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L^p-calculus approach to the random autonomous linear differential equation with discrete delay

J. Calatayud, J.-C. Cortés and M. Jornet

Abstract. In this paper, we provide a full probabilistic study of the random autonomous linear differential equation with discrete delay $\tau > 0$: $x'(t) = ax(t) + bx(t-\tau)$, $t \geq 0$, with initial condition $x(t) = g(t)$, $-\tau \leq t \leq 0$. The coefficients a and b are assumed to be random variables, while the initial condition $g(t)$ is taken as a stochastic process. By using L^p-calculus, we prove that, under certain conditions, the deterministic solution constructed with the method of steps that involves the delayed exponential function is an L^p-solution too. An analysis of L^p-convergence when the delay τ tends to 0 is also performed in detail.

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Keywords. Random autonomous linear differential equation with discrete delay, L^p random calculus, Method of steps, Uncertainty quantification.

1. Introduction

Delay differential equations can be viewed as generalizations of classical differential equations. The study of delay differential equations requires a distinctive treatment with respect to their classical counterpart [1]. This fact can be checked starting from introducing a delay in the basic linear ordinary differential equation that leads to richer qualitative and quantitative behaviors [2]. Regarding applications, the delays or lags into the formulation of classical differential equations expand the variety and complexity of possibles behavior regimes often allowing a better description of the real phenomenon [3]. In particular, delays play a key role in Biomathematics (population dynamics, infectious diseases, physiology, biotic population, immunology, neural networks and cell kinetics) [4–6], but also in other realms like Chemistry [7, Ch. 4], Engineering [8], Economics and Finance [9, 10].

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As it has been previously indicated, delay differential equations allow describing more complex dynamics than their classical counterpart. This fact is particularly convenient in dealing with modeling using real data where, in addition, it is necessary to perform a rigorous treatment of uncertainty (uncertainty quantification). This randomness usually comes from sampling or simply because of the inherent complexity of the phenomena under study. In this setting, stochastic and random delay differential equations are formulated instead.

On the one hand, stochastic delay differential equations are those in which uncertainty in driven by stochastic processes whose sample path behavior is irregular (typically Brownian motion, or more generally Wiener process, and Poisson process). Their mathematical study requires Itô or Malliavin calculus [12]. Under this approach, uncertainty is limited to specific probabilistic patterns. In the case of considering the Wiener process, then the underlying noise is of Gaussian type. An excellent overview of this approach can be found in [13]. While some recent theoretical and numerical advances using Gaussian and Poisson distributions are reported in the book [11, Ch. 1 and Ch. 10] and in the articles [14–18], for example. Stochastic delay differential equations have also been successfully applied to model real problems in different settings. For example, mathematical models to describe the dynamics of obesity and alcohol consumption have been proposed in [19] and [20], respectively. The stochastic Navier-Stokes with infinite delay has been recently addressed in [21]. A predator prey stochastic model with delay has been proposed in [22].

On the other hand, random delay differential equations are those in which random effects are directly manifested in their inputs (coefficients, initial/boundary conditions and/or source term). The sample path behavior of these inputs is regular (e.g. sample path continuous) with respect to time and space [23, p. 97]. The rigorous analysis of this type of differential equations can be conducted mainly by using two approaches, sample path calculus or mean square random calculus [24]. The former approach is strongly based upon the well-behavior (regularity) of the trajectories of the inputs involved in the random differential equations in order to take advantage of the power of deterministic calculus. In the latter case, results are formulated in the setting of the Hilbert space $(\mathrm{L}^2, \langle\cdot,\cdot\rangle)$ of real random variables on Ω having second-order moment (thus having mean and variance too) endowed with the inner product $\langle U, V\rangle = \mathbb{E}[UV]$, where $\mathbb{E}[\cdot]$ denotes the expectation operator and Ω is the sample space of an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ [24]. Mean square random calculus have been successfully applied to study random differential equations, see for example [24, 25]. However, to the best of our knowledge, in the context of random delay differential equations only a few theoretical results have been established. Some recently and very interesting contributions focusing on numerical methods instead are [26, 27]. In [26], a sparse grid stochastic Legendre spectral collocation

method is proposed to numerically solve linear systems of random differential equations with constant and pantograph delays. While in [27], the authors extend the generalized polynomial chaos method to study nonlinear random delay differential equations by taking advantage of orthogonality properties in the Hilbert space L^2. Nevertheless, there is lack of theoretical results for random delay differential equations, starting from the random autonomous linear differential equation with delay, in the general context of random Lebesgue spaces $(\mathrm{L}^p, \|\cdot\|_p)$, where $\|U\|_p = (\mathbb{E}[|U|^p])^{1/p}$, for $1 \leq p < \infty$, and $\|U\|_\infty = \inf\{C \geq 0 : |U(\omega)| \leq C \text{ for almost every } \omega\}$, for $U : \Omega \to \mathbb{R}$ being a random variable. And this is precisely the aim of this contribution.

Apart from stochastic and random delay differential equations, it must also be mentioned a complementary approach usually referred to as fuzzy delay differential equations, whose uncertainty is driven by particular stochastic processes like the fuzzy Liu process [28].

Finally, it must be pointed out that randomness is directly introduced in the delay instead of coefficients and/or forcing term in order to account for uncertainties associated to the time instant in which relevant factors determining the output of the mathematical model under study take place. Examples in this regard can be found in [29–31], for example.

The autonomous linear differential equation with discrete time delay $\tau > 0$ is given by

$$\begin{cases} x'(t) &= ax(t) + bx(t-\tau),\ t \geq 0, \\ x(t) &= g(t),\ -\tau \leq t \leq 0, \end{cases} \tag{1}$$

where a is the coefficient of the non-delay component, b is the parameter of the delay term, and the function $g(t)$ defined on $[-\tau, 0]$ is the initial condition. If $g \in C^1([-\tau, 0])$, then the unique solution to (1) is obtained with the method of steps and is given by [34, Th. 1],

$$x(t) = \mathrm{e}^{a(t+\tau)}\mathrm{e}_\tau^{b_1,t} g(-\tau) + \int_{-\tau}^{0} \mathrm{e}^{a(t-s)}\mathrm{e}_\tau^{b_1,t-\tau-s}(g'(s) - ag(s))\,\mathrm{d}s, \tag{2}$$

where $b_1 = \mathrm{e}^{-a\tau} b$,

$$\mathrm{e}_\tau^{c,t} = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \leq t < 0, \\ 1 + c\dfrac{t}{1!}, & 0 \leq t < \tau, \\ 1 + c\dfrac{t}{1!} + c^2\dfrac{(t-\tau)^2}{2!}, & \tau \leq t < 2\tau, \\ \vdots & \vdots \\ \displaystyle\sum_{k=0}^{n} c^k \frac{(t-(k-1)\tau)^k}{k!}, & (n-1)\tau \leq t < n\tau, \end{cases}$$

is the delayed exponential function [34, Def. 1], $c, t \in \mathbb{R}$, $\tau > 0$ and $n = \lfloor t/\tau \rfloor + 1$ (here $\lfloor \cdot \rfloor$ denotes the integer part defined by the so-called floor function).

