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Existence of Picard operator and iterated function system

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ABSTRACT

In this paper, we define weak θ_m- contraction mappings and give a new class of Picard operators for such class of mappings on a complete metric space. Also, we obtain some new results on the existence and uniqueness of attractor for a weak θ_m- iterated multifunction system. Moreover, we introduce $(\alpha, \beta, \theta_m)-$ contractions using cyclic $(\alpha, \beta)-$ admissible mappings and obtain some results for such class of mappings without the continuity of the operator. We also provide an illustrative example to support the concepts and results proved herein.

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KEYWORDS: *Picard operator; fixed point; weak θ_m-contraction; iterated function system.*

1. INTRODUCTION

The iterated function system (IFS) is the main generator of fractals. It is introduced by Hutchinson [7] and generalized by Barnsley [2]. An IFS is a finite family of contractions $\{f_i\}_{i=1}^N$ on a complete metric space (M, d). For an IFS there is always a non-empty set $A \subset M$ such that $A = \bigcup_{i=1}^N f_i(A)$, such A is known as attractor of the respective IFS.

In this paper, we study the concept of weak θ -contraction used by Imdad and Alfaqih [8] which is an extension of θ -contraction (or JS contraction) introduced by Jleli and Samet [9]. We consider the family $\Theta_{1,2,4}$ and introduce

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weak θ_m-contraction and prove that every (continuous) weak θ_m -contraction is a Picard operator in section 3. In section 4, we study about iterated multifunction system (IMS) and obtain some results on the existence and uniqueness of attractor for a weak θ_m- IMS. Also, we obtain some results on $(\alpha, \beta, \theta_m)-$ contractions using cyclic $(\alpha, \beta)-$ admissible mappings without the continuity of the operator in the last section.

2. Preliminaries

In this section, we recall some notations, basic definitions and results to be used in the sequel.

Definition 2.1 (see [12, 13]). Let (M, d) be a metric space and $f : M \to M$ be a self mapping. A sequence $\{u_n\}$ defined by $u_n = f^n u_0$ is called a *Picard sequence* based at the point $u_0 \in M$. A self-mapping f is said to be a *Picard operator* if it has a unique fixed point $z \in M$ and $z = \lim\limits_{n\to\infty} f^n u$ for all $u \in M$.

Definition 2.2 (see [12, 13]). Let (M, d) be a metric space, and let $K(M)$ be the class of all non-empty compact sets of M. The function $\eta : K(M) \times K(M) \to [0, \infty)$ define by $\eta(A, B) = \max\{D(A, B), D(B, A)\}$ where $D(A, B) = \sup_{a\in A} \inf_{b\in B} d(a, b)$ for all $A, B \in K(M)$ is a metric known as *Hausdorff-Pompeiu metric*. It is well known that if (M, d) is complete then $(K(M), \eta)$ is also complete.

Alizadeh et al. [1] introduced the notion of cyclic (α, β)-admissible mapping which is defined as follows:

Definition 2.3. Let M be a nonempty set, f be a self-mapping on M and $\alpha, \beta : M \to [0, \infty)$ be two mappings. We say that f is a cyclic (α, β)-admissible mapping if $x \in M$ with $\alpha(x) \geq 1$ implies $\beta(fx) \geq 1$ and $\beta(x) \geq 1$ implies $\alpha(fx) \geq 1$.

The following results will be needed in the proof of our main results.

Lemma 2.4 ([10]). *Let (M, d) be a metric space and let $\{x_n\}$ be a sequence in M such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.1}$$

If $\{x_n\}$ is not a Cauchy sequence in M, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to ε^+ when $k \to +\infty$:

$$d\left(x_{m(k)}, x_{n(k)}\right),\ d\left(x_{m(k)}, x_{n(k)+1}\right),\ d\left(x_{m(k)-1}, x_{n(k)}\right),$$
$$d\left(x_{m(k)-1}, x_{n(k)+1}\right),\ d\left(x_{m(k)+1}, x_{n(k)+1}\right). \tag{2.2}$$

Remark 2.5. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). If for all $n \in \mathbb{N}$ holds $d(x_{n+1}, x_n) < d(x_n, x_{n-1})$, then $n \neq m$ implies $x_n \neq x_m$ whenever $n, m \in \mathbb{N}$.

Lemma 2.6 ([14]). *Let $A, B, C \in K(M)$. Then we have the following:*
(i) $A \subset B$ if and only if $D(A, B) = 0$;
(ii) $D(A, B) \leq D(A, C) + D(C, B)$.

Lemma 2.7 ([15]). *If $\{E_i\}_{i\in\tau}$ and $\{F_i\}_{i\in\tau}$ are finite collection of elements in $K(M)$, then*

$$\eta(\bigcup_{i\in\tau} E_i, \bigcup_{i\in\tau} F_i) \leq \sup_{i\in\tau} \eta(E_i, F_i).$$

3. Weak θ_m-contraction

Now we use the definition of an auxiliary function and utilize the same to introduce weak θ_m-contraction.

Definition 3.1 (see [6, 8, 9]). Let $\theta : (0, \infty) \to (1, \infty)$ be a function and consider the following conditions:
$\Theta 1$: θ is non-decreasing.
$\Theta 2$: for each sequence $\{\alpha_n\}$ in $(0, \infty)$,

$$\lim_{n\to\infty} \theta(\alpha_n) = 1 \Leftrightarrow \lim_{n\to\infty} (\alpha_n) = 0.$$

$\Theta 3$: there exist $r \in (0, 1)$ and $l \in (0, \infty)$ such that $\lim_{\alpha\to 0^+} \frac{\theta(\alpha)-1}{\alpha^r} = l$;
$\Theta 4$: θ is continuous.

