FINITE INTERSECTION PROPERTY AND DYNAMICAL COMPACTNESS

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This paper is dedicated to Professor Ethan Akin on the occasion of his 70th birthday.

ABSTRACT. Dynamical compactness with respect to a family as a new concept of chaoticity of a dynamical system was introduced and discussed in [23]. In this paper we continue to investigate this notion. In particular, we prove that all dynamical systems are dynamically compact with respect to a Furstenberg family if and only if this family has the finite intersection property. We investigate weak mixing and weak disjointness by using the concept of dynamical compactness. We also explore further difference between transitive compactness and weak mixing. As a byproduct, we show that the $\omega_{\mathcal{F}}$-limit and the ω-limit sets of a point may have quite different topological structure. Moreover, the equivalence between multi-sensitivity, sensitive compactness and transitive sensitivity is established for a minimal system. Finally, these notions are also explored in the context of linear dynamics.

1. INTRODUCTION

By a *(topological) dynamical system* (X, T) we mean a compact metric space X with a metric d and a continuous self-surjection T of X. We say it is *trivial* if the space is a singleton. *Throughout this paper, we are only interested in a nontrivial dynamical system, where the state space is a compact metric space without isolated points.*

This paper is a continuation of the research carried out in [23], where the authors discussed a dynamical property called dynamical compactness and examined it firstly for transitive compactness. Some results of this paper can be considered as a contribution to an area of dynamical systems called *dynamical topology*, in which the topological properties of maps can be described in dynamical terms.

Let $\mathbb{Z}_+$ be the set of all nonnegative integers and $\mathbb{N}$ the set of all positive integers. Before going on, let us recall the notion of a Furstenberg family from [1]. Denote by $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$ the set of all subsets of $\mathbb{Z}_+$. A subset $\mathcal{F} \subset \mathcal{P}$ is a *(Furstenberg) family*, if it is *hereditary upward*, that is, $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. Any subset $\mathcal{A}$ of $\mathcal{P}$ clearly generates a family $\{F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A}\}$. Denote by $\mathcal{B}$ the family of all infinite subsets of $\mathbb{Z}_+$, and by $\mathcal{P}_+$ the family of all nonempty subsets of $\mathbb{Z}_+$. For a family $\mathcal{F}$, the *dual family* of $\mathcal{F}$, denoted by $k\mathcal{F}$, is defined as

$$\{F \in \mathcal{P} : F \cap F' \neq \varnothing \text{ for any } F' \in \mathcal{F}\}.$$

A family $\mathcal{F}$ is *proper* if it is a proper subset of $\mathcal{P}$, that is, $\mathbb{Z}_+ \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$. By a *filter* $\mathcal{F}$ we mean a proper family closed under intersection, that is, $F_1, F_2 \in \mathcal{F}$ implies $F_1 \cap F_2 \in \mathcal{F}$. A filter is *free* if the intersection of all its elements is empty. We extend this concept, a family $\mathcal{F}$ is called *free* if the intersection of all elements of $\mathcal{F}$ is empty.

For any $F \in \mathcal{P}$, every point $x \in X$ and each subset $G \subset X$, we define $T^F x = \{T^i x : i \in F\}$, $N_T(x, G) = \{n \in \mathbb{Z}_+ : T^n x \in G\}$. The ω-*limit set of x with respect to $\mathcal{F}$* (see [1]), or shortly the $\omega_{\mathcal{F}}$-*limit set of x*, denoted by $\omega_{\mathcal{F}}(x)$[1], is defined as

$$\bigcap_{F \in \mathcal{F}} \overline{T^F x} = \{z \in X : N_T(x, G) \in k\mathcal{F} \text{ for every neighborhood } G \text{ of } z\}.$$

Let us note that not always $\omega_{\mathcal{F}}(x)$ is a subset of the ω-*limit set* $\omega_T(x)$, which is defined as

$$\bigcap_{n=1}^{\infty} \overline{\{T^k x : k \geq n\}} = \{z \in X : N_T(x, G) \in \mathcal{B} \text{ for every neighborhood } G \text{ of } z\}.$$

For instance, if each element of $\mathcal{F}$ contains 0 then any point $x \in \omega_{\mathcal{F}}(x)$. Nevertheless, if a family $\mathcal{F}$ is free, then $\omega_{\mathcal{F}}(x) \subset \omega_T(x)$ for any point $x \in X$ and if (X, T) has a nonrecurrent point[2], then the converse is true (see Proposition 2.2).

A dynamical system (X, T) is called *compact with respect to $\mathcal{F}$*, or shortly *dynamically compact*, if the $\omega_{\mathcal{F}}$-limit set $\omega_{\mathcal{F}}(x)$ is nonempty for all $x \in X$.

H. Furstenberg started a systematic study of transitive systems in his paper on disjointness in topological dynamics and ergodic theory [14], and the theory was further developed in [16] and [15]. Recall that the system (X, T) is (*topologically*) *transitive* if $N_T(U_1, U_2) = \{n \in \mathbb{Z}_+ : U_1 \cap T^{-n} U_2 \neq \varnothing\}$ $(= \{n \in \mathbb{Z}_+ : T^n U_1 \cap U_2 \neq \varnothing\}) \in \mathcal{P}_+$ for any *opene*[3] subsets $U_1, U_2 \subset X$, equivalently, $N_T(U_1, U_2) \in \mathcal{B}$ for any opene subsets $U_1, U_2 \subset X$.

In [23] the authors consider one of possible dynamical compactness — transitive compactness, and its relations with well-known chaotic properties of dynamical systems. Let $\mathcal{N}_T$ be the set of all subsets of $\mathbb{Z}_+$ containing some $N_T(U, V)$, where U, V are opene subsets of X. A dynamical system (X, T) is called *transitive compact*, if for any point $x \in X$ the $\omega_{\mathcal{N}_T}$-limit set $\omega_{\mathcal{N}_T}(x)$ is nonempty, in other words, for any point $x \in X$ there exists a point $z \in X$ such that

$$N_T(x, G) \cap N_T(U, V) \neq \varnothing$$

for any neighborhood G of z and any opene subsets U, V of X.

Let (X, T) and (Y, S) be two dynamical systems and $k \in \mathbb{N}$. The product system $(X \times Y, T \times S)$ is defined naturally, and denote by $(X^k, T^{(k)})$ the product system of k copies of the system (X, T). Recall that the system (X, T) is *minimal* if it does not admit a nonempty, closed, proper subset K of X with $TK \subset K$, and is *weakly mixing* if the product system $(X^2, T^{(2)})$ is transitive. Any transitive compact system is obviously topologically transitive, and observe that each weakly mixing system is transitive compact ([4]). In fact, as it was shown in [23], each of notions are different in general and equivalent for minimal systems.

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[1]Note that the notation $\omega_{\mathcal{F}}(x)$ used here is different from the one used in [1] (the notation $\omega_{\mathcal{F}}(x)$ used here is in fact $\omega_{k\mathcal{F}}(x)$ introduced in [1]). As this paper is a continuation of the research in [23], in order to avoid any confusion of notation or concept, we will follow the ones used in [23].

[2]A point $x \in X$ is called *recurrent* if $x \in \omega_T(x)$.

[3]Because we so often have to refer to open, nonempty subsets, we will call such subsets *opene*.

Recall a very useful notion of weakly mixing subsets of a system, which was introduced in [9] and further discussed in [33] and [34]. The notion of weakly mixing subsets can be regraded as a local version of weak mixing. Among many very interesting properties let us mention just one of them – positive topological entropy of a dynamical system implies the existence of weakly mixing sets (see [29] for details). A nontrivial closed subset $A \subset X$ is called *weakly mixing* if for every $k \geq 2$ and any opene sets $U_1, \ldots, U_k, V_1, \ldots, V_k$ of X with $U_i \cap A \neq \varnothing$, $V_i \cap A \neq \varnothing$, for any $i = 1, \ldots, k$, one has that $\bigcap_{i=1}^k N_T(U_i \cap A, V_i) \neq \varnothing$. Let A be a weakly mixing subset of X and let $\mathcal{N}_T(A)$ be the set of all subsets of $\mathbb{Z}_+$ containing some $N_T(U \cap A, V)$, where U, V are opene subsets of X intersecting A.

The notion of sensitivity was first used by Ruelle [37], which captures the idea that in a chaotic system a small change in the initial condition can cause a big change in the trajectory. According to the works by Guckenheimer [20], Auslander and Yorke [6] a dynamical system (X, T) is called *sensitive* if there exists $\delta > 0$ such that for every $x \in X$ and every neighborhood U_x of x, there exist $y \in U_x$ and $n \in \mathbb{N}$ with $d(T^n x, T^n y) > \delta$. Such a δ is called a *sensitive constant* of (X, T). Recently in [31] Moothathu initiated a way to measure the sensitivity of a dynamical system, by checking how large is the set of nonnegative integers for which the sensitivity occurs (see also [30]). For a positive δ and a subset $U \subset X$ define

$$S_T(U, \delta) = \{n \in \mathbb{Z}_+ : \text{ there are } x_1, x_2 \in U \text{ such that } d(T^n x_1, T^n x_2) > \delta\}.$$

A dynamical system (X, T) is called *multi-sensitive* if there exists $\delta > 0$ such that $\bigcap_{i=1}^k S_T(U_i, \delta) \neq \varnothing$ for any finite collection of opene $U_1, \ldots, U_k \subset X$. Such a δ is called a *constant of multi-sensitivity* of (X, T).

Recall that a collection A of subsets of a set Y has the *finite intersection property* (FIP) if the intersection of all sets in any finite subcollection of A is nonempty. The FIP is useful in formulating an alternative definition of compactness of a topological space: a topological space is compact if and only if every collection of closed subsets satisfying the FIP has a nonempty intersection itself (see, for instance [13, 26]).

We recall that if (X, T) is weakly mixing then the family $\mathcal{N}_T$ is a filter and hence has FIP. More generally, if A a weakly mixing subset of (X, T) then the family $\mathcal{N}_T(A)$ also has FIP. Denote by $\mathcal{S}_T(\delta)$ the set of all subsets of $\mathbb{Z}_+$ containing $S_T(U, \delta)$ for some $\delta > 0$ and opene $U \subset X$. If (X, T) is a multi-sensitive system with a constant of multi-sensitivity $\delta > 0$ then obviously the family $\mathcal{S}_T(\delta)$ has FIP. Since all of these families are also free, actually they have the *strong finite intersection property* (SFIP), i.e., if the intersection over any finite subcollection of the family is infinite (see Proposition 2.2).

In fact we can say more — the FIP is useful in characterizing the dynamical compactness (see Theorem 3.1).

Theorem FIP. *All dynamical systems are dynamically compact with respect to $\mathcal{F}$ if and only if the family $\mathcal{F}$ has the finite intersection property.*

We also introduce two new stronger versions of sensitivity: sensitive compactness and transitive sensitivity. We will call the system (X, T) *transitively sensitive* if there exists $\delta > 0$ such that $S_T(W, \delta) \cap N_T(U, V) \neq \varnothing$ for any opene subsets U, V, W of X; and *sensitive compact*, if there exists $\delta > 0$ such that for any point $x \in X$ the $\omega_{\mathcal{S}_T(\delta)}$-limit set $\omega_{\mathcal{S}_T(\delta)}(x)$ is nonempty, in other words, for any point $x \in X$ there

exists a point $z \in X$ such that

$$N_T(x, G) \cap S_T(U, \delta) \neq \varnothing$$

for any neighborhood G of z and any opene U of X.

The paper is organized as follows. In Section 2 we recall some basic concepts and properties used in later discussions from topological dynamics. In Section 3 we obtain some general results concerning dynamical compactness. In particular we show that all dynamical systems are dynamically compact with respect to a Furstenberg family if and only if this family has the finite intersection property (Theorem 3.1).