The randomization of (1) consists in assuming that the system depends on an outcome ω of an experiment:

$$\begin{cases} x'(t,\omega) &= a(\omega)x(t,\omega) + b(\omega)x(t-\tau,\omega),\ t \geq 0, \\ x(t,\omega) &= g(t,\omega),\ -\tau \leq t \leq 0. \end{cases} \tag{3}$$

Here, the coefficients $a = a(\omega)$ and $b = b(\omega)$ are random variables, while $g(t) = g(t,\omega)$ is a stochastic process, all of them defined in an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The formal solution to (3) is obtained after randomization of (2):

$$\begin{aligned} x(t,\omega) &= \mathrm{e}^{a(\omega)(t+\tau)}\mathrm{e}_{\tau}^{b_1(\omega),t} g(-\tau,\omega) \\ &\quad + \int_{-\tau}^{0} \mathrm{e}^{a(\omega)(t-s)}\mathrm{e}_{\tau}^{b_1(\omega),t-\tau-s}(g'(s,\omega) - a(\omega)g(s,\omega))\,\mathrm{d}s, \end{aligned} \tag{4}$$

where $b_1(\omega) = \mathrm{e}^{-a(\omega)\tau}b(\omega)$. The stochastic process (4) is a solution to (3) in the sample path sense, under the assumption that the sample paths of g belong to $C^1([-\tau,0])$.

In this paper, we study conditions under which (4) is an L^p-solution to (3). This kind of delay differential equations appear in Engineering and Control problems [32, 33], for example. When they are applied to real data its coefficients a and b and its *preshape* function $g(t)$ need to be calibrated. Since data often involve uncertainty coming from errors measurements, it is more realistic to treat a and b as random variables and $g(t)$ as a stochastic process, as it will be assumed throughout this manuscript.

This paper is organized as follows. In Section 2, we state preliminary results on L^p-calculus that are required for the exposition. In Section 3, we prove that (4) is the unique L^p-solution to (3) under certain assumptions. In Section 4, we demonstrate that (4) tends as $\tau \to 0$ to the solution to (3) with $\tau = 0$, in the space L^p.

2. Preliminary results on L^p-calculus

In this section, we state some preliminary results on L^p-calculus that will be required in the coming sections.

Proposition 2.1 (Chain Rule Theorem). *Let $\{X(t) : t \in [a,b]\}$ be a stochastic process. Let f be a deterministic C^1 function on an open set that contains $X([a,b])$. Fix $1 \leq p < \infty$. Let $t \in [a,b]$ such that:*

(i) *X is L^{2p}-differentiable at t,*
(ii) *X is path continuous on $[a,b]$,*
(iii) *there exist $r > 2p$ and $\delta > 0$ such that $\sup_{s\in[-\delta,\delta]} \mathbb{E}[|f'(X(t+s))|^r] < \infty$.*

Then $f \circ X$ is L^p-differentiable at t and $(f \circ X)'(t) = f'(X(t))X'(t)$.

The proof of Proposition 2.1 is analogous to [36, Th. 3.19], but not restricted to mean square and mean fourth calculus. In the proof, instead of applying Hölder's inequality as $\|UV\|_2 \leq \|U\|_4\|V\|_4$, one uses the more

general version $\|UV\|_p \leq \|U\|_{2p}\|V\|_{2p}$ (here U and V represent random variables).

Lemma 2.1. *Let $Y_1(t,s)$, $Y_2(t,s)$ and $Y_3(t,s)$ be three stochastic processes and fix $1 \leq p < \infty$. If Y_1 and Y_2 are L^q-continuous for all $1 \leq q < \infty$, and Y_3 is $\mathrm{L}^{p+\eta}$-continuous for certain $\eta > 0$, then the product process $Y_1Y_2Y_3$ is L^p-continuous.*

On the other hand, if Y_1 and Y_2 are L^∞-continuous, and Y_3 is L^p-continuous, then the product process $Y_1Y_2Y_3$ is L^p-continuous.

Proof. Suppose that Y_1 and Y_2 are L^q-continuous for all $1 \leq q < \infty$, and Y_3 is $\mathrm{L}^{p+\eta}$-continuous. Notice that Y_1Y_2 is L^q-continuous for all $1 \leq q < \infty$. Indeed, if $t_n \to t$ and $s_n \to s$ as $n \to \infty$, then, by the triangular and Hölder's inequalities,

$$\begin{aligned}
&\|Y_1(t_n,s_n)Y_2(t_n,s_n) - Y_1(t,s)Y_2(t,s)\|_q \\
&\leq \|(Y_1(t_n,s_n) - Y_1(t,s))Y_2(t_n,s_n)\|_q + \|Y_1(t,s)(Y_2(t,s) - Y_2(t_n,s_n))\|_q \\
&\leq \|Y_1(t_n,s_n) - Y_1(t,s)\|_{2q}\|Y_2(t_n,s_n)\|_{2q} \\
&\quad + \|Y_1(t,s)\|_{2q}\|Y_2(t,s) - Y_2(t_n,s_n)\|_{2q} \xrightarrow{n\to\infty} 0\,. \qquad (5)
\end{aligned}$$

Now,

$$\begin{aligned}
&\|Y_1(t_n,s_n)Y_2(t_n,s_n)Y_3(t_n,s_n) - Y_1(t,s)Y_2(t,s)Y_3(t,s)\|_p \\
&\leq \|Y_1(t_n,s_n)Y_2(t_n,s_n)(Y_3(t_n,s_n) - Y_3(t,s))\|_p \\
&\quad + \|(Y_1(t_n,s_n)Y_2(t_n,s_n) - Y_1(t,s)Y_2(t,s))Y_3(t,s)\|_p \\
&\leq \|Y_1(t_n,s_n)Y_2(t_n,s_n)\|_q\|Y_3(t_n,s_n) - Y_3(t,s)\|_{p+\eta} \\
&\quad + \|Y_1(t_n,s_n)Y_2(t_n,s_n) - Y_1(t,s)Y_2(t,s)\|_q\|Y_3(t,s)\|_{p+\eta} \xrightarrow{n\to\infty} 0, \qquad (6)
\end{aligned}$$

where $q = \frac{p(p+\eta)}{\eta}$ has been chosen to apply Hölder's inequality (note that $\frac{1}{p} = \frac{1}{p+\eta} + \frac{1}{q}$). This proves the L^p-continuity of $Y_1Y_2Y_3$.

