instance	It1%(SM)	It1%(Conejo)	ItMax.	ItConejo
1	12	372	27	972
2	17	232	43	247
3	6	569	170	718
4	-	-	48	111
5	-	-	82	-
6	-	999	83	102
7	11	687	18	-
8	-	-	79	431
9	2	354	4	944

10	-	-	26	155

*It1%(SM)*Iteration with SM to 1% gap, *It1%(Conejo)* Iteration with Conejo et al. (2006) to 1% gap, *ItMax* iteration with maximum gap difference between methods, *ItConejo* iteration when Conejo et al. (2006) method has lower gap that the SM method

Table 5 UGAP and LGAP of instances.

Instance	UGAP10	UGAP50	UGAP100	UGAP500	UGAP1000	LGAP10	LGAP50	LGAP100	LGAP500	LGAP1000
1SM	0,58%	0,41%	0,41%	0,41%	0,41%	0,44%	0,21%	0,17%	0,14%	0,14%
1 Conejo	6,90%	3,07%	3,07%	0,45%	0,41%	6,43%	6,43%	4,58%	0,33%	0,13%
2 SM	27,53%	0,47%	0,47%	0,47%	0,47%	0,57%	0,30%	0,25%	0,10%	0,10%
2 Conejo	6,34%	5,52%	5,17%	0,47%	0,47%	6,73%	6,73%	1,10%	0,10%	0,10%
3 SM	0,03%	0,03%	0,03%	0,03%	0,03%	0,13%	0,11%	0,09%	0,03%	0,02%
3 Conejo	1,33%	1,33%	1,33%	0,70%	0,03%	1,21%	1,21%	1,21%	0,43%	0,02%
4 SM	22,35%	6,51%	6,51%	6,51%	6,51%	4,39%	0,35%	0,23%	0,19%	0,19%
4 Conejo	8,11%	8,11%	7,04%	4,54%	4,54%	5,32%	5,32%	2,34%	0,68%	0,26%
5 SM	3,32%	1,69%	1,69%	1,69%	1,69%	0,54%	0,21%	0,14%	0,07%	0,07%
5 Conejo	3,32%	3,32%	3,32%	2,02%	1,69%	9,54%	9,54%	3,10%	0,56%	0,17%
6 SM	2,38%	2,38%	2,38%	2,38%	2,38%	0,30%	0,15%	0,13%	0,12%	0,12%
6 Conejo	2,38%	2,38%	1,70%	1,16%	0,82%	6,83%	6,83%	2,27%	0,30%	0,20%
7 SM	0,65%	0,28%	0,28%	0,21%	0,21%	0,36%	0,15%	0,14%	0,12%	0,12%
7 Conejo	6,32%	4,12%	3,10%	0,82%	0,29%	10,31%	10,31%	10,31%	0,32%	0,16%
8 SM	11,88%	3,93%	3,93%	3,93%	3,93%	0,81%	0,23%	0,17%	0,15%	0,15%
8 Conejo	7,87%	7,67%	5,41%	1,05%	1,05%	22,62%	22,62%	12,84%	0,34%	0,19%
9 SM	0,14%	0,14%	0,14%	0,14%	0,14%	0,13%	0,08%	0,07%	0,06%	0,06%
9 Conejo	1,36%	0,91%	0,91%	0,14%	0,14%	0,77%	0,77%	0,77%	0,11%	0,06%
10 SM	5,43%	5,43%	5,43%	5,43%	5,43%	0,35%	0,12%	0,11%	0,10%	0,10%
10 Conejo	5,43%	3,86%	3,86%	3,86%	3,86%	11,49%	11,49%	11,49%	0,40%	0,16%
AvgSM	7,43%	2,13%	2,13%	2,12%	2,12%	0,80%	0,19%	0,15%	0,11%	0,11%
AvgConejo	4,94%	4,03%	3,49%	1,52%	1,33%	8,12%	8,12%	5,00%	0,36%	0,14%

UGAP10 Upper bound gap at iteration 10, *LGAP10* Lower bound gap at iteration 10, *1 Conejo* test one with Conejo et al. (2006) method, *1 SM* test one with *subgradient* method (Fisher, 1985)

What has been found from Table 4 is that less iteration to reduce the gap below 1% is required by the SM compared to the Conejo et al. (2006) method (Conejo method), but after several iterations the Conejo method is able to improve the gap obtained by the SM method.

Table 5 shows that the SM method can reduce the UGAP after the 50th iteration but no further improvement during the following iterations with the SM method. On the other hand, the UGAP with the Conejo method is reduced progressively by each iteration and can improve the SM method around the 500th iterations. Also, Table 5 shows that the LGAP is reduced drastically with the SM method before the 50th iteration and the Conejo method takes longer iterations to reduce de gap and can't improve the SM method on the 1000th iteration on average. Since all the values are positive, it is confirmed that the optimal solution (OP) proposed by Gurobi®, without applying the Lagrange decomposition, has not been improved by the lower and upper bound obtained in the different iterations.