In Section 4 we discuss two stronger versions of sensitivity: transitive sensitivity and sensitive compactness. It was shown that each weakly mixing system is transitively sensitive (Proposition 4.5), and in fact we can characterize transitive sensitivity of a general dynamical system in terms of dynamical compactness (Proposition 4.3). Furthermore, all of the multi-sensitivity, sensitive compactness and transitive sensitivity are equivalent for a minimal system (Theorem 4.1). Even though each minimal transitive compact system is multi-sensitive, there are many minimal multi-sensitive systems which are not transitive compact. We recall that the sensitivity of a dynamical system can be lifted up from a factor to an extension by an almost open factor map between transitive systems by [17, Corollary 1.7]. We prove that the transitive sensitivity can be lifted up to an extension from a factor by an almost one-to-one factor map and that the transitive sensitivity is projected from an extension to the sensitivity of a factor by a weakly almost one-to-one factor map (Lemma 4.4).

In Section 5 we show that dynamical compactness can be used to characterize the weak disjointness of dynamical systems (Theorem 5.2). We also extend the result of Jian Li [28]: weak mixing implies $\mathcal{F}_{ip}$-point transitivity in terms of transitive compactness (Proposition 5.4).

In Section 6 the further difference between weak mixing and transitive compactness is explored. Precisely, there is a totally transitive, non weakly mixing, transitive compact system (Theorem 6.1); and in fact any compact metric space can be realized as the $\omega_{\mathcal{N}_T}$-limit set of a non totally transitive, transitive compact system (X, T) (Theorem 6.4). As a byproduct, we show that the $\omega_{\mathcal{N}_T}$-limit sets and the ω-limit sets have quite different topological structures for a general dynamical compact system (X, T). At the end of this section we add one more chaotical property of transitive compact systems (in additional to already known from [23]): transitive compactness implies Li-Yorke chaos (Proposition 6.6).

In Section 7 we consider the dynamics of linear operators on infinite dimensional spaces in relation to the properties studied in previous sections. In particular, we show the equivalence of the topological weak mixing property with a weak version of transitive compactness (Theorem 7.1). Some results on sensitivity are also obtained.

2. Preliminaries

In this section we recall standard concepts and results used in later discussions.

2.1. **Basic concepts in topological dynamics.** Recall that $x \in X$ is a *fixed point* if $Tx = x$, and an $\mathcal{F}$-*transitive point of* (X,T) [28] if $N_T(x,U) \in \mathcal{F}$ for any opene subset U of X. It is a trivial observation that if a family $\mathcal{F}$ admits an $\mathcal{F}$-transitive dynamical system (X,T) without isolated points, then $\mathcal{F}$ is free. Since $k(k\mathcal{F}) = \mathcal{F}$, it is easy to see that $x \in X$ is an $\mathcal{F}$-transitive point of (X,T) if and only if $\omega_{k\mathcal{F}}(x) = X$. Denote by $\mathrm{Tran}_{\mathcal{F}}(X,T)$ the set of all $\mathcal{F}$-transitive points of (X,T). The system (X,T) is $\mathcal{F}$-*point transitive* if $\mathrm{Tran}_{\mathcal{F}}(X,T) \neq \varnothing$, and is $\mathcal{F}$-*transitive* if $N_T(U,V) \in \mathcal{F}$ for any opene subsets U,V of X. Write $\mathrm{Tran}(X,T) = \mathrm{Tran}_{\mathcal{P}_+}(X,T)$ for short, and we also call the point x *transitive* if $x \in \mathrm{Tran}(X,T)$, equivalently, its *orbit* $\mathrm{orb}_T(x) = \{T^n x : x = 0,1,2,\dots\}$ is dense in X. Since T is surjective, the system (X,T) is transitive if and only if $\mathrm{Tran}(X,T)$ is a dense G_δ subset of X.

In general, a subset A of X is T-*invariant* if $TA = A$, and *positively T-invariant* if $TA \subset A$. If A is a closed, nonempty, T-invariant subset then $(A, T|_A)$ is called the associated *subsystem*. A *minimal subset* of X is a closed, nonempty, T-invariant subset such that the associated subsystem is minimal. Clearly, (X,T) is minimal if and only if $\mathrm{Tran}(X,T) = X$, if and only if it admits no a proper, closed, nonempty, positively T-invariant subset. A point $x \in X$ is called *minimal* if it lies in some minimal subset. In this case, in order to emphasize the underlying system (X,T) we also say that $x \in X$ is a *minimal point of* (X,T). Zorn's Lemma implies that every closed, nonempty, positively T-invariant set contains a minimal set.

A pair of points $x,y \in X$ is called *proximal* if $\liminf_{n\to\infty} d(T^n x, T^n y) = 0$. In this case each of points from the pair is said to be *proximal* to another. Denote by $\mathrm{Prox}_T(X)$ the set of all proximal pairs of points. For each $x \in X$, denote by $\mathrm{Prox}_T(x)$, called the *proximal cell of* x, the set of all points which are proximal to x. Recall that a dynamical system (X,T) is called *proximal* if $\mathrm{Prox}_T(X) = X \times X$. The system (X,T) is proximal if and only if (X,T) has the unique fixed point, which is the only minimal point of (X,T) (e.g. see [4]).

The opposition to the notion of sensitivity is the concept of equicontinuity. Recall that $x \in X$ is an *equicontinuity point* of (X,T) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x,x') < \delta$ implies $d(T^n x, T^n x') < \varepsilon$ for any $n \in \mathbb{Z}_+$. Denote by $\mathrm{Eq}(X,T)$ the set of all equicontinuity points of (X,T). The system (X,T) is called *equicontinuous* if $\mathrm{Eq}(X,T) = X$. Each dynamical system admits a maximal equicontinuous factor. Recall that by a *factor map* $\pi : (X,T) \to (Y,S)$ between dynamical systems (X,T) and (Y,S), we mean that $\pi : X \to Y$ is a continuous surjection with $\pi \circ T = S \circ \pi$. In this case, we call $\pi : (X,T) \to (Y,S)$ an *extension*; and (X,T) an *extension* of (Y,S), (Y,S) a *factor* of (X,T).

2.2. **Basic concepts of Furstenberg families.** In this subsection we recall from [1] basic concepts about Furstenberg families.

Let $F \in \mathcal{P}$. Recall that a subset F is *thick* if it contains arbitrarily long runs of positive integers. Denote by $\mathcal{F}_{\text{thick}}$ the set of all thick subsets of $\mathbb{Z}_+$, and define $\mathcal{F}_{\text{syn}} = k\mathcal{F}_{\text{thick}}$. Each element of $\mathcal{F}_{\text{syn}}$ is said to be *syndetic*, equivalently, F is syndetic if and only if there is $N \in \mathbb{N}$ such that $\{i, i+1, \ldots, i+N\} \cap F \neq \varnothing$ for every $i \in \mathbb{Z}_+$. We say that F is *thickly syndetic* if for every $N \in \mathbb{N}$ the positions where length N runs begin form a syndetic set. Denote by $\mathcal{F}_{\text{cof}}$ the set of all cofinite subsets of $\mathbb{Z}_+$. Note that by the classic result of Gottschalk a point $x \in X$ is minimal if and only if $N_T(x, U) = \{n \in \mathbb{Z}_+ : T^n x \in U\}$ is syndetic for any neighborhood U of x. Hence, for any minimal system (X, T), the subset $N_T(U, V)$ is syndetic for any opene subsets U, V of X.

Recall that a family $\mathcal{F}$ is proper if it is a proper subset of $\mathcal{P}$, that is, $\mathbb{Z}_+ \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$. By a filter $\mathcal{F}$ we mean a proper family closed under intersection, that is, $F_1, F_2 \in \mathcal{F}$ implies $F_1 \cap F_2 \in \mathcal{F}$. For families $\mathcal{F}_1$ and $\mathcal{F}_2$, we define the family $\mathcal{F}_1 \cdot \mathcal{F}_2 := \{F_1 \cap F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$ and call it the *interaction of $\mathcal{F}_1$ and $\mathcal{F}_2$*. Thus we have $\mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{F}_1 \cdot \mathcal{F}_2$; and it is easy to check that $\mathcal{F}$ is a filter if and only if $\mathcal{F} = \mathcal{F} \cdot \mathcal{F}$, and $\mathcal{F}_1 \cdot \mathcal{F}_2$ is proper if and only if $\mathcal{F}_2 \subset k\mathcal{F}_1$.

For each $i \in \mathbb{Z}_+$, we define $g^i : \mathbb{Z}_+ \to \mathbb{Z}_+, j \mapsto i + j$. Let $\mathcal{F}$ be a family. Recall that $\mathcal{F}$ is *positively invariant* if for every $i \in \mathbb{Z}_+$, $F \in \mathcal{F}$ implies $g^i(F) \in \mathcal{F}$; *negatively invariant* if for every $i \in \mathbb{Z}_+$, $F \in \mathcal{F}$ implies $g^{-i}(F) \in \mathcal{F}$, where $g^{-i}(F) = (g^i)^{-1}(F) = \{j - i : j \in F, j \geq i\}$; and *translation invariant* if it is both positively and negatively invariant, equivalently, for every $i \in \mathbb{Z}_+$, $F \in \mathcal{F}$ if and only if $g^{-i}(F) \in \mathcal{F}$.

As $g^{-i}(g^i A) = A$ and $g^i(g^{-i}A) \subset A$ for any $i \in \mathbb{Z}_+$, it is easy to obtain that the family $\mathcal{F}$ is positively (negatively, translation, respectively) invariant if and only if $k\mathcal{F}$ is negatively (positively, translation, respectively) invariant (see for example [1, Proposition 2.5.b]). And then we have:

Proposition 2.1. *Let $x \in X$. Then $T\omega_{\mathcal{F}}(x) \subset \omega_{\mathcal{F}}(Tx)$. Additionally, if $\mathcal{F}$ is negatively (positively, translation, respectively) invariant then $\omega_{\mathcal{F}}(Tx) \subset (\supset, =,$ respectively) $\omega_{\mathcal{F}}(x)$.*

Proof. Since the other items are alternative versions of [1, Proposition 3.6] in our notations, it suffices to prove that if $\mathcal{F}$ is positively invariant then $\omega_{\mathcal{F}}(Tx) \supset \omega_{\mathcal{F}}(x)$.

For each $y \in \omega_{\mathcal{F}}(x)$ take an arbitrary neighborhood U of y, and let $F \in \mathcal{F}$. Then $g^1(F) = \{i + 1 \in \mathbb{Z}_+ : i \in F\} \in \mathcal{F}$ as $\mathcal{F}$ is positively invariant, and hence $N_T(x, U) \cap g^1(F) \neq \varnothing$, thus $\varnothing \neq g^{-1}(N_T(x, U) \cap g^1(F)) = N_T(Tx, U) \cap F$. It follows $y \in \omega_{\mathcal{F}}(Tx)$ from the arbitrariness of U and F, which finishes the proof. $\square$

Proposition 2.2. *Let (X, T) be a dynamical system and let $\mathcal{F}$ be a family.*

(i) *If $\mathcal{F}$ is free, then $\omega_{\mathcal{F}}(x) \subset \omega_T(x)$ for any $x \in X$. Moreover, if (X, T) has a nonrecurrent point, then the converse implication is true.*

(ii) *If $\mathcal{F}$ is free and has FIP then it has SFIP.*

Proof. (i) Suppose $\omega_{\mathcal{F}}(x) \neq \varnothing$ and take a point $y \in \omega_{\mathcal{F}}(x) := \bigcap_{F \in \mathcal{F}} \overline{T^F x}$. Let us show that $y \in \omega_T(x)$. If $\text{orb}_T(x)$ is a finite set (equivalently $T^i x = T^j x$ for some $0 \leq i < j$) then for each $F \in \mathcal{F}$ there exists $n_F \in F$ with $y = T^{n_F} x$. Since $\mathcal{F}$ is free, there exist different elements F and F' of $\mathcal{F}$ such that $n_F < n_{F'}$, which implies $y = T^{n_F + k(n_{F'} - n_F)} x$ for all $k \in \mathbb{Z}_+$ and hence $y \in \omega_T(x)$.