The following notations to be used in the sequel.

- $\Theta_{1,2,3}$ the family of all θ that satisfy $\Theta 1 - \Theta 3$.
- $\Theta_{1,2,4}$ the family of all θ that satisfy $\Theta 1, \Theta 2$ and $\Theta 4$.
- $\Theta_{2,3}$ the family of all θ that satisfy $\Theta 2$ and $\Theta 3$.
- $\Theta_{2,4}$ the family of all θ that satisfy $\Theta 2$ and $\Theta 4$.
- Θ_2 the family of all θ that satisfy $\Theta 2$.

Example 3.2 ([6]). Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\alpha) = e^{\sqrt{\alpha}}$, for all $\alpha \in (0, \infty)$.Then $\theta \in \Theta_{1,2,3,4}$.

Example 3.3 ([6]). Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\alpha) = e^{\alpha}$, for all $\alpha \in (0, \infty)$.Then $\theta \in \Theta_{1,2,3}$.

Example 3.4 ([8]). The following function $\theta : (0, \infty) \to (1, \infty)$ are in $\Theta_{2,4}$:
(1) $\theta(\alpha) = e^{\frac{\alpha}{2} + sin\alpha}$;
(2) $\theta(\alpha) = \alpha^r + 1, r \in (0, \infty)$.

Example 3.5. Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\alpha) = 2^{\sqrt{\alpha} 2^{-\frac{1}{\sqrt{\alpha}}}}$, for all $\alpha \in (0, \infty)$.Then $\theta \in \Theta_{1,2,4}$.

Now, we define weak θ_m-contraction mapping.

Definition 3.6. Let (M, d) be a metric space and $f : M \to M$ is a self-mapping. A mapping f is called a weak θ_m-contraction if there exist a $\theta \in \Theta_{2,4}$ (or $\theta \in \Theta_{1,2,4}$) and $h \in (0, 1)$, such that for all $u, v \in M$, we have

$$d(fu, fv) > 0 \Rightarrow \quad \theta(d(fu, fv)) \leq [\theta(M(u, v))]^h, \tag{3.1}$$

where $M(u,v) = \max\{d(u,fu), d(v,fv), d(u,v)\}$.

Remark 3.7. Here, we note that weak θ_m-contraction mapping has at most one fixed point. Assume that f has another fixed point say $v \in M$, $d(u,v) > 0$. Using (3.1) we have

$$\begin{aligned}\theta(d(u,v)) &= \theta(d(fu,fv))\\ &\leq [\theta(\max\{d(u,fu), d(v,fv), d(u,v)\})]^h\\ &= [\theta(d(u,v))]^h,\end{aligned}$$

which is a contradiction.

Lemma 3.8. *Let (M,d) be a metric space and $f : M \to M$ is a weak θ_m-contraction. Suppose that there exists a Picard sequence $\{u_n\} \subseteq M$ defined by $u_{n+1} = f^n u_0 = fu_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then $d(u_n, u_{n+1}) \to 0$ as $n \to \infty$, where $u_n \neq u_{n+1}$ (Here $\theta \in \Theta_{2,4}$ or $\Theta_{1,2,4}$).*

Proof. Let $u_0 \in M$ be an arbitrary point. Define the Picard sequence as $\{u_n\} \subseteq M$ by $u_{n+1} = f^n u_0 = fu_n$ for all $n \in \mathbb{N} \cup \{0\}$. Assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Applying (3.1) we have, for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}\theta(d(u_n,u_{n+1})) &= \theta(d(fu_{n-1},fu_n))\\ &\leq [\theta(\max\{d(u_{n-1},fu_{n-1}), d(u_n,fu_n), d(u_{n-1},u_n)\})]^h\\ &= [\theta(\max\{d(u_n,u_{n+1}), d(u_n,u_{n-1})\})]^h\end{aligned}$$

Case 1: When $d(u_n,u_{n+1}) > d(u_n,u_{n-1})$, then we have $\theta(d(fu_{n-1},fu_n)) = \theta(d(u_n,u_{n+1})) \leq [\theta(d(u_n,u_{n+1})]^h$, but $\alpha \geq \alpha^h, \forall \alpha \in \mathbb{R}^+, h \in (0,1)$. Thus we get contradiction.
Case 2: When $d(u_n,u_{n-1}) > d(u_n,u_{n+1})$, we have $\theta(d(fu_{n-1},fu_n)) \leq [\theta(d(u_n,u_{n-1})]^h$. Hence on the same lines, we have

$[\theta(d(fu_{n-1},fu_{n-2}))]^h \leq [\theta(\max\{d(u_{n-1},fu_{n-1}), d(u_{n-2},fu_{n-2}), d(u_{n-1},u_{n-2})\})]^{h^2} = [\theta(\max\{d(u_{n-1},u_n), d(u_{n-1},u_{n-2})\})]^{h^2} \leq [\theta(d(u_{n-1},u_{n-2}))]^{h^2}$.

Proceeding on these lines, we get

$$\begin{aligned}\theta(d(fu_n,fu_{n-1})) &\leq [\theta(d(fu_{n-1},fu_{n-2}))]^h\\ &\leq [\theta(d(fu_{n-2},fu_{n-3}))]^{h^2}\\ &\leq \ldots \leq [\theta(d(fu_0,u_0))]^{h^n}.\end{aligned}$$

Thus, we have $\theta(d(u_n,u_{n+1})) \leq [\theta(d(u_1,u_0))]^{h^n}$. Now, taking $n \to \infty$ we have, $\lim_{n\to\infty} \theta(d(u_n,u_{n+1})) = 1$. Using Θ_2, we have $\lim_{n\to\infty} d(u_n,u_{n+1}) = 0$.

