RACSAM (2019) 113:1739–1746
https://doi.org/10.1007/s13398-018-0574-0

ORIGINAL PAPER

Centralizer's applications to the (b, c)-inverses in rings

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Received: 2 February 2018 / Accepted: 28 August 2018 / Published online: 6 September 2018

Abstract
We give several conditions in order that the absorption law for one sided (b, c)-inverses in rings holds. Also, by using centralizers, we obtain the absorption law for the (b, c)-inverse and the reverse order law of the (b, c)-inverse in rings. As applications, we obtain the related results for the inverse along an element, Moore–Penrose inverse, Drazin inverse, group inverse and core inverse.

Keywords Centralizer · (b, c)-inverse · Absorption law · Reverse order law

Mathematics Subject Classification 16W10 · 15A09

1 Introduction

Throughout this paper, R denotes a unital ring. The following notations $aR = \{ax \mid x \in R\}$, $Ra = \{xa \mid x \in R\}$ and $[a, b] = ab - ba$ will be used in the sequel for $a, b \in R$. In [9, Definition 1.3], Drazin introduced a new class of outer inverse in the setting of semigroups or rings, namely, the (b, c)-inverse. Let $a, b, c \in R$, we say that a is (b, c)-invertible if exists $y \in R$ such that

$$y \in bRy \cap yRc, \quad yab = b \quad \text{and} \quad cay = c.$$

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If such y exists, then it is unique, denoted by $a^{\|(b,c)}$, and said to be the (b, c)-inverse of a. Many existence criteria and properties of the (b, c)-inverse can be found in, for example, [3,4,9,10,13,14,19,21–23]. In [10, Definition 1.2] and [14, Definition 2.1], the authors independently introduced the one-sided (b, c)-inverses in rings. Let $a, b, c \in R$. We call that $x \in R$ is a *left* (b, c)*-inverse* of a if $Rx \subseteq Rc$ and $xab = b$. We call that $y \in R$ is a *right* (b, c)*-inverse* of a if $yR \subseteq bR$ and $cay = c$.

In [16], Mary introduced a new type of generalized inverse, namely, the inverse along an element. Let $a, d \in R$. We say that a is *invertible along* d if there exists $y \in R$ such that

$$yad = d = day, \quad yR = dR \quad \text{and} \quad Ry = Rd.$$

If such y exists, then it is unique and denoted by $a^{\|d}$. Many existence criteria and properties of the inverse along an element can be found in, for example, [2,16,17,24–26]. By the definition of the inverse along d, we have that $a^{\|d}$ is the (d, d)-inverse of a. The definitions of left and right inverses along an element can be found in [24].

An element $a \in R$ is said to be *Drazin invertible* if there exists $x \in R$ such that $ax = xa$, $xax = x$ and $a^k = a^{k+1}x$ for some nonnegative integer k. The element x above is unique if it exists and denoted by a^D [8]. The smallest positive integer k is called the *Drazin index* of a, denoted by ind(a). If ind$(a) = 1$, then a is group invertible and the *group inverse* of a is denoted by $a^\#$. Thus, $a^\#$ satisfies $a^\# a a^\# = a^\#$, $a^\# a = a a^\#$ and $a a^\# a = a$.

An involutory ring R means that R is a unital ring with involution, i.e., a ring with unity 1, and a mapping $a \mapsto a^*$ from R to R such that $(a^*)^* = a$, $(ab)^* = b^* a^*$ and $(a + b)^* = a^* + b^*$, for all $a, b \in R$. Let $a, x \in R$. If $axa = a$, $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$, then x is called a *Moore–Penrose inverse* of a. If such an element x exists, then it is unique and denoted by $a^\dagger$. We call that $x \in R$ is an *inner inverse* of a if $axa = a$.

The notion of the core inverse for a complex matrix was introduced by Baksalary and Trenkler [1]. In [20], Rakić et al. generalized the core inverse of a complex matrix to the case of an element in R with involution. More precisely, let $a, x \in R$, if $axa = a$, $xR = aR$ and $Rx = Ra^*$, then x is called a *core inverse* of a. If such an element x exists, then it is unique and denoted by $a^{\textcircled{\#}}$. Also, in [20] the authors defined a related inner inverse in a ring with an involution. If $a \in R$, then $x \in R$ is called a *dual core inverse* of a if $axa = a$, $xR = a^* R$ and $Rx = Ra$. If such an element x exists, then it is unique and denoted by $a_{\textcircled{\#}}$. It is evident that $a \in R^{\textcircled{\#}}$ if and only $a^* \in R_{\textcircled{\#}}$, and in this case, one has $(a^{\textcircled{\#}})^* = (a^*)_{\textcircled{\#}}$.