Now assume that $T^i x \neq T^j x$ whenever $0 \leq i \neq j$. Since $\mathcal{F}$ is free, $\bigcap_{F \in \mathcal{F}} T^F x = \varnothing$. Otherwise there is $m \in \mathbb{Z}_+$ such that $T^m x \in T^F x$ for all $F \in \mathcal{F}$, in other words

$m \in \bigcap_{F \in \mathcal{F}} F$, a contradiction. It means that there is $F \in \mathcal{F}$ such that $y \notin T^F x$. So $y \in \overline{T^F x} \setminus T^F x$, and F is infinite. Therefore there exists an infinite sequence $T^{n_1} x, \dots, T^{n_i} x, \dots$, which converges to y. Hence $y \in \omega_T(x)$.

Now, let (X, T) be a dynamical system with a nonrecurrent point $x_0 \in X$, let $\mathcal{F}$ be a family and $\omega_{\mathcal{F}}(x) \subset \omega_T(x)$ for any $x \in X$. Suppose $\mathcal{F}$ is not free. It means there is a $k \in \mathbb{Z}_+$ that lies in each element of $\mathcal{F}$. Then obviously that $T^k(x) \in \omega_{\mathcal{F}}(x)$ for any $x \in X$. Since x_0 is nonrecurrent, $x_0 \notin \omega_T(x_0) \neq X$. It is well known that $\omega_T(x)$ is T-invariant, therefore $\omega_T(y) = \omega_T(x_0)$ for any $y \in \{T^{-i}(x_0) : i \in \mathbb{Z}_+\}$, and $y \notin \omega_T(y)$. Take a point $x_k \in X$ with $T^k(x_k) = x_0$. As we know $x_0 \in \omega_{\mathcal{F}}(x_k)$. But $\omega_{\mathcal{F}}(x_k) \subset \omega_T(x_k) = \omega_T(x_0)$, a contradiction.

(ii) Suppose that for some $F_1, \dots F_k \in \mathcal{F} \bigcap_{i=1}^{k} F_i = \{n_1, n_2, \dots n_m\}$. Since $\mathcal{F}$ is free for each $k \in \mathbb{Z}_+$ there is $G_k \in \mathcal{F}$ such that $k \notin G_k$. Then $\bigcap_{i=1}^{k} F_i \cap \bigcap_{j=1}^{m} G_{n_j} = \varnothing$, contradiction. $\qquad \square$

2.3. **The concept of an almost one-to-one map.** Let $\phi : X \to Y$ be a continuous surjective map from a compact metric space X onto a compact Hausdorff space Y. Recall that ϕ is *almost open* if $\phi(U)$ has a nonempty interior in Y for any opene $U \subset X$. Note that each factor map between minimal systems is almost open [5, Theorem 1.15], in particular, for a minimal system (X, T) the map $T : X \to X$ is almost open [27]. Denote by $Y_0 \subset Y$ the set of all points $y \in Y$ whose fiber is a singleton. Then Y_0 is a G_δ subset of Y, because

$$Y_0 = \{y \in Y : \phi^{-1}(y) \text{ is a singleton}\} = \bigcap_{n \in \mathbb{N}} \left\{y \in Y : \text{diam}(\phi^{-1}(y)) < \frac{1}{n}\right\}$$

and the map $y \mapsto \text{diam}(\phi^{-1}(y))$ is upper semi-continuous. Here, we denote by $\text{diam}(A)$ the diameter of a subset $A \subset X$. Recall that the function $f : Y \to \mathbb{R}_+$ is *upper semi-continuous* if $\limsup_{y \to y_0} f(y) \leq f(y_0)$ for each $y_0 \in Y$. Denote by $X_0 \subset X$ the set of all points $x \in X$ such that the pre-image of $\phi(x)$ is a singleton. Then $X_0 = \pi^{-1}(Y_0)$ is a G_δ subset of X.

We call ϕ *weakly almost one-to-one* if Y_0 is dense in Y, and *almost one-to-one*[4] if X_0 is dense in X. It is not hard to show that: if ϕ is weakly almost one-to-one, then for any $\delta > 0$ and any opene subset U of Y there exists opene $V \subset U$ with $\text{diam}(\phi^{-1}V) < \delta$; and if ϕ is almost one-to-one, then for any opene subset U^* of X there exists an opene subset V^* of Y with $\phi^{-1}V^* \subset U^*$. Clearly almost one-to-one is much stronger than weakly almost one-to-one. For example, let X be the closed unit interval, define $T(x) = 2x$ for $x \in [0, \frac{1}{2}]$ and $T(x) = 1$ for $x \in [\frac{1}{2}, 1]$, and then $T : X \to X$ is clearly not almost one-to-one but weakly almost one-to-one.

For each minimal system (X, T), the map $T : X \to X$ is weakly almost one-to-one [27, Theorem 2.7], and in fact almost one-to-one [24, Proposition 2.3]. The following result characterizes the relationship between weakly almost one-to-one and almost one-to-one, which extends [24, Proposition 2.3].

Proposition 2.3. *Let $\phi : X \to Y$ be a continuous surjective map from a compact metric space X onto a compact Hausdorff space Y. Then ϕ is almost one-to-one if and only if it is almost open and weakly almost one-to-one.*

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[4]Here we use the concept of *almost one-to-one* following [3], and the concept of *almost one-to-one* used in [12, 24, 27] is in fact our weakly almost one-to-one.

Proof. Firstly assume that ϕ is almost one-to-one. Let $U \subset X$ be an arbitrary opene subset. Take $x_0 \in U$ such that the pre-image of $\phi(x_0)$ is a singleton. From this it is easy to see that $\phi(x_0)$ is contained in the interior of $\phi(U)$. This implies that ϕ is almost open. The map ϕ is clearly weakly almost one-to-one.

Now assume that ϕ is almost open and weakly almost one-to-one. Let $U \subset X$ be an arbitrary opene subset. Since ϕ is almost open, $\phi(U)$ has a nonempty interior in Y, and then $\phi^{-1}(y_0)$ is a singleton for some $y_0 \in \phi(U)$, as ϕ is weakly almost one-to-one. This shows $U \cap X_0 \neq \varnothing$, which finishes the proof. $\qquad\square$

As a direct corollary, we have:

Corollary 2.4. *Let* $\phi : X \to Y$ *and* $\pi : Y \to Z$ *be continuous surjective maps between compact metric spaces. Then the composition map* $\pi \circ \phi : X \to Z$ *is almost one-to-one if and only if both* ϕ *and* π *are almost one-to-one.*

Proof. Denote by X_0 (X_1, respectively) the set of all points $x \in X$ such that the pre-image of $(\pi \circ \phi)(x)$ ($\phi(x)$, respectively) is a singleton. Denote by Z_0 (Z_1, respectively) the set of all points $z \in Z$ whose $\pi \circ \phi$-fibers (π-fibers, respectively) are singletons. All of them are G_δ subsets. Moreover, $X_0 = X_1 \cap \phi^{-1}(\pi^{-1}Z_1)$. In fact, $x \in X_0$ if and only if $\{x\} = (\pi \circ \phi)^{-1}(\pi \circ \phi(x)) = \phi^{-1}(\pi^{-1}(\pi(\phi x)))$, if and only if $\pi^{-1}(\pi(\phi x)) = \{\phi(x)\}$ and $\phi^{-1}(\phi x) = \{x\}$, if and only if $\pi(\phi x) \in Z_1$ and $x \in X_1$.

First assume that $\pi \circ \phi$ is almost one-to-one, and then by Proposition 2.3: X_0 is a dense subset of X, Z_0 is a dense subset of Z and the map $\pi \circ \phi$ is almost open. Note that $X_0 \subset X_1$ and $Z_0 \subset Z_1$, we have that X_1 is dense in X and Z_1 is dense in Z. Hence ϕ is almost one-to-one. Furthermore, as the map $\pi \circ \phi$ is almost open, for any opene $V \subset Y$ one has that $\pi(V) = (\pi \circ \phi)(\phi^{-1}V)$ has a nonempty interior in Z, which implies that π is almost one-to-one by Proposition 2.3.

Now assume that both ϕ and π are almost one-to-one. Then X_1 is a dense G_δ subset of X and Z_1 is a dense G_δ subset of Z. Moreover, by Proposition 2.3 both ϕ and π are almost open, and then the continuous surjection $\pi \circ \phi$ is also almost open, which implies that $(\pi \circ \phi)^{-1}(Z_1)$ is also a dense G_δ subset of X. Thus, $X_0 = X_1 \cap \phi^{-1}(\pi^{-1}Z_1)$ is a dense G_δ subset of X, that is, the composition map $\pi \circ \phi : X \to Z$ is almost one-to-one. This finishes the proof. $\qquad\square$

Let $\pi : (X, T) \to (Y, S)$ be a factor map between dynamical systems. If the map $\pi : X \to Y$ is almost one-to-one (weakly almost one-to-one, respectively), then we also call (X, T) an *almost one-to-one extension* (a *weakly almost one-to-one extension*, respectively) *of* (Y, S). The main result of [24] states that a minimal system is either multi-sensitive or a weakly almost one-to-one extension of its maximal equicontinuous factor. This is an analog of the well-known Auslander-Yorke dichotomy theorem: a minimal system is either sensitive or equicontinuous.

2.4. **Symbolic dynamics.** Let A be a nonempty finite set. We call A the *alphabet* and elements of A are *symbols*. The *full (one-sided) A-shift* is defined as

$$\Sigma = \{x = \{x_i\}_{i=0}^\infty : x_i \in A \text{ for all } i \in \mathbb{Z}_+\},$$

where we equip A with the discrete topology and Σ with the product topology, and the *shift map* $\sigma : \Sigma \to \Sigma$ is a continuous surjection given by

$$x = \{x_i\}_{i=0}^\infty \mapsto \sigma x = \{x_{i+1}\}_{i=0}^\infty,$$

that is, $\sigma(x)$ is the sequence obtained by dropping the first symbol of x. Usually we write an element of Σ as $x = \{x_i\}_{i=0}^\infty = x_0 x_1 x_2 x_3 \ldots$

A *block* w over Σ is a finite sequence of symbols and its *length* is the number of its symbols (denoted by $|w|$). An *n-block* stands for a block of length n. In general we are only interested in a block w with $|w| \geq 1$ if without any special statement, and denote by Σ^* the set of all blocks over Σ. The block w is a *subblock* of a block $v = v_1 \ldots v_m$ with $v_1, \ldots, v_m \in A$ if there exists $1 \leq i \leq j \leq m$ with $w = v_i \ldots v_j$. The concatenation of two blocks $u = a_1 \ldots a_k$ and $v = b_1 \ldots b_l$ is the block $uv = a_1 \ldots a_k b_1 \ldots b_l$. We write u^n for the concatenation of $n \geq 1$ copies of a block u and u^∞ for the sequence $uuu \cdots \in \Sigma$. By $x_{[i,j]}$ we denote the block $x_i x_{i+1} \ldots x_j$, where $0 \leq i \leq j$ and $x = \{x_k\}_{k=0}^\infty \in \Sigma$. The subset $X \subset \Sigma$ is called a *subshift* if it is a closed, nonempty, σ-invariant subset of Σ. A *cylinder* of an n-block $w \in \Sigma^*$ in a subshift X is the set $C[w] = \{x \in X : x_{[0,n-1]} = w\}$. The collection of all cylinders forms a basis of the topology of X.

3. Dynamical compactness with respect to an arbitrary family

Recall that a family $\mathcal{F}$ has the finite intersection property (FIP) if the intersection of all sets in any finite subcollection of $\mathcal{F}$ is nonempty. The following theorem shows that the FIP is useful in characterizing the dynamical compactness.