Suppose that Y_1 and Y_2 are L^∞-continuous, and Y_3 is L^p-continuous. Then Y_1Y_2 is L^∞-continuous, by (5) with $q = \infty$. Statement (6) holds with $q = \infty$ and p in lieu of $p+\eta$. This demonstrates the L^p-continuity of $Y_1Y_2Y_3$. □

Lemma 2.2. *Let $Y_1(t)$, $Y_2(t)$ and $Y_3(t)$ be three stochastic processes, and $1 \leq p < \infty$. If Y_1 and Y_2 are L^q-differentiable for all $1 \leq q < \infty$, and Y_3 is $\mathrm{L}^{p+\eta}$-differentiable for certain $\eta > 0$, then the product process $Y_1Y_2Y_3$ is L^p-differentiable and $\frac{\mathrm{d}}{\mathrm{d}t}(Y_1(t)Y_2(t)Y_3(t)) = Y_1'(t)Y_2(t)Y_3(t) + Y_1(t)Y_2'(t)Y_3(t) + Y_1(t)Y_2(t)Y_3'(t)$.*

Additionally, if Y_1 and Y_2 are assumed to be L^∞-differentiable, and Y_3 is L^p-differentiable, then $Y_1Y_2Y_3$ is L^p-differentiable, with $\frac{\mathrm{d}}{\mathrm{d}t}(Y_1(t)Y_2(t)Y_3(t)) = Y_1'(t)Y_2(t)Y_3(t) + Y_1(t)Y_2'(t)Y_3(t) + Y_1(t)Y_2(t)Y_3'(t)$.

The proof of this lemma follows the same reasoning as Lemma 2.1, but working with incremental quotients instead. Similar reasonings are also given in [24, p. 96 (4)], [36, Lemma 3.14]. We omit the details.

Proposition 2.2 (L^p-differentiation under the L^p-Riemann integral sign). *Let $F(t,s)$ be a stochastic process on $[a,b]\times[c,d]$. Fix $1\le p<\infty$. Suppose that $F(t,\cdot)$ is L^p-continuous on $[c,d]$, for each $t\in[a,b]$, and that there exists the L^p-partial derivative $\frac{\partial F}{\partial t}(t,s)$ for all $(t,s)\in[a,b]\times[c,d]$, which is L^p-continuous on $[a,b]\times[c,d]$. Let $G(t)=\int_c^d F(t,s)\,\mathrm{d}s$ (the integral is understood as an L^p-Riemann integral). Then G is L^p-differentiable on $[a,b]$ and $G'(t)=\int_c^d \frac{\partial F}{\partial t}(t,s)\,\mathrm{d}s$.*

Proof. We have, for $h\neq 0$,

$$\begin{aligned}
&\left\|\frac{G(t+h)-G(t)}{h}-\int_c^d\frac{\partial F}{\partial t}(t,s)\,\mathrm{d}s\right\|_p\\
&=\left\|\int_c^d\left(\frac{F(t+h,s)-F(t,s)}{h}-\frac{\partial F}{\partial t}(t,s)\right)\mathrm{d}s\right\|_p\\
&\le\int_c^d\left\|\frac{F(t+h,s)-F(t,s)}{h}-\frac{\partial F}{\partial t}(t,s)\right\|_p\mathrm{d}s,
\end{aligned}\tag{7}$$

where the last inequality comes from [24, p. 102] in the general setting of L^p-calculus. We know that

$$\lim_{h\to 0}\left\|\frac{F(t+h,s)-F(t,s)}{h}-\frac{\partial F}{\partial t}(t,s)\right\|_p=0,$$

by definition of L^p-partial derivative. We bound $\|\frac{F(t+h,s)-F(t,s)}{h}-\frac{\partial F}{\partial t}(t,s)\|_p$ in order to apply the Dominated Convergence Theorem in (7).

On the one hand,

$$\left\|\frac{\partial F}{\partial t}(t,s)\right\|_p\le M,\tag{8}$$

for all $(t,s)\in[a,b]\times[c,d]$, by L^p-continuity of $\frac{\partial F}{\partial t}(t,s)$. On the other hand, by Barrow's rule [24, p. 104], an inequality from [24, p. 102], and (8),

$$\begin{aligned}
\left\|\frac{F(t+h,s)-F(t,s)}{h}\right\|_p&=\frac{1}{|h|}\left\|\int_t^{t+h}\frac{\partial F}{\partial t}(t',s)\,\mathrm{d}t'\right\|_p\\
&\le\frac{1}{|h|}\left|\int_t^{t+h}\left\|\frac{\partial F}{\partial t}(t',s)\right\|_p\mathrm{d}t'\right|\le M.
\end{aligned}\tag{9}$$

By the triangular inequality, (8) and (9),

$$\begin{aligned}
&\left\|\frac{F(t+h,s)-F(t,s)}{h}-\frac{\partial F}{\partial t}(t,s)\right\|_p\\
&\le\left\|\frac{F(t+h,s)-F(t,s)}{h}\right\|_p+\left\|\frac{\partial F}{\partial t}(t,s)\right\|_p\le 2M,
\end{aligned}$$

so the use of the Dominated Convergence Theorem to conclude that (7) tends to 0 as $h\to 0$ is justified.

□

3. L^p-solution to the random autonomous linear differential equation with discrete delay

In this section, we solve (3) in the L^p-sense. We will establish its uniqueness of solution, and we will prove that (4) is an L^p-solution under certain conditions (the integral from (4) will be understood as an L^p-Riemann integral).

Theorem 3.1 (Uniqueness). *The stochastic system (3) has at most one* L^p*-solution, for* $1 \leq p < \infty$.

Proof. Suppose that $x(t)$ and $y(t)$ are two L^p-solutions to (3). Let $z(t) = x(t) - y(t)$, which satisfies the random differential equation problem with delay

$$\begin{cases} z'(t,\omega) = a(\omega)z(t,\omega) + b(\omega)z(t-\tau,\omega), \ t \geq 0, \\ z(t,\omega) = 0, \ -\tau \leq t \leq 0. \end{cases}$$

If $t \in [0,\tau]$, then $t - \tau \in [-\tau, 0]$, therefore $z(t-\tau) = 0$. Thus, $z(t)$ satisfies a random differential equation problem with no delay:

$$\begin{cases} z'(t,\omega) = a(\omega)z(t,\omega), \ t \geq 0, \\ z(0,\omega) = 0. \end{cases} \tag{10}$$

In [37], it was proved that any L^p-solution to a random initial value problem has a product measurable representative which is an absolutely continuous solution in the sample path sense. Since the sample path solution to (10) must be 0 (from the deterministic theory), we conclude that $z(t) = 0$, as wanted. □

Proposition 3.1 (L^p-derivative of the delayed exponential function). *Consider the stochastic system with discrete delay*

$$\begin{cases} x'(t,\omega) = c(\omega)x(t-\tau,\omega), \ t \geq 0, \\ x(t,\omega) = 1, \ -\tau \leq t \leq 0, \end{cases} \tag{11}$$

where $c(\omega)$ *is a random variable.*

If c *has centered absolute moments of any order, then* $\mathrm{e}_\tau^{c,t}$ *is the unique* L^p*-solution to (11), for all* $1 \leq p < \infty$.

On the other hand, if c *is bounded, then* $\mathrm{e}_\tau^{c,t}$ *is the unique* L^∞*-solution to (11).*

Proof. Suppose that c has centered absolute moments of any order, and let $x(t) = \mathrm{e}_\tau^{c,t}$. Fix $t_0 \geq 0$. We want to prove that x is L^p-differentiable at t_0, for all $1 \leq p < \infty$, with $x'(t_0) = cx(t_0 - \tau)$.