If $a \in R$ are both Moore–Penrose invertible and group invertible and $a^\dagger = a^\#$, we call that a is an *EP element*.

2 Absorption laws for the (b, c)-inverse

Let $a, b \in R$ be two invertible elements. It is well known that

$$a^{-1} + b^{-1} = a^{-1}(a + b)b^{-1}. \tag{2.1}$$

The equality (2.1) is known as the *absorption law* of invertible elements. In general, the absorption law does not hold for generalized inverses, for example, [11,15]. In this section, the absorption laws for one-sided (b, c)-inverses are obtained.

Lemma 2.1 *Let $a, b, c, d \in R$. Then*

(1) *If $a_l^{\|(b,c)}$ is a left (b, c)-inverse of a and $d_r^{\|(b,c)}$ is a right (b, c)-inverse of d, then $a_l^{\|(b,c)} a d_r^{\|(b,c)} = d_r^{\|(b,c)}$ and $a_l^{\|(b,c)} d d_r^{\|(b,c)} = a_l^{\|(b,c)}$;*

(2) *If* $a_r^{\|(b,c)}$ *is a right* (b, c)*-inverse of* a *and* $d_l^{\|(b,c)}$ *is a left* (b, c)*-inverse of* d, *then* $d_l^{\|(b,c)} d a_r^{\|(b,c)} = a_r^{\|(b,c)}$ *and* $d_l^{\|(b,c)} a a_r^{\|(b,c)} = d_l^{\|(b,c)}$.

Proof (1) Let $x = a_l^{\|(b,c)}$ and $y = d_r^{\|(b,c)}$, then $x = rc$ and $y = bs$ for some $r, s \in R$. Thus, $xay = xabs = bs = y$ by $xab = b$ and $xdy = rcdy = rc = x$ by $cdy = c$.

(2) Can be proved by changing the roles of a and d in (1). □

By $a^{\|d}$ is the (d, d)-inverse of a, [26, Lemma 2.1] is a corollary of Lemma 2.1.

Theorem 2.2 *Let* $a, b, c, d \in R$. *Then*

(1) *If* $a_l^{\|(b,c)}$ *is a left* (b, c)*-inverse of* a *and* $d_r^{\|(b,c)}$ *is a right* (b, c)*-inverse of* d, *then* $a_l^{\|(b,c)} + d_r^{\|(b,c)} = a_l^{\|(b,c)}(a + d)d_r^{\|(b,c)}$;
(2) *If* $a_r^{\|(b,c)}$ *is a right* (b, c)*-inverse of* a *and* $d_l^{\|(b,c)}$ *is a left* (b, c)*-inverse of* d, *then* $a_r^{\|(b,c)} + d_l^{\|(b,c)} = d_l^{\|(b,c)}(a + d)a_r^{\|(b,c)}$.

Proof (1) Let $x = a_l^{\|(b,c)}$ and $y = d_r^{\|(b,c)}$, then by Lemma 2.1, we have $xay = y$ and $xdy = x$. Thus,

$$x(a + d)y = xay + xdy = x + y.$$

(2) Can be proved by changing the roles of a and d in (1). □

By Theorem 2.2, we have the following corollary.

Corollary 2.3 *Let* $a, b, c, d \in R$. *Then*

(1) *If* a *is* (b, c)*-invertible and* d *is* (b, c)*-invertible, then* $a^{\|(b,c)} + d^{\|(b,c)} = a^{\|(b,c)}(a + d)d^{\|(b,c)}$;
(2) [26, Proposition 2.2] *If* $a_r^{\|d}$ *is a right inverse along* d *of* a *and* $b_l^{\|d}$ *is a left inverse along* d *of* b, *then* $a_r^{\|d} + b_l^{\|d} = b_l^{\|d}(a + b)a_r^{\|d}$;
(3) [26, Corollary 2.3] *If* a *is invertible along* d *and* b *is invertible along* d, *then* $a^{\|d} + b^{\|d} = a^{\|d}(a + b)b^{\|d}$.