Theorem 3.1. *All dynamical systems are dynamically compact with respect to the family $\mathcal{F}$ if and only if $\mathcal{F}$ has the finite intersection property.*

Proof. Sufficiency. Suppose that $\mathcal{F}$ has FIP. Take arbitrary dynamical system (X, T) and let $x \in X$. Obviously the family $\{\overline{T^F x} : F \in \mathcal{F}\}$ also has FIP, and then by compactness of X the family $\{\overline{T^F x} : F \in \mathcal{F}\}$ has a nonempty intersection itself, i.e., $\omega_\mathcal{F}(x) \neq \varnothing$. Thus (X, T) is dynamically compact with respect to $\mathcal{F}$.

Necessity. Suppose that the family $\mathcal{F}$ has no FIP, and then there is a collection $\{F_1, \ldots, F_k\} \subset \mathcal{F}$ with $\bigcap_{i=1}^k F_i = \varnothing$. Let $A = \{a_1, \ldots, a_k\}$ be an alphabet and let $(X, T) := (\Sigma, \sigma)$ be the full (one-sided) A-shift. We are going to define a point $x \in X$ with $\omega_\mathcal{F}(x) = \varnothing$. Let $x_0 = a_1$. For any $n \geq 1$ there is i with $n \notin F_i$, else the intersection of $F_1, \ldots, F_k$ would be nonempty. Then define $x_n := a_i$. Finally, let $x = x_0 x_1 x_2 x_3 \ldots$ and the construction is finished.

Assume the contrary that we can take $z \in \omega_\mathcal{F}(x)$, and that z begins with $a_i \in A$. Take $G_z = C[a_i]$. As $z \in \omega_\mathcal{F}(x)$ we have $N_T(x, G_z) \cap F_i \neq \varnothing$. But if $n \in N_T(x, G_z)$, then $x_n = a_i$ and so $n \notin F_i$ by the construction, a contradiction. $\qquad\square$

As we have mentioned in Introduction, any filter has FIP; if (X, T) is weakly mixing then the family $\mathcal{N}_T$ is a filter; if A a weakly mixing subset of (X, T) then the family $\mathcal{N}_T(A)$ has FIP; and if (X, T) is a multi-sensitive system with a constant of multi-sensitivity $\delta > 0$ then the family $\mathcal{S}_T(\delta)$ also has FIP.

Let $\mathcal{F}$ have the finite intersection property. Then there exists an ultrafilter $\mathcal{U}$ (in $\mathcal{P}$) such that $\mathcal{F} \subset \mathcal{U}$. This result is known as Ultrafilter Lemma (see details and proof in [21]). Recall that an ultrafilter is maximal among all proper filters. As a consequence of this fact we have a natural open question:

Question A. *Let (X, T) be a dynamically compact system with respect to a family $\mathcal{F}$ and $\mathcal{F}$ has FIP. When is $\mathcal{F}$ a filter, or contains a nontrivial filter?*

Especially we address this question to the family $\mathcal{S}_T(\delta)$. More precisely, when a system (X, T) is dynamically compact with respect to the family $\mathcal{N}_T$ and $\mathcal{N}_T$ has FIP, then, as well known, the systems is weakly mixing and $\mathcal{N}_T$ is a filter. Now, let a system (X, T) is dynamically compact with respect to the family $\mathcal{S}_T(\delta)$ for some

$\delta > 0$ and $\mathcal{S}_T(\delta)$ has FIP, then the systems is multi-sensitive. But, the following question is still open – when is $\mathcal{S}_T(\delta)$ a filter?

A collection $\mathcal{H} \subset \mathcal{F}$ will be called a *base* for $\mathcal{F}$ if for any $F \in \mathcal{F}$ there is $H \in \mathcal{H}$ with $H \subset F$. We are interested in those families which have *a countable base*, that is, there exists a base $\mathcal{H}$ which is countable.

Note that not every Furstenberg family $\mathcal{F}$ has a countable base, for example, the family $\mathcal{B}$. Assume the contrary that $\mathcal{B}$ admits a countable base $\{F_n : n \in \mathbb{N}\}$. We take $k_1 \in F_1$, and once $k_m \in F_m, m \in \mathbb{N}$ is defined we choose $k_{m+1} \in F_{m+1}$ with $k_{m+1} > k_m + m + 1$. Set $E = \{k_n : n \in \mathbb{N}\}$ and $F = \mathbb{Z}_+ \setminus E$. Then $E \cap F_n \neq \varnothing$ for all $n \in \mathbb{N}$, and $F \supset \{k_m + m : m \in \mathbb{N}\}$ and hence $F \in \mathcal{B}$, in particular, there exists no $n \in \mathbb{N}$ with $F_n \subset F$, a contradiction.

It is not hard to show even the existence of a family with FIP, but without a countable base. Nevertheless the families $\mathcal{N}_T$ and $\mathcal{S}_T(\delta)$ have countable bases. Indeed, we can consider a countable base $\mathcal{U}$ of open sets for the space X. Note that $U_1 \subset U$, $V_1 \subset V$ implies $N_T(U_1, V_1) \subset N_T(U, V)$ and $S_T(U_1, \delta) \subset S_T(U, \delta)$. Then $\{N_T(U, V) : U, V \in \mathcal{U}\}$ and $\{S_T(U, \delta) : U \in \mathcal{U}\}$ are countable bases for $\mathcal{N}_T$ and $\mathcal{S}_T(\delta)$, respectively.

The following is a general result that will be especially useful for families with countable bases.

Proposition 3.2. *Let (X, T) be a dynamical system and let $\mathcal{F}$ be a family such that there exists $x \in \mathrm{Tran}_{\mathcal{F}}(X, T)$. Then $\mathrm{orb}_T(x) \subset \mathrm{Tran}_{\mathcal{F}}(X, T)$.*

Proof. By assumption, given an arbitrary opene $U \subset X$ we have that $N_T(x, U) \in \mathcal{F}$. Thus, for any $m \in \mathbb{N}$,

$$N_T(T^m x, U) = N_T(x, T^{-m}(U)) \in \mathcal{F},$$

and we conclude that $T^m x \in \mathrm{Tran}_{\mathcal{F}}(X, T)$. $\qquad\square$

Proposition 3.3. *Assume that $\mathcal{F}$ admits a countable base $\mathcal{H}$. Then $\mathrm{Tran}_{k\mathcal{F}}(X, T)$ is a G_δ subset of X. Moreover, the following are equivalent:*

(1) *The system (X, T) is $k\mathcal{F}$-transitive,*
(2) $\mathrm{Tran}_{k\mathcal{F}}(X, T)$ *is a dense G_δ subset of X,*
(3) $\mathrm{Tran}_{k\mathcal{F}}(X, T) \neq \varnothing$.

Proof. Let $\mathcal{U}$ be a countable base of the family of all opene subsets of X. Then the class $\mathcal{U} \times \mathcal{H}$ is countable, and we enumerate it as $\{(U_i, F_i) : i \in \mathbb{N}\}$. Denote by $T^{-F}U = \bigcup_{n \in F} T^{-n}U$ for any $F \subset \mathbb{Z}_+$ and each $U \subset X$. Then it is easy to obtain

$$(3.1) \qquad \mathrm{Tran}_{k\mathcal{F}}(X, T) = \bigcap_{i=1}^{\infty} T^{-F_i}U_i.$$

In fact, given arbitrary point $x \in X$, $x \in \mathrm{Tran}_{k\mathcal{F}}(X, T)$ if and only if $N_T(x, U) \in k\mathcal{F}$ for any opene subset U of X, if and only if $N_T(x, U) \cap F \neq \varnothing$ for any opene subset U of X and each $F \in \mathcal{F}$, if and only if $N_T(x, U_i) \cap F_i \neq \varnothing$ for each $i \in \mathbb{N}$ by the construction. In particular, $\mathrm{Tran}_{k\mathcal{F}}(X, T)$ is a G_δ subset of X.

Thus (X, T) is $k\mathcal{F}$-transitive, if and only if for any $F \in \mathcal{F}$ and arbitrary opene subsets U, V of X we have $N_T(V, U) \cap F \neq \varnothing$ and equivalently $T^{-F}U \cap V \neq \varnothing$, if and only if $T^{-F}U$ is an opene dense subset of X for any $F \in \mathcal{F}$ and each opene subset U of X, if and only if $\mathrm{Tran}_{k\mathcal{F}}(X, T)$ is a dense G_δ subset of X by (3.1).

Now we assume $\text{Tran}_{k\mathcal{F}}(X,T) \neq \varnothing$. Let $x \in \text{Tran}_{k\mathcal{F}}(X,T)$. By Proposition 3.2 $\text{orb}_T(x) \subset \text{Tran}_{k\mathcal{F}}(X,T)$, and hence $\text{Tran}_{k\mathcal{F}}(X,T)$ is a dense G_δ subset of X since $x \in \text{Tran}(X,T)$. This finishes the proof. $\qquad\square$

Remark 3.4. *Observe that when the state space X is a compact metric space without isolated points, $x \in \text{Tran}(X,T)$ if and only if $x \in \text{Tran}_{\mathcal{B}}(X,T)$. The family $\mathcal{F}_{cof}$ is clearly translation invariant (and hence positively invariant) and admits a countable base, and $k\mathcal{F}_{cof} = \mathcal{B}$. Thus by Proposition 3.3 one has: (X,T) is transitive if and only if $\text{Tran}(X,T)$ is a dense G_δ subset of X if and only if $\text{Tran}(X,T) \neq \varnothing$.*

4. Transitive sensitivity and sensitive compactness

Recall that a dynamical system (X,T) is transitively sensitive if there exists $\delta > 0$ such that $S_T(W,\delta) \cap N_T(U,V) \neq \varnothing$ for any opene subsets U, V, W of X; and sensitive compact if there exists $\delta > 0$ such that for any point $x \in X$ the set $\omega_{\mathcal{S}_T(\delta)}(x)$ is nonempty. Sometimes in that cases we will say also (X,T) is transitively sensitive with a sensitive constant δ and (X,T) is sensitive compact with a sensitive constant δ. The main result of this section is the following

Theorem 4.1. *Let (X,T) be a minimal system. Then the following conditions are equivalent:*

   (1) *(X,T) is multi-sensitive.*
   (2) *(X,T) is sensitive compact.*
   (3) *There exists $\delta > 0$ such that $\omega_{\mathcal{S}_T(\delta)}(x) = X$ for each $x \in X$.*
   (4) *There exist $\delta > 0$ and $x \in X$ with $\omega_{\mathcal{S}_T(\delta)}(x) = X$.*
   (5) *(X,T) is transitively sensitive.*

Before proceeding, we need:

Lemma 4.2. *Let $\delta > 0$ and $x \in X$. If $T : X \to X$ is almost open, then the family $\mathcal{S}_T(\delta)$ is negatively invariant, and the subset $\omega_{\mathcal{S}_T(\delta)}(x)$ is positively T-invariant.*

Proof. By Proposition 2.1 it suffices to prove that $\mathcal{S}_T(\delta)$ is a negatively invariant family. Take arbitrary $F \in \mathcal{S}_T(\delta)$ and any $i \in \mathbb{Z}_+$. Then there exists opene subset U of X with $S_T(U,\delta) \subset F$. As $T : X \to X$ is almost open, $T^i : X \to X$ is also almost open, and then we can choose opene $V \subset T^i U$. One has $g^{-i}(F) \supset g^{-i} S_T(U,\delta) = S_T(T^i U, \delta) \supset S_T(V,\delta)$, which implies that the family $\mathcal{S}_T(\delta)$ is negatively invariant. $\qquad\square$

The following result gives a characterization of transitive sensitivity for a general dynamical system in terms of dynamical compactness.

