Asymptotic estimates on the von Neumann inequality for homogeneous polynomials

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Abstract. By the von Neumann inequality for homogeneous polynomials there exists a positive constant $C_{k,q}(n)$ such that for every k-homogeneous polynomial p in n variables and every n-tuple of commuting operators $(T_1, \ldots, T_n)$ with $\sum_{i=1}^{n} \|T_i\|^q \leq 1$ we have

$$\|p(T_1, \ldots, T_n)\|_{\mathcal{L}(\mathcal{H})} \leq C_{k,q}(n) \sup\left\{ |p(z_1, \ldots, z_n)| : \sum_{i=1}^{n} |z_i|^q \leq 1 \right\}.$$

For fixed k and q, we study the asymptotic growth of the smallest constant $C_{k,q}(n)$ as n (the number of variables/operators) tends to infinity. For $q = \infty$, we obtain the correct asymptotic behavior of this constant (answering a question posed by Dixon in the 1970s). For $2 \leq q < \infty$ we improve some lower bounds given by Mantero and Tonge, and prove the asymptotic behavior up to a logarithmic factor. To achieve this we provide estimates of the norm of homogeneous unimodular Steiner polynomials, i.e. polynomials such that the multi-indices corresponding to the nonzero coefficients form partial Steiner systems.

1. Introduction

A classical inequality in operator theory, due to von Neumann [30], asserts that if T is a linear contraction on a complex Hilbert space $\mathcal{H}$ (i.e. its operator norm is less than or equal to one), then

$$\|p(T)\|_{\mathcal{L}(\mathcal{H})} \leq \sup\{|p(z)| : z \in \mathbb{C}, \, |z| \leq 1\},$$

for every polynomial p in one (complex) variable. Note that, as a direct consequence of von Neumann's inequality, we can define a functional calculus on the disk algebra. There are many other consequences of this important inequality in functional analysis; we refer the reader

to [25, Chapter 1] and the references therein for a fuller treatment of this inequality and its applications.

For some time, it was very natural to ask whether the von Neumann inequality could be extended to polynomials in two or more commuting contractions. For polynomials in two contractions Ando [2], using "dilation theory" (see [28]), provided a positive answer. However, in the mid seventies, Varopoulos [29] showed that von Neumann's inequality cannot be extended to three or more contractions. For this, he used the metric theory of tensor products together with probabilistic tools to construct a polynomial and operators that violate the inequality. The work of Varopoulos has since been simplified and extended by several authors [5, 9, 15, 21, 22].

It is an open problem of great interest in operator theory (see [6, 25]) to determine whether there exists a constant $K(n)$ that adjusts von Neumann's inequality. More precisely, it is unknown whether or not for every n there exists a constant $K(n)$ such that

$$(1.1) \qquad \|p(T_1, \ldots, T_n)\|_{\mathcal{L}(\mathcal{H})} \leq K(n) \sup\{|p(z_1, \ldots, z_n)| : |z_i| \leq 1\},$$

for every polynomial p in n variables and every n-tuple $(T_1, \ldots, T_n)$ of commuting contractions in $\mathcal{L}(\mathcal{H})$.

Dixon in [15] gave lower estimates for the optimal $K(n)$ and showed that, if such a constant verifying (1.1) exists, then it must grow faster than any power of n. He did this by considering the problem in the smaller class of k-homogeneous polynomials. More precisely, he studied the asymptotic behavior (as n, the number of variables/operators, tends to infinity) of the smallest constant $C_{k,\infty}(n)$ such that

$$(1.2) \qquad \|p(T_1, \ldots, T_n)\|_{\mathcal{L}(\mathcal{H})} \leq C_{k,\infty}(n) \sup\{|p(z_1, \ldots, z_n)| : |z_i| \leq 1\},$$

for every k-homogeneous polynomial p in n variables and every n-tuple of commuting contractions $(T_1, \ldots, T_n)$. In [15, Theorem 1.2] he showed that

$$(1.3) \qquad n^{\frac{1}{2}\left[\frac{k-1}{2}\right]} \ll C_{k,\infty}(n) \ll n^{\frac{k-2}{2}},$$

where $[x]$ denotes the integer part of x. For the lower bound Dixon used probabilistic techniques (the Kahane–Salem–Zygmund theorem) and combinatorial ideas (Steiner systems) along with an ingenious construction of the operators and the Hilbert space involved.

This problem was taken up by Mantero and Tonge in [21]. Among other problems, for each $1 \leq q < \infty$ they consider $C_{k,q}(n)$, the smallest constant such that

$$(1.4) \qquad \|p(T_1, \ldots, T_n)\|_{\mathcal{L}(\mathcal{H})} \leq C_{k,q}(n) \sup\left\{|p(z_1, \ldots, z_n)| : \sum_{j=1}^{n} |z_j|^q \leq 1\right\},$$

for every k-homogeneous polynomial p in n variables and every n-tuple of commuting contractions $(T_1, \ldots, T_n)$ with $\sum_{i=1}^{n} \|T_i\|_{\mathcal{L}(\mathcal{H})}^q \leq 1$. They give upper and lower estimates for the growth of $C_{k,q}(n)$ (see [21, Propositions 11 and 17]) (here q' denotes the conjugate of q; see below):

$$(1.5) \qquad n^{\frac{k-1}{q'} - \frac{1}{2}\left[\frac{k}{2}\right]} \ll C_{k,q}(n) \ll n^{\frac{k-2}{q'}} \quad \text{for } 1 \leq q \leq 2,$$

$$(1.6) \qquad n^{\frac{k}{2} - \frac{1}{2}\left(\left[\frac{k}{2}\right]+1\right)} \ll C_{k,q}(n) \ll n^{\frac{k-2}{2}} \quad \text{for } 2 \leq q < \infty.$$

It is worth noting that the upper bounds here hold for every n-tuple $(T_1, \ldots, T_n)$ satisfying $\sum_{i=1}^{n} \|T_i\|^q \leq 1$ (and even a weaker condition), not necessarily commuting. If we do not ask the contractions to commute, this bound is shown to be optimal in [21, Proposition 15].

Based on the combinatorial methods from [15] (i.e. considering polynomials whose monomials are determined by Steiner blocks) we change the construction of the Hilbert space and the operators given there to find the exact asymptotic growth of $C_{k,\infty}(n)$, answering a question that was explicitly posed by Dixon.

