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Additional Information

ON p -DUNFORD INTEGRABLE FUNCTIONS WITH VALUES IN BANACH SPACES

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ABSTRACT. Let (Ω, Σ, μ) be a complete probability space, X a Banach space and $1 \leq p < \infty$. In this paper we discuss several aspects of p -Dunford integrable functions $f : \Omega \rightarrow X$. Special attention is paid to the compactness of the Dunford operator of f . We also study the p -Bochner integrability of the composition $u \circ f : \Omega \rightarrow Y$, where u is a p -summing operator from X to another Banach space Y . Finally, we also provide some tests of p -Dunford integrability by using w^* -thick subsets of X^* .

1. INTRODUCTION

Throughout this paper (Ω, Σ, μ) is a complete probability space, X is a Banach space and $1 \leq p < \infty$. Dunford and Pettis integrable functions $f : \Omega \rightarrow X$ have been widely studied over the years and their properties are nowadays well understood, see e.g. [22, 23, 34]. However, it seems that this is not the case for p -Dunford and p -Pettis integrable functions when $p > 1$. This paper aims to contribute to fill in this gap. The following definition goes back to Pettis [26].

Definition 1.1. A function $f : \Omega \rightarrow X$ is called:

- (i) p -Dunford integrable if $\langle f, x^* \rangle \in L^p(\mu)$ for every $x^* \in X^*$;
- (ii) p -Pettis integrable if it is p -Dunford integrable and Pettis integrable.

As usual, for any $f : \Omega \rightarrow X$ and $x^* \in X^*$, the scalar function $\langle f, x^* \rangle : \Omega \rightarrow \mathbb{R}$ is defined by $\langle f, x^* \rangle(\omega) := \langle f(\omega), x^* \rangle$ for all $\omega \in \Omega$.

Some scattered results about p -Dunford and p -Pettis integrable functions can be found in [5, 10, 16, 17, 18, 26]. Most of them are restricted to the case of *strongly measurable* functions or, equivalently, *separable* Banach spaces. For instance, if $p > 1$, then every strongly measurable p -Dunford integrable function $f : \Omega \rightarrow X$ is p -Pettis integrable, see [26, Corollary 5.31] (cf. [22, Corollary 5.2]). In this paper we deal with p -Dunford and p -Pettis integrable functions which are not necessarily strongly measurable.

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Let us summarize the content of this work.

In Section 2 we introduce the basic terminology and include some preliminaries on p -Dunford and p -Pettis integrable functions.

In Section 3 we discuss the compactness of the *Dunford operator*

$$T_f^p : L^{p'}(\mu) \rightarrow X^{**},$$

associated to a p -Dunford integrable function $f : \Omega \rightarrow X$. As usual, $1 < p' \leq \infty$ denotes the conjugate exponent of p , i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. The operator T_f^p is defined as the adjoint of the operator

$$(1.1) \quad S_f^p : X^* \rightarrow L^p(\mu), \quad S_f^p(x^*) := \langle f, x^* \rangle.$$

Compactness of the Dunford operator is important for applications and some authors add it to the definition of p -Pettis integrable function, see [16]. If f is p -Pettis integrable, then T_f^p takes values in X . In this case, the compactness of T_f^p is equivalent to the approximation of f by simple functions in the p -Pettis norm (Theorem 3.3). An example of Pettis [26] (cf. Example 3.1) already showed that T_f^p might not be compact for $p > 1$ even for a strongly measurable p -Pettis integrable function f . In the case $p = 1$, the counterexamples for Pettis integrable functions involve necessarily non strongly measurable functions and are far from elementary, the first one being constructed by Fremlin and Talagrand [15]. We show that for a p -Dunford integrable function f , the operator T_f^p is compact if and only if (i) $T_f^1 : L^\infty(\mu) \rightarrow X^{**}$ is compact, and (ii) the family of real-valued functions

$$Z_f^p := \{ |\langle f, x^* \rangle|^p : x^* \in B_{X^*} \},$$

is a uniformly integrable subset of $L^1(\mu)$ (Theorem 3.4). Condition (i) follows automatically from (ii) whenever f is strongly measurable, but also in many other cases, e.g. if μ is perfect or if $X \not\cong \ell^1(\aleph_1)$, where \aleph_1 denotes the first uncountable cardinal (see Corollary 3.7).

In Section 4 we study the integrability of the composition $u \circ f : \Omega \rightarrow Y$, where u is a p -summing operator from X to another Banach space Y and $f : \Omega \rightarrow X$ is a p -Dunford integrable function. For $p = 1$, Diestel [6] proved that $u \circ f$ is Bochner integrable whenever f is strongly measurable and Pettis integrable. As remarked in [7, p. 56], his argument can be easily modified for arbitrary $1 \leq p < \infty$ to obtain that $u \circ f$ is p -Bochner integrable whenever f is strongly measurable and p -Dunford integrable. Several papers discussed such type of questions for $p = 1$ beyond the strongly measurable case, see [3, 4, 19, 31, 32]. An unpublished result of Lewis [19], rediscovered independently in [31, Theorem 2.3], states that for $p = 1$ the composition $u \circ f$ is at least *scalarly equivalent* to a Bochner integrable function if f is Dunford integrable. Here we generalize this result for the range $1 \leq p < \infty$ (Theorem 4.12) and provide many examples of Banach spaces X for which $u \circ f$ is actually p -Bochner integrable if f is p -Dunford integrable (Corollary 4.10). To this end we need some auxiliary results on the p -variation and p -semivariation of a vector measure which might be of independent interest.

Finally, in Section 5 we give some criteria to check the p -Dunford integrability of a function $f : \Omega \rightarrow X$ by looking at the family of real-valued functions

$$Z_{f,\Gamma} := \{\langle f, x^* \rangle : x^* \in \Gamma\},$$

for some set $\Gamma \subseteq X^*$. Fonf proved that f is Dunford integrable if X is separable, $X \not\cong c_0$ and $\langle f, x^* \rangle \in L^1(\mu)$ for every extreme point x^* of B_{X^*} (see [14, Theorem 4]). This result is based on the striking fact that if $X \not\cong c_0$, then the set of all extreme points of B_{X^*} (or, more generally, any James boundary of B_{X^*}) is w^* -thick, see [14, Theorem 1] (cf. [24, Theorem 2.3]). A set $\Gamma \subseteq X^*$ is said to be w^* -thick if whenever $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ for some increasing sequence (Γ_n) of sets, there is $n \in \mathbb{N}$ such that $\overline{\text{aco}}^{w^*}(\Gamma_n)$ contains a ball centered at 0. This concept is useful to check several properties without testing on the whole dual, see [24] for more information. In [1] it was pointed out that if X is separable, then a function $f : \Omega \rightarrow X$ is Dunford integrable whenever $Z_{f,\Gamma} \subseteq L^1(\mu)$ for some w^* -thick set $\Gamma \subseteq X^*$. As an application of our main theorem of this section (Theorem 5.1), we extend the result of [1] to the range $1 \leq p < \infty$ and a wide class of Banach spaces, namely, those having Efremov's property (\mathcal{E}) (Corollary 5.2). This also complements similar results in [30] dealing with scalarly bounded functions.