Figure 1 shows the test 1 with the SM method and its Lagrangian multiplier values on figure 2. We also can observe the fast update of the lower bound (dual) and the only three steps that update the upper bound. On figure 2, we observe that the Lagrangian multipliers get stable at the end of the figure.

Figure 3 shows the test 1 with the Conejo method and figure 4 its Lagrangian multiplier values. In this case we observe the oscillation of the lower bound and its best value at 58th iteration. Also, that the upper bound has been improved even when the lower bound has not been improved. Therefore, that the upper

bound or the solution of the primal function has been improved more frequently with the Conejo et al. (2006) method. Also, the fast gap reduction can be confirmed with the SM method with the figure 5 and the improvement of the Conejo et al. (2006) method at high iterations.

Fig. 1 Results of the Dual/Primal GMOP with SM.

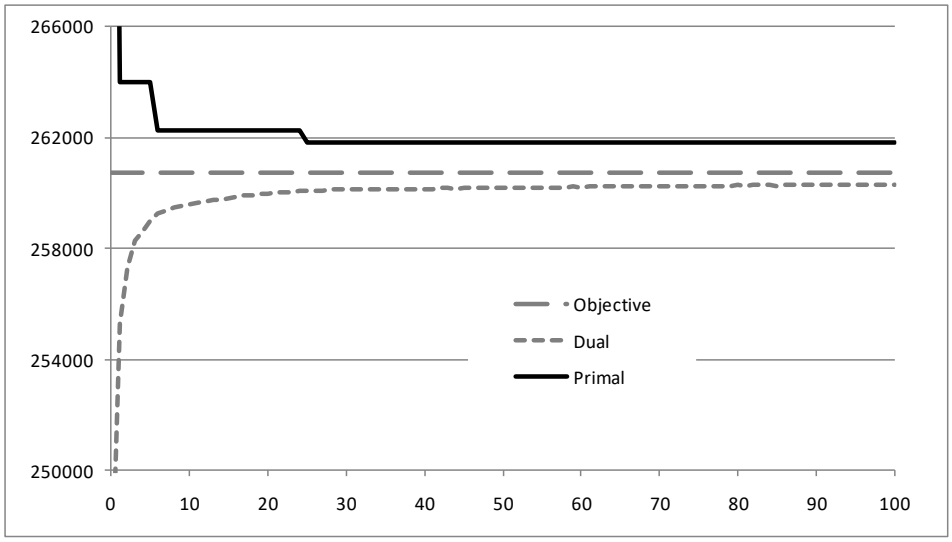


Fig. 2 Evolution of Lagrangian multiplier vector values with the SM.

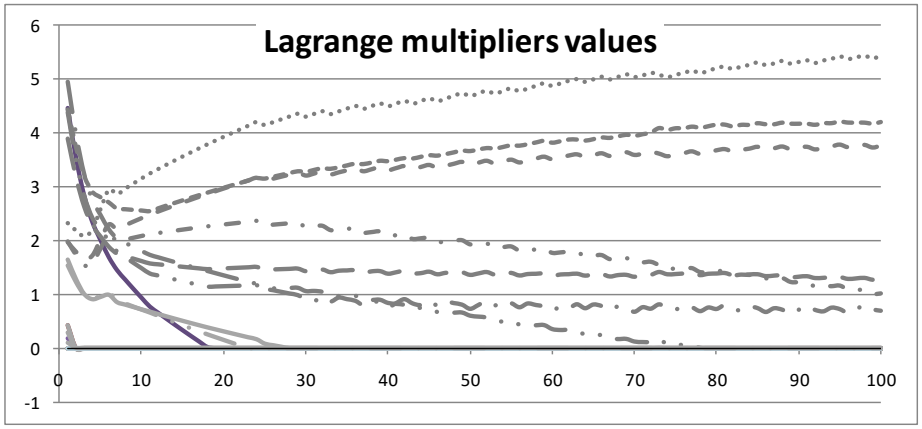


Fig. 3 Results of the Dual/Primal GMOP with Conejo method.

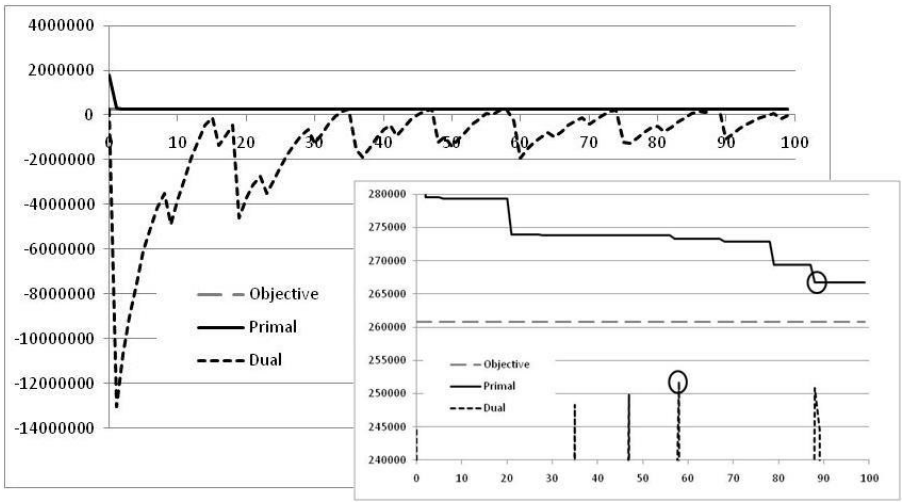


Fig. 4 Evolution of Lagrangian multipliers values with Conejo method .