On the other hand, by applying some probabilistic tools used by Bayart in [3], we are able to control the increments of a Rademacher process and in this way we in this way we manage to narrow the range in (1.6), showing that the exponent in the power of n is indeed optimal. We collect this in our main result.

Theorem 1.1. *For $k \geq 3$ and $1 \leq q \leq \infty$, let $C_{k,q}(n)$ be the smallest constant such that*

$$\|p(T_1, \ldots, T_n)\|_{\mathscr{L}(\mathscr{H})} \leq C_{k,q}(n) \sup\{|p(z_1, \ldots, z_n)| : \|(z_j)_j\|_q \leq 1\},$$

for every k-homogeneous polynomial p in n variables and every n-tuple of commuting contractions $(T_1, \ldots, T_n)$ with $\sum_{i=1}^{n} \|T_i\|_{\mathscr{L}(\mathscr{H})}^q \leq 1$. Then:

 (i) $C_{k,\infty}(n) \sim n^{\frac{k-2}{2}}$,

 (ii) *for $2 \leq q < \infty$ we have*

$$\log^{-\frac{3}{q}}(n) n^{\frac{k-2}{2}} \ll C_{k,q}(n) \ll n^{\frac{k-2}{2}}.$$

In particular,

$$n^{\frac{k-2}{2}-\varepsilon} \ll C_{k,q}(n) \ll n^{\frac{k-2}{2}}$$

for every $\varepsilon > 0$.

The proof of this result will be given in Section 3.

2. Steiner unimodular polynomials

The systematic study of norms of random homogeneous polynomials started with the Kahane–Salem–Zygmund theorem [17, Chapter 6], which is found very useful in Fourier analysis. More recently, applications of norms of random polynomials with unimodular coefficients were found in complex and functional analysis (see for example [3, 7, 8, 13]).

The philosophy in this problem and in many others of the same kind (e.g. to compute the Sidon constant for polynomials [11, 23]) is to find polynomials which have "big" (or "many") coefficients, but whose maximum modulus on the unit ball is "small".

In this section we are going to relax the number of terms appearing in the polynomials, by allowing them to have some zero coefficients. In this way we will find a special class of tetrahedral unimodular polynomials having many terms, but keeping the maximum modulus quite small.

Let us first start with some notation and preliminaries. As usual we will denote ℓ_q^n for $\mathbb{C}^n$ with the norm

$$\|(z_1, \ldots, z_n)\|_q = \left(\sum_{i=1}^{n} |z_i|^q \right)^{\frac{1}{q}} \quad \text{if } 1 \leq q < \infty,$$

$$\|(z_1, \ldots, z_n)\|_\infty = \max_{i=1,\ldots,n} |z_i| \quad \text{for } q = \infty.$$

A k-homogeneous polynomial in n variables is a function $p : \mathbb{C}^n \to \mathbb{C}$ of the form

$$p(z_1, \ldots, z_n) = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| = k}} a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} = \sum_{\substack{J = (j_1, \ldots, j_k) \\ 1 \leq j_1 \leq \ldots \leq j_k \leq n}} c_J z_{j_1} \cdots z_{j_k},$$

where $a_\alpha \in \mathbb{C}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Given α we have

$$a_\alpha = c_J,$$

where $J = (1, \overset{\alpha_1}{\ldots}, 1, \ldots, n, \overset{\alpha_k}{\ldots}, n)$. We will write $z_1^{\alpha_1} \cdots z_n^{\alpha_n} = z^\alpha$ and $z_{j_1} \cdots z_{j_k} = z_J$. For $1 \leq q \leq \infty$ we denote by $\mathcal{P}(^k \ell_q^n)$ the Banach space of all k-homogeneous polynomials on n variables with the norm

$$\|p\|_{\mathcal{P}(^k \ell_q^n)} = \sup\{|p(z_1, \ldots, z_n) : \|(z_1, \ldots, z_n)\|_q \leq 1\}.$$

It is a well-known fact (see e.g. [14, Chapter 1]) that for every k-homogeneous polynomial there is a unique symmetric k-linear form L on $\mathbb{C}^n$ such that $p(z) = L(z, \ldots, z)$ for all $z \in \mathbb{C}^n$. Also for each $1 \leq q \leq \infty$ and $k \geq 2$ there exists a constant $\lambda(k, q) > 0$ such that

$$(2.1) \qquad \|p\|_{\mathcal{P}(^k \ell_q^n)} \leq \sup\{L(z^{(1)}, \ldots, z^{(k)}) : \|z^{(j)}\|_q \leq 1, \ j = 1, \ldots, k\}$$
$$\leq \lambda(k, q)\|p\|_{\mathcal{P}(^k \ell_q^n)}.$$

In general,

$$\lambda(k, q) \leq \frac{k^k}{k!},$$

but improvements in concrete cases include

$$\lambda(k, 2) = 1 \quad \text{and} \quad \lambda(k, \infty) \leq \frac{k^{\frac{k}{2}}(k+1)^{\frac{k+1}{2}}}{2^k k!}$$

(see [14, Propositions 1.44 and 1.43]).

If $(a_n)_n$ and $(b_n)_n$ are two sequences of real numbers, we will write $a_n \ll b_n$ if there exists a constant $C > 0$ (independent of n) such that $a_n \leq C b_n$ for every n. We will write $a_n \sim b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$.

Given a set A we will denote its cardinality by $|A|$.

For an index $1 < q < \infty$ we denote by q' its conjugate, i.e. $1 = \frac{1}{q} + \frac{1}{q'}$.