2. PRELIMINARIES

We follow standard Banach space terminology as it can be found in [8] and [13]. All our Banach spaces are real. An *operator* between Banach spaces is a continuous linear map. Given a Banach space Z , its norm is denoted by $\|\cdot\|_Z$ or simply $\|\cdot\|$ if no confusion arises. We write $B_Z = \{z \in Z : \|z\| \leq 1\}$ (the closed unit ball of Z) and $S_Z = \{z \in Z : \|z\| = 1\}$ (the unit sphere of Z). The topological dual of Z is denoted by Z^* . The weak topology on Z and the weak* topology on Z^* are denoted by w and w^* , respectively. The evaluation of $z^* \in Z^*$ at $z \in Z$ is denoted by either $\langle z, z^* \rangle$ or $\langle z^*, z \rangle$. A *subspace* of Z is a closed linear subspace. Given another Banach space Y , we write $Z \not\cong Y$ if Z contains no subspace isomorphic to Y . The absolutely convex hull of a set $S \subseteq Z$ is denoted by $\text{aco}(S)$.

The characteristic function of $A \in \Sigma$ is denoted by χ_A . A set $H \subseteq L^1(\mu)$ is called *uniformly integrable* if it is bounded and for every $\varepsilon > 0$ there is $\delta > 0$ such that $\sup_{h \in H} \int_A |h| d\mu \leq \varepsilon$ for every $A \in \Sigma$ with $\mu(A) \leq \delta$. This is equivalent to saying that H is relatively weakly compact in $L^1(\mu)$, see e.g. [8, p. 76, Theorem 15]. By using Hölder's inequality it is easy to check, for $p > 1$, that any bounded subset of $L^p(\mu)$ is uniformly integrable as a subset of $L^1(\mu)$.

A function $f : \Omega \rightarrow X$ is called:

- *simple* if it can be written as $f = \sum_{i=1}^n x_i \chi_{A_i}$, where $n \in \mathbb{N}$, $x_i \in X$ and $A_i \in \Sigma$ for every $i = 1, \dots, n$;
- *scalarly bounded* if there is a constant $M > 0$ such that, for each $x^* \in X^*$, we have $|\langle f, x^* \rangle| \leq M \|x^*\|$ μ -a.e. (the exceptional set depending on x^*);
- *scalarly measurable* if $\langle f, x^* \rangle$ is measurable for every $x^* \in X^*$;
- *strongly measurable* if there is a sequence of simple functions $f_n : \Omega \rightarrow X$ such that $f_n(\omega) \rightarrow f(\omega)$ in norm for μ -a.e. $\omega \in \Omega$.

The celebrated Pettis' measurability theorem (see e.g. [8, p. 42, Theorem 2]) states that f is strongly measurable if and only if it is scalarly measurable and there is $A \in \Sigma$ with $\mu(\Omega \setminus A) = 0$ such that $f(A)$ is separable.

Two functions $f, g : \Omega \rightarrow X$ are said to be *scalarly equivalent* if for each $x^* \in X^*$ we have $\langle f, x^* \rangle = \langle g, x^* \rangle$ μ -a.e. (the exceptional set depending on x^*).

Given any Dunford (i.e. 1-Dunford) integrable function $f : \Omega \rightarrow X$, there is a finitely additive measure $\nu_f : \Sigma \rightarrow X^{**}$ satisfying

$$\langle \nu_f(A), x^* \rangle = \int_A \langle f, x^* \rangle d\mu \quad \text{for all } A \in \Sigma \text{ and } x^* \in X^*.$$

As usual, we also write $\int_A f d\mu := \nu_f(A)$. Recall that f is said to be *Pettis integrable* if $\int_A f d\mu \in X$ for all $A \in \Sigma$.

Remark 2.1. Let $f : \Omega \rightarrow X$ be a p -Dunford integrable function. Then:

- (i) f is Dunford integrable.
- (ii) A standard closed graph argument shows that S_f^p (defined in (1.1)) is an operator (see e.g. [8, p. 52, Lemma 1] for a proof of the case $p = 1$), hence

$$\|f\|_{\mathcal{D}_p(\mu, X)} := \sup_{x^* \in B_{X^*}} \left(\int_{\Omega} |\langle f, x^* \rangle|^p d\mu \right)^{1/p} < \infty.$$

In particular, the family of real-valued functions

$$Z_f := \{ \langle f, x^* \rangle : x^* \in B_{X^*} \},$$

is uniformly integrable in $L^1(\mu)$ whenever $p > 1$.

- (iii) For each $g \in L^{p'}(\mu)$ the product $gf : \Omega \rightarrow X$ is Dunford integrable and

$$T_f^p(g) = \int_{\Omega} gf d\mu.$$

In particular, $T_f^p(\chi_A) = \int_A f d\mu$ for all $A \in \Sigma$.

- (iv) If f is p -Pettis integrable, then T_f^p takes values in X and gf is Pettis integrable for every $g \in L^{p'}(\mu)$. Moreover, in this case Z_f is uniformly integrable in $L^1(\mu)$ even for $p = 1$ (see e.g. [22, Corollary 4.1]).
- (v) By Schauder's theorem, the compactness of T_f^p is equivalent to that of S_f^p .

In general, scalarly measurable bounded functions might not be Pettis integrable. This is an interesting phenomenon occurring in the Pettis integral theory of non strongly measurable functions. The space X is said to have the *Pettis Integral Property* with respect to μ (shortly μ -PIP) if every scalarly bounded and scalarly measurable function $f : \Omega \rightarrow X$ is Pettis integrable. The μ -PIP is equivalent to the following (apparently stronger) condition: a function $f : \Omega \rightarrow X$ is Pettis integrable if (and only if) it is Dunford integrable and Z_f is uniformly integrable in $L^1(\mu)$. The space X is said to have the *Pettis Integral Property* (PIP) if it has the μ -PIP for any complete probability space (Ω, Σ, μ) . The class of Banach spaces having the PIP is rather wide and includes, for instance, all spaces having Corson's property (C), all spaces having Mazur's property and all spaces which are weakly measure-compact. In particular, every weakly compactly generated space has the PIP. We refer the

reader to [22, Chapter 7] and [23, Section 8] for more information on the PIP. The following connection is immediate:

Corollary 2.2. *The following statements are equivalent:*

- (i) X has the μ -PIP;
- (ii) for some/any $1 < p < \infty$, every p -Dunford integrable function $f : \Omega \rightarrow X$ is p -Pettis integrable.

Proof. (ii) \Rightarrow (i): Note that any scalarly bounded and scalarly measurable function $f : \Omega \rightarrow X$ is p -Dunford integrable, for any $1 \leq p < \infty$.

(i) \Rightarrow (ii): Let $f : \Omega \rightarrow X$ be a p -Dunford integrable function for some $1 < p < \infty$. According to Remark 2.1(ii), Z_f is uniformly integrable in $L^1(\mu)$ and so the μ -PIP of X ensures that f is Pettis integrable. \square

For any strongly measurable function $f : \Omega \rightarrow X$ there is a separable subspace $Y \subseteq X$ such that $f(\omega) \in Y$ for μ -a.e. $\omega \in \Omega$. Since separable Banach spaces have the PIP, from Corollary 2.2 we get the classical result mentioned in the introduction:

Corollary 2.3 (Pettis). *Suppose $1 < p < \infty$. Then every strongly measurable p -Dunford integrable function $f : \Omega \rightarrow X$ is p -Pettis integrable.*

3. COMPACTNESS OF THE DUNFORD OPERATOR

We first revisit, with an easier proof, Pettis' example (see [26, 9.3]) of a strongly measurable 2-Pettis integrable function having non-compact Dunford operator.