Proposition 4.3. *Let (X,T) be a dynamical system. Then the family $\mathcal{S}_T(\delta)$ is positively invariant for any $\delta > 0$. Furthermore, the following conditions are equivalent:*

   (1) *(X,T) is transitively sensitive.*
   (2) *There exist a $\delta > 0$ and a dense G_δ subset $X_0 \subset X$ such that $\omega_{\mathcal{S}_T(\delta)}(x) = X$ for each $x \in X_0$.*
   (3) *There exist a $\delta > 0$ and a point $x \in X$ with $\omega_{\mathcal{S}_T(\delta)}(x) = X$.*

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Proof. Firstly, we show that $\mathcal{S}_T(\delta)$ is a positively invariant family. In fact, take any $F \in \mathcal{S}_T(\delta)$ and any $i \in \mathbb{Z}_+$. We choose opene subsets U, V of X with $F \supset S_T(U, \delta)$ and $V \subset T^{-i}U$ satisfying $\mathrm{diam}(T^j V) < \delta$ for all $j = 0, 1, \ldots, i$. Thus $g^i(F) \supset g^i S_T(U, \delta) \supset S_T(V, \delta)$ from the construction, and then $g^i(F) \in \mathcal{S}_T(\delta)$. This implies the positive invariance of the family $\mathcal{S}_T(\delta)$.

Observe that (X, T) is transitively sensitive with a sensitive constant δ, if and only if (X, T) is $k\mathcal{S}_T(\delta)$-transitive; and that the family $\mathcal{S}_T(\delta)$ has a countable base: let $\mathcal{U}$ be a countable base of the family of all opene subsets of X, then $\{S_T(U, \delta) : U \in \mathcal{U}\}$ is a countable base of $\mathcal{S}_T(\delta)$. Then applying Proposition 3.3 the equivalence of $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ follows from the fact that $x \in \mathrm{Tran}_{k\mathcal{S}_T(\delta)}(X, T)$ if and only if $\omega_{\mathcal{S}_T(\delta)}(x) = X$. $\qquad\square$

We recall that by [17, Corollary 1.7] the sensitivity of a dynamical system can be lifted up from a factor to an extension by an almost open factor map between transitive systems. The following result shows that the transitive sensitivity can be lifted up to an extension from a factor by an almost one-to-one factor map and that the transitive sensitivity is projected from an extension to the sensitivity of a factor by a weakly almost one-to-one factor map.

Lemma 4.4. *Let $\pi : (X, T) \to (Z, R)$ be a factor map between dynamical systems.*
  (1) *Assume that π is almost one-to-one. If (Z, R) is transitively sensitive with a sensitive constant $\delta > 0$ then (X, T) is also transitively sensitive.*
  (2) *Assume that there exists $z \in Z$ whose fiber is a singleton. If (X, T) is transitively sensitive then (Z, R) is sensitive, in particular, $Eq(Z, R) = \varnothing$.*

Proof. (1) We take a compatible metric ρ over Z and let $\varepsilon > 0$ such that $d(x_1, x_2) \leq \varepsilon$ implies $\rho(\pi x_1, \pi x_2) \leq \delta$ for any $x_1, x_2 \in X$. Now let U, V, W be arbitrary opene subsets of X. As the map $\pi : X \to Z$ is almost one-to-one, we may take opene subsets U_Z, V_Z, W_Z of Z with $\pi^{-1}U_Z \subset U$, $\pi^{-1}V_Z \subset V$ and $\pi^{-1}W_Z \subset W$. Observe that if $n \in S_R(W_Z, \delta) \cap N_R(U_Z, V_Z)$, then: on one hand, there exist $z_1, z_2 \in W_Z$ with $\rho(R^n z_1, R^n z_2) > \delta$, and so $d(T^n x_1, T^n x_2) > \varepsilon$ for any $x_1 \in \pi^{-1}(z_1)$ and $x_2 \in \pi^{-1}(z_2)$, hence $\mathrm{diam}(T^n W) > \varepsilon$; on the other hand, $U_Z \cap R^{-n}V_Z \neq \varnothing$, and then $U \cap S^{-n}V \supset \pi^{-1}U_Z \cap \pi^{-1}(R^{-n}V_Z) \neq \varnothing$. This implies $S_T(W, \varepsilon) \cap N_T(U, V) \supset S_R(W_Z, \delta) \cap N_R(U_Z, V_Z) \neq \varnothing$, as (Z, R) is transitively sensitive. Thus, by the arbitrariness of U, V and W, we have that (X, T) is also transitively sensitive.

(2) As (X, T) is transitively sensitive (with a sensitive constant $\delta > 0$), it is clear that (Z, R) is transitive, and then by the refined Auslander-York dichotomy the system (Z, R) is sensitive if and only if $Eq(Z, R) = \varnothing$ (see [6], [17], [2] and the book [1]). Thus it suffices to prove $Eq(Z, R) = \varnothing$. Assume to the contrary that $Eq(Z, R)$ is a nonempty set. Let ρ be a compatible metric over Z, and take a point $z \in Eq(Z, R)$. By the assumption that there exists a point of Z whose fiber is a singleton, we may take an opene subset W in Z with $\mathrm{diam}(\pi^{-1}W) < \delta$, and an opene subset $W_* \subset W$ and $\delta_* > 0$ such that if the distance between a point of Z and W_* is smaller than δ_* then the point belongs to W. Since $z \in Eq(Z, R)$, there exists an open neighborhood U_* of z with $\mathrm{diam}(R^n U_*) < \delta_*$ for all $n \in \mathbb{Z}_+$. As (X, T) is transitively sensitive with a sensitive constant δ, take $m \in N_T(\pi^{-1}U_*, \pi^{-1}W_*) \cap S_T(\pi^{-1}U_*, \delta)$. Thus $T^m(\pi^{-1}U_*) \cap \pi^{-1}W_* \neq \varnothing$, and then $R^m U_* \cap W^* \neq \varnothing$, which implies $R^m U_* \subset W$ by the construction of U_*, W_* and W. Hence

$$\mathrm{diam}(T^m(\pi^{-1}U_*)) \leq \mathrm{diam}(\pi^{-1}(R^m U_*)) \leq \mathrm{diam}(\pi^{-1}W) < \delta,$$

a contradiction to the selection of $m \in S_T(\pi^{-1}U_*, \delta)$. This finishes the proof. $\qquad\square$

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. $(1) \Rightarrow (2)$ follows directly from the definitions. As the system (X, T) is minimal, the map $T : X \to X$ is almost open. Observing that $\omega_{\mathcal{S}_T(\delta)}(x)$ is a closed subset of X for each $x \in X$, the implication of $(2) \Rightarrow (3)$ follows from Lemma 4.2 and the minimality of (X, T). The implication of $(3) \Rightarrow (4) \Rightarrow (5)$ follows from Proposition 4.3. Since a minimal system is either multi-sensitive or a weakly almost one-to-one extension of its maximal equicontinuous factor by [24], then $(5) \Rightarrow (1)$ follows from Lemma 4.4. This finishes the proof. $\qquad\square$

Clearly each multi-sensitive system is sensitive compact. Observe that each non-proximal, transitive compact system is multi-sensitive by [23, Theorem 4.7]. In particular, each minimal transitive compact system is multi-sensitive, as each minimal proximal system is trivial by [4] and all dynamical systems considered are assumed to be nontrivial. Nevertheless, there are many minimal, non transitive compact, multi-sensitive systems. For example, consider the classical dynamical system (X, T) given by $X = \mathbb{R}^2/\mathbb{Z}^2$ and $T : (x, y) \mapsto (x + \alpha, x + y)$ with $\alpha \notin \mathbb{Q}$ (see [15, Chapter 1]). As commented in [23, Page 1816], (X, T) is an invertible minimal multi-sensitive system; note that (X, T) is not weakly mixing, since (X, T) admits an irrational rotation as its nontrivial equicontinuous factor and any equicontinuous factor of a weakly mixing system is trivial. Recall that by [23, Corollary 3.10] for a minimal system the system is transitive compact if and only if it is weakly mixing, and then the constructed system (X, T) is not transitive compact.

Proposition 4.5. *Each nontrivial weakly mixing system (X, T) is transitively sensitive.*

Proof. As the system (X, T) is nontrivial, we take $0 < \delta < \mathrm{diam}(X)$. Choose opene subsets W_1, W_2 of X such that the distance between W_1 and W_2 is strictly larger than δ. Now take arbitrary opene subsets U, V, W of X. As (X, T) is weakly mixing, $(X^3, T^{(3)})$ is transitive by [14], and then $N_T(U, V) \cap S_T(W, \delta) \supset N_{T^{(3)}}(U \times W \times W, V \times W_1 \times W_2) \neq \varnothing$. This implies that (X, T) is transitively sensitive with a sensitive constant $\delta > 0$. $\qquad\square$

We give a sufficient condition for a dynamical system being transitively sensitive (by Proposition 4.3) as the last result of this section.

Lemma 4.6. *Assume $\omega_{\mathcal{S}_T(\varepsilon)}(x) = X$ for some $x \in X$ and $\varepsilon > 0$. Then there is $\delta > 0$ such that for any opene subset U of X and each neighbourhood U_x of x there are $y \in U_x$ and $n \in N_T(x, U)$ with $d(T^n x, T^n y) > \delta$. If in addition, the map $T : X \to X$ is almost one-to-one, then the converse holds.*

Proof. Fix an opene subset U of X and a neighborhood U_x of x. As $\omega_{\mathcal{S}_T(\varepsilon)}(x) = X$, there is $n \in N_T(x, U) \cap S_T(U_x, \varepsilon)$, and then there are points $x_1, x_2 \in U_x$ with $d(T^n x_1, T^n x_2) > \varepsilon$. We have either $d(T^n x, T^n x_1) > \frac{\varepsilon}{2}$ or $d(T^n x, T^n x_2) > \frac{\varepsilon}{2}$, and then obtain the desired statement for $\delta = \frac{\varepsilon}{2}$.

Now suppose that there is $\delta > 0$ such that for any opene subset U of X and each neighbourhood U_x of x there are $y \in U_x$ and $n \in N_T(x, U)$ with $d(T^n x, T^n y) > \delta$, and that the map $T : X \to X$ is almost one-to-one. Let U, W be arbitrary opene subsets of X. It is clear that $x \in \mathrm{Tran}(X, T)$, and so there is $k \in N_T(x, W)$. Note that $T : X \to X$ is almost one-to-one, so the map $T^k : X \to X$ is also almost

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one-to-one by Corollary 2.4, and then we may take an opene subset V of X with $T^{-k}V = (T^k)^{-1}V \subset U$. Observe that $T^{-k}W$ is a neighborhood of x, and then by the assumption there exists a point $y \in T^{-k}W$ and an integer $n \in \mathbb{Z}_+$ such that $n \in N_T(x, V)$ and $d(T^n x, T^n y) > \delta$. In fact, we may assume $n > k$, else we can replace $T^{-k}W$ by a small enough open neighbourhood $G_x \subset T^{-k}W$ of x with $\mathrm{diam}(T^i G_x) < \delta$ for all $0 \le i \le k$. Then $\mathrm{diam}(T^{n-k}W) > \delta$ as $T^k x \in W$ and $T^k y \in W$, and $T^{n-k}x \in T^{-k}(T^n x) \subset T^{-k}V \subset U$. In particular, $N_T(x, U) \cap S_T(W, \delta) \ne \varnothing$. Thus $\omega_{\mathcal{S}_T(\delta)}(x) = X$ by the arbitrariness of U and W. $\qquad\square$

5. Weakly disjointness and weakly mixing

Recall that dynamical systems (X, T) and (Y, S) are *weakly disjoint* if the product system $(X \times Y, T \times S)$ is transitive. The following theorem characterizes weak disjointness, which is proved firstly by Weiss [39] in some special class and then is generalized by Akin and Glasner [3]. We say that $\mathcal{F}$ is *thick* if $\tau\mathcal{F} = \mathcal{F}$, where

$$\tau\mathcal{F} = \left\{ F \subset \mathbb{Z}_+ : \bigcap_{j=1}^{n} g^{-i_j} F \in \mathcal{F} \text{ for each } n \in \mathbb{N} \text{ and all } i_1, \dots, i_n \in \mathbb{Z}_+ \right\}.$$

Weiss-Akin-Glasner Theorem. *Let $\mathcal{F}$ be a proper, translation invariant, thick family. A dynamical system is $k\mathcal{F}$-transitive if and only if it is weakly disjoint from every $\mathcal{F}$-transitive system.*