Let $\mathcal{C} \subset \mathbb{N}_0^n$ denote any set of multi-indices α with $|\alpha| = k$. Then as a consequence of the Kahane–Salem–Zygmund theorem [17, Chapter 6] there exists a k-homogeneous polynomial, with unimodular coefficients a_α for $\alpha \in \mathcal{C}$ and $a_\alpha = 0$ if $\alpha \notin \mathcal{C}$, of small maximum modulus on the n-polydisk. More precisely, let $(\varepsilon_\alpha)_{\alpha \in \mathcal{C}}$ be independent Bernoulli variables on a probability space $(\Omega, \Sigma, \mathbb{P})$; then we have

$$(2.2) \qquad \mathbb{P}\left\{\omega \in \Omega : \left\|\sum_{\alpha \in \mathcal{C}} \varepsilon_\alpha(\omega) z^\alpha\right\|_{\mathcal{P}(^k \ell_\infty^n)} \geq D\big(n \log(k)|\mathcal{C}|\big)^{\frac{1}{2}}\right\} \leq \frac{1}{k^2 e^n},$$

where $D > 0$ is an absolute constant which is less than 8. In particular, there are signs $(a_\alpha)_{\alpha \in \mathcal{C}}$ such that the k-homogeneous unimodular polynomial

$$p(z) = \sum_{\alpha \in \mathcal{C}} a_\alpha z^\alpha$$

satisfies

$$(2.3) \qquad \|p\|_{\mathcal{P}(^k \ell_\infty^n)} \leq D\big(n \log(k)|\mathcal{C}|\big)^{\frac{1}{2}}.$$

We are going to work with polynomials with many zero coefficients, expecting that this will make the norm of the polynomial small enough. The presence of $|\mathcal{C}|$ in (2.3) is sufficient for our needs when the norm of the polynomial is computed in ℓ_∞^n but not when we consider the norm in ℓ_p^n and then we need different tools. The relevant results we have to hand [3, 7, 12, 13] do not take into account the number of nonzero coefficients, so considering our tetrahedral polynomials does not improve these estimates. We deal with polynomials with a particular combinatorial configuration in order to get useful estimates for our purposes. We modify some arguments from [3], reflecting this configuration.

To achieve our goal we consider special subsets of multi-indices: partial Steiner systems on the set $\{1, \ldots, n\}$. An $S_p(t, k, n)$ *partial Steiner system* is a collection of subsets of size k of $\{1, \ldots, n\}$ such that every subset of t elements is contained in at most one member of the collection of subsets of size k.

Definition 2.1. A k-homogeneous polynomial of n variables is a *Steiner unimodular polynomial* if there exists an $S_p(t, k, n)$ partial Steiner system $\mathcal{S}$ such that

$$p(z_1, \ldots, z_n) = \sum_{J \in \mathcal{S}} c_J z_J \quad \text{and} \quad c_J = \pm 1.$$

Observe that our Steiner unimodular polynomials are tetrahedral, i.e. in every term z_J each variable z_{j_0} appears at most once. In other words, no term in the polynomial contains a factor of degree 2 or higher in any of the variables $z_1, \ldots, z_n$.

The first one to consider Steiner unimodular polynomials was Dixon [15], who used $S_p([\frac{k-1}{2}], k, n)$ partial Steiner systems. He used this to obtain lower bounds for (1.2). The combinatorial property was only applied to define some Hilbert space operators that violate the inequality, but not to estimate the norm of the polynomial, which he did using (2.3) and the number of nonzero coefficients.

In the following lemmas, in ℓ_q^n, $1 \leq q < \infty$, we will strongly use the fact that the multi-indices of the nonzero coefficients form a partial Steiner system to estimate the maximum modulus. We use an entropy argument due to Pisier to control the increments of a Rademacher process and subsequently apply an interpolation argument.

Let us first recall some definitions and a result on regularity of random process. A complete account on these can be found in [20, Chapters 4 and 11]. A Young function ψ is a convex increasing function defined on $[0, \infty[$ such that $\lim_{t \to \infty} \psi(t) = \infty$ and $\psi(0) = 0$. For a probability space $(\Omega, \Sigma, \mathbb{P})$, the Orlicz space $L_\psi = L_\psi(\Omega, \Sigma, \mathbb{P})$ is defined as the space of all real-valued random variables Z for which there exists $c > 0$ such that $\mathbb{E}(\psi(\frac{|Z|}{c})) < \infty$. It is a Banach space with the norm $\|Z\|_{L_\psi} = \inf\{c > 0 : \mathbb{E}(\psi(\frac{|Z|}{c})) \leq 1\}$.

Let (X, d) be a metric space. Given $\varepsilon > 0$, the entropy number $N(X, d; \varepsilon)$ is defined as the smallest number of open balls of radius ε in the metric d, which form a covering of the metric space X.

With this, the entropy integral of (X, d) with respect to ψ is given by

$$J_\psi(X, d) := \int_0^{\text{diam}(X)} \psi^{-1}(N(X, d; \varepsilon)) \, d\varepsilon.$$

We are going to define a random process $(Y_z)_{z \in B_{\ell_2^n}}$ and we will need to estimate the expectation of $\sup_z Y_z$. To do so, we use the following theorem due to Pisier [24] (see

also [20, Theorem 11.1]) that bounds this expectation with the entropy integral, provided that the random process satisfies a certain contraction condition.

Theorem 2.2. *Let* $Z = (Z_x)_{x \in X}$ *be a random process indexed by* (X, d) *in* L_ψ *such that, for every* $x, x' \in X$,

$$\|Z_x - Z_{x'}\|_{L_\psi} \le d(x, x').$$

Then, if $J_\psi(X, d)$ *is finite,* Z *is almost surely bounded and*

$$\mathbb{E}\left(\sup_{x,x' \in X} |Z_x - Z_{x'}| \right) \le 8 J_\psi(X, d).$$

Let now $k \ge 2$ and let $\mathcal{S}$ be an $S_p(k-1, k, n)$ partial Steiner system. We consider a family of independent Bernoulli variables $(\varepsilon_J)_{J \in \mathcal{S}}$ on the probability space $(\Omega, \Sigma, \mathbb{P})$. For $z \in B_{\ell_2^n}$ we define the following Rademacher process indexed by $B_{\ell_2^n}$ as

$$(2.4) \qquad\qquad Y_z = \frac{1}{k} \sum_{J \in \mathcal{S}} \varepsilon_J z_J.$$

We view it as a random process in the Orlicz space defined by the Young function

$$\psi_2(t) = e^{t^2} - 1.$$

Lemma 2.3. *The Rademacher process defined in* (2.4) *fulfils the following Lipschitz condition:*

$$\|Y_z - Y_{z'}\|_{L_{\psi_2}} \le C \|z - z'\|_\infty,$$

for some universal constant $C \ge 1$ *and every* $z, z' \in B_{\ell_2^n}$.