Example 3.1 (Pettis). Let (f_n) be an orthonormal system in $L^2[0, 1]$ and let us consider the strongly measurable function

$$f : [0, 1] \rightarrow L^2[0, 1], \quad f(t) := \sum_{n=1}^{\infty} 2^n f_n \cdot \chi_{I_n}(t),$$

where $I_n := (1/2^n, 1/2^n + 1/4^n)$ for all $n \in \mathbb{N}$ (so that the I_n 's are pairwise disjoint). For each $g \in L^2[0, 1]^* = L^2[0, 1]$ we have

$$\left(\int_0^1 |\langle f, g \rangle(t)|^2 dt \right)^{1/2} = \left(\sum_{n=1}^{\infty} |\langle f_n, g \rangle|^2 \right)^{1/2} \leq \|g\|_{L^2[0, 1]},$$

and so $\langle f, g \rangle \in L^2[0, 1]$. Thus, f is 2-Dunford integrable and hence 2-Pettis integrable (apply Corollary 2.3). Let us check that the operator $S_f^2 : L^2[0, 1] \rightarrow L^2[0, 1]$ is not compact. Indeed, observe that $S_f^2(f_n) = \langle f, f_n \rangle = 2^n \chi_{I_n}$ for all $n \in \mathbb{N}$. Since $(2^n \chi_{I_n})$ is an orthonormal system in $L^2[0, 1]$, it does not admit any norm convergent subsequence. Therefore, S_f^2 is not compact.

The previous construction can be generalized, as we show in Example 3.2 below. By a *Köthe function space* over μ we mean an order ideal Z of $L^1(\mu)$ containing all simple functions which is equipped with a complete lattice norm. The space Z is said to be q -convex, for a given $1 \leq q < \infty$, if there is a constant $C > 0$ such that

$$\left\| \left(\sum_{i=1}^n |z_i|^q \right)^{1/q} \right\|_Z \leq C \left(\sum_{i=1}^n \|z_i\|_Z^q \right)^{1/q},$$

for every $n \in \mathbb{N}$ and $z_1, \dots, z_n \in Z$. For instance, $L^q(\mu)$ is a q -convex Köthe function space over μ . We refer the reader to [20] for more information on q -convexity and related notions in Banach lattices.

Example 3.2. Suppose that there is an infinite sequence (A_i) of pairwise disjoint elements of Σ with $\mu(A_i) > 0$ for all $i \in \mathbb{N}$. Let $1 < p < \infty$. Let Z be a Köthe function space over μ which is p' -convex and order continuous. Then there is a strongly measurable p -Pettis integrable function $\phi : \Omega \rightarrow Z$ such that its Dunford operator $T_\phi^p : L^{p'}(\mu) \rightarrow Z$ is not compact.

Proof. Since Z is order continuous, its topological dual Z^* coincides with the Köthe dual of Z , i.e. the set of all $g \in L^1(\mu)$ such that $fg \in L^1(\mu)$ for all $f \in Z$, the duality being given by $\langle f, g \rangle = \int_\Omega fg \, d\mu$ (see e.g. [20, p. 29]). On the other hand, since Z is p' -convex, Z^* is p -concave (see e.g. [20, Proposition 1.d.4]), i.e. there is a constant $M > 0$ such that

$$(3.1) \quad \left(\sum_{i=1}^n \|h_i\|_{Z^*}^p \right)^{1/p} \leq M \left\| \left(\sum_{i=1}^n |h_i|^p \right)^{1/p} \right\|_{Z^*},$$

for every $n \in \mathbb{N}$ and $h_1, \dots, h_n \in Z^*$. For each $i \in \mathbb{N}$, we fix $f_i \in S_Z$ such that $f_i = f_i \chi_{A_i}$ (e.g. $f_i = \|\chi_{A_i}\|_Z^{-1} \chi_{A_i}$) and we choose $g_i \in B_{Z^*}$ such that $\langle f_i, g_i \rangle = 1$. Note that $g_i \chi_{A_i} \in B_{Z^*}$ also satisfies this equality, so we can assume without loss of generality that $g_i = g_i \chi_{A_i}$. Define a strongly measurable function $\phi : \Omega \rightarrow Z$ by

$$\phi(\omega) := \sum_{i=1}^{\infty} \mu(A_i)^{-1/p} f_i \cdot \chi_{A_i}(\omega).$$

Let us check first that ϕ is p -Pettis integrable. To this end, take any $g \in Z^*$ and $n \in \mathbb{N}$. Observe that

$$\begin{aligned} \int_{\bigcup_{i=1}^n A_i} |\langle \phi, g \rangle|^p \, d\mu &= \sum_{i=1}^n \int_{A_i} |\mu(A_i)^{-1/p} \langle f_i, g \rangle|^p \, d\mu = \sum_{i=1}^n |\langle f_i, g \rangle|^p \\ &= \sum_{i=1}^n \left| \int_\Omega f_i g \, d\mu \right|^p = \sum_{i=1}^n |\langle f_i, g \chi_{A_i} \rangle|^p \leq \sum_{i=1}^n \|g \chi_{A_i}\|_{Z^*}^p \\ &\stackrel{(3.1)}{\leq} M^p \left\| \left(\sum_{i=1}^n |g \chi_{A_i}|^p \right)^{1/p} \right\|_{Z^*}^p = M^p \left\| \sum_{i=1}^n |g \chi_{A_i}| \right\|_{Z^*}^p \leq M^p \|g\|_{Z^*}^p. \end{aligned}$$

As $n \in \mathbb{N}$ is arbitrary, it follows that $\int_\Omega |\langle \phi, g \rangle|^p \, d\mu < \infty$. Thus, ϕ is p -Dunford integrable. Since ϕ is strongly measurable and $p > 1$, we conclude that ϕ is p -Pettis integrable (Corollary 2.3).

To finish the proof we will check that the operator $S_\phi^p : Z^* \rightarrow L^p(\mu)$ is not compact. Indeed, since $\langle f_i, g_i \rangle = 1$ for all $i \in \mathbb{N}$ and

$$\langle f_j, g_i \rangle = \int_\Omega f_j g_i \, d\mu = \int_{A_j \cap A_i} f_j g_i \, d\mu = 0 \quad \text{whenever } i \neq j,$$

we have $S_\phi^p(g_i) = \mu(A_i)^{-1/p} \chi_{A_i}$ for all $i \in \mathbb{N}$. Thus, $(S_\phi^p(g_i))$ is a sequence of norm one vectors in $L^p(\mu)$ having pairwise disjoint supports, so it does not have norm convergent subsequences. Therefore, S_ϕ^p is not compact. \square

The proof of the following result is similar to that of the case $p = 1$ (see e.g. [22, Theorem 9.1]) and is included for the sake of completeness.

Theorem 3.3. *Let $f : \Omega \rightarrow X$ be a p -Pettis integrable function. The following statements are equivalent:*

- (i) T_f^p is compact;
- (ii) for every $\varepsilon > 0$ there is a simple function $h : \Omega \rightarrow X$ such that

$$\|f - h\|_{\mathcal{D}_p(\mu, X)} \leq \varepsilon.$$

Proof. (ii) \Rightarrow (i): Note that for any simple function $h : \Omega \rightarrow X$ the operator S_h^p has finite rank (hence it is compact) and

$$\|S_f^p - S_h^p\| = \|S_{f-h}^p\| = \|f - h\|_{\mathcal{D}_p(\mu, X)}.$$

From (ii) and the previous comments it follows at once that S_f^p is compact.