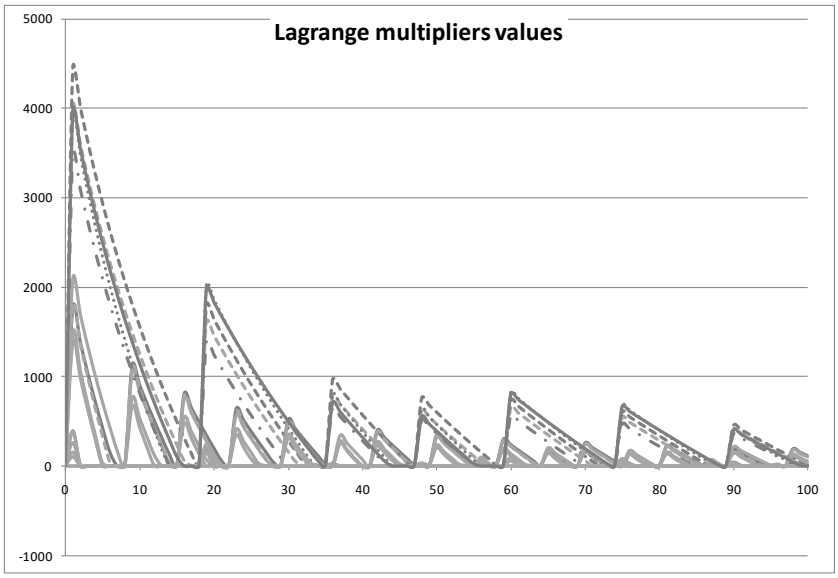
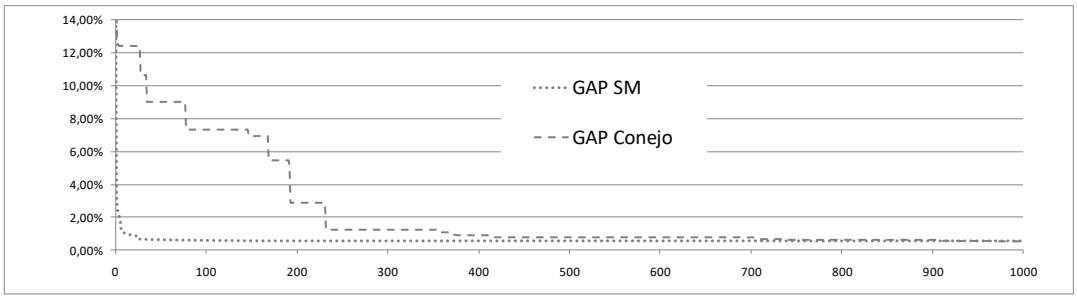


Fig. 5 Convergence comparison between SM and Conejo method.



4 Conclusion and future work.

In this paper, the procedures have been presented to implement the lagrangian relaxation to the Generic Materials and Operations Planning formulation and some numerical experiments with the lagrangian multiplier update with the subgradient method and Conejo et al. (2006) method. The literature has demonstrated that there are different proposals to update the Lagrangian multiplier with no unique better method. The subgradient method has shown, with the numerical experiments, its rapid gap reduction with little iteration and it can generate good lower bound on less that 10 iterations bellow 1% on average. The Conejo et al. (2006) method shows a slower convergence, but can improve the SM method on longer iteration. Thus, Conejo et al. (2006) method shows better generation of valid solution to the main function, specially on high number of iterations, on average for 500 iterations. And Conejo method approach the lower bound generation by subgradient method after 1.000 iterations.

On the other hand, it should be noted that the method of updating the lagrange multiplier of the subgradient tends to stabilize the Lagrange multipliers as opposed to the oscillations presented by the method proposed by Conejo et al. (2006). This property can be applied as a coordination mechanism to align the objectives, since the subgradient provides stable values for the shared resources capacity.

GMOP's Lagrangian relaxation will allow managers to solve complex industry modeling with the versatility of the GMOP formulation. In addition, since Lagrange relaxation is one of the tools frequently

used as a coordination mechanism in distributed decisions, it allows aligning models with different objectives but shares a restriction.

Future research should be done with higher complexity problems and real industrial cases to evaluate both processes. Also, it opens the investigation to other heuristics to update the Lagrangian multipliers as the augmented methods or relaxing a duplicated variable in order to apply a Lagrangian Decomposition or the Dantzig-Wolfe method (1960) or the Benders cuts (1962) or heuristics to improve the upper bound results to the GMOP models. In addition, the evaluation of these methods can be performed in relation to the calculation time in comparison with the commercial solvers, with respect to different problems sizes.

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