Proof. As a consequence of Khintchine inequalities (see e.g. [10, Section 8.5]), the ψ_2-norm of a Rademacher process is comparable to its L_2-norm. Now,

$$\|Y_z - Y_{z'}\|_{L_2} = \frac{1}{k} \left(\int_\Omega \left| \sum_{J \in \mathcal{S}} \varepsilon_J(\omega)(z_J - z'_J) \right|^2 d\mathbb{P}(\omega) \right)^{\frac{1}{2}} = \frac{1}{k} \left(\sum_{J \in \mathcal{S}} |z_J - z'_J|^2 \right)^{\frac{1}{2}}$$

$$= \frac{1}{k} \left(\sum_{J \in \mathcal{S}} \left| \sum_{u=1}^{k} z_{j_1} \cdots z_{j_{u-1}} (z_{j_u} - z'_{j_u}) z'_{j_{u+1}} \cdots z'_{j_k} \right|^2 \right)^{\frac{1}{2}}$$

$$\le \frac{1}{k} \sum_{u=1}^{k} \left(\sum_{J \in \mathcal{S}} |z_{j_1} \cdots z_{j_{u-1}} (z_{j_u} - z'_{j_u}) z'_{j_{u+1}} \cdots z'_{j_k}|^2 \right)^{\frac{1}{2}}$$

$$\le \frac{1}{k} \sum_{u=1}^{k} \|z - z'\|_\infty \left(\sum_{J \in \mathcal{S}} |z_{j_1} \cdots z_{j_{u-1}} z'_{j_{u+1}} \cdots z'_{j_k}|^2 \right)^{\frac{1}{2}}.$$

Since $\mathcal{S}$ is an $S_p(k-1, k, n)$ partial Steiner system, given $j_1, \ldots, j_{u-1}, j_{u+1}, \ldots, j_k$ for a fixed u, there is at most one index j_u such that $(j_1, \ldots, j_k)$ belongs to $\mathcal{S}$. Therefore the sum $\sum_{J \in \mathcal{S}} |z_{j_1} \cdots z_{j_{u-1}} z'_{j_{u+1}} \cdots z'_{j_k}|^2$ can be bounded by

$$\left(\sum_{l_1=1}^{n} |z_{l_1}|^2 \right) \cdots \left(\sum_{l_{u-1}=1}^{n} |z_{l_{u-1}}|^2 \right) \left(\sum_{l_{u+1}=1}^{n} |z'_{l_{u+1}}|^2 \right) \cdots \left(\sum_{l_k=1}^{n} |z'_{l_k}|^2 \right),$$

and this is less than or equal to one (since $z, z' \in B_{\ell_2^n}$). This combined with the previous inequality concludes the proof.   □

We are now in a position to use Theorem 2.2 with L_{ψ_2}, $X = B_{\ell_2^n}$ and $d = \|\cdot\|_\infty$ to bound the expectation of the supremum. For this, we are going to estimate the entropy integral $J_{\psi_2}(B_{\ell_2^n}, \|\cdot\|_\infty)$. Note that $\psi_2^{-1}(t) = \log^{\frac{1}{2}}(t+1)$; we use instead $\log^{\frac{1}{2}}(t)$, which does not change the computation of the integral. We estimate the integral in the following result, which is a version of [3, Lemma 2.1]; the proof is essentially the same and we include it here for the sake of completeness.

Lemma 2.4. *There exists a constant $C > 0$ such that for every $n \geq 2$ we have*

$$J_{\psi_2}(B_{\ell_2^n}, \|\cdot\|_\infty) \leq C \log^{\frac{3}{2}}(n).$$

Proof. We fix n and for each m we consider the number

$$e_m = \inf\left\{\sigma > 0 : B_{\ell_2^n} \subset \bigcup_{i=1}^{2^m} x_i + \sigma B_{\ell_\infty^n}\right\}.$$

By result of Schütt [27, Theorem 1] there exists a constant K, independent of n and m, such that

$$(2.5) \qquad e_m \leq K \times \begin{cases} 1 & \text{if } m \leq \log(n), \\ \left(\dfrac{\log(1 + \frac{2n}{m})}{m}\right)^{\frac{1}{2}} & \text{if } \log(n) \leq m \leq 2n, \\ 2^{-\frac{m}{2n}} n^{-\frac{1}{2}} & \text{if } m \geq 2n. \end{cases}$$

Let us note that Schütt's result is stated for real spaces. Since the $(2n)$-dimensional real euclidean space is isometrically isomorphic to ℓ_2^n, we get (2.5).

For $m \geq 2n$, if $K2^{-\frac{m+1}{2n}} n^{-\frac{1}{2}} \leq \varepsilon < K2^{-\frac{m}{2n}} n^{-\frac{1}{2}}$, then by (2.5) we have

$$N(B_{\ell_2^n}, \|\cdot\|_\infty; \varepsilon) \leq 2^{m+1} \leq 2\frac{K^{2n}}{\varepsilon^{2n} n^n}$$

and

$$\int_0^{\frac{K}{2\sqrt{n}}} \log^{\frac{1}{2}}\left(N(B_{\ell_2^n}, \|\cdot\|_\infty; \varepsilon)\right) d\varepsilon \leq \int_0^{\frac{K}{2\sqrt{n}}} n^{\frac{1}{2}} \log^{\frac{1}{2}}\left(\frac{2K^2}{\varepsilon^2 n}\right) d\varepsilon$$

$$= \int_0^{\frac{K}{2}} \log^{\frac{1}{2}}\left(\frac{2C^2}{u^2}\right) du = K_1 < \infty.$$

With the same argument, if $\frac{K}{2\sqrt{n}} \leq \varepsilon K \sqrt{\frac{\log 2}{2n}}$, then

$$N(B_{\ell_2^n}, \|\cdot\|_\infty; \varepsilon) \leq 2^{2n}$$

and with this we can bound the integral from $\frac{K}{2\sqrt{n}}$ to $K\sqrt{\frac{\log 2}{2n}}$ by some K_2.