□

Lemma 3.9. *Let (M,d) be a metric space and $f : M \to M$ is a weak θ_m-contraction. Suppose that there exists a Picard sequence $\{u_n\} \subseteq M$ defined by $u_{n+1} = f^n u_0 = fu_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then Picard sequence $\{u_n\}$ is a Cauchy sequence (Here $\theta \in \Theta_{2,4}$ or $\Theta_{1,2,4}$).*

Proof. Let $u_0 \in M$ be an arbitrary point. Define the Picard sequence as $\{u_n\} \subseteq M$ by $u_{n+1} = f^n u_0 = f u_n$ for all $n \in \mathbb{N} \cup \{0\}$. Assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using Lemma 3.8, we have $\lim_{n\to\infty} d(u_n, u_{n+1}) = 0$. Now we have to prove that $\{u_n\}$ is a Cauchy sequence. We'll prove this by contradiction. Assume that $\{u_n\}$ is not a Cauchy sequence.

Now, since the sequence $\{u_n\}$ is not a Cauchy sequence, then by Lemma 2.4, we have $d\left(u_{m(k)}, u_{n(k)}\right)$ and $d\left(u_{m(k)+1}, u_{n(k)+1}\right)$ tend to $\varepsilon > 0$, as $k \to \infty$. Using (3.1), we have

$$\begin{aligned} \theta(d(u_{m(k)}, u_{n(k)})) &= \theta(d(fu_{m(k)-1}, fu_{n(k)-1})) \\ &\leq [\theta(\max\{d(u_{m(k)-1}, fu_{m(k)-1}), d(u_{n(k)-1}, fu_{n(k)-1}), \\ &\quad d(u_{m(k)-1}, u_{n(k)-1})\})]^h. \end{aligned}$$

Case 1: If $\max\{d(u_{m(k)-1}, fu_{m(k)-1}), d(u_{n(k)-1}, fu_{n(k)-1}), d(u_{m(k)-1}, u_{n(k)-1})\} = d(u_{m(k)-1}, fu_{m(k)-1})$, then we have $\theta(d(u_{m(k)}, u_{n(k)})) \leq [\theta(d(u_{m(k)-1}, fu_{m(k)-1})]^h$. Letting $k \to \infty$, from Lemma 2.4 and $\Theta 4$, we have

$$\theta(\epsilon) \leq [\theta(0)]^h,$$

which is a contradiction.

Case 2: If $\max\{d(u_{m(k)-1}, fu_{m(k)-1}), d(u_{n(k)-1}, fu_{n(k)-1}), d(u_{m(k)-1}, u_{n(k)-1})\} = d(u_{n(k)-1}, fu_{n(k)-1})$, then proceeding the same way as in Case 1 we again get a contradiction.

Case 3: If $\max\{d(u_{m(k)-1}, fu_{m(k)-1}), d(u_{n(k)-1}, fu_{n(k)-1}), d(u_{m(k)-1}, u_{n(k)-1})\} = d(u_{m(k)-1}, u_{n(k)-1})$, then we have

$$\theta(d(u_{m(k)}, u_{n(k)})) \leq [\theta(d(u_{m(k)-1}, u_{n(k)-1})]^h.$$

Letting $k \to \infty$ and using Lemma 2.4 and $\Theta 4$, we obtain $\theta(\epsilon) \leq [\theta(\epsilon)]^h$, which is again a contradiction. Hence Picard sequence $\{u_n\}$ is a Cauchy sequence. ☐

Theorem 3.10. *Every weak θ_m-contraction on a complete metric space is a Picard operator.* [*Here, we consider $\theta \in \Theta_{1,2,4}$.*]

Proof. Let $u_0 \in M$ be an arbitrary point. Define the Picard sequence as $\{u_n\} \subseteq M$ by $u_{n+1} = f^n u_0 = fu_n$ for all $n \in \mathbb{N} \cup \{0\}$. If there exist $n_0 \in \mathbb{N} \cup \{0\}$ such that $u_{n_0} = fu_{n_0}$, then we are done. Assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using Lemma 3.9, we have $\{u_n\}$ is a Cauchy sequence. Now as (M, d) is a complete metric space so there exist $u \in M$ such that $\{u_n\}$ converges to u. From (Θ_1) and (3.1), it is easy to conclude that

$$\begin{aligned} \theta(d(fu, fv)) &\leq [\theta(\max\{d(u, fu), d(v, fv), d(u, v)\})]^h \\ &\leq \theta(\max\{d(u, fu), d(v, fv), d(u, v)\}) \end{aligned}$$

for all $u, v \in M$ with $d(fu, fv) > 0$. Using (Θ_1) and above inequality, we have $d(fu, fv) \leq \max\{d(u, fu), d(v, fv), d(u, v)\}$. Suppose that $u \neq fu$. Therefore, we have

$$\begin{aligned} d(u_{n+1}, fu) = d(fu_n, fu) &\leq \max\{d(u_n, fu_n), d(u, fu), d(u_n, u)\} \\ &= \max\{d(u_n, u_{n+1}), d(u, fu), d(u_n, u)\}. \end{aligned}$$

Taking $n \to \infty$, using Lemma 3.8 we have $d(u, fu) \leq d(u, fu)$, which is a contradiction. Hence $fu = u$, thus we get a fixed point.