For $n = \lfloor t_0/\tau \rfloor + 1$, t_0 belongs to $[(n-1)\tau, n\tau)$. We distinguish two cases: $(n-1)\tau < t_0 < n\tau$ and $t_0 = (n-1)\tau$.

In the former case, $\mathrm{e}_\tau^{c,t} = \sum_{k=0}^{n} c^k \frac{(t-(k-1)\tau)^k}{k!}$ for all $t \in ((n-1)\tau, n\tau)$, which is a neighborhood of t_0. Each addend $c^k \frac{(t-(k-1)\tau)^k}{k!}$ is L^p-differentiable,

with derivative $\frac{\mathrm{d}}{\mathrm{d}t}\{c^k \frac{(t-(k-1)\tau)^k}{k!}\} = c^k \frac{(t-(k-1)\tau)^{k-1}}{(k-1)!}$:

$$\left\| c^k \frac{\frac{(t+h-(k-1)\tau)^k}{k!} - \frac{(t-(k-1)\tau)^k}{k!}}{h} - c^k \frac{(t-(k-1)\tau)^{k-1}}{(k-1)!} \right\|_p$$

$$= \|c^k\|_p \left| \frac{\frac{(t+h-(k-1)\tau)^k}{k!} - \frac{(t-(k-1)\tau)^k}{k!}}{h} - \frac{(t-(k-1)\tau)^{k-1}}{(k-1)!} \right| \xrightarrow{h\to 0} 0,$$

since c has centered absolute moments of any order and by the classical derivative of $\frac{(t-(k-1)\tau)^k}{k!}$. Then $x(t) = \mathrm{e}_\tau^{c,t}$ is L^p-differentiable on $((n-1)\tau, n\tau)$ and $x'(t) = \sum_{k=1}^n c^k \frac{(t-(k-1)\tau)^{k-1}}{(k-1)!} = c\sum_{k=0}^{n-1} c^k \frac{(t-k\tau)^k}{k!} = cx(t-\tau)$. Notice that if c were bounded, the limit computed as $h \to 0$ holds with $p = \infty$, so $x(t)$ is L^∞-differentiable on $((n-1)\tau, n\tau)$.

In the latter case $t_0 = (n-1)\tau$, we need to compute the left and right derivatives of $x(t) = \mathrm{e}_\tau^{c,t}$ at t_0, and check that both are equal to $cx(t_0-\tau) = c\sum_{k=0}^{n-1} c^k \frac{(t_0-k\tau)^k}{k!} = \sum_{k=1}^n c^k \frac{(t_0-(k-1)\tau)^{k-1}}{(k-1)!}$. On the one hand, for $h > 0$,

$$\left\| \frac{x(t_0+h)-x(t_0)}{h} - \sum_{k=1}^n c^k \frac{(t_0-(k-1)\tau)^{k-1}}{(k-1)!} \right\|_p$$

$$\leq \sum_{k=1}^n \left\| c^k \frac{\frac{(t_0+h-(k-1)\tau)^k}{k!} - \frac{(t_0-(k-1)\tau)^k}{k!}}{h} - c^k \frac{(t_0-(k-1)\tau)^{k-1}}{(k-1)!} \right\|_p$$

$$= \sum_{k=1}^n \|c^k\|_p \left| \frac{\frac{(t_0+h-(k-1)\tau)^k}{k!} - \frac{(t_0-(k-1)\tau)^k}{k!}}{h} - \frac{(t_0-(k-1)\tau)^{k-1}}{(k-1)!} \right| \xrightarrow{h\to 0^+} 0,$$

by the classical derivative of $\frac{(t-(k-1)\tau)^k}{k!}$ and the boundedness of the absolute moments of c. On the other hand, for $h < 0$,

$$\left\| \frac{x(t_0+h)-x(t_0)}{h} - \sum_{k=1}^n c^k \frac{(t_0-(k-1)\tau)^{k-1}}{(k-1)!} \right\|_p$$

$$= \left\| \sum_{k=1}^{n-1} c^k \frac{\frac{(t_0+h-(k-1)\tau)^k}{k!} - \frac{(t_0-(k-1)\tau)^k}{k!}}{h} - \sum_{k=1}^{n-1} c^k \frac{(t_0-(k-1)\tau)^{k-1}}{(k-1)!} \right\|_p$$

$$\leq \sum_{k=1}^{n-1} \|c^k\|_p \left| \frac{\frac{(t_0+h-(k-1)\tau)^k}{k!} - \frac{(t_0-(k-1)\tau)^k}{k!}}{h} - \frac{(t_0-(k-1)\tau)^{k-1}}{(k-1)!} \right| \xrightarrow{h\to 0^-} 0.$$

This proves that $x(t) = \mathrm{e}_\tau^{c,t}$ is L^p-differentiable at t_0, with $x'(t_0) = cx(t-t_0)$. Again, note that if c were bounded, the limits computed as $h \to 0$ hold with $p = \infty$, so $x(t)$ is L^∞-differentiable at t_0. □

In what follows, we denote the moment-generating function of a random variable a as $\phi_a(\zeta) = \mathbb{E}[\mathrm{e}^{a\zeta}]$, $\zeta \in \mathbb{R}$.

Theorem 3.2 (Existence and uniqueness). *Fix $1 \leq p < \infty$. Suppose that $\phi_a(\zeta) < \infty$ for all $\zeta \in \mathbb{R}$, b has centered absolute moments of any order,*

and g belongs to $C^1([-\tau,0])$ in the $\mathrm{L}^{p+\eta}$-sense, for certain $\eta>0$. Then the stochastic process $x(t)$ defined by (4) is the unique L^p-solution to (3).

Proof. Let us see that $b_1 = \mathrm{e}^{-a\tau}b$ has centered absolute moments of any order. For $m \geq 0$ and by Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}[|b_1|^m] &= \mathbb{E}[\mathrm{e}^{-ma\tau}|b|^m] \leq \left(\mathbb{E}[\mathrm{e}^{-2ma\tau}]\right)^{\frac{1}{2}} \left(\mathbb{E}[b^{2m}]\right)^{\frac{1}{2}} \\ &= \left(\phi_a(-2m\tau)\right)^{\frac{1}{2}} \left(\mathbb{E}[b^{2m}]\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

By Proposition 3.1, $\mathrm{e}_\tau^{b_1,t}$ is L^q-differentiable, for each $1 \leq q < \infty$, and $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}_\tau^{b_1,t} = b_1\mathrm{e}_\tau^{b_1,t-\tau}$.