Let $a, b, c, d \in R$. If a and d are both (b, c)-invertible, then the absorption law for the (b, c)-inverse holds by Corollary 2.3. A natural question: if a is (b, c)-invertible and d is (u, v)-invertible for some $u, v \in R$, does the absorption law for $a^{\|(b,c)}$ and $d^{\|(u,v)}$ holds? That is, does the relation

$$a^{\|(b,c)} + d^{\|(u,v)} = a^{\|(b,c)}(a + d)d^{\|(u,v)} \tag{2.2}$$

hold for arbitrary $b, c, u, v \in R$?

Example 2.4 Let $\mathbb{C}^{2\times 2}$ denotes the set of all 2×2 complex matrices over the complex field $\mathbb{C}$. The involution in $\mathbb{C}^{2\times 2}$ is the conjugate transposition. Consider $a = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $d = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $b = c = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $u = v = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Note that $a^{\|(b,c)} = a^{\dagger} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ and $d^{\|(u,v)} = d^{\|u} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. It is easily to check that the relation in (2.2) does not hold in general. In fact, $a^{\|(b,c)} + d^{\|(u,v)} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = a^{\|(b,c)}(a + d)d^{\|(u,v)}$.

Let σ be a map from R to R. If $\sigma(ab) = \sigma(a)b$ for all $a, b \in R$, we call that σ is a *left centralizer* [12]. If $\sigma(ab) = a\sigma(b)$ for all $a, b \in R$, we call that σ is a *right centralizer* [12]. We call that σ is a *centralizer* if it is both a left and a right centralizer, that is, σ is a mapping that satisfies $\sigma(ab) = \sigma(a)b = a\sigma(b)$ for all $a, b \in R$. It is well-known that if σ is a bijective centralizer, then so is σ^{-1}. The tool of centralizers is useful in the theory of generalized inverses, for example, [26,27]. This tool is also useful in Hopf algebra, for example, [5].

Before investigate the absorption law for $a^{\|(b,c)}$ and $d^{\|(u,v)}$ by using centralizers, the following two lemmas are necessary.

Lemma 2.5 [21, Proposition 3.3] *Let $a, b, c \in R$. If a is (b, c)-invertible, then b and c are regular.*

Lemma 2.6 [4, Remark 2.2(i)] *Let $a, d, u, v \in R$. If $bR = uR$ and $Rc = Rv$, then a is (b, c)-invertible if and only if a is (u, v)-invertible. In this case, we have $a^{\|(b,c)} = a^{\|(u,v)}$.*

Theorem 2.7 *Let $\sigma, \tau : R \to R$ be two bijective centralizers and let $a, b, c, d, u, v \in R$ with $b = \sigma(u)$ and $c = \tau(v)$. If $a^{\|(b,c)}$ and $d^{\|(u,v)}$ exist, then $a^{\|(b,c)} + d^{\|(u,v)} = a^{\|(b,c)}(a + d)d^{\|(u,v)}$.*

Proof Since $\sigma : R \to R$ is a bijective centralizers, thus

$$b = \sigma(u) = \sigma(u1) = u\sigma(1);$$
$$u = \sigma^{-1}(b) = \sigma^{-1}(b1) = b\sigma^{-1}(1).$$

That is $bR = uR$. The condition $Rc = Rv$ can be proved in a similar way. Then, $a^{\|(b,c)} = a^{\|(u,v)}$ by Lemma 2.6. Therefore, we have $a^{\|(b,c)} + d^{\|(u,v)} = a^{\|(b,c)}(a + d)d^{\|(u,v)}$ by Corollary 2.3. □

Let R have an involution and $a \in R$. By [9], we have that a is Moore–Penrose invertible if and only if a is (a^*, a^*)-invertible, a is Drazin invertible if and only if there exists $k \in \mathbb{N}$ such that a is (a^k, a^k)-invertible and a is group invertible if and only if a is (a, a)-invertible. By [20, Theorem 4.4], we have that the (a, a^*)-inverse coincides with the core inverse of a and the (a^*, a)-inverse coincides with the dual core inverse of a. By [16, Lemma 3], we have that a is invertible along d if and only if a is (d, d)-invertible. As applications of Theorem 2.7, we have the following corollary. The item (1) in the following corollary can be found in [26, Theorem 2.6]. The items (2), (3) and (4) in the following corollary can be found in [26, Corollary 2.8].