We define now

$$\varepsilon_m = \left(\frac{\log\left(1 + \frac{2n}{m}\right)}{m}\right)^{\frac{1}{2}} \quad \text{for } [\log n] \leq m < 2n.$$

Again by (2.5), if $\varepsilon_{m+1} \leq \varepsilon < \varepsilon_m$, then $N(B_{\ell_2^n}, \|\cdot\|_\infty; \varepsilon) \leq 2^{m+1}$. Then

$$\int_{\varepsilon_{2n}}^{\varepsilon_{[\log(n)]}} \log^{\frac{1}{2}}\big(N(B_{\ell_2^n}, \|\cdot\|_\infty; \varepsilon)\big) \, d\varepsilon \leq \sum_{m=[\log(n)]}^{2n-1} (m+1)^{\frac{1}{2}} (\varepsilon_m - \varepsilon_{m+1}) \log^{\frac{1}{2}}(2).$$

We write

$$(\varepsilon_m - \varepsilon_{m+1}) = K\left[\frac{\log^{\frac{1}{2}}(1 + \frac{2n}{m})}{m^{\frac{1}{2}}} - \frac{\log^{\frac{1}{2}}(1 + \frac{2n}{m})}{(m+1)^{\frac{1}{2}}} + \frac{\log^{\frac{1}{2}}(1 + \frac{2n}{m})}{(m+1)^{\frac{1}{2}}} - \frac{\log^{\frac{1}{2}}(1 + \frac{2n}{m+1})}{(m+1)^{\frac{1}{2}}}\right]$$

and we get

$$\int_{\varepsilon_{2n}}^{\varepsilon_{[\log(n)]}} \log^{\frac{1}{2}}\big(N(B_{\ell_2^n}, \|\cdot\|_\infty; \varepsilon)\big) \, d\varepsilon$$

$$\leq K\left(\log^{\frac{1}{2}}(n) \sum_{s=[\log(n)]}^{2s-1} \frac{(s+1)^{\frac{1}{2}}}{s^{\frac{3}{2}}} + \sum_{s=[\log(n)]}^{2s-1} \log^{\frac{1}{2}}\left(1 + \frac{2n}{s}\right) - \log^{\frac{1}{2}}\left(1 + \frac{2n}{s+1}\right)\right)$$

$$\leq K_3 \log^{\frac{3}{2}}(n).$$

Finally, for the remaining subinterval we have that, by (2.5), if $\varepsilon \geq \varepsilon_{[\log(n)]}$, then

$$N(B_{\ell_2^n}, \|\cdot\|_\infty; \varepsilon) \leq 2^{\log(n)}.$$

Hence

$$\int_{\varepsilon_{[\log(n)]}}^{1} \log^{\frac{1}{2}}\big(N(B_{\ell_2^n}, \|\cdot\|_\infty; \varepsilon)\big) \, d\varepsilon \leq K_4 \int_0^1 \log^{\frac{1}{2}}(n) \, d\varepsilon \leq K_4 \log^{\frac{1}{2}}(n).$$

This completes the proof.   □

We can now find Steiner unimodular polynomials that have small norm in $\mathcal{P}(^k\ell_q^n)$, for every $2 \leq q \leq \infty$ simultaneously.

Theorem 2.5. *Let $k \geq 2$ and $\mathcal{S}$ an $S_p(k-1, k, n)$ partial Steiner system. Then there exist signs $(c_J)_{J\in\mathcal{S}}$ and a constant $A_{k,q} > 0$ independent of n such that the k-homogeneous polynomial $p = \sum_{J\in\mathcal{S}} c_J z_J$ satisfies*

$$\|p\|_{\mathcal{P}(^k\ell_q^n)} \leq A_{k,q} \cdot \begin{cases} \log^{\frac{3}{q}}(n) n^{\frac{k}{2}\left(\frac{q-2}{q}\right)} & \text{for } 2 \leq q < \infty, \\ \log^{\frac{3q-3}{q}}(n) & \text{for } 1 \leq q \leq 2. \end{cases}$$

Moreover, the constant $A_{k,q}$ may be taken independent of k for $q \neq 2$.

Proof. To prove this theorem we will first find a polynomial with small norm both in $\mathcal{P}(^k\ell_2^n)$ and in $\mathcal{P}(^k\ell_\infty^n)$. For this we use an interesting technique borrowed from the proof of [8, Lemma 2.1], followed by an interpolation argument.

Note first that any $S_p(k-1, k, n)$ partial Steiner system $\mathcal{S}$ satisfies $|\mathcal{S}| \leq \frac{1}{k}\binom{n}{k-1}$. We use $\mathcal{S}$ to define a Rademacher process $(Y_z)_{z\in B_{\ell_2^n}}$ as in (2.4). By Lemma 2.3, Theorem 2.2 and Lemma 2.4 there is a constant $K > 0$ such that

$$\mathbb{E}\left(\sup_{z\in B_{\ell_2^n}} |Y_z|\right) \leq K \log^{\frac{3}{2}}(n).$$

Therefore, by Markov's inequality we have

$$(2.6) \qquad \mathbb{P}\left\{\omega \in \Omega : \left\|\sum_{J \in \mathcal{S}} \varepsilon_J(\omega) z_J\right\|_{\mathcal{P}(^k \ell_2^n)} \geq MkK \log^{\frac{3}{2}}(n)\right\} \leq \frac{1}{M},$$

where M is some constant to be determined. On the other hand, recall that by (2.2) we have

$$\mathbb{P}\left\{\omega \in \Omega : \left\|\sum_{J \in \mathcal{S}} \varepsilon_J(\omega) z_J\right\|_{\mathcal{P}(^k \ell_\infty^n)} \geq D(n \log(k)|\mathcal{S}|)^{\frac{1}{2}}\right\} \leq \frac{1}{k^2 e^n},$$

Therefore, if $M > 1 + \frac{1}{k^2 e^n - 1}$ (note that we can take $M = 2$ here), we have the following inequalities for ω in a positive measure set:

$$(2.7) \qquad \begin{cases} \left\|\displaystyle\sum_{J \in \mathcal{S}} \varepsilon_J(\omega) z_J\right\|_{\mathcal{P}(^k \ell_2^n)} \leq MkK, \log^{\frac{3}{2}}(n), \\[4mm] \left\|\displaystyle\sum_{J \in \mathcal{S}} \varepsilon_J(\omega) z_J\right\|_{\mathcal{P}(^k \ell_\infty^n)} \leq D(n \log(k)|\mathcal{S}|)^{\frac{1}{2}} \leq D\left(\frac{\log(k)}{k}\binom{n}{k-1}n\right)^{\frac{1}{2}} \\[5mm] \qquad\qquad\qquad\qquad\qquad \leq D\left(\frac{\log(k)}{k!}n^k\right)^{\frac{1}{2}}. \end{cases}$$