(i) \Rightarrow (ii): Let Π be the collection of all partitions of Ω into finitely many measurable sets, which becomes a directed set when ordered by refinement. For each $\mathcal{P} \in \Pi$, we define an operator $U_{\mathcal{P}} : L^p(\mu) \rightarrow L^p(\mu)$ by

$$U_{\mathcal{P}}(g) := \sum_{A \in \mathcal{P}} \frac{\int_A g d\mu}{\mu(A)} \cdot \chi_A,$$

(with the convention $\frac{0}{0} = 0$). We have $\sup_{\mathcal{P} \in \Pi} \|U_{\mathcal{P}}\| \leq 1$ and

$$\lim_{\mathcal{P}} \|U_{\mathcal{P}}(g) - g\|_{L^p(\mu)} = 0 \quad \text{for every } g \in L^p(\mu)$$

(the proof of the case $p = 1$ given in [8, pp. 67–68, Lemma 1] can be easily adapted to the general case). Therefore, for any relatively norm compact set $K \subseteq L^p(\mu)$ we have

$$\limsup_{\mathcal{P}} \sup_{g \in K} \|U_{\mathcal{P}}(g) - g\|_{L^p(\mu)} = 0.$$

Fix $\varepsilon > 0$. Since S_f^p is compact, the set $K := S_f^p(B_{X^*})$ is relatively norm compact in $L^p(\mu)$ and, therefore, there is $\mathcal{P}_0 \in \Pi$ such that

$$(3.2) \quad \sup_{x^* \in B_{X^*}} \|U_{\mathcal{P}}(\langle f, x^* \rangle) - \langle f, x^* \rangle\|_{L^p(\mu)} \leq \varepsilon \quad \text{for any } \mathcal{P} \in \Pi \text{ finer than } \mathcal{P}_0.$$

Since f is Pettis integrable, for each $\mathcal{P} \in \Pi$ the simple function

$$h_{\mathcal{P}} : \Omega \rightarrow X, \quad h_{\mathcal{P}} := \sum_{A \in \mathcal{P}} \frac{1}{\mu(A)} \int_A f d\mu \cdot \chi_A,$$

satisfies $\langle h_{\mathcal{P}}, x^* \rangle = U_{\mathcal{P}}(\langle f, x^* \rangle)$ for all $x^* \in X^*$. Hence (3.2) reads as

$$\|h_{\mathcal{P}} - f\|_{\mathcal{D}_p(\mu, X)} \leq \varepsilon \quad \text{for any } \mathcal{P} \in \Pi \text{ finer than } \mathcal{P}_0.$$

This finishes the proof. □

Any p -Dunford integrable function $f : \Omega \rightarrow X$ is Dunford integrable and we have a factorization

$$\begin{array}{ccc} L^\infty(\mu) & \xrightarrow{T_f^1} & X^{**} \\ \downarrow i & \nearrow T_f^p & \\ L^{p'}(\mu) & & \end{array}$$

where i is the inclusion operator. Our next result clarifies the relationship between the compactness of T_f^p and that of T_f^1 .

Theorem 3.4. *Let $f : \Omega \rightarrow X$ be a p -Dunford integrable function. The following statements are equivalent:*

- (i) T_f^p is compact;
- (ii) T_f^1 is compact and

$$Z_f^p = \{ |\langle f, x^* \rangle|^p : x^* \in B_{X^*} \},$$

is relatively norm compact in $L^1(\mu)$;

- (iii) T_f^1 is compact and Z_f^p is uniformly integrable in $L^1(\mu)$.

The proof of Theorem 3.4 requires a couple of lemmas.

Lemma 3.5. *Let $f : \Omega \rightarrow X$ be a p -Dunford integrable function. If T_f^p is compact, then for every sequence (x_n^*) in B_{X^*} there exist a subsequence $(x_{n_k}^*)$ and $x^* \in B_{X^*}$ such that $\langle f, x_{n_k}^* \rangle \rightarrow \langle f, x^* \rangle$ μ -a.e.*

Proof. The set $S_f^p(B_{X^*})$ is relatively norm compact in $L^p(\mu)$. Hence there exist a subsequence $(x_{n_k}^*)$ and $h \in L^p(\mu)$ such that $\|\langle f, x_{n_k}^* \rangle - h\|_{L^p(\mu)} \rightarrow 0$. By passing to a further subsequence, not relabeled, we can assume that $\langle f, x_{n_k}^* \rangle \rightarrow h$ μ -a.e. Let $x^* \in B_{X^*}$ be a w^* -cluster point of $(x_{n_k}^*)$. Then $\langle f(\omega), x^* \rangle$ is a cluster point of the sequence of real numbers $(\langle f(\omega), x_{n_k}^* \rangle)$ for each $\omega \in \Omega$. It follows that $h = \langle f, x^* \rangle$ μ -a.e. and $\langle f, x_{n_k}^* \rangle \rightarrow \langle f, x^* \rangle$ μ -a.e. \square

We include a proof of the following well-known fact since we did not find a suitable reference for it.

Lemma 3.6. *The map $L^p(\mu) \rightarrow L^1(\mu)$ given by $h \mapsto |h|^p$ is norm-to-norm continuous.*

Proof. Let (h_n) be a norm convergent sequence in $L^p(\mu)$, with limit $h \in L^p(\mu)$. If $p = 1$, then we have

$$\| |h_n| - |h| \|_{L^1(\mu)} = \int_{\Omega} ||h_n| - |h|| \, d\mu \leq \int_{\Omega} |h_n - h| \, d\mu \rightarrow 0.$$

Suppose now that $p > 1$. Fix a constant $C > 0$ such that $\|h_n\|_{L^p(\mu)} \leq C$ for all $n \in \mathbb{N}$. Bearing in mind that

$$|a^p - b^p| \leq p \cdot |a^{p-1} + b^{p-1}| \cdot |a - b| \quad \text{for every } a, b \geq 0,$$

Hölder's inequality and the triangle inequality in $L^{p'}(\mu)$, we obtain

$$\begin{aligned} \int_{\Omega} \left| |h_n|^p - |h|^p \right| d\mu &\leq \int_{\Omega} p \cdot \left| |h_n|^{p-1} + |h|^{p-1} \right| \cdot \left| |h_n| - |h| \right| d\mu \\ &\leq p \cdot \left\| |h_n|^{p-1} + |h|^{p-1} \right\|_{L^{p'}(\mu)} \cdot \left(\int_{\Omega} \left| |h_n| - |h| \right|^p d\mu \right)^{1/p} \\ &\leq 2p C^{p/p'} \cdot \left(\int_{\Omega} |h_n - h|^p d\mu \right)^{1/p} \rightarrow 0. \end{aligned}$$

This proves that $|h_n|^p \rightarrow |h|^p$ in $L^1(\mu)$. \square

Proof of Theorem 3.4. (i) \Rightarrow (ii): Clearly, T_f^1 is compact as it factors through T_f^p . On the other hand, $S_f^p(B_{X^*}) = \{\langle f, x^* \rangle : x^* \in B_{X^*}\}$ is relatively norm compact in $L^p(\mu)$. Therefore, an appeal to Lemma 3.6 ensures that Z_f^p is relatively norm compact in $L^1(\mu)$.

(ii) \Rightarrow (iii): Obvious.

(iii) \Rightarrow (i): We will check that S_f^p is compact. Let (x_n^*) be a sequence in B_{X^*} . By Lemma 3.5 (applied to T_f^1), there exist a subsequence $(x_{n_k}^*)$ and $x^* \in B_{X^*}$ such that $\langle f, x_{n_k}^* \rangle \rightarrow \langle f, x^* \rangle$ μ -a.e. For each $k \in \mathbb{N}$, define $h_k := |\langle f, x_{n_k}^* - x^* \rangle|^p \in L^1(\mu)$. Then $h_k \rightarrow 0$ μ -a.e. and (h_k) is uniformly integrable in $L^1(\mu)$ (bear in mind that $2^{-p} \cdot h_k \in Z_f^p$ for all $k \in \mathbb{N}$). We can apply Vitali's convergence theorem to conclude that $\|h_k\|_{L^1(\mu)} \rightarrow 0$, that is, $S_f^p(x_{n_k}^*) = \langle f, x_{n_k}^* \rangle \rightarrow S_f^p(x^*) = \langle f, x^* \rangle$ in the norm topology of $L^p(\mu)$. This proves that S_f^p is compact. \square

Let $f : \Omega \rightarrow X$ be a Dunford integrable function such that

$$Z_f = \{\langle f, x^* \rangle : x^* \in B_{X^*}\},$$

is uniformly integrable in $L^1(\mu)$. The Dunford operator T_f^1 is known to be compact in many cases, for instance, if μ is perfect (e.g. a Radon measure) or if $X \not\cong \ell^1(\mathbb{N}_1)$, see e.g. [34, Theorem 4-1-6]. In particular, the compactness of T_f^1 is guaranteed if either μ is the Lebesgue measure on $[0, 1]$ or X is weakly compactly generated.