Consider the stochastic process e^{at}. Let us check the conditions of Proposition 2.1 (Chain Rule Theorem) with $f(x) = \mathrm{e}^x$, $X(t) = at$ and any $1 \leq q < \infty$. Notice that, from $\phi_a(\zeta) < \infty$ for all $\zeta \in \mathbb{R}$, we know that a has centered absolute moments of any order and that ϕ_a is an analytic real function with $\phi_a(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{n!}\mathbb{E}[a^n]$. These facts give conditions (i), (ii) and (iii) from Proposition 2.1. Indeed, (i) holds because a has centered absolute moments of any order, so in particular $a \in \mathrm{L}^{2q}$, which gives the L^{2q}-differentiability of at; (ii) is clear since at is path-continuous; and (iii) requires, for each $t \geq 0$, an $r > 2q$ and a $\delta > 0$ such that $\sup_{s\in[-\delta,\delta]} \phi_a(r(t+s)) < \infty$, but this is clear by continuity of ϕ_a on $\mathbb{R}$ and its boundedness on any compact set. The conclusion is that e^{at} is L^q-differentiable, for each $1 \leq q < \infty$, and $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{at} = a\mathrm{e}^{at}$.

We apply Lemma 2.2 with $Y_1(t) = \mathrm{e}^{a(t+\tau)}$, $Y_2(t) = \mathrm{e}_\tau^{b_1,t}$ and $Y_3(t) = g(-\tau)$. We saw that Y_1 and Y_2 are L^q-differentiable for all $1 \leq q < \infty$, and $Y_3(t) \in \mathrm{L}^{p+\eta}$ by hypothesis, therefore the product process $Y_1(t)Y_2(t)Y_3(t) = \mathrm{e}^{a(t+\tau)}\mathrm{e}_\tau^{b_1,t}g(-\tau)$ is L^p-differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathrm{e}^{a(t+\tau)}\mathrm{e}_\tau^{b_1,t}g(-\tau)\right\} = \left\{a\mathrm{e}^{a(t+\tau)}\mathrm{e}_\tau^{b_1,t} + \mathrm{e}^{a(t+\tau)}b_1\mathrm{e}_\tau^{b_1,t-\tau}\right\}g(-\tau). \quad (12)$$

Let $Y_1(t,s) = \mathrm{e}^{a(t-s)}$, $Y_2(t,s) = \mathrm{e}_\tau^{b_1,t-\tau-s}$ and $Y_3(t,s) = g'(s) - ag(s)$. Set $F(t,s) = Y_1(t,s)Y_2(t,s)Y_3(t,s)$. On the one hand, since e^{at} and $\mathrm{e}_\tau^{b_1,t}$ are L^q-continuous processes, for each $1 \leq q < \infty$, $Y_1(t,s)$ and $Y_2(t,s)$ are L^q-continuous at (t,s). On the other hand, from $g \in C^1([-\tau,0])$ in the $\mathrm{L}^{p+\eta}$-sense, a having absolute moments of any order, and Hölder's inequality, we derive that $Y_3(t,s)$ is $\mathrm{L}^{p+\mu}$-continuous, for $0 < \mu < \eta$. By Lemma 2.1, $F(t,s)$ is L^p-continuous at (t,s).

Fixed s, let $Y_1(t) = \mathrm{e}^{a(t-s)}$, $Y_2(t) = \mathrm{e}_\tau^{b_1,t-\tau-s}$ and $Y_3(t) = g'(s) - ag(s)$. We know that $Y_1(t)$ and $Y_2(t)$ are L^q-differentiable, for each $1 \leq q < \infty$. Also, from $g \in C^1([-\tau,0])$ in the $\mathrm{L}^{p+\eta}$-sense, a having absolute moments of any order, and Hölder's inequality, the random variable $Y_3(t)$ belongs to $\mathrm{L}^{p+\mu}$, for all $0 < \mu < \eta$. By Lemma 2.2, $F(\cdot,s)$ is L^p-differentiable at t, with

$$\frac{\partial F}{\partial t}(t,s) = \left\{a\mathrm{e}^{a(t-s)}\mathrm{e}_\tau^{b_1,t-\tau-s} + \mathrm{e}^{a(t-s)}b_1\mathrm{e}_\tau^{b_1,t-2\tau-s}\right\}\left(g'(s) - ag(s)\right). \quad (13)$$

Let us see that $\frac{\partial F}{\partial t}(t,s)$ is L^p-continuous at (t,s). Since a has centered absolute moments of any order and $\mathrm{e}^{a(t-s)}$ is L^q-continuous at (t,s), for

each $1 \leq q < \infty$, we derive that $a\mathrm{e}^{a(t-s)}$ is L^q-continuous at (t,s), for each $1 \leq q < \infty$, by Hölder's inequality. We have that $Y_1(t,s) = a\mathrm{e}^{a(t-s)}$ and $Y_2(t,s) = \mathrm{e}_\tau^{b_1,t-\tau-s}$ are L^q-continuous at (t,s), for each $1 \leq q < \infty$, while $Y_3(t,s) = g'(s) - ag(s)$ is $\mathrm{L}^{p+\mu}$-continuous, for $0 < \mu < \eta$. By Lemma 2.1, $a\mathrm{e}^{a(t-s)}\mathrm{e}_\tau^{b_1,t-\tau-s}(g'(s) - ag(s))$ is L^p-continuous at each (t,s). Analogously, $\mathrm{e}^{a(t-s)}b_1\mathrm{e}_\tau^{b_1,t-2\tau-s}(g'(s)-ag(s))$ is L^p-continuous at (t,s). Therefore, $\frac{\partial F}{\partial t}(t,s)$ is L^p-continuous at (t,s).

Set $G(t) = \int_{-\tau}^0 F(t,s)\,\mathrm{d}s$. By Lemma 2.2 and (13), G is L^p-differentiable and

$$\begin{aligned} G'(t) &= \int_{-\tau}^0 \frac{\partial F}{\partial t}(t,s)\,\mathrm{d}s \\ &= \int_{-\tau}^0 \left\{a\mathrm{e}^{a(t-s)}\mathrm{e}_\tau^{b_1,t-\tau-s} + \mathrm{e}^{a(t-s)}b_1\mathrm{e}_\tau^{b_1,t-2\tau-s}\right\}(g'(s) - ag(s))\,\mathrm{d}s. \end{aligned} \tag{14}$$

By combining (12) and (14) and taking into account expression (4) for $x(t)$, we derive that $x(t)$ is L^p-differentiable and $x'(t) = ax(t) + bx(t-\tau)$, after simple operations.