There is a choice of signs $(c_J)_{J \in \mathcal{S}}$ such that the polynomial $p(z) := \sum_{J \in \mathcal{S}} c_J z_J$ satisfies the inequalities in (2.7). We now use an interpolation argument to obtain a bound of the norm of p in $\mathcal{P}(^k \ell_q^n)$ for $2 < q < \infty$. We consider the k-linear form associated to p; then [4, Theorem 4.4.1], together with (2.1) and (2.7), give

$$\|p\|_{\mathcal{P}(^k \ell_q^n)} \leq (MkK)^{2/q}\left(D\lambda(k,\infty)\frac{\log^{\frac{1}{2}}(k)}{\sqrt{k!}}\right)^{\frac{q-2}{q}} \log^{\frac{3}{q}}(n) n^{\frac{k}{2}\left(\frac{q-2}{q}\right)}$$

$$\leq \max\{MK, D\}\underbrace{\left(\frac{k^{\frac{k}{2}}(k+1)^{\frac{k+1}{2}}\sqrt{\log k}}{2^k k!\sqrt{k!}}\right)^{\frac{q-2}{q}}}_{A_{k,q}} k^{\frac{2}{q}} \log^{\frac{3}{q}}(n) n^{\frac{k}{2}\left(\frac{q-2}{q}\right)}.$$

Note that for $q > 2$, $A_{k,q} \to 0$ as $k \to \infty$, and thus we may take a constant independent of k in this case.

For $q = 1$, it is immediately seen that every Steiner unimodular polynomial has norm less than or equal to one. Actually, more can be said. Let $P(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$ be any k-homogeneous polynomial. Then

$$(2.8) \qquad |P(z)| \leq \sum_{|\alpha|=k} |a_\alpha z^\alpha|$$

$$\leq \sup_{|\alpha|=k}\left\{|a_\alpha|\frac{\alpha!}{k!}\right\} \sum_{|\alpha|=k}\left|\frac{k!}{\alpha!}z^\alpha\right| = \sup_{|\alpha|=k}\left\{|a_\alpha|\frac{\alpha!}{k!}\right\}\left(\sum_{j=1}^n |z_j|\right)^k.$$

In particular, the polynomial p considered above satisfies

$$\|p\|_{\mathcal{P}(^k \ell_1^n)} \leq \frac{1}{k!}.$$

Finally, proceeding by interpolation between the ℓ_1^n and ℓ_2^n cases we obtain that for $1 < q < 2$,

$$\|p\|_{\mathscr{P}(^k\ell_q^n)} \leq \left(\frac{k^k}{(k!)^2} \right)^{\frac{2-q}{q}} \left(MkK \log^{\frac{3}{2}}(n) \right)^{\frac{2q-2}{q}} = A_{k,q} \log^{\frac{3q-3}{q}}(n).$$

Note that also in this case, for every $1 \leq q < 2$ we have $A_{k,q} \to 0$ as $k \to \infty$.   □

As was already noted in [12, Corollary 6.5], the argument in (2.8) improves the estimates given in [7] and [3, Corollary 3.2] for the $q = 1$ case.

Remark 2.6. It is not difficult to prove that every 2-homogeneous Steiner unimodular polynomial has norm in $\mathscr{P}(^2\ell_2^n)$ less than or equal to $\frac{1}{2}$. It would be interesting to know if there exists a constant C, perhaps depending on $k \geq 3$ and not on n, such that given any $S_p(k-1,k,n)$ partial Steiner system $\mathcal{S}$, we can find a k-homogeneous unimodular polynomial $p(z) := \sum_{J \in \mathcal{S}} c_J z_J$ with $\|p\|_{\mathscr{P}(^k\ell_2^n)} \leq C$. An affirmative answer to this question would in particular give that the upper bound (by Mantero–Tonge) in (1.6) for $C_{k,q}(n)$ with $2 \leq q < \infty$ is actually optimal.

The last ingredient we need for our applications is the existence of nearly optimal partial Steiner systems, in the sense that they have many elements. This translates to many unimodular coefficients of the Steiner polynomials. It is well known that any partial Steiner system $S_p(t,k,n)$ has cardinality less than or equal to $\binom{n}{t}/\binom{k}{t}$. A conjecture of Erdős and Hanani [16], proved positively by Rödl [26], states that there exist partial Steiner systems $S_p(t,k,n)$ of cardinality at least $(1-o(1))\binom{n}{t}/\binom{k}{t}$, where $o(1)$ tends to zero as n goes to infinity. This bound was improved in [1] (see also [19] for a panoramic overview of the subject), where it is proved that there exists a constant $c > 0$ such that there exist partial Steiner systems $S_p(k-1,k,n)$ of cardinality at least

$$(2.9) \qquad \begin{cases} \dfrac{\binom{n}{k-1}}{k}\left(1 - \dfrac{c}{n^{\frac{1}{k-1}}}\right) & \text{for } k > 3, \\[4mm] \dfrac{\binom{n}{k-1}}{k}\left(1 - \dfrac{c\log^{\frac{3}{2}} n}{n^{\frac{1}{k-1}}}\right) & \text{for } k = 3. \end{cases}$$

Taking partial Steiner systems of this cardinality in Theorem 2.5 we have the following.

Corollary 2.7. *Let $k \geq 3$. Then there exists a k-homogeneous Steiner unimodular polynomial p of n complex variables with at least $\psi(k,n)$ (defined in (2.9)) coefficients satisfying the estimates in Theorem 2.5. Note that in this case $\psi(k,n) \gg n^{k-1}$.*

Remark 2.8. Very recently, a longstanding open problem in combinatorial design theory was solved by Keevash [18]. A *Steiner system* $S(t,k,n)$ is a collection of subsets of size k of $\{1,\ldots,n\}$ such that every subset of t elements is contained in exactly one member of the collection of subsets of size k. Keevash's result implies the asymptotic existence of Steiner systems, that is, that given $t < k$, Steiner systems $S(t,k,n)$ exist for every sufficiently large n that satisfies some natural divisibility conditions. In particular, for an infinite number of n's we may take

$$\psi(k,n) = \frac{\binom{n}{k-1}}{k}$$

in the above corollary.