Let us say that X has the *Dunford Compactness Property* with respect to μ (shortly μ -DCP) if T_f^1 is compact whenever $f : \Omega \rightarrow X$ is Dunford integrable and Z_f is uniformly integrable in $L^1(\mu)$. For more information on this property, we refer to [22, Chapter 9], [23, Section 5] and [34, Chapter 4].

Corollary 3.7. *Suppose X has the μ -DCP. Let $f : \Omega \rightarrow X$ be a p -Dunford integrable function. The following statements are equivalent:*

- (i) T_f^p is compact;
- (ii) Z_f^p is uniformly integrable in $L^1(\mu)$.

Proof. Note that (ii) implies that Z_f is uniformly integrable in $L^1(\mu)$, because Hölder's inequality yields

$$\int_A |\langle f, x^* \rangle| d\mu \leq \left(\int_A |\langle f, x^* \rangle|^p d\mu \right)^{1/p} \quad \text{for every } A \in \Sigma \text{ and } x^* \in B_{X^*}.$$

The result now follows at once from Theorem 3.4. \square

Bearing in mind that separable Banach spaces have the μ -DCP, we get:

Corollary 3.8. *Let $f : \Omega \rightarrow X$ be a strongly measurable p -Dunford integrable function. The following statements are equivalent:*

- (i) T_f^p is compact;
- (ii) Z_f^p is uniformly integrable in $L^1(\mu)$.

4. p -SUMMING OPERATORS AND p -DUNFORD INTEGRABLE FUNCTIONS

Vector measures of bounded p -variation play a relevant role in the duality theory of Lebesgue-Bochner spaces, see [8, p. 115] and [9, §13]. We begin this section by giving some auxiliary results on the p -variation of a vector measure which will be helpful when studying the composition of p -Dunford integrable functions with p -summing operators.

We denote by Π the collection of all finite partitions of Ω into measurable sets. Given a Banach space Z , we write $V(\mu, Z)$ for the set of all finitely additive vector measures $\nu : \Sigma \rightarrow Z$ such that $\nu(A) = 0$ whenever $\mu(A) = 0$.

Definition 4.1. Let Z be a Banach space. The *total p -variation* of $\nu \in V(\mu, Z)$ is defined by

$$\begin{aligned} |\nu|_p(\Omega) &:= \sup \left\{ \left(\sum_{A \in \mathcal{P}} \frac{\|\nu(A)\|_Z^p}{\mu(A)^{p-1}} \right)^{1/p} : \mathcal{P} \in \Pi \right\} \\ &= \sup \left\{ \sum_{A \in \mathcal{P}} |\alpha_A| \|\nu(A)\|_Z : \mathcal{P} \in \Pi, \sum_{A \in \mathcal{P}} \alpha_A \chi_A \in B_{L^{p'}(\mu)} \right\} \in [0, \infty], \end{aligned}$$

(with the convention $\frac{0}{0^{p-1}} = 0$). We say that ν has *bounded p -variation* if $|\nu|_p(\Omega)$ is finite.

A function $f : \Omega \rightarrow X$ is said to be *p -Bochner integrable* if it is strongly measurable and the real-valued function $\|f(\cdot)\|_X$ belongs to $L^p(\mu)$. In this case, f is p -Pettis integrable and it is known that the countably additive vector measure

$$\nu_f : \Sigma \rightarrow X, \quad \nu_f(A) = \int_A f \, d\mu,$$

has bounded p -variation and

$$|\nu_f|_p(\Omega) = \left(\int_{\Omega} \|f(\cdot)\|_X^p \, d\mu \right)^{1/p} =: \|f\|_{L^p(\mu, X)}.$$

In Theorem 4.3 below we show that this equality holds for any strongly measurable p -Dunford integrable function $f : \Omega \rightarrow X$. Of course, in this more general case the total p -variation of the corresponding measure ν_f (which belongs to $V(\mu, X^{**})$) can be infinite. We should point out that Theorem 4.3 for $p = 1$ is a particular case of a well-known result (see e.g. [22, Theorem 4.1 and Remark 4.2]).

The following lemma is folklore (see e.g. [8, p. 42, Corollary 3]).

Lemma 4.2. *A function $f : \Omega \rightarrow X$ is strongly measurable if and only if for every $\varepsilon > 0$ there exist a sequence (A_n) of pairwise disjoint measurable sets and a sequence (x_n) in X such that the function $g := \sum_{n=1}^{\infty} x_n \chi_{A_n}$ satisfies $\|f(\omega) - g(\omega)\|_X \leq \varepsilon$ for μ -a.e. $\omega \in \Omega$.*

Theorem 4.3. *Let $f : \Omega \rightarrow X$ be a strongly measurable p -Dunford integrable function. Then*

$$(4.1) \quad |\nu_f|_p(\Omega) = \left(\int_{\Omega} \|f(\cdot)\|_X^p d\mu \right)^{1/p}.$$

In particular, f is p -Bochner integrable if and only if ν_f has bounded p -variation.

Proof. Let us prove the inequality “ \leq ” in (4.1). Of course, we may (and do) assume that f is p -Bochner integrable since otherwise the inequality is obvious. Take $\mathcal{P} \in \Pi$ and $g \in B_{L^{p'}(\mu)}$ of the form $g = \sum_{A \in \mathcal{P}} \alpha_A \chi_A$, where $\alpha_A \in \mathbb{R}$. Fix $\varepsilon > 0$. Then for each $A \in \mathcal{P}$ there exists $x_A^* \in S_{X^*}$ such that

$$(4.2) \quad |\alpha_A| \|\nu_f(A)\|_{X^{**}} \leq |\alpha_A| \langle \nu_f(A), x_A^* \rangle + \frac{\varepsilon}{\#\mathcal{P}} = |\alpha_A| \int_A \langle f, x_A^* \rangle d\mu + \frac{\varepsilon}{\#\mathcal{P}},$$

(where $\#\mathcal{P}$ stands for the cardinality of \mathcal{P}). Let $\tilde{g} : \Omega \rightarrow X^*$ be the simple function defined by $\tilde{g} := \sum_{A \in \mathcal{P}} |\alpha_A| x_A^* \chi_A$. By applying Hölder’s inequality to the non-negative measurable functions $\|f(\cdot)\|_X$ and $\|\tilde{g}(\cdot)\|_{X^*} = |\tilde{g}(\cdot)|$, we have:

$$\begin{aligned} \sum_{A \in \mathcal{P}} |\alpha_A| \|\nu_f(A)\|_{X^{**}} &\stackrel{(4.2)}{\leq} \sum_{A \in \mathcal{P}} |\alpha_A| \int_A \langle f, x_A^* \rangle d\mu + \varepsilon \\ &= \int_{\Omega} \langle f(\cdot), \tilde{g}(\cdot) \rangle d\mu + \varepsilon \leq \int_{\Omega} \|f(\cdot)\|_X \|\tilde{g}(\cdot)\|_{X^*} d\mu + \varepsilon \\ &\leq \left(\int_{\Omega} \|f(\cdot)\|_X^p d\mu \right)^{1/p} \|g\|_{L^{p'}(\mu)} + \varepsilon \leq \left(\int_{\Omega} \|f(\cdot)\|_X^p d\mu \right)^{1/p} + \varepsilon. \end{aligned}$$