Further, now we prove the uniqueness of the fixed point. Assume that f has another fixed point say $v \in M$, $v \neq u$. Using (3.1) we have

$$\begin{aligned} \theta(d(fu, fv)) &\leq [\theta(\max\{d(u, fu), d(v, fv), d(u, v)\})]^h \\ &= [\theta(d(u, v))]^h, \end{aligned}$$

which is a contradiction. Hence the result. □

Theorem 3.11. *Every continuous weak θ_m-contraction on a complete metric space is a Picard operator. [Here, we consider $\theta \in \Theta_{2,4}$.]*

Proof. Let $u_0 \in M$ be an arbitrary point. Define the Picard sequence as $\{u_n\} \subseteq M$ by $u_{n+1} = f^n u_0 = f u_n$ for all $n \in \mathbb{N} \cup \{0\}$. If there exist $n_0 \in \mathbb{N} \cup \{0\}$ such that $u_{n_0} = f u_{n_0}$, then we are done. Assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Proceeding as in Theorem 3.10, we have Picard sequence $\{u_n\}$ is a Cauchy sequence. Now as (M, d) is a complete metric space so there exist $u \in M$ such that $\{u_n\}$ converges to u. The continuity of f and uniqueness of limit implies $fu = u$, thus we get a fixed point. Hence every continuous weak θ_m-contraction on a complete metric space is a Picard operator. □

Example 3.12. Let $M = \{1, 2, 3\}$. Define the metric $d : M \times M \to [0, \infty)$ by $d(x, y) = |x - y|$, for all $x, y \in M$. Define a function $f : M \to M$ as $f(1) = 2, f(2) = 2, f(3) = 1$.

Define a function $\theta : (0, \infty) \to (1, \infty)$ by $\theta(t) = e^{\sqrt{t}}$. So $\theta \in \Theta_{1,2,3,4}$.

Case 1. Consider $(u, v) = (1, 3)$. We have $\theta(d(f1, f3)) = \theta(d(2, 1)) = \theta(1) = e$. Also, $[\theta(\max\{d(1, f1), d(3, f3), d(1, 3)\})]^h = [\theta(d(1, 2), d(1, 3))]^h = [\theta(2)]^h = [e^{\sqrt{2}}]^h$. Therefore $\theta(d(f1, f3)) \leq [\theta(\max\{d(1, f1), d(3, f3), d(1, 3)\})]^h$, for all $h \in [\frac{1}{\sqrt{2}}, 1)$.

Case 2. Consider $(u, v) = (2, 3)$. We have $\theta(d(f2, f3)) = \theta(d(2, 1)) = \theta(1) = e$. Also, $[\theta(\max\{d(2, f2), d(3, f3), d(2, 3)\})]^h = [\theta(d(3, 1), d(2, 3))]^h = [\theta(2)]^h = [e^{\sqrt{2}}]^h$. Therefore $\theta(d(f1, f3)) \leq [\theta(\max\{d(2, f2), d(3, f3), d(2, 3)\})]^h$, for all $h \in [\frac{1}{\sqrt{2}}, 1)$.

Thus all the conditions of Theorem 3.10 are satisfied and 2 is a unique fixed point of f.

Here is to note that when $(u, v) = (2, 3)$ in the above example, then
(a) f is not Banach contraction;
(b) f is not weak θ-contraction of Imdad et al. [8];
(c) f is weak θ_m-contraction.
(d) f is a Picard operator.

Theorem 3.13. *Let (M, d) be a complete metric space and let $f : M \to M$ be a self mapping. If there exist $n \in \mathbb{N}$ such that f^n is a weak θ_m -contraction, then f is a Picard operator.*

Proof. From Theorem 3.10, it is obvious that f^n is a Picard operator, thus there exists a unique $z \in M$ such that $f^n z = z$ and $\lim\limits_{m \to \infty} t_{m+1} = (f^n)^m u = z$,

for all $u \in M$. Also, we observe that $f^{n+1}z = f^n z$, that is $f^n(fz) = fz$, thus fz is also a fixed point of f^n. Thus $fz = z$.
Further, if z^* is another fixed point of f, then it must be a fixed point of f^n. Hence $z = z^*$. Therefore f has a unique fixed point.

Now, let m be a positive integer greater than n. Then there exist $l \geq 1$ and $s \in \{0, 1, 2, ..., n-1\}$ such that $m = nl + s$. Here, we notice that for all $u \in M$, we have

$$\lim_{m\to\infty} u_{m+1} = \lim_{m\to\infty} f^m u = \lim_{l\to\infty} f^{nl}(f^s u) = \lim_{l\to\infty} (f^n)^l (f^s u) = z.$$

Hence the result. □

Haghi et al. [5], in 2011, proved a lemma by using the axiom of choice as follows:

Lemma 3.14. *Let M be a nonempty set and $f : M \to M$ a function. Then there exist a set $E \subseteq M$ such that $f(E) = f(M)$ and $f : E \to M$ is one-to-one.*

By using above lemma, we prove common fixed point theorems for two self mappings on M as follows:

Theorem 3.15. *Let (M, d) be a complete metric space and f, g be two self maps on M satisfying*

$$d(fu, fv) > 0 \Rightarrow \quad \theta(d(fu, fv)) \leq [\theta(\max\{d(gu, fu), d(gv, fv), d(gu, gv)\})]^h. \tag{3.2}$$

for all $u, v \in M$ and $\theta \in \Theta_{2,4}$ (or $\theta \in \Theta_{1,2,4}$). If $f(M) \subseteq g(M)$ and $g(M)$ is a complete subset of M then f and g have a unique common fixed point in M.