3. Estimates on the multivariable von Neumann inequality

In this section we estimate the asymptotic failure of different versions of the multivariable von Neumann inequality for homogeneous polynomials. Before we prove Theorem 1.1, let us observe that we modify Dixon's original proof of the lower bound in (1.3) in several ways.

Dixon considered partial Steiner systems $S_p([\frac{k-1}{2}], k, n)$, for which the number of non-zero coefficients is of the order $n^{[\frac{k-1}{2}]}$. This is not enough to find a good lower bound. Instead, we use partial Steiner systems $S_p(k - 1, k, n)$. This allows us to have more nonzero coefficients, but also forces us to make a new construction of the Hilbert space and the operators which we feel is closer to that given by Varopoulos in [29].

Proof of Theorem 1.1 (i).   The upper bound was proved in [15, Theorem 1.2]. Thus we only have to construct a polynomial, a Hilbert space and commuting contractions that show that the asymptotic growth of this bound is optimal.

Let $n \geq k \geq 3$ and choose a partial Steiner system $S_p(k - 1, k, n)$, denoted by $\mathcal{S}$, such that $|\mathcal{S}| = \psi(k, n)$ as in (2.9). By Theorem 2.5, see also (2.7), there exists a k-homogeneous polynomial $p(z) = \sum_{J \in \mathcal{S}} c_J z_J$ with $c_J = \pm 1$ for every $J \in \mathcal{S}$ and such that

$$(3.1) \qquad \|p\|_{\mathcal{P}(^k \ell_\infty^n)} \leq D \left(\frac{\log(k)}{k} \binom{n}{k-1} n \right)^{\frac{1}{2}}.$$

Let $\mathcal{H}$ be the (finite-dimensional) Hilbert space which has as orthonormal basis the following vectors:

$$\begin{cases} e, \\ e(j_1, \ldots, j_m) & \text{for } 0 \leq m \leq k - 2 \text{ and } 1 \leq j_1 \leq \cdots \leq j_m \leq n, \\ f_i & \text{for } i = 1, \ldots, n, \\ g. \end{cases}$$

Given any subset $\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$, we denote by $[i_1, \ldots, i_r]$ its nondecreasing reordering. We define, for $l = 1, \ldots, n$, the operators that act as follows on the basis of $\mathcal{H}$,

$$T_l e = e(l),$$
$$T_l e(j_1, \ldots, j_m) = e[l, j_1, \ldots, j_m] \quad \text{if } 0 \leq m < k - 2,$$
$$T_l e(j_1, \ldots, j_{k-2}) = \sum_i \gamma_{\{i, l, j_1, \ldots, j_{k-2}\}} f_i,$$
$$T_l f_i = \delta_{li} g,$$
$$T_l g = 0,$$

where

$$\gamma_{\{i_1, \ldots, i_k\}} = \begin{cases} c_{\{i_1, \ldots, i_k\}} & \text{if } \{i_1, \ldots, i_k\} \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{S}$ is an $S_p(k - 1, k, n)$ partial Steiner system, we have $\|T_l\| = 1$ for $l = 1, \ldots, n$. It is easily checked that the operators commute. We have

$$p(T_1, \ldots, T_n)e = \sum_{\{i_1, \ldots, i_k\} \in \mathcal{S}} c_{\{i_1, \ldots, i_k\}} T_{i_1} T_{i_2} \ldots T_{i_k} e = \sum_{\{i_1, \ldots, i_k\} \in \mathcal{S}} c_{\{i_1, \ldots, i_k\}}^2 g$$
$$= |\mathcal{S}|g = \psi(k, n)g.$$

Now, using (3.1) we get

$$
\begin{aligned}
\| p(T_1, \ldots, T_n) \|_{\mathcal{L}(\mathcal{H})} &\geq \| p(T_1, \ldots, T_n)e \|_{\mathcal{H}} \\
&= \psi(k, n) \\
&\geq \frac{1}{D} \left(\frac{\binom{n}{k-1}}{nk \log(k)} \right)^{\frac{1}{2}} (1 - o(1)) \| p \|_{\mathcal{P}(^k \ell_\infty^n)} \\
&\gg n^{\frac{k-2}{2}} \| p \|_{\mathcal{P}(^k \ell_\infty^n)}.
\end{aligned}
$$

This gives the desired conclusion.   □

Proof of Theorem 1.1 (ii).   The upper bound was proved in [21, Corollary 11]. For the lower bound, we take the Hilbert space and the operators $T_1, \ldots, T_n$ defined in the proof of Theorem 1.1 (i). Then

$$
R_j = \frac{T_j}{n^{\frac{1}{q}}} \quad \text{for } j = 1, \ldots, n
$$

clearly satisfy $\sum_{i=1}^{n} \| R_i \|^q \leq 1$. Taking the polynomial p given by Theorem 2.5 we have

$$
\begin{aligned}
\| p(R_1, \ldots, R_n) \|_{\mathcal{L}(\mathcal{H})} &\geq \frac{1}{n^{\frac{k}{q}}} \| p(T_1, \ldots, T_n)e \|_{\mathcal{H}} \\
&= \frac{|\mathcal{S}|}{n^{\frac{k}{q}}} \\
&\geq \frac{\| p \|_{\mathcal{P}(^k \ell_q^n)} |\mathcal{S}|}{A_{k,q} \log^{\frac{3}{q}}(n) n^{\frac{k}{2}\left(\frac{q-2}{q}\right)} n^{k/q}} \\
&\geq A_{k,q}^{-1} C_k \log^{-\frac{3}{q}}(n) n^{\frac{k-2}{2}} \| p \|_{\mathcal{P}(^k \ell_2^n)}.
\end{aligned}
$$

This concludes the proof of the theorem.   □

3.1. Other possible extensions of the von Neumann inequality for homogeneous polynomials: Some particular cases.   Mantero and Tonge [21, Proposition 17] also obtained lower bounds for $C_{k,q,r}(n)$, defined as the least constant C such that

$$
(3.2) \qquad \| p(T_1, \ldots, T_n) \|_{\mathcal{L}(\mathcal{H})} \leq C \sup \left\{ |p(z_1, \ldots, z_n)| : \sum_{j=1}^{n} |z_j|^q \leq 1 \right\},
$$

for every k-homogeneous polynomial p in n variables and every n-tuple of commuting contractions $(T_1, \ldots, T_n)$ with $\sum_{i=1}^{n} \| T_i \|_{\mathcal{L}(\mathcal{H})}^r \leq 1$. Proceeding as in the proof of Theorem 1.1 (ii), we can show the following.