Finally, for the initial condition, we put $t \in [-\tau, 0]$ into (4). For the first addend, note that $\mathrm{e}_\tau^{b_1,t} = 1$ by definition of exponential delay function. For the second addend, we need to work more. Let $s \in [-\tau, 0]$. If $-\tau \leq t < s \leq 0$, then $t - \tau \leq t - \tau - s < -\tau$, so $\mathrm{e}_\tau^{b_1,t-\tau-s} = 0$. If $-\tau \leq s \leq t \leq 0$, then $-\tau \leq t - \tau - s \leq t$, therefore $\mathrm{e}_\tau^{b_1,t-\tau-s} = 1$. Thus, $x(t) = \mathrm{e}^{a(t+\tau)}g(-\tau) + \int_{-\tau}^t \mathrm{e}^{a(t-s)}(g'(s) - ag(s))\,\mathrm{d}s$. Fixed t, let $Y_1(s) = 1$, $Y_2(s) = \mathrm{e}^{a(t-s)}$ and $Y_3(s) = g(s)$. By Lemma 2.2, the product process $Y_1(s)Y_2(s)Y_3(s) = \mathrm{e}^{a(t-s)}g(s)$ is L^p-differentiable, with $\frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{e}^{a(t-s)}g(s)) = \mathrm{e}^{a(t-s)}(g'(s)-ag(s))$, which is L^p-continuous. By Barrow's rule for L^p-calculus, see [24, p. 104] (in the setting of mean square calculus),

$$\begin{aligned} x(t) &= \mathrm{e}^{a(t+\tau)}g(-\tau) + \int_{-\tau}^t \mathrm{e}^{a(t-s)}(g'(s) - ag(s))\,\mathrm{d}s \\ &= \mathrm{e}^{a(t+\tau)}g(-\tau) + \int_{-\tau}^t \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{a(t-s)}g(s))\,\mathrm{d}s = g(t). \end{aligned}$$

□

Example 3.3. *Consider the delay $\tau = 2$. Set $a \sim$ Normal$(2,1)$ and $b \sim$ Gamma$(2,2)$. Let $g(t) = \sin(dt^2)$, where $d \sim$ Beta$(10,9)$. We know that $\phi_a(\zeta) < \infty$ for all $\zeta \in \mathbb{R}$, and that b has centered absolute moments of any order. On the other hand, $g(t)$ is L^p-differentiable on $\mathbb{R}$, for each $1 \leq p < \infty$, as a consequence of Proposition 2.1. Indeed, in the notation of Proposition 2.1, take $f(x) = \sin x$ and $X(t) = dt^2$. Condition (i) holds because d is bounded, so it has absolute moments of any order. Condition (ii) is obvious. Finally, condition (iii) follows since $|f'(x)| = |\cos x| \leq 1$. Therefore $g(t)$ is L^p-differentiable, for each $1 \leq p < \infty$, and $g'(t) = 2dt\cos(dt^2)$. In fact, with*

the same reasoning by applying Proposition 2.1 *and the product rule differentiation, g is $C^\infty(\mathbb{R})$. The assumptions of Theorem* 3.2 *are satisfied, so $x(t)$ defined by (4) is the unique L^p-solution to (3), for each $1 \leq p < \infty$.*

To understand the main probabilistic features of $x(t)$ (uncertainty quantification), one may approximate the statistical moments of $x(t)$ (since $x(t) \in \mathrm{L}^p$ for $1 \leq p < \infty$). The main statistics of $x(t)$ are its expectation, $\mathbb{E}[x(t)]$, and its variance, $\mathbb{V}[x(t)]$. Table 1 *shows the approximation of these two statistics for different times $t \geq 0$, $\mathbb{E}_{MC}[x(t)]$ and $\mathbb{V}_{MC}[x(t)]$, by using Monte Carlo simulations with* $2,000,000$ *realizations. We have executed Monte Carlo simulations twice. Observe that the approximations agree, although they deteriorate for large t and the second-order moment.*

t	0.1	0.2	0.4	1	1.5
$\mathbb{E}_{MC}[x(t)]$	0.366665	0.852564	2.25884	14.3022	64.3984
$\mathbb{E}_{MC}[x(t)]$	0.366769	0.852833	2.25973	14.3123	64.4420
$\mathbb{V}_{MC}[x(t)]$	0.0784624	0.411983	3.06961	318.432	25090.4
$\mathbb{V}_{MC}[x(t)]$	0.0784137	0.412158	3.07703	321.213	25008.9

TABLE 1. Approximations of $\mathbb{E}[x(t)]$ and $\mathbb{V}[x(t)]$ with Monte Carlo simulations ($2,000,000$ realizations).

Theorem 3.4 (Existence and uniqueness). *Fix $1 \leq p < \infty$. Suppose that a and b are bounded random variables, and g belongs to $C^1([-\tau, 0])$ in the L^p-sense. Then the stochastic process $x(t)$ defined by (4) is the unique L^p-solution to (3).*

Proof. First, note that $b_1 = \mathrm{e}^{-a\tau} b$ is bounded, from the assumed boundedness of both a and b. By Proposition 3.1, $\mathrm{e}_\tau^{b_1,t}$ is L^∞-differentiable and $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}_\tau^{b_1,t} = b_1 \mathrm{e}_\tau^{b_1,t-\tau}$.

On the other hand, e^{at} is L^∞-differentiable and $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{at} = a\mathrm{e}^{at}$. Indeed, given $h \neq 0$, by the deterministic Mean Value Theorem and the boundedness of a,

$$\begin{aligned}
&\left\| \frac{\mathrm{e}^{a(t+h)} - \mathrm{e}^{at}}{h} - a\mathrm{e}^{at} \right\|_\infty = \left\| \mathrm{e}^{at} \left(\frac{\mathrm{e}^{ah} - 1}{h} - a \right) \right\|_\infty \\
&\leq \mathrm{e}^{\|a\|_\infty t} \left\| \frac{\mathrm{e}^{ah} - 1}{h} - a \right\|_\infty = \mathrm{e}^{\|a\|_\infty t} \left\| a(\mathrm{e}^{a\xi_h} - 1) \right\|_\infty \\
&\leq \|a\|_\infty \mathrm{e}^{\|a\|_\infty t} \|\mathrm{e}^{a\xi_h} - 1\|_\infty = \|a\|_\infty \mathrm{e}^{\|a\|_\infty t} \|a\mathrm{e}^{a\delta_h}\xi_h\|_\infty \\
&\leq \|a\|_\infty^2 \mathrm{e}^{\|a\|_\infty t} \mathrm{e}^{\|a\|_\infty |h|} |h| \xrightarrow{h \to 0} 0,
\end{aligned}$$

where ξ_h and δ_h are random variables, that arise from applying twice the deterministic Mean Value Theorem, which depend on h and $|\delta_h| < |\xi_h| < |h|$.

The rest of the proof is completely analogous to Theorem 3.2, by applying the second part of both Lemma 2.1 and Lemma 2.2. □

Remark 3.5. *The condition of boundedness for a and b in Theorem 3.4 is necessary if we only assume that $g \in C^1([-\tau, 0])$ in the L^p-sense. See [37, Example p. 541], where it is proved that, in order for a random autonomous and homogeneous linear differential equation of first-order to have an L^p-solution for every initial condition in L^p, one needs the random coefficient to be bounded.*

4. L^p-convergence to a random autonomous linear differential equation when the delay tends to 0

Given a discrete delay $\tau > 0$, we denote the L^p-solution to (3) as $x_\tau(t)$, which is given by (4) (we make the dependence on τ explicitly). If we put $\tau = 0$ into (3), we obtain a random differential equation problem:

$$\begin{cases} x_0'(t,\omega) = (a(\omega) + b(\omega))x_0(t,\omega), \ t \geq 0, \\ x_0(0,\omega) = g(0). \end{cases} \tag{15}$$

Under certain conditions that imitate those from Theorem 3.2 and Theorem 3.4, there exists a unique solution to (15), see the forthcoming Proposition 4.1 and subsequent Corollary 4.1. Our objective in this section will be to demonstrate that $\lim_{\tau \to 0} x_\tau(t) = x_0(t)$ in L^p.