Since both $\varepsilon > 0$ and g are arbitrary we obtain

$$(4.3) \quad |\nu_f|_p(\Omega) \leq \left(\int_{\Omega} \|f(\cdot)\|_X^p d\mu \right)^{1/p}.$$

In order to prove the inequality “ \geq ” in (4.1) we can assume that $|\nu_f|_p(\Omega)$ is finite. We begin with the following:

Particular case. Suppose that $f = \sum_{n=1}^{\infty} x_n \chi_{A_n}$, where (A_n) is a sequence of pairwise disjoint measurable sets and (x_n) is a sequence in X . The inequality “ \geq ” in (4.1) is obvious if $f = 0$ μ -a.e. Otherwise, we have $\sum_{n=1}^N \|x_n\|^p \mu(A_n) > 0$ for large enough $N \in \mathbb{N}$. Fix such an N and define

$$\alpha_i := \begin{cases} \|x_i\|^{p-1} \left(\sum_{n=1}^N \|x_n\|^p \mu(A_n) \right)^{-1/p'} & \text{if } x_i \neq 0 \\ 0 & \text{if } x_i = 0 \end{cases}$$

for all $i = 1, \dots, n$. It is easy to see that $g = \sum_{i=1}^N \alpha_i \chi_{A_i}$ belongs to $S_{L^{p'}(\mu)}$. Therefore

$$|\nu_f|_p(\Omega) \geq \sum_{i=1}^N |\alpha_i| \|\nu_f(A_i)\|_{X^{**}} = \sum_{i=1}^N |\alpha_i| \|x_i\| \mu(A_i) = \left(\sum_{n=1}^N \|x_n\|^p \mu(A_n) \right)^{1/p},$$

and by taking limits when $N \rightarrow \infty$ we obtain $|\nu_f|_p(\Omega) \geq \left(\int_{\Omega} \|f(\cdot)\|_X^p d\mu \right)^{1/p}$.

General case. Fix $\varepsilon > 0$. Lemma 4.2 ensures the existence of a sequence (A_n) of pairwise disjoint measurable sets and a sequence (x_n) in X such that the function $g := \sum_{n=1}^{\infty} x_n \chi_{A_n}$ satisfies $\|f(\cdot) - g(\cdot)\|_X \leq \varepsilon$ μ -a.e. Since the functions f and $f - g$

are p -Dunford integrable, so is g , with $\nu_g = \nu_f - \nu_{f-g}$. On one hand, (4.3) applied to $f - g$ yields

$$(4.4) \quad |\nu_g|_p(\Omega) \leq |\nu_{f-g}|_p(\Omega) + |\nu_f|_p(\Omega) \leq \varepsilon + |\nu_f|_p(\Omega).$$

On the other hand, we can apply the *Particular case* to g , so that

$$\left(\int_{\Omega} \|g(\cdot)\|_X^p d\mu \right)^{1/p} \leq |\nu_g|_p(\Omega) \stackrel{(4.4)}{\leq} \varepsilon + |\nu_f|_p(\Omega).$$

In particular, g is p -Bochner integrable, and so does $f = (f - g) + g$. Moreover,

$$\begin{aligned} \left(\int_{\Omega} \|f(\cdot)\|_X^p d\mu \right)^{1/p} &\leq \left(\int_{\Omega} \|f(\cdot) - g(\cdot)\|_X^p d\mu \right)^{1/p} + \left(\int_{\Omega} \|g(\cdot)\|_X^p d\mu \right)^{1/p} \\ &\leq 2\varepsilon + |\nu_f|_p(\Omega). \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, $|\nu_f|_p(\Omega) \geq \left(\int_{\Omega} \|f(\cdot)\|_X^p d\mu \right)^{1/p}$. The proof is finished. \square

Definition 4.4. Let Z be a Banach space. The *total p -semivariation* of $\nu \in V(\mu, Z)$ is defined by

$$\|\nu\|_p(\Omega) := \sup \{ |\langle \nu, z^* \rangle|_p(\Omega) : z^* \in B_{Z^*} \},$$

where $\langle \nu, z^* \rangle \in V(\mu, \mathbb{R})$ stands for the composition of z^* and ν . We say that ν has *bounded p -semivariation* if $\|\nu\|_p(\Omega)$ is finite.

Remark 4.5. Let Z be a Banach space and $\nu \in V(\mu, Z)$. If Δ is any w^* -dense subset of B_{Z^*} , then $\|\nu\|_p(\Omega) = \sup \{ |\langle \nu, z^* \rangle|_p(\Omega) : z^* \in \Delta \}$.

Proof. Write $\gamma := \sup \{ |\langle \nu, z^* \rangle|_p(\Omega) : z^* \in \Delta \}$. Fix $\varepsilon > 0$ and $z^* \in B_{Z^*}$. Take $\mathcal{P} \in \Pi$ and $g \in B_{L^{p'}(\mu)}$ of the form $g = \sum_{A \in \mathcal{P}} \alpha_A \chi_A$, where $\alpha_A \in \mathbb{R}$. Since Δ is w^* -dense in B_{Z^*} , there is $z_0^* \in \Delta$ such that

$$\sum_{A \in \mathcal{P}} |\alpha_A| |\langle \nu(A), z^* \rangle| \leq \sum_{A \in \mathcal{P}} |\alpha_A| |\langle \nu(A), z_0^* \rangle| + \varepsilon \leq |\langle \nu, z_0^* \rangle|_p(\Omega) + \varepsilon \leq \gamma + \varepsilon.$$

It follows that $|\langle \nu, z^* \rangle|_p(\Omega) \leq \gamma + \varepsilon$. As ε and z^* are arbitrary, $\|\nu\|_p(\Omega) = \gamma$. \square

Corollary 4.6. If $f : \Omega \rightarrow X$ is p -Dunford integrable, then $\nu_f \in V(\mu, X^{**})$ has bounded p -semivariation and $\|\nu_f\|_p(\Omega) = \|f\|_{\mathcal{D}^p(\mu, X)}$.

Proof. For each $x^* \in X^*$ the composition $\langle \nu_f, x^* \rangle$ is the indefinite integral of $\langle f, x^* \rangle \in L^p(\mu)$, hence $|\langle \nu_f, x^* \rangle|_p(\Omega) = \|\langle f, x^* \rangle\|_{L^p(\mu)}$ (apply Theorem 4.3 in the real-valued case to $\langle f, x^* \rangle$). Now, Remark 4.5 applied to ν_f with $Z := X^{**}$ and $\Delta := B_{X^*}$ (which is w^* -dense in $B_{X^{***}}$ by Goldstine's theorem) ensures that

$$\begin{aligned} \|\nu_f\|_p(\Omega) &= \sup \{ |\langle \nu_f, x^* \rangle|_p(\Omega) : x^* \in B_{X^*} \} \\ &= \sup \{ \|\langle f, x^* \rangle\|_{L^p(\mu)} : x^* \in B_{X^*} \} = \|f\|_{\mathcal{D}^p(\mu, X)}. \end{aligned}$$

\square

Throughout the rest of this section Y is a Banach space. Recall that an operator $u : X \rightarrow Y$ is said to be *p -summing* if there exists a constant $K \geq 0$ such that

$$\left(\sum_{i=1}^n \|u(x_i)\|_Y^p \right)^{1/p} \leq K \sup \left\{ \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \right)^{1/p} : x^* \in B_{X^*} \right\},$$

for every $n \in \mathbb{N}$ and every $x_1, \dots, x_n \in X$. The least constant K satisfying this condition is usually denoted by $\pi_p(u)$.