Observe that a dynamical system is weakly mixing if and only if it is weakly disjoint from itself, and then weak disjointness is characterized by [23, Proposition 3.8] in some special case. Now we discuss weak disjointness using dynamical compactness which will be some generalization of [23, Proposition 3.8]. We will need the following

Lemma 5.1. *Let (X, T) and (Y, S) be dynamical systems and let $x \in X$. Then the family $\mathcal{N}_S$ is translation invariant and $\omega_{\mathcal{N}_S}(x) = \omega_{\mathcal{N}_S}(Tx)$.*

Proof. By Proposition 2.1 it suffices to prove that $\mathcal{N}_S$ is a translation invariant family. We also suppose that $\mathcal{N}_S$ is proper (i.e., (Y, S) is a transitive system) since, otherwise, the result is trivial. Take arbitrary $F \in \mathcal{N}_S$ and any $i \in \mathbb{Z}_+$. Then there exist opene subsets U, V of Y with $N_S(U, V) \subset F$. As the non-singleton space Y contains no isolated points, we can take suitable opene $V_1 \subset V$ and $U_1 \subset U$ such that $U_1 \cap \bigcup_{k=0}^{i} S^k \overline{V_1} = \varnothing$. One has $g^i N_S(U, V) \supset N_S(S^{-i}U_1, V_1)$, which implies that the family $\mathcal{N}_S$ is positively invariant: in fact, if $n \in N_S(S^{-i}U_1, V_1)$ then $n > i$ by the selection, and so $n - i \in N_S(U_1, V_1) \subset N_S(U, V)$. Moreover, it is clear $g^{-i}(F) \supset g^{-i}N_S(U, V) \supset N_S(U, S^{-i}V)$, and then the family $\mathcal{N}_S$ is negatively invariant. This finishes the proof. $\qquad\square$

Theorem 5.2. *The following conditions are equivalent:*
  (1) *The systems (X, T) and (Y, S) are weakly disjoint.*
  (2) *Both $\mathrm{Tran}_{k\mathcal{N}_S}(X, T)$ and $\mathrm{Tran}_{k\mathcal{N}_T}(Y, S)$ are dense G_δ subsets.*
  (3) *The set $\mathrm{Tran}_{k\mathcal{N}_S}(X, T)$ is a dense G_δ subset of X.*
  (4) *Both $\mathrm{Tran}_{k\mathcal{N}_S}(X, T)$ and $\mathrm{Tran}_{k\mathcal{N}_T}(Y, S)$ are nonempty subsets.*
  (5) *The set $\mathrm{Tran}_{k\mathcal{N}_S}(X, T)$ is a nonempty subset of X.*

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (3)$: It is clear from the definition that: the system (X, T) is $k\mathcal{N}_S$-transitive, if and only if the systems (X, T) and (Y, S) are weakly disjoint, if

and only if (X,T) is $k\mathcal{N}_S$-transitive and (Y,S) is $k\mathcal{N}_T$-transitive. As both $\mathcal{N}_T$ and $\mathcal{N}_S$ are families admitting a countable base, it is direct to obtain the equivalence of $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ by applying Proposition 3.3.

The implication $(2) \Rightarrow (4) \Rightarrow (5)$ is obvious. To finish the proof, we only need to show $(5) \Leftrightarrow (3)$. By Lemma 5.1 the family $\mathcal{N}_S$ is translation invariant, and then the family $k\mathcal{N}_S$ is also translation invariant. Thus the equivalence of $(5) \Leftrightarrow (3)$ follows from Proposition 3.3. $\qquad\square$

Note that we have a characterization of weak mixing by using dynamical compactness [23, Proposition 3.8]. Now we extend [23, Proposition 3.8] as follows.

Recall that $\mathcal{S} \subset \mathbb{N}$ is an *IP set* if there exists $\{p_k : k \in \mathbb{N}\} \subset \mathbb{N}$ with $FS\{p_i\}_{i=1}^\infty \subset \mathcal{S}$, where $FS\{p_i\}_{i=1}^\infty = \{p_{i_1} + \cdots + p_{i_k} : k \in \mathbb{N} \text{ and } 1 \le i_1 < \cdots < i_k\}$. Analogously, for each $n \in \mathbb{N}$ we define $FS\{p_i\}_{i=1}^n = \{p_{i_1} + \cdots + p_{i_k} : k \in \mathbb{N} \text{ and } 1 \le i_1 < \cdots < i_k \le n\}$. Denote by $\mathcal{F}_{ip}$ the family of all IP sets.

By [28, Theorem 3.2], the subset $\mathrm{Tran}_{\mathcal{F}_{ip}}(X,T)$ contains a dense G_δ subset of X for any weakly mixing system (X,T), while $\mathrm{Tran}_{\mathcal{F}_{ip}}(X,T) \neq \varnothing$ does not imply the weak mixing of the system (X,T) by [28, Proposition 3.4]. We will extend that in the following Proposition 5.4. Before proceeding, we make the following

Lemma 5.3. *Let (X,T) be a dynamical system and $\mathcal{F}$ be a family.*

    (1) *Let $\delta > 0$. If the family $\mathcal{S}_T(\delta) \cdot \mathcal{F}$ is proper then $\mathcal{S}_T(\delta) \cdot \mathcal{F} \subset \mathcal{B}$.*
    (2) *If the family $\mathcal{N}_T \cdot \mathcal{F}$ is proper then $\mathcal{N}_T \cdot \mathcal{F} \subset \mathcal{B}$.*

Proof. (1) Assume the contrary that there exists an opene subset U in X and $F \in \mathcal{F}$ such that $S_T(U,\delta) \cap F$ is finite, and so we may choose $m \in \mathbb{N}$ such that $n \notin S_T(U,\delta) \cap F$ for any integer $n > m$. Since $T : X \to X$ is uniformly continuous one can find opene $V \subset U$ small enough such that $\mathrm{diam}(T^iV) < \delta$ for all $0 \le i \le m$. Then $S_T(V,\delta) \subset S_T(U,\delta)$, which implies $S_T(V,\delta) \cap F = \varnothing$. A contradiction.

(2) Assume the contrary that there exist opene subsets U, V in X and $F \in \mathcal{F}$ such that $N_T(U,V) \cap F$ is finite, say $N_T(U,V) \cap F = \{n_1, \ldots, n_k\}$. As the non-singleton space X contains no isolated points, we can take opene $U_1 \subset U$ small enough such that $V_1 := V \setminus \bigcup_{i=1}^k T^{n_i}\overline{U_1}$ is an opene subset of X. By the construction we have $N_T(U_1,V_1) \subset N_T(U,V)$ and then $N_T(U_1,V_1) \cap F = \varnothing$, a contradiction. $\qquad\square$

Proposition 5.4. *The following conditions are equivalent:*

    (1) *(X,T) is weakly mixing.*
    (2) *There exists a dense G_δ subset X' of X such that, for each $x \in X'$, $N_T(x,G) \cap N_T(U,V) \in \mathcal{F}_{ip}$ for all opene subsets G, U, V of X.*

Proof. $(2) \Rightarrow (1)$: Just observe from the assumption that $\omega_{\mathcal{N}_T}(x) = X$ for all $x \in X'$, and hence the system (X,T) is weakly mixing by [23, Proposition 3.8].

$(1) \Rightarrow (2)$: Since (X,T) is weakly mixing, $(X^2, T^{(2)})$ is also weakly mixing by [14] and [35], and hence by [23, Proposition 3.8] there is a dense G_δ subset $Y \subset X^2$ such that $\omega_{\mathcal{N}_{T^{(2)}}}((x_1,x_2)) = X^2$ for each $(x_1,x_2) \in Y$. Applying the well-known Ulam Lemma there is a dense G_δ subset $X' \subset X$ such that, for any $x \in X'$, $\{y : (x,y) \in Y\}$ is a dense G_δ subset of X. Now we show that X' is the desired set.

Let $x \in X'$ and fix any opene subsets G, U, V of X. Choose $y \in G$ with $(x,y) \in Y$ and then $\omega_{\mathcal{N}_{T^{(2)}}}((x,y)) = X^2$, in particular, $(y,y) \in \omega_{\mathcal{N}_{T^{(2)}}}((x,y))$. Thus

$$N_T(x,G) \cap N_T(y,G) \cap N_T(U,V) \cap N_T(V,V) \neq \varnothing,$$

and take $p_1 \in \mathbb{N}$ from this set by Lemma 5.3. We have $p_1 \in N_T(x, G) \cap N_T(U, V)$ and $T^{p_1} y \in G$, $T^{p_1} V \cap V \neq \varnothing$. Define opene subsets $G_1 = G \cap T^{-p_1} G \ni y$ and $V_1 = V \cap T^{-p_1} V$. Now we proceed inductively. Suppose that we are given a sequence $\{p_1, \ldots, p_k\} \subset \mathbb{N}$ with $FS\{p_i\}_{i=1}^{k} \subset N_T(x, G) \cap N_T(U, V)$, and opene subsets

$$G_k = \bigcap_{s \in FS\{p_i\}_{i=1}^{k} \cup \{0\}} T^{-s} G \ni y, V_k = \bigcap_{s \in FS\{p_i\}_{i=1}^{k} \cup \{0\}} T^{-s} V.$$

As $(y, y) \in \omega_{\mathcal{N}_{T^{(2)}}}((x, y))$, we may take $p_{k+1} \in \mathbb{N}$ by Lemma 5.3 from the set $N_T(x, G_k) \cap N_T(y, G_k) \cap N_T(U, V_k) \cap N_T(V_k, V_k)$. It is not hard to check that

$$G_{k+1} = G_k \cap T^{-p_{k+1}} G_k = \bigcap_{s \in FS\{p_i\}_{i=1}^{k+1} \cup \{0\}} T^{-s} G \ni y,$$

$$V_{k+1} = V_k \cap T^{-p_{k+1}} V_k = \bigcap_{s \in FS\{p_i\}_{i=1}^{k+1} \cup \{0\}} T^{-s} V$$

are both opene subsets of X, and that $FS\{p_i\}_{i=1}^{k+1} \subset N_T(x, G) \cap N_T(U, V)$, which completes the induction. Finally, $N_T(x, G) \cap N_T(U, V) \in \mathcal{F}_{ip}$ with $FS\{p_i\}_{i=1}^{\infty} \subset N_T(x, G) \cap N_T(U, V)$. This finishes the proof. $\square$

6. Transitive compact (non weakly mixing) systems

Recall that the system (X, T) is *totally transitive* if (X, T^k) is transitive for each $k \in \mathbb{N}$; and is *topologically mixing* if $N_T(U, V) \in \mathcal{F}_{\mathrm{cof}}$ for any opene subsets U, V in X. Note that (X, T) is weakly mixing if and only if $N_T(U, V) \in \mathcal{F}_{\mathrm{thick}}$ for any opene sets U, V in X by [14, 35], and so any weakly mixing system is totally transitive. It is direct to check that each weakly mixing system is transitive compact. In [23] the authors showed the existence of non totally transitive, transitive compact systems in both proximal and non-proximal cases. We extend it as follows:

Theorem 6.1. *There is a totally transitive, transitive compact system (X, T) which is not weakly mixing.*

Proof. Let $(\mathbb{S}^1, R_\alpha)$ be the standard irrational rotation on the unit circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with $\alpha \notin \mathbb{Q}$. Take a nontrivial proximal, topologically mixing system (Y, S) (for the existence of such a system see some example [22]). Note that a dynamical system is proximal if and only if it contains the unique fixed point, which is the only minimal point of the system [4]. Denote by p_Y the unique minimal point (fixed point) of (Y, S). Observe that the system $(Y \times \mathbb{S}^1, S \times R_\alpha)$ is totally transitive: for each $n \in \mathbb{N}$, the system (Y, S^n) is topologically mixing by the definition and it is standard that the system $(\mathbb{S}^1, R_\alpha^n)$ is minimal, then it is direct to see that these two systems are weakly disjoint.