Proposition 4.1. *Consider the random differential equation problem*

$$\begin{cases} x_0'(t,\omega) = a(\omega)x_0(t,\omega), \ t \geq 0, \\ x_0(0,\omega) = y_0(\omega), \end{cases} \tag{16}$$

where $a(\omega)$ and $y_0(\omega)$ are random variables. Fix $1 \leq p < \infty$.

If $\phi_a(\zeta) < \infty$ for all $\zeta \in \mathbb{R}$, and $y_0 \in \mathrm{L}^{p+\eta}$ for certain $\eta > 0$, then the stochastic process $x_0(t) = y_0 \mathrm{e}^{at}$ is the unique L^p-solution to (16).

On the other hand, if a is a bounded random variable and $y_0 \in \mathrm{L}^p$, then the stochastic process $x_0(t) = y_0 \mathrm{e}^{at}$ is the unique L^p-solution to (16).

Proof. From $\phi_a(\zeta) < \infty$ for all $\zeta \in \mathbb{R}$, we know that a has centered absolute moments of any order and that ϕ_a is an analytic real function. From these facts, the conditions of Proposition 2.1 with $f(x) = \mathrm{e}^x$ and $X(t) = at$ are easy to check. Hence, e^{at} is an L^q-differentiable stochastic process, for each $1 \leq q < \infty$, with $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{at} = a\mathrm{e}^{at}$. On the other hand, since $y_0 \in \mathrm{L}^{p+\eta}$, we can turn to Lemma 2.2 with $Y_1(t) = 1$, $Y_2(t) = \mathrm{e}^{at}$ and $Y_3(t) = y_0$ to derive that $x_0(t) = y_0\mathrm{e}^{at}$ is L^p-differentiable and $x_0'(t) = ax_0(t)$.

Finally, if a is bounded, then e^{at} is L^∞-differentiable, check the proof of Theorem 3.4. As $y_0 \in \mathrm{L}^p$, we can turn to the second part of Lemma 2.2 with $Y_1(t) = 1$, $Y_2(t) = \mathrm{e}^{at}$ and $Y_3(t) = y_0$ to conclude that $x_0(t) = y_0\mathrm{e}^{at}$ is L^p-differentiable and $x_0'(t) = ax_0(t)$. □

Corollary 4.1. *Fix $1 \leq p < \infty$. If $\phi_a(\zeta) < \infty$ and $\phi_b(\zeta) < \infty$ for all $\zeta \in \mathbb{R}$, and $g(0) \in \mathrm{L}^{p+\eta}$ for certain $\eta > 0$, then the stochastic process $x_0(t) = g(0)\mathrm{e}^{(a+b)t}$ is the unique L^p-solution to (15).*

On the other hand, if a and b are bounded random variables and $g(0) \in \mathrm{L}^p$, then the stochastic process $x_0(t) = y_0 \mathrm{e}^{(a+b)t}$ is the unique L^p-solution to (15).

From now on, we try to demonstrate that $\lim_{\tau\to 0} x_\tau(t) = x_0(t)$ in L^p. First, we prove that the delayed exponential function tends to the classical exponential function in a random setting (Lemma 4.3 and Lemma 4.4), from a well-known deterministic inequality (Lemma 4.2).

Lemma 4.2. *[35, Th. A.3] Let $c \in \mathbb{R}$, $T > 0$, $\tau_0 > 0$ and $\alpha = 1 + |c|\mathrm{e}^{\tau_0|c|}$. Then, for all $\tau \in (0, \tau_0]$, $|\mathrm{e}_\tau^{c,t-\tau} - \mathrm{e}^{ct}| \le \tau \mathrm{e}^{\alpha T|c|}$, for all $t \in [0, T]$.*

Lemma 4.3. *Let c be a bounded random variable, $T > 0$ and $\tau_0 > 0$. Set $k \ge \|c\|_\infty$. Then, $|\mathrm{e}_\tau^{c(\omega),t-\tau} - e^{c(\omega)t}| \le C_{T,\tau_0,k} \cdot \tau$, for almost every ω, for all $t \in [0, T]$ and $\tau \in (0, \tau_0]$, for some real constant $C_{T,\tau_0,k} > 0$ that only depends on T, τ_0 and k.*

Proof. By Lemma 4.2, for all $\tau \in (0, \tau_0]$, $|\mathrm{e}_\tau^{c(\omega),t-\tau} - \mathrm{e}^{c(\omega)t}| \le \tau \mathrm{e}^{\alpha(\omega)T|c(\omega)|}$, for all ω and $t \in [0, T]$. Note that $\alpha(\omega) = 1 + |c(\omega)|\mathrm{e}^{\tau_0|c(\omega)|} \le 1 + k\mathrm{e}^{\tau_0 k}$. Then $|\mathrm{e}_\tau^{c(\omega),t-\tau} - \mathrm{e}^{c(\omega)t}| \le \tau \mathrm{e}^{(1+k\mathrm{e}^{\tau_0 k})Tk}$. Now simply set $C_{T,\tau_0,k} = \mathrm{e}^{(1+k\mathrm{e}^{\tau_0 k})Tk}$, and we are done. □

Lemma 4.4. *Let c be a bounded random variable, $T > 0$ and $\tau_0 > 0$. Set $k \ge \|c\|_\infty$. Then, $|\mathrm{e}_\tau^{c(\omega),t} - e^{c(\omega)t}| \le C_{T,\tau_0,k} \cdot \tau$, for almost every ω, for all $t \in [-\tau, T]$ and $\tau \in (0, \tau_0]$, for some real constant $C_{T,\tau_0,k} > 0$ that only depends on T, τ_0 and k.*

Proof. By Lemma 4.3, $|\mathrm{e}_\tau^{c(\omega),t} - e^{c(\omega)(t+\tau)}| \le C'_{T,\tau_0,k} \cdot \tau$, for almost every ω, for all $t \in [-\tau, T]$ and $\tau \in (0, \tau_0]$. By the triangular inequality, $|\mathrm{e}_\tau^{c(\omega),t} - \mathrm{e}^{c(\omega)t}| \le |\mathrm{e}_\tau^{c(\omega),t} - e^{c(\omega)(t+\tau)}| + |e^{c(\omega)(t+\tau)} - \mathrm{e}^{c(\omega)t}|$. The former addend is bounded by $C'_{T,\tau_0,k} \cdot \tau$, as previously justified. The latter addend is bounded via the deterministic Mean Value Theorem: $|e^{c(\omega)(t+\tau)} - \mathrm{e}^{c(\omega)t}| = \mathrm{e}^{\xi_{\omega,t,\tau}}|c(\omega)|\tau \le \mathrm{e}^{k(T+\tau_0)}k\tau$, where $\xi_{\omega,t,\tau}$ is between ct and $c(t+\tau)$. Let $C_{T,\tau_0,k} = C'_{T,\tau_0,k} + \mathrm{e}^{k(T+\tau_0)}k$, and we are done. □

Theorem 4.5. *Fix $1 \le p < \infty$. Let a and b be bounded random variables and let g be a stochastic process that belongs to $C^1([-\tau, 0])$ in the L^p-sense. Then, $\lim_{\tau\to 0} x_\tau(t) = x_0(t)$ in L^p, uniformly on $[0, T]$, for each $T > 0$.*

Proof. Fix $[0, T]$. Let $\tau_0 = 1$, and $k \ge \|b_1\|_\infty = \|\mathrm{e}^{-a\tau}b\|_\infty$ for all $\tau \in (0, 1]$.