Corollary 2.8 *Let $\sigma, \tau : R \to R$ be two bijective centralizers and let $a, b, d_1, d_2 \in R$. Then*

(1) *If $a^{\|d_1}$ and $b^{\|d_2}$ exist with $d_1 = \sigma(d_2)$, then $a^{\|d_1} + b^{\|d_2} = a^{\|d_1}(a + b)b^{\|d_2}$;*
(2) *If $a^{\#}$ and $b^{\#}$ exist with $a = \sigma(b)$, then $a^{\#} + b^{\#} = a^{\#}(a + b)b^{\#}$;*
(3) *If a^D and b^D exist with $a^n = \sigma(b^m)$, where $\operatorname{ind}(a) = n$ and $\operatorname{ind}(b) = m$, then $a^D + b^D = a^D(a + b)b^D$;*
(4) *If $a^{\dagger}$ and $b^{\dagger}$ exist with $a^* = \sigma(b^*)$, then $a^{\dagger} + b^{\dagger} = a^{\dagger}(a + b)b^{\dagger}$;*
(5) *If $a^{\#\!\!\!\!\bigcirc}$ and $b^{\#\!\!\!\!\bigcirc}$ exist with $a = \sigma(b)$ and $a^* = \tau(b^*)$, then $a^{\#\!\!\!\!\bigcirc} + b^{\#\!\!\!\!\bigcirc} = a^{\#\!\!\!\!\bigcirc}(a + b)b^{\#\!\!\!\!\bigcirc}$;*
(6) *If $a_{\#\!\!\!\!\bigcirc}$ and $b_{\#\!\!\!\!\bigcirc}$ exist with $a^* = \sigma(b^*)$ and $a = \tau(b)$, then $a_{\#\!\!\!\!\bigcirc} + b_{\#\!\!\!\!\bigcirc} = a_{\#\!\!\!\!\bigcirc}(a + b)b_{\#\!\!\!\!\bigcirc}$.*

Recall that if an element in a ring is invertible and Hermite, we call such an element a positive element. Let R be a unitary ring with an involution and consider $a \in R$ and two positive element $m, n \in R$. Then by [2, Theorem 3.2], we have a is weighted Moore–Penrose invertible relative to m and n if and only if a is invertible along $n^{-1}a^*m$. Furthermore, in this case, $a^{\|n^{-1}a^*m} = a^{\dagger}_{m,n}$. Thus, by Corollary 2.8(1), we can obtain an absorption law of the weighted Moore–Penrose inverse.

3 Reverse order laws for the (b, c)-inverse

Let $a, b \in R$ be two invertible elements. It is well known that

$$(ab)^{-1} = b^{-1}a^{-1}. \tag{3.1}$$

The equality (3.1) is known as the *reverse order law* of invertible elements. In general, the reverse order law does not hold for generalized inverses, for example, [6,7,18,25]. The following two lemmas will be useful in the sequel.

Lemma 3.1 [9, Theorem 2.1 (ii) and Proposition 6.1] *Let $a, b, c \in R$. Then $y \in R$ is the (b, c)-inverse of a if and only if $yay = y$, $yR = bR$ and $Ry = Rc$.*

Lemma 3.2 [10, Theorem 2.1] *Let $a, b, c \in R$. If a is both left and right (b, c)-invertible, then the left (b, c)-inverse of a and the right (b, c)-inverse of a are unique. Moreover, the left (b, c)-inverse of a coincides with the right (b, c)-inverse of a.*

Theorem 3.3 *Let $a, b, c, d \in R$ such that $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist. If $a^{\|(b,c)}a = aa^{\|(b,c)}$, then $z(ad)z = z$, $zR = bR$ and $Rz \subseteq Rc$, where $z = d^{\|(b,c)}a^{\|(b,c)}$. In particular, ad is left (b, c)-invertible and z is a left (b, c)-inverse of ad.*

Proof Let $x = a^{\|(b,c)}$ and $y = d^{\|(b,c)}$, then $x \in xRc$ and $ydb = b$. Then $x \in xRc$ implies $z \in Rc$, that is $Rz \subseteq Rc$. From Lemma 2.1, we can get $yax = y$. Then $zadb = yxadb = yaxdb = ydb = b$ by $xa = ax$. Since $d^{\|(b,c)}$ exists, then $yR = bR$ by Lemma 3.1 and b is regular by Lemma 2.5. If b^- is an inner inverse of b, then

$$y = bs = bb^-bs = bb^-y \quad \text{for some } s \in R. \tag{3.2}$$

Then by $yax = y$, $ax = xa$ and (3.2), we have

$$\begin{aligned} z(ad)z &= yx(ad)yx = yaxdyx = ydyx = yx = z;\\ z &= yx = bb^-yx \in bR;\\ b &= ydb = yaxdb = yxadb \in zR. \end{aligned}$$

Thus, we have $z(ad)z = z$ and $zR = bR$. The conditions $Rz \subseteq Rc$ and $zadb = b$ imply that ad is left (b, c)-invertible and z is a left (b, c)-inverse of ad. □

The following Theorem 3.4 is the corresponding result of Theorem 3.3.