Proof. By using Lemma 3.14, there exist $E \subseteq M$ such that $g(E) = g(M)$ and $g : E \to M$ is one-to-one. Define $h : g(E) \to g(E)$ by $h(gu) = fu$. Clearly, h is well defined as g is one-to-one on E. Also,

$$\theta(d(h(gu), h(gv))) \leq [\theta(\max\{d(gu, fu), d(gv, fv), d(gu, gv)\})]^h,$$

for all $gx, gy \in g(E)$. Since $g(E) = g(M)$ is complete, then by using Theorem 3.10, we can easily prove that f and g have a unique common fixed point in M. □

4. Weak θ_m iterated multifunction system

As application of results proved in the last section, we obtain some results on the existence and uniqueness of attractor of iterated multifunction system composed by weak θ_m-contraction in the setting of complete metric space in this section.

In the following section, we consider (M, d) is a complete metric space, $N \in \mathbb{N}$ and $\theta \in \Theta_{1,2,4}$.

Definition 4.1. Let $\{f_i\}_{i=1}^N$ be a finite family of self mappings on M. If $f_i : M \to M$ is a weak θ_m-contraction (for each i), then the family $\{f_i\}_{i=1}^N$ is called a *weak θ_m- iterated function system* (weak θ_m-IFS).

The set function $G : K(M) \to K(M)$ define by $G(B) = \bigcup_{i=1}^{N} f_i(Y)$ (for all $Y \in K(M)$) is said to be *associated Hutchinson operator*. A set $A \in K(M)$ is called an *attractor* of the weak θ_m- IFS if $G(A) = A$.

Let (M, d) be a metric space and $F_1, F_2, ...F_n : M \to K(M)$ be multivalued operator. Then the system $F = (F_1, F_2, ..., F_n)$ is called an iterated multifunction system (abbreviated as IMS).

Definition 4.2. Let $\{F_i\}_{i=1}^{N}$ be a finite family of iterated multifunction system. If $F_i : M \to K(M)$ is a weak θ_m-contraction (for each i), then the family $\{F_i\}_{i=1}^{N}$ is called a *weak θ_m- iterated multifunction system* (weak θ_m-IMS).

Define $P(M) = \{Y \subset M : Y \text{ is nonempty}\}$. If $T : M \to P(M)$ is a multivalued operator then $T(Y) := \bigcup_{x \in Y} T(x), Y \in P(M)$. Let $F_1, F_2, ...F_m : M \to K(M)$ be a finite family of multivalued operators, we define multifractal operator T_F generated by the iterated multifunction system $F = (F_1, F_2, ...F_m)$ by $G_F : K(M) \to K(M), G_F(Y) = \bigcup_{i=1}^{m} F_i(Y)$. In this framework, a nonempty compact subset A^* of M is said to be a multivalued fractal with respect to the iterated multifunctions system $F = (F_1, F_2, ...F_m)$ if and only if it is a fixed point for the associated multifractal operator.

In particular, if the operators $F_i = f_i$ are singlevalued, then a fixed point for the fractal operator $G_f : K(M) \to K(M)$, $G_f(Y) = \cup_{i=1}^{m} f_i(Y)$ generated by generated by iterated function system $f = (f_1, f_2, ...f_m)$ is said to be a self similar set or a fractal. Throughout, $Fix(f)$ denotes the set of fixed points of f (see [2, 4, 7]).

Definition 4.3. If $\{F_i\}_{i=1}^{N}$ is weak θ_m- IMS such that $F_i : M \to K(M)$ is continuous for $i = 1, 2, \ldots, N$ then the operator

$$G_F : K(M) \to K(M),\ G_F(Y) = \bigcup_{i=1}^{N} F_i(Y)$$

is well defined and is called weak θ_m- multi-fractal operator. A fixed point of G_F is called a multivalued fractal.

Now we will use the following lemma to show that a weak θ_m- multi-fractal operator has a unique multivalued fractal.

Lemma 4.4. *Let $f : M \to K(M)$ is a continuous weak θ_m- multivalued operator. Then the mapping $A \mapsto f(A)$ is also a weak θ_m-multivalued operator from $K(M)$ into itself.*

Proof. Let $A, B \in K(M)$ be such that $\eta(f(A), f(B)) > 0$. Assume that

$$\begin{aligned} \eta(f(A), f(B)) &= D(f(A), f(B)) \\ &= \sup_{u \in A} \inf_{v \in B} D(fu, fv), \quad \text{for all } \ A, B \in K(M). \end{aligned} \tag{4.1}$$

As f is a continuous weak θ_m-multivalued operator so there exist $h \in (0, 1)$ such that

$$\theta(D(fu, fv)) \le [\theta(\max\{D(u, fu), D(v, fv), d(u, v)\})]^h,$$

for all $u, v \in M$.

Now using (4.1), compactness of A, and continuity of f, we can find $a \in A$ such that $D(f(A), f(B)) = \inf_{v\in B} D(fa, fv) > 0$, so that $D(fa, fv) > 0$, for all $v \in B$. Hence, for all $v \in B$, we have

$$\begin{aligned} \theta(\inf_{v\in B} D(fa, fv)) &\leq \theta(D(fa, fv) \\ &\leq [\theta(max\{D(a, fa), D(v, fv), d(a, v)\})]^h. \end{aligned}$$

Therefore, for all $v \in B$ we get

$$\theta(\eta(f(A), F(B)) \leq [\theta(max\{D(a, fa), D(v, fv), d(a, v)\})]^h. \tag{4.2}$$

Case 1: If $\max\{D(a, fa), D(v, fv), d(a, v)\} = D(a, fa)$, then we have:

$$\theta(\inf_{v\in B} D(fa, fv)) \leq [\theta(D(a, fa))]^h,$$

Now from (4.2) we have

$$\begin{aligned} \theta(\eta(f(A), f(B))) &\leq [\theta(D(a, fa'))]^h \\ &\leq [\theta(\sup_{a\in A} \inf_{fa\in f(A)} d(a, fa))]^h \\ &= [\theta(D(A, A))]^h, \end{aligned}$$

which is a contradiction.