Recall that e^{at} is L^∞-continuous (check the proof of Theorem 3.4), so $\lim_{\tau\to 0} \mathrm{e}^{a(t+\tau)} = \mathrm{e}^{at}$ in L^∞, uniformly on $[0, T]$. We also have $\lim_{\tau\to 0} g(-\tau) = g(0)$ in L^p, by hypothesis. By Lemma 4.4, $|\mathrm{e}_\tau^{b_1(\omega),t} - e^{b_1(\omega)t}| \le C_{T,k} \cdot \tau$, so $\|\mathrm{e}_\tau^{b_1,t} - e^{b_1 t}\|_\infty \le C_{T,k} \cdot \tau$. Since $\lim_{\tau\to 0} \mathrm{e}^{b_1 t} = \mathrm{e}^{bt}$ in L^∞ uniformly on $[0, T]$, we derive that $\lim_{\tau\to 0} \mathrm{e}_\tau^{b_1,t} = \mathrm{e}^{bt}$ in L^∞ uniformly on $[0, T]$. We conclude that

$$\lim_{\tau\to 0} \mathrm{e}^{a(t+\tau)}\mathrm{e}_\tau^{b_1,t} g(-\tau) = g(0)\mathrm{e}^{(a+b)t} = x_0(t) \tag{17}$$

in L^p, with uniform convergence on $[0, T]$.

Now, for $\tau \in (0,1]$ and $t \in [0,T]$,

$$\begin{aligned}
&\left\| \int_{-\tau}^{0} \mathrm{e}^{a(t-s)} \mathrm{e}_{\tau}^{b_1,t-\tau-s} (g'(s) - ag(s))\, \mathrm{d}s \right\|_p \\
&\leq \int_{-\tau}^{0} \left\| \mathrm{e}^{a(t-s)} \mathrm{e}_{\tau}^{b_1,t-\tau-s} (g'(s) - ag(s)) \right\|_p \mathrm{d}s \\
&\leq \int_{-\tau}^{0} \left\| \mathrm{e}^{a(t-s)} \right\|_\infty \left\| \mathrm{e}_{\tau}^{b_1,t-\tau-s} \right\|_\infty \left\| g'(s) - ag(s) \right\|_p \mathrm{d}s. \qquad (18)
\end{aligned}$$

We bound the three terms inside the integral from (18). First, $\|\mathrm{e}^{a(t-s)}\|_p \leq \mathrm{e}^{\|a\|_\infty (T+1)}$. Secondly, $\|g'(s) - ag(s)\|_p \leq \|g'(s)\|_p + \|a\|_\infty \|g(s)\|_p \leq C_{\|a\|_\infty, g}$, where $C_{\|a\|_\infty, g} > 0$ is a constant. Finally, we bound $\|\mathrm{e}_{\tau}^{b_1,t-\tau-s}\|_\infty$. We have $t \geq 0$ and $s \in [-\tau, 0]$. Then $t - s \in [-\tau, \infty)$. If $t - s \in [-\tau, 0]$, then $t - s - \tau \in [-2\tau, -\tau]$, so $\mathrm{e}_{\tau}^{b_1,t-\tau-s} = 0$ by definition of delayed exponential function. Otherwise, if $t - s \geq 0$, then Lemma 4.3 applies: $\|e_{\tau}^{b_1,t-\tau-s} - \mathrm{e}^{b_1(t-s)}\|_\infty \leq C_{T,k} \cdot \tau$, for $t \in [0,T]$, $s \in [-\tau, 0]$ and $\tau \in (0,1]$. Since $\|\mathrm{e}^{b_1(t-s)}\|_\infty \leq \mathrm{e}^{k(T+1)}$, we conclude that $\|e_{\tau}^{b_1,t-\tau-s}\|_\infty \leq C_{T,k} \cdot \tau + \mathrm{e}^{k(T+1)}$, by the triangular inequality. Thus, all terms inside the integral from (18) are bounded for $t \in [0,T]$ and $s \in [-\tau, 0]$, therefore

$$\lim_{\tau \to 0} \int_{-\tau}^{0} \mathrm{e}^{a(t-s)} \mathrm{e}_{\tau}^{b_1,t-\tau-s} (g'(s) - ag(s))\, \mathrm{d}s = 0 \qquad (19)$$

in L^p, uniformly on $[0,T]$.

By combining both (17) and (19), we conclude that $\lim_{\tau \to 0} x_\tau(t) = x_0(t)$ in L^p, uniformly on $[0,T]$. □

5. Conclusions

In this paper, we have addressed the analysis of the random autonomous linear differential equation with discrete delay. The coefficients have been assumed to be random variables, while the initial condition has been taken as a stochastic process. Although the sample path approach is the easiest extension of the deterministic results to a random framework, an L^p-random calculus approach is usually the most appropriate method. Uncertainty quantification for stochastic systems requires the computation or approximation of the statistical moments of the solution stochastic process (for instance, via Monte Carlo simulations). Only if we know that the solution process belongs to L^p, we guarantee that the computation or approximation of its statistical moments makes sense. This paper establishes general conditions under which the random autonomous linear differential equation with discrete delay has a unique L^p-solution. An analysis of L^p-convergence when the delay tends to 0 has also been performed in detail. Our methodology could be extended to other random differential equations with some sort of delay. This will be done in future contributions.

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Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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J. Calatayud
Instituto Universitario de Matemática Multidisciplinar,
Building 8G, access C, *2*nd floor,
Universitat Politècnica de València,
Camino de Vera s/n,
46022, Valencia,
Spain
e-mail: `jucagre@doctor.upv.es`

J.-C. Cortés
Instituto Universitario de Matemática Multidisciplinar,
Building 8G, access C, *2*nd floor,
Universitat Politècnica de València,
Camino de Vera s/n,
46022, Valencia,
Spain
e-mail: `jccortes@imm.upv.es`

M. Jornet
Instituto Universitario de Matemática Multidisciplinar,
Building 8G, access C, 2nd floor,
Universitat Politècnica de València,
Camino de Vera s/n,
46022, Valencia,
Spain
e-mail: marjorsa@doctor.upc.es