Theorem 3.4 *Let $a, b, c, d \in R$ such that $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist. If $d^{\|(b,c)}d = dd^{\|(b,c)}$, then $z(ad)z = z$, $zR \subseteq bR$ and $Rz = Rc$, where $z = d^{\|(b,c)}a^{\|(b,c)}$. In particular, ad is right (b, c)-invertible and z is a right (b, c)-inverse of ad.*

Theorem 3.5 *Let $a, b, c, d \in R$ such that $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist. If $a^{\|(b,c)}a = aa^{\|(b,c)}$ and $d^{\|(b,c)}d = dd^{\|(b,c)}$, then ad is (b, c)-invertible and*

$$(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}.$$

Proof It is easy to check that $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$ by Lemma 3.2, Theorems 3.3 and 3.4. □

Lemma 3.6 *Let $a, b, c \in R$, $\sigma : R \to R$ be a right centralizer and $\tau : R \to R$ be a left centralizer with $ab = \sigma(ba)$ and $ca = \tau(ac)$. If $a^{\|(b,c)}$ exists, then $a^{\|(b,c)}a = aa^{\|(b,c)}$.*

Proof Since $a^{\|(b,c)}$ exists, $\sigma : R \to R$ is a right centralizer and $\tau : R \to R$ is a left centralizer, we have

$$ab = \sigma(ba) = \sigma(a^{\|(b,c)}aba) = a^{\|(b,c)}a\sigma(ba) = a^{\|(b,c)}a^2b, \tag{3.3}$$

$$ca = \tau(ac) = \tau(acaa^{\|(b,c)}) = \tau(ac)aa^{\|(b,c)} = ca^2a^{\|(b,c)}. \tag{3.4}$$

Thus, by $a^{\|(b,c)}$ exists and by Lemma 3.1, we can get $a^{\|(b,c)}R = bR$ and $Ra^{\|(b,c)} = Rc$. Then $a^{\|(b,c)} = br = sc$ for some $r, s \in R$. Post-multiplying by r on (3.3) gives

$$aa^{\|(b,c)} = abr = a^{\|(b,c)}a^2br = a^{\|(b,c)}a^2a^{\|(b,c)}. \tag{3.5}$$

Pre-multiplying by s on (3.4) gives

$$a^{\|(b,c)}a = sca = sca^2a^{\|(b,c)} = a^{\|(b,c)}a^2a^{\|(b,c)}. \tag{3.6}$$

Therefore, we have that $a^{\|(b,c)}a = aa^{\|(b,c)}$ by (3.5) and (3.6). □

As applications of Lemma 3.6, we have the following corollary.

Corollary 3.7 [26, Lemma 3.1] *Let $a, d \in R$ and let $\sigma : R \to R$ be a bijective centralizer with $ad = \sigma(da)$. If $a^{\|d}$ exists, then $a^{\|d}a = aa^{\|d}$.*

Theorem 3.8 *Let $a, b, c \in R$, $\sigma : R \to R$ be a right centralizer and $\tau : R \to R$ be a left centralizer with $ab = \sigma(ba)$ and $ca = \tau(ac)$. If $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist, then ad is (b, c)-invertible and*

$$(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}.$$

Proof Let $x = a^{\|(b,c)}$ and $y = d^{\|(b,c)}$, then $ax = xa$ by Lemma 3.6. Thus, by Theorem 3.3, we have $z(ad)z = z$, $zR = bR$ and $Rz \subseteq Rc$, where $z = d^{\|(b,c)}a^{\|(b,c)}$. Since

$$c = cax = \tau(ac)x = \tau(a)cx = \tau(a)(cdy)x = \tau(a)cdz \in Rz,$$

Thus, $Rz = Rc$. The proof is completed by Lemma 3.1. □

If we let $\sigma = \tau = I$ in Theorem 3.8, then we can get the following corollary.

Corollary 3.9 [6, Corollary 2.5] *Let $a, b, c, d \in R$ and $ab = ba$ and $ca = ac$. If $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist, then ad is (b, c)-invertible and*

$$(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}.$$

If we let $b = c = d$ in Theorem 3.8, then we can get the following corollary.