Case 2: If $\max\{D(a, fa), D(v, fv), d(a, v)\} = D(v, fv)$, then proceeding in the same way as in Case 1 we again get a contradiction.

Case 3: If $\max\{D(a, fa), D(v, fv), d(a, v)\} = d(a, v)$, then for all $v \in B$ we have $\theta(\eta(f(A), f(B))) \leq [\theta(d(a, v))]^h$.

Now let $v \in B$ be such that $d(a, v) = \inf_{v\in B} d(a, v)$. From (4.2) we have,

$$\begin{aligned} \theta(\eta(f(A), f(B))) &\leq [\theta(d(a, v))]^h, \\ &= [\theta(\inf_{b\in B} d(a, v))]^h, \\ &\leq [\theta(\sup_{a\in A} \inf_{v\in B} d(a, v))]^h \\ &= [\theta(D(A, B))]^h \\ &\leq [\eta(A, B)]^h. \end{aligned}$$

Hence we get the result. □

Theorem 4.5. *Let (M, d) be a complete metric space and $F_i : M \to K(M)$, $i = \{1, 2, ..., m\}$ be continuous multivalued operator satisfying*

$$\theta(\eta(F_i u, F_i v)) \leq [\theta(\max\{d(u, F_i u), d(v, F_i v), d(u, v)\})]^h,$$

for all $u, v \in M$ and $h \in (0, 1)$. Then there exists a unique multivalued fractal with respect to the iterated multifunction system $F = (F_1, F_2, ...F_m)$, that is, $Fix(G_F) = \{A^*\}$ and $\{G_F^n(A)\}_{n\in\mathbb{N}}$ converges to A^*, for each $A \in K(M)$.

Proof. First we prove that the operator $G_F : K(M) \to K(M)$, $G_F(Y) = \cup_{i=1}^m F_i(Y)$ satisfies the conditions of Theorem 3.10. Let $B, C \in K(M)$ such that

$$0 < \eta(G_F(B), G_F(C)) = \eta(\bigcup_{i=1}^{m} F_i(B), \bigcup_{i=1}^{m} F_i(C)).$$

Now Lemma 2.7 implies that

$$\begin{aligned} \eta(G_F(B), G_F(C)) = \eta(\bigcup_{i=1}^{m} F_i(B), \bigcup_{i=1}^{m} F_i(C)) &\leq \sup_{1\leq i\leq N} \eta(F_i(B), F_i(C)) \\ &= \eta(F_{i_0}(B), F_{i_0}(C)), \end{aligned}$$

for some $i_0 \in \{1, 2, 3, \dots, N\}$. Using $\Theta 1$ and Lemma 4.4, we have

$$\theta(\eta(G_F(B), G_F(C)) \leq \theta(\eta(F_{i_0}(B), F_{i_0}(C))) \leq [\theta(\eta(B, C))]^{h_{i_0}}.$$

Therefore G_F is also a continuous weak θ_m contraction on the complete metric space $(K(M), \eta)$. Theorem 3.11 ensures the existence and uniqueness of $A^* \in K(M)$ such that $G_F(A^*) = A^*$ and $A^* = \lim_{n\to\infty} G_F^n(B)$ for all $B \in K(M)$. This completes the proof. □

In particular, when the operators are single valued, we have the following result.

Theorem 4.6. *If $\{f_i\}_{i=1}^{N}$ is a continuous weak θ_m -IFS, then it has unique attractor. Moreover, $A = \lim_{n\to\infty} G^n(B)$ for all $B \in K(M)$, the limit being taken with respect to the Hutchinson-Pompeiu metric.*

Proof. For each $i \in \{1, 2, \dots N\}$, let h_i be constant such that $h_i \in (0, 1)$ and is associated with f_i. Let $B, C \in K(M)$ such that

$$0 < \eta(G(B), G(C)) = \eta(\bigcup_{i=1}^{N} f_i(B), \bigcup_{i=1}^{N} f_i(C)).$$

Now Lemma 2.7 implies that

$$\begin{aligned} \eta(G(B), G(C)) = \eta(\bigcup_{i=1}^{N} f_i(B), \bigcup_{i=1}^{N} f_i(C)) &\leq \sup_{1\leq i\leq N} \eta(f_i(B), f_i(C)) \\ &= \eta(f_{i_0}(B), f_{i_0}(C)), \end{aligned}$$

for some $i_0 \in \{1, 2, 3, \dots, N\}$. Using $\Theta 1$ and Lemma 4.4, we have

$$\theta(\eta(G(B), G(C)) \leq \theta(\eta(f_{i_0}(B), f_{i_0}(C))) \leq [\theta(\eta(B, C))]^{h_{i_0}}.$$

Therefore G is also a continuous weak θ_m contraction on the complete metric space $(K(M), \eta)$. Theorem 3.11 ensures the existence and uniqueness of $A \in K(M)$ such that $G(A) = A$ and $A = \lim_{n\to\infty} G^n(B)$ for all $B \in K(M)$. This completes the proof. □

Example 4.7. Let M=$[0, 1] \subset \mathbb{R}$, with the metric given by the usual metric. We define, $F : K(M) \to K(M)$ by

$$F(A) = f_1(A) \cup f_2(A),$$

where

$$f_1(x) = \tfrac{1}{3}x,\ f_2(x) = \tfrac{1}{3}x + \tfrac{2}{3},\ 0\leq \text{x} \leq 1.$$