Proposition 3.1.   *Let $k \geq 3$. Then the following hold:*
(i) $\log^{-\frac{3}{q}}(n) n^{k(\frac{1}{2}+\frac{1}{q}-\frac{1}{r})-1} \ll C_{k,q,r}(n)$, *for $q \geq 2$ and $1 \leq r \leq \infty$,*
(ii) $\log^{-\frac{3}{q'}}(n) n^{\frac{k}{r'}-1} \ll C_{k,q,r}(n)$, *for $q \leq 2$ and $1 \leq r \leq \infty$.*

Remark 3.2.   The above proposition improves the lower bounds for $C_{k,q,r}$ given in [21, Proposition 17] in all cases but $q \leq 2$ and $k = 3$.

Another possible multivariable extension of the von Neumann inequality (also studied in [21]) is by considering polynomials on commuting operators $T_1, \ldots, T_n$ satisfying that for any pair h, g of norm one vectors in the Hilbert space,

$$(3.3) \qquad \sum_{j=1}^{n} |\langle T_j h, g \rangle|^q \leq 1,$$

or, equivalently, that for any vector $\alpha \in \mathbb{C}^n$ such that $\|\alpha\|_{\ell_{q'}^n} = 1$, we have

$$\left\| \sum_{j=1}^{n} \alpha_j T_j \right\| \leq 1.$$

Let $D_{k,q}(n)$ denote the smallest constant such that

$$(3.4) \qquad \|p(T_1, \ldots, T_n)\|_{\mathcal{L}(\mathcal{H})} \leq D_{k,q}(n) \sup \left\{ |p(z_1, \ldots, z_n)| : \sum_{j=1}^{n} |z_j|^q \leq 1 \right\},$$

for every k-homogeneous polynomial p in n variables and every n-tuple of commuting contractions $(T_1, \ldots, T_n)$ satisfying (3.3). The upper bound obtained in [21, Proposition 20] is

$$D_{k,q}(n) \ll \begin{cases} n^{(k-1)\left(\frac{1}{2} + \frac{1}{q}\right)} & \text{for } q \geq 2, \\ n^{(k-1)\left(\frac{1}{2} + \frac{1}{q'}\right)} & \text{for } q \leq 2. \end{cases}$$

For $k = 3$ and $q = 2$ we show that this is optimal up to a logarithmic factor.

Proposition 3.3. *We have the following asymptotic behavior:*

$$\frac{n^2}{\log^{\frac{15}{4}} n} \ll D_{3,2}(n) \ll n^2.$$

Proof. Let $p(z) = \sum_{J \in \mathcal{S}} c_J z_J$ be a 3-homogeneous Steiner unimodular polynomial as in Theorem 2.5 and let $T_1, \ldots, T_n$ be the operators defined in the proof of Theorem 1.1 (i). We prove first that

$$\frac{T_1}{\|p\|_{\mathcal{P}(^3\ell_2^n)}^{\frac{1}{2}}}, \ldots, \frac{T_n}{\|p\|_{\mathcal{P}(^3\ell_2^n)}^{\frac{1}{2}}}$$

satisfy (3.3). Note that these operators are defined on a $(2n+2)$-dimensional Hilbert space $\mathcal{H}$ with orthonormal basis $\{e, e_1, \ldots, e_n, f_1, \ldots, f_n, g\}$.

For $\alpha \in \ell_2^n$ and $h \in \mathcal{H}$, (below we take some β in the unit ball of ℓ_2^n)

$$\left\| \sum_j \alpha_j T_j h \right\|^2 = \sum_j |\alpha_j \langle h, e \rangle|^2 + \sum_i \left| \sum_{j,l} \alpha_j \langle h, e_l \rangle a_{\{i,j,l\}} \right|^2 + \left| \sum_j \alpha_j \langle h, f_j \rangle \right|^2$$

$$= |\langle h, e \rangle|^2 \|\alpha\|_{\ell_2^n}^2 + \left(\sum_i \beta_i \sum_{j,l} \alpha_j \langle h, e_l \rangle a_{\{i,j,l\}} \right)^2 + \left| \sum_j \alpha_j \langle h, f_j \rangle \right|^2$$

$$\leq |\langle h, e \rangle|^2 \|\alpha\|_{\ell_2^n}^2 + \|p\|_{\mathcal{P}(^3\ell_2^n)} \|\alpha\|_{\ell_2^n}^2 \|(\langle h, e_l \rangle)_l\|_{\ell_2^n}^2 + \|\alpha\|_{\ell_2^n}^2 \|(\langle h, f_j \rangle)_j\|_{\ell_2^n}^2$$

$$\leq \|p\|_{\mathcal{P}(^3\ell_2^n)} \|\alpha\|_{\ell_2^n}^2 \|h\|_{\mathcal{H}}^2.$$

Therefore,

$$\left\| p\left(\frac{T_1}{\|p\|_{\mathscr{P}(^3\ell_2^n)}^{\frac{1}{2}}}, \dots, \frac{T_n}{\|p\|_{\mathscr{P}(^3\ell_2^n)}^{\frac{1}{2}}} \right) \right\|_{\mathcal{L}(\mathcal{H})} \geq \|p\|_{\mathscr{P}(^3\ell_2^n)}^{-\frac{3}{2}} \|p(T_1, \dots, T_n)e\|_{\mathcal{H}}$$

$$= \|p\|_{\mathscr{P}(^3\ell_2^n)}^{-\frac{3}{2}} |\mathcal{S}|$$

$$\gg \|p\|_{\mathscr{P}(^3\ell_2^n)} \frac{n^2}{\log^{\frac{15}{4}} n},$$

and this concludes the proof.   □

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