Let (X, T) be the quotient system $Y \times \mathbb{S}^1 / \sim$ equipped with the action T induced naturally from $S \times R_\alpha$, where the equivalence relation $\sim$ is defined via: given $x, y \in X$, $x \sim y$ if and only if either $x = y$ or x and y both have p_Y in the first coordinate. In other words the space X looks like a cone space, where the vertex of the cone is a point p, each "horizontal" fiber spaces are the space Y, the vertical fiber spaces are the circles (see Figure 1). Clearly, (X, T) is totally transitive.
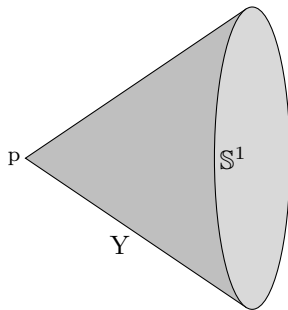
Figure 1.

Denote by $q : Y \times \mathbb{S}^1 \to X$ the corresponding quotient map, then $q : Y_\infty \times \mathbb{S}^1 \to X \setminus \{p\}$ is a homeomorphism, where we set $Y_\infty = Y \setminus \{p_Y\}$. It is standard that the system $(\mathbb{S}^1, R_\alpha)$ is not weakly mixing, and then there exist opene subsets U_*, V_* of $\mathbb{S}^1$ with $N_{R_\alpha}(U_*, V_*) \notin \mathcal{F}_{\text{thick}}$, hence

$$N_T(q(Y_\infty \times U_*), q(Y_\infty \times V_*)) = N_{S \times R_\alpha}(Y_\infty \times U_*, Y_\infty \times V_*) \subset N_{R_\alpha}(U_*, V_*)$$

is not thick. This implies that the system (X, T) is not weakly mixing.

Now let U, V be arbitrary opene subsets of X. We can choose opene subsets $U_1, V_1 \subset Y_\infty$ and $U_2, V_2 \subset \mathbb{S}^1$ with $U_1 \times U_2 \subset q^{-1}U$ and $V_1 \times V_2 \subset q^{-1}V$. As (Y, S) is topologically mixing and $(\mathbb{S}^1, R_\alpha)$ is minimal, $N_S(U_1, V_1) \in \mathcal{F}_{\text{cof}}$ and $N_{R_\alpha}(U_2, V_2) \in \mathcal{F}_{\text{syn}}$, and then $N_S(U_1, V_1) \cap N_{R_\alpha}(U_2, V_2) \in \mathcal{F}_{\text{syn}}$, thus

$$N_T(U, V) = N_{S \times R_\alpha}(q^{-1}U, q^{-1}V) \supset N_{S \times R_\alpha}(U_1 \times U_2, V_1 \times V_2)$$

is a syndetic set. Observe from the construction that the system (X, T) is proximal with p as its unique fixed point, then $N_T(x, U_p)$ is a thickly syndetic subset for each point $x \in X$ and any neighbourhood U_p of p (see [23, Lemma 3.12]). This implies $p \in \omega_{\mathcal{N}_T}(x)$ for each $x \in X$, and then the system (X, T) is transitive compact. □

The following result is proved independently in [11] and [38].

Lemma 6.2. *Any ω-limit set $\omega_T(x)$ can not be decomposed into α disjoint closed, nonempty, positively T-invariant subsets, where $2 \leq \alpha \leq \aleph_0$.*

Before proceeding, we need the following example, for which we fail to find a reference and hence provide a detailed construction, as it is crucial in our arguments.

Proposition 6.3. *For any given compact metric space Z, there exists a topologically mixing system (X, T) such that, Z can be realized as the set of all of its minimal points, furthermore, its each minimal point is a fixed point.*

Proof. The construction is divided into two steps.

In the first step we shall construct a topologically mixing system (Y, F) but with two fixed points, which are the only minimal points of the system. Let $\Sigma = \{0, 1\}^{\mathbb{Z}_+}$ and $\sigma : \Sigma \to \Sigma$ be the full (one-sided) shift. We are going to find the system (Y, F) of the form $(\overline{\text{orb}_\sigma(x)}, \sigma)$ for some $x \in \Sigma$.

In order to define $x \in \Sigma$, firstly we represent each $W \in \Sigma^*$ with $|W| \geq 1$ in the form of $W = a^i Q b^j$ with $a, b \in \{0, 1\}$ and $i \in \mathbb{N}, j \in \mathbb{Z}_+$ as follows. Assume $W = w_1 \ldots w_{|W|}$ with $w_1, \ldots, w_{|W|} \in \{0, 1\}$. If $w_1 = \cdots = w_{|W|}$ then we set Q to be the empty subblock and $a = w_1, i = |W|, b = 0, j = 0$. If $w_i \neq w_j$ for some $1 \leq i < j \leq |W|$ then we set $a = w_1, b = w_{|W|}$ and choose $i, j \in \mathbb{N}$ such that a^i and

b^j are the longest segments of equal digits which we can take at the beginning and at the end of W, respectively, whereas Q is the rest, possible the empty subblock. In particular, if $W = a^k$ with $a \in \{0, 1\}$ and $k \in \mathbb{N}$ then we set $i = k$, $b = 0$, $j = 0$.

Now starting with $A_1 = 10$, we construct inductively the blocks $A_2, A_3, \ldots$, and then define $x \in \Sigma$ as the limit of the starting blocks A_k. Suppose that we have defined $A_k, k \in \mathbb{N}$. Since A_k has finitely many subblocks, there is a finite number of different pairs of these subblocks. For any pair (W_1, W_2) of subblocks of A_k we define a block $c(W_1, W_2)$ by using their combination as follows. The definition $c(W_1, W_2)$ depends on the structure of W_1 and W_2. Let us write W_1 and W_2 in the form as in the previous paragraph: $W_1 = a^{i_1}Q_1 b^{j_1}$ and $W_2 = c^{i_2}Q_2 d^{j_2}$ with $a, b, c, d \in \{0, 1\}$ and $i_1, i_2 \in \mathbb{N}, j_1, j_2 \in \mathbb{Z}_+$, whereas Q_1 and Q_2 are possible empty subblocks. *The combination block $c(W_1, W_2)$ of the pair W_1, W_2 is defined as*

$$c(W_1, W_2) = a^{k+i_1}Q_1 b^{j_1+k} c^{k+i_2}Q_2 d^{k+j_2} a^{k+i_1}Q_1 b^{j_1+k+1} c^{k+i_2}Q_2 d^{k+j_2}.$$

Now let us define A_{k+1}: at the beginning of A_{k+1} we write $A_k 0^k 1^k$, and then all possible blocks $c(W_1, W_2)$ of pairs (W_1, W_2) of subbloks of A_k in any fixed order.

We see from the construction that A_{k+1} is presented as a sequence of blocks with length not longer than $|A_k|$, which are separated from each other with some sequences of blocks of consecutive 1's or 0's of length not less than k. In fact, for all $m > k$ this property holds for A_m (and hence in any subblock of A_m with length more than $|A_k| + 2k - 1$ one can find 0^k or 1^k). Suppose that A_m may be presented in this form. Since a block W is presented as $a^i Q b^j$ and the property holds for W, it holds for Q. Indeed, if $i \geq k$ and $j \geq k$ then everything is clear. Else either a^i or b^j is a part of the "bad" block in the beginning (or in the end) of W. Hence Q begins (terminates) with a "bad" block of length not more then $|A_k| - i$ ($|A_k| - j$). Consequently, by the construction if the property is true for W_1 and W_2 then it is true for (W_1, W_2). Now observe that A_{m+1} is obtained by adding to $A_m 0^m 1^m$ combination blocks $c(W_1^*, W_2^*)$ of all pairs (W_1^*, W_2^*) of subblocks of A_m concatenated in a proper way. So, the property holds for A_{m+1}.

Put $(Y, F) := (\overline{\mathrm{orb}_\sigma(x)}, \sigma)$. Now let us check that it has the required properties. In any subblock of x with length more than $|A_k| + 2k - 1$ one can find 0^k or 1^k. Fix $y \in Y$. Any starting block of y can be definitely found as a block in x. Hence for any $k \in \mathbb{N}$ there exist big enough M such that the starting block of y consists either 0^k or 1^k. Moreover, for each $k \in \mathbb{N}$ both 0^k and 1^k appear in x. Hence, both 0^∞ and 1^∞ are fixed points of $(\overline{\mathrm{orb}_\sigma(x)}, \sigma)$, and there is no other minimal sets in it.

Recall that the base for the open sets in Σ is given by the collection of all cylinder sets $C[c_0 c_1 c_2 \ldots c_m] = \{x \in \Sigma : x_i = c_i \text{ for } 0 \leq i \leq m\}$. We are going to prove that for any $U, V \subset Y$ $N_F(U, V)$ is cofinite. By the latter, there exist subblocks W_1, W_2 of x such that $Y \cap C[W_1] \subset U$ and $Y \cap C[W_2] \subset V$. Both of W_1, W_2 belong to A_m for big enough m. We write W_1 and W_2 in the form as above: $W_1 = a^{i_1}Q_1 b^{j_1}$ and $W_2 = c^{i_2}Q_2 d^{j_2}$ with $a, b, c, d \in \{0, 1\}$ and $i_1, i_2 \in \mathbb{N}, j_1, j_2 \in \mathbb{Z}_+$, whereas Q_1 and Q_2 are possible empty subblocks. If $b = c$, then for each $l \geq m$ there exists a combination block $c_l(W_1, W_2)$ containing the subblock $W_1 b^l W_2$, and hence $N_F(C[W_1] \cap Y, C[W_2] \cap Y) \supset \{m + |W_1|, m + |W_1| + 1, \ldots\}$; if $b \neq c$, then for each $l \geq m$ there exists a combination block $c_l(W_1, W_2)$ containing the subblocks $W_1 b^l c^l W_2$ and $W_1 b^l 1 c^l W_2$, and hence $N_F([W_1] \cap Y, [W_2] \cap Y) \supset \{2m + |W_1|, 2m + |W_1| + 1, \ldots\}$. This shows that the system (Y, F) is topologically mixing.

Now we shall finish the construction by the second step. Define $(X', T') = \prod_1^\infty (Y, F)$. Let us check that the system (X', T') is topologically mixing, for which the middle-third Cantor set C is the set of all its minimal points and its each minimal point is a fixed point. Note that there exists a continuous surjection $h : C \to Z$ (see for example [26, Page 165-166, Problem O]), and then we consider the quotient system (X, T) with $X = X'/\sim$ equipped with the action induced naturally from T', where the closed positively $T' \times T'$-invariant equivalence relation $\sim$ is defined via $x \sim y$ if and only if $x = y \in X' \setminus Z$ or $h(x) = h(y)$ for $x, y \in Z$. Then the system (X, T) has the required properties. $\qquad\square$

The following result shows that in general there is no topological structure similar to Lemma 6.2 for the $\omega_{\mathcal{N}_T}$-limit sets.

Theorem 6.4. *For any given compact metric space Z, there exists a non totally transitive, transitive compact system (X, T) such that, Z can be realized as the set of all its minimal points with its each minimal point being a fixed point, furthermore, Z is realized as $\omega_{\mathcal{N}_T}(x)$ for some $x \in X$.*

Proof. The idea of the proof is very similar to that of the first part (proximal case) of [23, Theorem 3.14]. We fix the given compact metric space Z.