First we verify that f_1 and f_2 are weak θ_m contraction.
Take $\theta = e^x$ and $d(x,y) = |x-y|$, thus $d(x, f_1x) = \left|x - \frac{x}{3}\right| = \left|\frac{2x}{3}\right|$ for all $x \in M$. Therefore, $\max\{d(x, f_1x), d(y, f_1y), d(x,y)\} = \max\{\frac{2x}{3}, \frac{2y}{3}, \frac{|x-y|}{3}\}$.
Case 1: If $x > y$, $\max\{d(x, f_1x), d(y, f_1y), d(x,y)\} = \frac{2x}{3}$. We know that

$$\frac{x-y}{3} \leq \frac{2xy}{3} \text{ for all } x, y \in M. \tag{4.3}$$

Therefore,

$$e^{\frac{x-y}{3}} \leq e^{\frac{2xy}{3}} = [e^{\frac{2x}{3}}]^y = [e^{\frac{2x}{3}}]^h, \text{ where } h = y \in (0,1).$$

Hence we have $\theta(d(f_1x.f_1y)) = e^{\frac{x-y}{3}} \leq [\theta d(x, f_1x)]^h$, $h = y \in (0,1)$.

Now choose $f_2(x) = \frac{1}{3}x + \frac{2}{3}$, $d(x, f_2x) = \left|x - \left(\frac{1}{3}x + \frac{2}{3}\right)\right| = \left|\frac{2x}{3} - \frac{2}{3}\right|$ for all $x \in M$. As we know that

$$\frac{x-y}{3} - \frac{2}{3} \leq \frac{2xy}{3} - \frac{2}{3} \text{ for all } x, y \in M. \tag{4.4}$$

Therefore,

$$e^{\frac{x-y}{3} - \frac{2}{3}} \leq e^{\frac{2xy}{3} - \frac{2}{3}} \leq e^{\frac{2xy}{3}} = [e^{\frac{2x}{3}}]^y = [e^{\frac{2x}{3}}]^h, \text{ we have } h = y \in (0,1).$$

Thus we have $\theta(d(f_2x.f_2y)) = e^{\frac{x-y}{3} - \frac{2}{3}} \leq [\theta d(x, f_2x)]^h, h = y \in (0,1)$.
Case 2: Now take $y > x$, we have $\max\{d(x, f_1x), d(y, f_1y), d(x,y)\} = d(y, f_1y)$. In this case we also obtain same conclusion as in Case 1.

Therefore, $d(x,y) \neq \max\{d(x, f_1x), d(y, f_1y), d(x,y)\}$, for any value of $x, y \in [0,1]$. Hence from both cases we can say that f_1 is a weak θ_m-contraction for $\theta = e^x$.

In the similar way, we can prove that f_2 is also a weak θ_m-contraction for $\theta = e^x$. Thus $F = (f_1, f_2)$ is iterated multifunction system. The unique fixed point of F must satisfy

$$A = F(A) = f_1(A) \cup f_2(A).$$

Considering the nature of the two transformations, we get a unique fractal $A \subset K(M)$ which is Cantor subset of $[0,1]$.

5. Cyclic (α, β)-admissible mappings

Definition 5.1. Let (M, d) be a complete metric space, $f : M \to M$ be a mapping and $\alpha, \beta : \mathbb{R} \to [0, \infty)$ be two functions. Then S is said to be a generalized $(\alpha, \beta, \theta_m)-$ contraction mapping if f satisfies the following conditions:

(1) f is cyclic (α, β)-admissible;

(2) there exits a $\theta \in \Theta_{2,4}$ and $h \in (0,1)$ such that for all $u, v \in M$, we have

$$\alpha(u)\beta(v) \geq 1, d(fu, fv) > 0 \Rightarrow \theta(d(fu, fv)) \leq [\theta(M(u,v))]^h, \tag{5.1}$$

where $M(u,v) = \max\{d(u, fu), d(v, fv), d(u,v)\}$.

Theorem 5.2. *Let (M, d) be a complete metric space, $f : M \to M$ be a mapping and $\alpha, \beta : M \to [0,1)$ be two functions. Suppose that the following conditions hold.*

(1) *f is a generalized $(\alpha, \beta, \theta_m)-$ contraction mapping;*

(2) *There exists an element* $x_0 \in M$ *such that* $\alpha(x_0) \geq 1$ *and* $\beta(x_0) \geq 1$*;*
(3) f *is continuous;*
or
If sequence $\{x_n\}$ *in* M *converges to* $x \in M$ *with the property* $\alpha(x_n) \geq 1$ *(or* $\beta(x_n) \geq 1$*) for all* $n \in \mathbb{N}$*, then* $\alpha(x) \geq 1$ *(or* $\beta(x) \geq 1$*).*

Then f *is a Picard operator.*

Proof. Assume that there exist $x_0 \in M$ such that $\alpha(x_0) \geq 1$. Define a Picard sequence $\{x_n\}$ by $x_{n+1} = fx_n = f^n x_0$, for all $n \in \mathbb{N} \cup \{0\}$. If there exist $n_0 \in \mathbb{N} \cup \{0\}$ such that $u_{n_0} = fu_{n_0}$, then we are done. Assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Assume that there exist $x_0, x_1 \in M$ such that $\alpha(x_0) \geq 1 \implies \beta(fx_0) = \beta(x_1) \geq 1$ and $\beta(x_0) \geq 1 \implies \alpha(fx_0) = \alpha(x_1) \geq 1$. By continuing above process, we have $\alpha(x_n) \geq 1 \implies \beta(fx_n) = \beta(x_{n+1}) \geq 1$ and $\beta(x_n) \geq 1 \implies \alpha(fx_n) = \alpha(x_{n+1}) \geq 1$.