Our results on the composition of p -summing operators and p -Dunford integrable functions (Theorems 4.9 and 4.12 below) will be obtained with the help of Theorem 4.3 and the following two easy lemmas.

Lemma 4.7. *Let $u : X \rightarrow Y$ be an operator and $f : \Omega \rightarrow X$ a p -Dunford integrable function. Then $u \circ f$ is p -Dunford integrable and $\nu_{u \circ f} = u^{**} \circ \nu_f$.*

Proof. The first statement is obvious. On the other hand, for every $A \in \Sigma$ and $y^* \in Y^*$ we have

$$\begin{aligned} \langle \nu_{u \circ f}(A), y^* \rangle &= \int_A \langle u \circ f, y^* \rangle d\mu = \int_A \langle f, u^*(y^*) \rangle d\mu \\ &= \langle \nu_f(A), u^*(y^*) \rangle = \langle (u^{**} \circ \nu_f)(A), y^* \rangle, \end{aligned}$$

and so $\nu_{u \circ f} = u^{**} \circ \nu_f$. \square

Lemma 4.8. *Let $u : X \rightarrow Y$ be a p -summing operator. If $\nu \in V(\mu, X)$ has bounded p -semivariation, then $u \circ \nu \in V(\mu, Y)$ has bounded p -variation and*

$$|u \circ \nu|_p(\Omega) \leq \pi_p(u) \|\nu\|_p(\Omega).$$

Proof. For any $\mathcal{P} \in \Pi$ we have

$$\begin{aligned} &\left(\sum_{A \in \mathcal{P}} \frac{\|u(\nu(A))\|_Y^p}{\mu(A)^{p-1}} \right)^{1/p} = \left(\sum_{A \in \mathcal{P}} \left\| u \left(\frac{\nu(A)}{\mu(A)^{1/p'}} \right) \right\|_Y^p \right)^{1/p} \\ &\leq \pi_p(u) \sup \left\{ \left(\sum_{A \in \mathcal{P}} \left| \left\langle \frac{\nu(A)}{\mu(A)^{1/p'}}, x^* \right\rangle \right|^p \right)^{1/p} : x^* \in B_{X^*} \right\} \\ &= \pi_p(u) \sup \left\{ \left(\sum_{A \in \mathcal{P}} \frac{|\langle \nu(A), x^* \rangle|^p}{\mu(A)^{p-1}} \right)^{1/p} : x^* \in B_{X^*} \right\} \leq \pi_p(u) \|\nu\|_p(\Omega). \end{aligned}$$

\square

Theorem 4.9. *Let $u : X \rightarrow Y$ be a p -summing operator and $f : \Omega \rightarrow X$ a p -Dunford integrable function such that $u \circ f$ is strongly measurable. Then $u \circ f$ is p -Bochner integrable and $\|u \circ f\|_{L^p(\mu, Y)} \leq \pi_p(u) \|f\|_{\mathcal{D}^p(\mu, X)}$.*

Proof. We know that $\nu_f \in V(\mu, X^{**})$ satisfies $\|\nu_f\|_p(\Omega) = \|f\|_{\mathcal{D}^p(\mu, X)} < \infty$ (Corollary 4.6). On the other hand, the p -summability of u guarantees that of u^{**} (and in fact $\pi_p(u) = \pi_p(u^{**})$), see e.g. [7, Proposition 2.19]. So Lemma 4.8 applied to u^{**} and ν_f ensures that $u^{**} \circ \nu_f$ has bounded p -variation and, moreover, $|u^{**} \circ \nu_f|_p(\Omega) \leq \pi_p(u) \|\nu_f\|_p(\Omega)$.

Observe that $u \circ f$ is strongly measurable, p -Dunford integrable and satisfies $\nu_{u \circ f} = u^{**} \circ \nu_f$ (Lemma 4.7). Therefore, Theorem 4.3 applied to $u \circ f$ tells us that $u \circ f$ is p -Bochner integrable and

$$\|u \circ f\|_{L^p(\mu, Y)} = |\nu_{u \circ f}|_p(\Omega) = |u^{**} \circ \nu_f|_p(\Omega) \leq \pi_p(u) \|\nu_f\|_p(\Omega) = \pi_p(u) \|f\|_{\mathcal{D}^p(\mu, X)},$$

as we wanted to prove. \square

The celebrated Pietsch factorization theorem (see e.g. [7, 2.13]) states that for every p -summing operator $u : X \rightarrow Y$ there is a regular Borel probability measure η on (B_{X^*}, w^*) such that u factors as

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ i \downarrow & & \uparrow \hat{u} \\ i(X) & \xrightarrow{j} & Z \end{array}$$

where:

- i is the canonical isometric embedding of X in the Banach space $C(B_{X^*})$ of all real-valued continuous functions on (B_{X^*}, w^*) (which is given by $i(x)(x^*) := x^*(x)$ for every $x \in X$ and $x^* \in B_{X^*}$);
- Z is a subspace of $L^p(\eta)$;
- j is the restriction of the identity operator which maps each element of $C(B_{X^*})$ to its equivalence class in $L^p(\eta)$;
- \hat{u} is an operator.

Note that $u(X)$ is separable whenever Z is separable. Therefore, in this case, for every scalarly measurable function $f : \Omega \rightarrow X$ the composition $u \circ f$ is strongly measurable.

The previous discussion and Theorem 4.9 lead to Corollary 4.10 below. Recall that a probability measure η (defined on a σ -algebra) is said to be *separable* if its measure algebra equipped with the Fréchet-Nikodým metric is separable or, equivalently, if $L^p(\eta)$ is separable for some/all $1 \leq p < \infty$. We say that a compact Hausdorff topological space K belongs to the class MS if every regular Borel probability measure on K is separable.

Corollary 4.10. *Suppose (B_{X^*}, w^*) belongs to the class MS . Let $u : X \rightarrow Y$ be a p -summing operator. If $f : \Omega \rightarrow X$ is p -Dunford integrable, then $u \circ f$ is p -Bochner integrable and $\|u \circ f\|_{L^p(\mu, Y)} \leq \pi_p(u) \|f\|_{\mathcal{D}^p(\mu, X)}$.*

The class MS is rather wide and contains all compact spaces which are Eberlein, Rosenthal, weakly Radon-Nikodým, linearly ordered, etc. From the Banach space point of view, (B_{X^*}, w^*) belongs to the class MS whenever X is weakly countably determined, weakly precompactly generated (e.g. $X \not\cong \ell^1$), etc. For more information on the class MS we refer to [31, Section 3.1] and the references therein. Some recent works on this topic are [2, 21, 27].

Remark 4.11. One of the consequences of the aforementioned Pietsch theorem is that every p -summing operator is *completely continuous*, see e.g. [7, Theorem 2.17], i.e. it maps weakly compact sets to norm compact sets. *If X is weakly precompactly generated, then every completely continuous operator $u : X \rightarrow Y$ has separable range.* Indeed, $X = \overline{\text{span}}(G)$ for some weakly precompact set $G \subseteq X$ and so $u(X) \subseteq \overline{\text{span}}(u(G))$, where $u(G)$ is relatively norm compact (hence separable). The same assertion holds if X is weakly countably determined, but in this case the proof is more involved, see [33, Theorem 7.1].

In the following result (which is stronger than Theorem 4.9) we remove the strong measurability condition for $u \circ f$ at the cost of obtaining a weaker conclusion.