Corollary 3.10 [26, Theorem 3.2] *Let $a, b, d \in R$ and let $\sigma : R \to R$ be a bijective centralizer with $ad = \sigma(da)$. If $a^{\|d}$ and $b^{\|d}$ exist, then ab is invertible along d and*

$$(ab)^{\|d} = b^{\|d}a^{\|d}.$$

Lemma 3.11 [6, Theorem 2.3] *Let $a, b, c \in R$ such that $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist. Then ad is (b, c)-invertible and $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$ if and only if $d^{\|(b,c)}a^{\|(b,c)}adb = b$ and $cadd^{\|(b,c)}a^{\|(b,c)} = c$ both hold.*

Theorem 3.12 *Let $a, b, c \in R$ and let $\sigma, \tau : R \to R$ be two bijective centralizers with $db = \sigma(bd)$ and $ca = \tau(ac)$. If $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist, then ad is (b, c)-invertible and*

$$(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}.$$

Proof Let $x = a^{\|(b,c)}$ and $y = d^{\|(b,c)}$. We have that b and c are regular by Lemma 2.5. Let b^- and c^- be an inner inverse of b and c, respectively. Then

$$db = \sigma(bd) = \sigma(bb^-bd) = bb^-\sigma(bd) = bb^-db, \tag{3.7}$$

$$ca = \tau(ac) = \tau(acc^-c) = \tau(ac)c^-c = cac^-c. \tag{3.8}$$

Let $z = yx$. Then by (3.7), (3.8), $xab = b$ and $cdy = c$, we have

$$z(ad)b = yxadb = yxa(bb^-db) = y(xab)b^-db = ybb^-db = ydb = b;$$
$$c(ad)z = cadyx = (cac^-c)dyx = cac^-(cdy)x = cac^-cx = cax = c.$$

Thus, ad is (b, c)-invertible and $(ad)^{\|(b,c)} = z$ by Lemma 3.11. □

Corollary 3.13 [6, Corollary 2.5] *Let $a, b, c, d \in R$ and $db = bd$ and $ca = ac$. If $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist, then ad is (b, c)-invertible and*

$$(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}.$$

If $\sigma : R \to R$ is a bijective centralizer, then $b = \sigma(b)\sigma^{-1}(1)$. In fact, observe that $\sigma(b) = \sigma(b \cdot 1) = b\sigma(1)$. In addition, if we let $w = \sigma^{-1}(1)$, then $1 = \sigma(w) = \sigma(w \cdot 1) = w\sigma(1)$ and $1 = \sigma(1 \cdot w) = \sigma(1)w$, which imply that $\sigma(1)$ is invertible and $\sigma(1)^{-1} = w = \sigma^{-1}(1)$. From $\sigma(b) = b\sigma(1)$ we get $b = \sigma(b)\sigma(1)^{-1} = \sigma(b)\sigma^{-1}(1)$. The above facts will be used in the next theorem.

Theorem 3.14 *Let $a, b, d \in R$ and let $\sigma, \tau : R \to R$ be two bijective centralizers. Then $a^{\|(b,c)}$ exists if and only if $a^{\|(\sigma(b),\tau(c))}$ exists. In this case,*

$$a^{\|(b,c)} = a^{\|(\sigma(b),\tau(c))}.$$

Proof $(\Rightarrow)$. From the existence of the (b, c)-inverse of a, we have

$$\sigma(b) = \sigma(b1) = b\sigma(1) \in bR;$$
$$\tau(c) = \tau(1c) = \tau(1)c \in Rc. \tag{3.9}$$

From $b = \sigma(b)\sigma^{-1}(1)$ and $c = \tau^{-1}(1)\tau(c)$, we have $bR \subseteq \sigma(b)R$ and $Rc \subseteq R\tau(c)$, thus by (3.9), we have $bR = \sigma(b)R$ and $Rc = R\tau(c)$. Thus, $a^{\|(\sigma(b),\tau(c))}$ exists and $a^{\|(b,c)} = a^{\|(\sigma(b),\tau(c))}$ by Lemma 2.6.

$(\Leftarrow)$. Since σ^{-1} and τ^{-1} are bijective centralizers, we can get the equivalence by the manner in the first part of the proof of this theorem. □

Acknowledgements This research is supported by the National Natural Science Foundation of China (no. 11771076 and no. 11871301). The first author is grateful to China Scholarship Council for giving him a scholarship for his further study in Universitat Politècnica de València, Spain.

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