Since $\alpha(x_m) \geq 1 \implies \beta(fx_m) = \beta(x_{m+1}) \geq 1$ and $\beta(x_m) \geq 1 \implies \alpha(fx_m) = \alpha(x_{m+1}) \geq 1$, for all $m, n \in \mathbb{N}$ with $n < m$. Moreover, since $\alpha(x_m) \geq 1 \implies \beta(x_{m+2}) \geq 1$ and $\beta(x_m) \geq 1 \implies \alpha(x_{m+2}) \geq 1$, for all $m, n \in \mathbb{N}$ with $n < m$.

By continuing this process, we have $\alpha(x_n) \geq 1 \implies \beta(x_m) \geq 1$ and $\beta(x_n) \geq 1 \implies \alpha(x_m) \geq 1$, for all $m, n \in \mathbb{N}$. Thus $\alpha(x_n)\beta(x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Therefore, using (5.1) we have

$$
\begin{aligned}
&\theta(d(fx_n, fx_{n+1})) \\
&\leq [\theta(\max\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), d(x_n, x_{n+1})\})]^h \\
&= [\theta(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})\})]^h \\
&= [\theta(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\})]^h \qquad (5.2)
\end{aligned}
$$

Analysis similar to that in the proof of Theorem 3.11 shows that $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$.

Now, we prove that $\{x_n\}$ is a Cauchy sequence. On the contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. By Lemma 2.4, there exist $\varepsilon > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the sequences $\{d(x_{m(k)}, x_{n(k)})\}$ and $\{d(x_{m(k)+1}, x_{n(k)+1})\}$ tend to $\varepsilon^+ > 0$ as $k \to \infty$. Substituting $x = x_{m(k)}$ and $y = x_{n(k)}$ into the inequality (5.1), we obtain

$$
\begin{aligned}
\alpha\left(x_{m(k)}\right)\beta\left(x_{n(k)}\right) \geq 1 \Rightarrow \quad &\theta\left(d\left(fx_{m(k)}, fx_{n(k)}\right)\right) \\
&\leq \left[\theta\left(M\left(x_{m(k)}, x_{n(k)}\right)\right)\right]^h, \qquad (5.3)
\end{aligned}
$$

where

$$
M\left(x_{m(k)}, x_{n(k)}\right) = \max\left\{d\left(x_{m(k)}, x_{n(k)}\right), d\left(x_{m(k)}, x_{m(k)+1}\right), d\left(x_{n(k)}, x_{n(k)+1}\right)\right\}.
$$

Since $d\left(x_{m(k)}, x_{m(k)+1}\right) \to 0$ and $d\left(x_{n(k)}, x_{n(k)+1}\right) \to 0$ as $k \to \infty$. Then using the fact that $\alpha\left(x_{m(k)}\right)\beta\left(x_{n(k)}\right) \geq 1$ holds and that $d\left(x_{m(k)+1}, x_{n(k)+1}\right)$ and $d\left(x_{m(k)}, x_{n(k)}\right)$ are both positive numbers, by using the property Θ4, Lemma

2.4 and similar arguments as in Theorem 3.11, we obtain

$$\alpha\left(x_{m(k)}\right)\beta\left(x_{n(k)}\right) \geq 1 \Rightarrow \theta\left(d\left(fx_{m(k)}, fx_{n(k)}\right)\right) \leq \left[\theta\left(d\left(x_{m(k)}, x_{n(k)}\right)\right)\right]^{h}.$$

For sufficiently large k, $k \to \infty$, we get $\theta(\varepsilon) \leq [\theta(\varepsilon)]^h$, which is a contradiction.

Hence, $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence.

Now as (M, d) is a complete metric space so there exist $x \in M$ such that $\{x_n\}$ converges to x.

The continuity of f and uniqueness of limit implies $fx = x$, thus we get a fixed point.

Now, suppose that the sequence $\{x_n\}$ in M converges to $x \in M$ with the property $\alpha(x_n) \geq 1$ (or $\beta(x_n) \geq 1$) for all $n \in \mathbb{N}$, then $\alpha(x) \geq 1$ (or $\beta(x) \geq 1$). Hence $\alpha(x)\beta(x) \geq 1$

Further, we claim that $fx = x$. Suppose not, that is, $fx \neq x$. So $d(fx, x) > 0$ and $\lim_{n\to\infty} d(x_{n+1}, fx) \neq 0$. Using (5.1) we have

$$\begin{aligned} \theta(d(x_{n+1}, fx)) &= \theta(d(fx_n, fx)) \\ &\leq [\theta(\max\{d(x_n, fx_n), d(x, fx), d(x_n, x)\})]^h \\ &= [\theta(\max\{d(x_n, x_{n+1}), d(x, fx), d(x_n, x)\})]^h. \end{aligned} \tag{5.4}$$

Taking $n \to \infty$ and using property $\Theta4$, we have $\theta(d(x, fx)) \leq [d(x, fx)]^h$, which is a contradiction. We, thus, obtain that f has a fixed point $fx = x$. It is easy to prove the uniqueness of fixed point. □

Remark 5.3.

- Note that, throughout this paper, Lemma 2.4 and the contractive conditions imply that the iterative sequence, i.e. Picard sequence is a Cauchy.
- For different variants of inequality (3.1), we have many interesting results. For example, when, we replace $M(u, v)$ in (2.1) and (5.1) with $M(u, v) = \max\{d(u, f(u)), d(v, f(v)))\}$ (type of Bianchini [3]), we may extend Theorem 3.11, Theorem 3.13, Theorem 3.15, Theorem 4.6 and Theorem 5.2 to these different variants of inequality. Also when, we replace $M(u, v)$ in (2.1) with $M(u, v) = d(u, v)$, we have the corresponding results of Imdad et al. [8].

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