Theorem 4.12. *Let $u : X \rightarrow Y$ be a p -summing operator and $f : \Omega \rightarrow X$ a p -Dunford integrable function. Then $u \circ f$ is scalarly equivalent to a p -Bochner integrable function $g : \Omega \rightarrow Y$ and $\|g\|_{L^p(\mu, Y)} \leq \pi_p(u)\|f\|_{\mathcal{D}^p(\mu, X)}$.*

Proof. Since u is p -summing, it is also weakly compact (see e.g. [7, Theorem 2.17]) and hence $Z := \overline{u(X)}$ is weakly compactly generated. In particular, (Z, w) is Lindelöf (see e.g. [13, Theorem 14.31]) and so measure-compact. A result of Edgar (see [11, Proposition 5.4], cf. [34, Theorem 3-4-6]) ensures the existence of a strongly measurable function $g : \Omega \rightarrow Z$ such that $u \circ f$ and g are scalarly equivalent.

Since $u \circ f$ is p -Dunford integrable, the same holds for g , with $\nu_g = \nu_{u \circ f} = u^{**} \circ \nu_f$ (Lemma 4.7). On the other hand, in a similar way as we did in Theorem 4.9, we deduce that ν_g has bounded p -variation and $|\nu_g|_p(\Omega) \leq \pi_p(u)\|f\|_{\mathcal{D}^p(\mu, X)}$. The result now follows from Theorem 4.3 applied to g . \square

We finish this section by pointing out that in [25] a general approach is developed to obtain further results on the improvement of the integrability of a strongly measurable function by a summing operator.

5. TESTING p -DUNFORD INTEGRABILITY

In this section we study the p -Dunford integrability of a function $f : \Omega \rightarrow X$ via the family of real-valued functions

$$Z_{f, \Gamma} = \{\langle f, x^* \rangle : x^* \in \Gamma\},$$

for some $\Gamma \subseteq X^*$. To deal with our main result, Theorem 5.1, we use some ideas from the proof of [30, Theorem 9].

Theorem 5.1. *Suppose X has the μ -PIP. Let $f : \Omega \rightarrow X$ be a scalarly measurable function for which there is a w^* -thick set $\Gamma \subseteq X^*$ such that $Z_{f, \Gamma} \subseteq L^p(\mu)$. Then f is p -Dunford integrable.*

Proof. For each $n \in \mathbb{N}$, define the absolutely convex set

$$C_n := \left\{ x^* \in X^* : \left(\int_{\Omega} |\langle f, x^* \rangle|^p d\mu \right)^{1/p} \leq n \right\}.$$

We will prove that C_n is w^* -closed.

To this end, we first use the scalar measurability of f to find an increasing sequence (E_m) of measurable sets with $\Omega = \bigcup_{m \in \mathbb{N}} E_m$ such that $f_m := f \chi_{E_m}$ is scalarly bounded for all $m \in \mathbb{N}$ (see e.g. [22, Proposition 3.1]). The μ -PIP of X ensures that each f_m is p -Pettis integrable.

Fix $x^* \in \overline{C_n}^{w^*}$ and $m \in \mathbb{N}$. By the p -Pettis integrability of f_m , the operator $S_{f_m}^p : X^* \rightarrow L^p(\mu)$ is w^* - w -continuous (just make the obvious changes to the proof of the case $p = 1$, see e.g. [22, Theorem 4.3]). Hence

$$S_{f_m}^p(x^*) \in \overline{S_{f_m}^p(C_n)}^w = \overline{S_{f_m}^p(C_n)}^{\|\cdot\|_{L^p(\mu)}}$$

(bear in mind that $S_{f_m}^p(C_n)$ is convex). Therefore, there is a sequence (x_k^*) in C_n such that

$$\lim_{k \rightarrow \infty} \|\langle f_m, x_k^* \rangle - \langle f_m, x^* \rangle\|_{L^p(\mu)} \rightarrow 0.$$

In particular, $\|\langle f_m, x^* \rangle\|_{L^p(\mu)} \leq n$. As $m \in \mathbb{N}$ is arbitrary, an appeal to the monotone convergence theorem yields

$$\int_{\Omega} |\langle f, x^* \rangle|^p d\mu = \lim_{m \rightarrow \infty} \int_{\Omega} |\langle f_m, x^* \rangle|^p d\mu \leq n^p,$$

so that $x^* \in C_n$. This proves that C_n is w^* -closed.

Note that $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma \cap C_n$ and $C_n \subseteq C_{n+1}$ for all $n \in \mathbb{N}$. Since Γ is w^* -thick, there is $n \in \mathbb{N}$ such that $\overline{\text{aco}}^{w^*}(\Gamma \cap C_n)$ contains a ball centered at 0, and so does C_n (because it is absolutely convex and w^* -closed). That is, there is $\delta > 0$ such that $\delta x^* \in C_n$ for every $x^* \in B_{X^*}$. This clearly implies that f is p -Dunford integrable. The proof is finished. \square

The space X is said to have property (\mathcal{E}) (of Efremov) if for every convex bounded set $C \subseteq X^*$, any element of the w^* -closure of C is the w^* -limit of a sequence contained in C . This class of Banach spaces has been studied in [28, 29]. It contains all Banach spaces having w^* -angelic dual and, in particular, all weakly compactly generated spaces. Every Banach space having property (\mathcal{E}) also satisfies the so-called Mazur's property and, therefore, has the PIP (see [12]).

For Banach spaces having property (\mathcal{E}) the scalar measurability assumption in Theorem 5.1 is redundant and we have the following result.

Corollary 5.2. *Suppose X has property (\mathcal{E}) . Let $f : \Omega \rightarrow X$ be a function for which there is a w^* -thick set $\Gamma \subseteq X^*$ such that $Z_{f,\Gamma} \subseteq L^p(\mu)$. Then f is p -Dunford integrable.*

Proof. Since Γ is w^* -thick, the set $\overline{\text{aco}}^{w^*}(\Gamma)$ contains a ball centered at 0 and, in particular, Γ separates the points of X . Since $\langle f, x^* \rangle$ is measurable for all $x^* \in \Gamma$ and X has property (\mathcal{E}) , we can apply [28, Proposition 12] to conclude that f is scalarly measurable. Theorem 5.1 now ensures that f is p -Dunford integrable. \square

Remark 5.3. For $1 < p < \infty$, the conclusion of Theorem 5.1 and Corollary 5.2 can be strengthened to “ f is p -Pettis integrable” (by Corollary 2.2).

Our last example is based on a construction given in [30, Example 8] and shows that Corollary 5.2 does not work for arbitrary Banach spaces. Here $\ell^1([0, 1])$ is seen as the dual of $c_0([0, 1])$, so that the set $c_0([0, 1]) \subseteq \ell^1([0, 1])^* (= \ell^\infty([0, 1]))$ is w^* -thick (see e.g. [24, Theorem 1.5, (a) \Leftrightarrow (c)]).

Example 5.4. There is a function $f : [0, 1] \rightarrow \ell^1([0, 1])$ such that:

- (i) $\langle f, \varphi \rangle = 0$ a.e. for every $\varphi \in c_0([0, 1])$;
- (ii) f is not Dunford integrable.

Proof. For each $t \in [0, 1]$, let $e_t \in \ell^1([0, 1])$ be defined by $e_t(s) := 1$ if $t = s$ and $e_t(s) := 0$ if $t \neq s$. Define

$$f : [0, 1] \rightarrow \ell^1([0, 1]), \quad f(t) := h(t)e_t,$$

where $h : [0, 1] \rightarrow \mathbb{R}$ is any function such that $h \notin L^1[0, 1]$. Condition (i) holds because $\langle f, \varphi \rangle$ vanishes outside of a countable subset of $[0, 1]$. On the other hand, f is not Dunford integrable because the functional $\chi_{[0,1]} \in \ell^\infty([0, 1])$ satisfies $\langle f, \chi_{[0,1]} \rangle = h$. \square

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