By Proposition 6.3 there exists a topologically mixing system (Y, F), with Z realized as the set of all of its minimal points where its each minimal point is a fixed point. We take a copy of it: the system (Y_c, F_c), with Z_c realized as the set of all of its minimal points where its each minimal point is a fixed point, in particular, $Y_c = \{y_c : y \in Y\}$ with $F_c(y_c) = (Fy)_c$ for all $y \in Y$ and $Z_c = \{z_c : z \in Z\}$. Suppose that Y and Y_c are disjoint and consider the wedge sum $X := Y \vee Y_c$, i.e., the quotient space of the disjoint union of Y and Y_c by identifying z and z_c for each $z \in Z$, and then both topological spaces Y and Y_c look like subspaces of the wedge sum with the subspace topology. A self-map T over X is defined as follows:

$$T : x \mapsto \begin{cases} F_c y_c & \text{if } x = y \in Y \\ Fy & \text{if } x = y_c \in Y_c \end{cases}, \forall x \in X.$$

From the above construction it is not hard to show that the map $T : X \to X$ is a continuous surjection, and the system (X, T) is not totally transitive by observing that $T^2 Y \subset Y$ and $T^2 Y_c \subset Y_c$. Then we obtain a non totally transitive system (X, T) such that Z is realized as the set of all of its minimal points with its each minimal point being a fixed point.

Firstly we show that $Z \supset \omega_{\mathcal{N}_T}(x)$ for all $x \in X$. It is easy to check $\omega_{\mathcal{N}_T}(x) = \{x\}$ for the case of $x \in Z$. Now assume that $x \in Y \setminus Y_c$. For any $z \in Y \setminus Y_c$, take a neighborhood $G_z \subset Y \setminus Y_c$. From the construction of T one has that $N_T(x, G_z)$ consists of only even numbers. Now in Y we take an opene subset U of X and in Y_c we take an opene subset V of X. Again from the construction of T one has that $N_T(U, V)$ consists of only odd numbers. In particular, $N_T(x, G_z) \cap N_T(U, V) = \varnothing$, and then $z \notin \omega_{\mathcal{N}_T}(x)$. We can prove similarly that $\omega_{\mathcal{N}_T}(x) \cap Y_c \setminus Y = \varnothing$. Summing up, we obtain $\omega_{\mathcal{N}_T}(x) \subset Z$. The case of $x \in Y_c \setminus Y$ can be done similarly.

Now we prove the following claim:

Claim. For each $x \in X$, if $x = y \in Y$ and $p \in Z \cap \omega_{\mathcal{N}_{F^2}}(y)$ then, as a point in X, p belongs to $\omega_{\mathcal{N}_T}(x)$.

Proof of Claim. Let U_p be an open subset of X containing p, and clearly U_p may be also viewed as an open subset of Y containing p. Now for any given opene subsets U and V of X: if both U and V can be viewed as opene subsets of Y, then we can take $n \in N_{F^2}(y, U_p) \cap N_{F^2}(U, V)$ by the assumption $p \in \omega_{\mathcal{N}_{F^2}}(y)$ and hence $2n \in N_T(y, U_p) \cap N_T(U, V)$; if both U and V can be viewed as opene subsets of Y_c, then both $T^{-1}U$ and $T^{-1}V$ can be viewed as opene subsets of Y and hence $N_T(y, U_p) \cap N_T(U, V) = N_T(y, U_p) \cap N_T(T^{-1}U, T^{-1}V) \neq \varnothing$; if U and V can be viewed as opene subsets of Y and Y_c, respectively, noting $p \in Z$ and hence $Tp = p$, there is an opene subset V_p of X containing p such that $TV_p \subset U_p$, and then by the above reasoning we may take $n \in N_T(y, V_p) \cap N_T(U, T^{-1}V)$, and hence $n+1 \in N_T(y, U_p) \cap N_T(U, V)$; it can be treated similarly the other case that U and V can be viewed as opene subsets of Y_c and Y, respectively. $\qquad \square$

We continue our proof. As (Y, F) is topologically mixing, the system (Y, F^2) is weakly mixing, and then by [23, Proposition 3.8] we may choose $x^* \in Y$ such that $\omega_{\mathcal{N}_{F^2}}(x^*) = Y$, in particular, $\omega_{\mathcal{N}_{F^2}}(x^*) \supset Z$. Thus, by the above Claim, we obtain $Z \subset \omega_{\mathcal{N}_T}(x^*)$ and hence $Z = \omega_{\mathcal{N}_T}(x^*)$.

Finally we prove that the system (X, T) is transitive compact, which finishes the proof. Let $x \in X$. If $x \in Z$ then $\omega_{\mathcal{N}_T}(x) \neq \emptyset$ as $\omega_{\mathcal{N}_T}(x) = \{x\}$. If $x \in Y \setminus Y_c$, applying [23, Proposition 3.5] to the weakly mixing system (Y, F^2) one has that $\omega_{\mathcal{N}_{F^2}}(x)$ is a nonempty F^2-invariant subset of Y, and then $\omega_{\mathcal{N}_{F^2}}(x) \cap Z \neq \varnothing$ as Z is the set of all of minimal points in (Y, F), hence $\omega_{\mathcal{N}_T}(x) \neq \emptyset$ follows from the above Claim. The case of $x \in Y_c \setminus Y$ can be done similarly. $\qquad \square$

Note that a dynamical system is proximal if and only if it contains the unique fixed point, which is the only minimal point of the system [4]. Thus, as a direct corollary of Lemma 6.2 and Theorem 6.4, we have:

Corollary 6.5. *There exists a non-proximal, non totally transitive, transitive compact system (X, T) and a point $x_0 \in X$ such that $\omega_{\mathcal{N}_T}(x_0) \neq \omega_T(x)$ for all $x \in X$.*

Nevertheless is still open the following

Question B. *Let (X, T) be a weakly mixing system. Is there a point $x \in X$ and $2 \leq \alpha \leq \aleph_0$ such that $\omega_{\mathcal{N}_T}(x)$ can be decomposed into α disjoint closed, nonempty, positively T-invariant subsets?*

At the end of this section let us prove one more chaotical property of transitive compact systems in additional to already known in [23].

Recall that a pair of points $x, y \in X$ is *asymptotic* if $\lim_{n \to \infty} d(T^n x, T^n y) = 0$. Denote by $\mathrm{Asym}_T(X)$ the set of all asymptotic pairs of points. Any pair $(x, y) \in \mathrm{Prox}_T(X) \setminus \mathrm{Asym}_T(X)$ is called a *Li-Yorke pair*. Recall that a dynamical system (X, T) is *Li-Yorke chaotic* if there exists an uncountable set $S \subset X$ with $(S \times S) \setminus \Delta_2(X) \subset \mathrm{Prox}_T(X) \setminus \mathrm{Asym}_T(X)$, where $\Delta_2(X) = \{(x, x) : x \in X\}$.

Proposition 6.6. *Each transitive compact system (X, T) is Li-Yorke chaotic.*

Proof. Clearly (X, T) is transitive. Observe that we have assumed the state space to be not a singleton and in fact a compact metric space without isolated points, then (X, T) is a transitive system with X infinite. Thus, the subset $\mathrm{Asym}_T(X)$ is a fist category subset of $X \times X$ by [25, Corollary 2.2]. It is easy to show that $\mathrm{Prox}_T(X)$ is a G_δ subset of $X \times X$, and applying [23, Proposition 3.7] to the transitive compact system (X, T) we have that $\mathrm{Prox}_T(x)$ is a dense subset of X for each $x \in X$. Thus

Prox$_T$(X) is a dense G_δ subset of $X \times X$, and then Prox$_T$(X) \ Asym$_T$(X) is a second category subset of $X \times X$. Now applying the well-known Mycielski Theorem [32, Theorem 1] we obtain an uncountable subset $S \subset X$ with $(S \times S) \setminus \Delta_2(X) \subset$ Prox$_T$(X) \ Asym$_T$(X). That is, (X,T) is Li-Yorke chaotic. $\qquad\square$

7. Weak transitive compactness and sensitivity for linear operators

In this section we are considering the dynamics of linear operators on infinite dimensional spaces in relation to the properties studied in previous sections. More precisely, we will show the equivalence of the topological weak mixing property with a weak version of transitive compactness. We obtain some results on transitive sensitivity too.

One should keep in mind that, for a linear dynamical system (X,T), where X is an infinite dimensional space, neither compactness nor even local compactness of X is satisfied. In particular, we are interested in the case where X is an infinite dimensional separable Banach space and $T : X \to X$ is a continuous linear map (in short, operator). In this framework, we will just write (X,T) is an infinite dimensional linear dynamical system. We recall that X is a Banach space if it is a vector space endowed with a norm $\|\cdot\|$ such that X with the associated distance $d(x,y) := \|x - y\|$ becomes a complete metric space. It is well known that $T : X \to X$ is an operator if and only if $\|T\| := \sup\{\|Tx\| : \|x\| \le 1\} < \infty$. We refer the reader to the books [7] and [19] for the theory of linear dynamics.

Note that all notations and concepts discussed in previous sections can be introduced into linear dynamics. We also introduce a weak version of dynamical compactness. A linear system (X,T) is called *weakly dynamically compact with respect to the family $\mathcal{F}$* if there exists a dense subset $X_0 \subset X$ such that the $\omega_\mathcal{F}$-limit set $\omega_\mathcal{F}(x)$ is nonempty for all $x \in X_0$. In particular, (X,T) is called *weakly transitive compact*, if there exists a dense subset $X_0 \subset X$ such that for any point $x \in X_0$ the $\omega_{\mathcal{N}_T}$-limit set $\omega_{\mathcal{N}_T}(x)$ is nonempty, in other words, for any point $x \in X_0$ there exists a point $z \in X$ such that

$$N_T(x,G) \cap N_T(U,V) \ne \varnothing$$

for any neighborhood G of z and any opene subsets U, V of X.

Theorem 7.1. *Let (X,T) be an infinite dimensional linear system. Then (X,T) is weakly mixing if and only if it is weakly transitive compact.*

Proof. Sufficiency. Suppose that (X,T) is weakly transitive compact. Let $X_0 \subset X$ be a dense subset such that, for each $x \in X_0$, there exists $z(x) \in X$ such that

$$N_T(x,G) \cap N_T(U,V) \ne \varnothing$$

for any neighborhood G of $z(x)$ and opene $U, V \subset X$. As (X,T) is obviously transitive, by [18, Theorem 5] (see also [19, Theorem 2.45]) to obtain the weak mixing property we just need to show that, for each opene $U \subset X$ and 0-neighbourhood W, there is a continuous map $S : X \to X$ commuting with T such that

(7.1) $$S(U) \cap W \ne \varnothing \quad \text{and} \quad S(W) \cap U \ne \varnothing.$$

Given an opene subset U of X and a 0-neighborhood W, we fix $x \in U \cap X_0$ and $z(x) \in X$ accordingly to the weak transitive compactness of (X,T). Since

0-neighbourhoods are absorbing, we find a scalar $\lambda \neq 0$ such that $\lambda z(x) \in W$. Let G be a neighbourhood of $z(x)$ such that $\lambda G \subset W$. By the hypothesis we can find

$$m \in N_T(x, G) \cap N_T(\lambda W, U).$$

That is, $T^m x \in G$ and so $\lambda T^m x \in W$; additionally, there exists $w \in W$ with $T^m \lambda w \in U$. Now pick $S := \lambda T^m$, we have that S commutes with T and the property (7.1) is satisfied, therefore the system is weakly mixing.

Necessity. Conversely, under the assumption of the weak mixing property for (X, T), we know by [8, Theorem 2.3] (see also [19, Theorem 3.15]) that there exists an increasing sequence $\{n_k : k \in \mathbb{N}\} \subset \mathbb{N}$ and a dense subset $X_0 \subset X$ such that $T^{n_k} x \to 0$ for each $x \in X_0$ and, for arbitrary opene $U, V \subset X$, we can find $k \in \mathbb{N}$ such that $T^{n_k}(U) \cap V \neq \varnothing$. Thus, we obtain easily that (X, T) is weakly transitive compact by selecting $z(x) = 0$ for every $x \in X_0$. $\qquad\square$

Concerning sensitivity, the situation is more complicated and, although we obtain some advances, three related problems are left open.

Proposition 7.2. *Let (X, T) be an infinite dimensional linear, topologically transitive system. Then (X, T) is thickly multi-sensitive, that is, there exists $\delta > 0$ such that $\bigcap_{i=1}^k S_T(U_i, \delta)$ is thick for any finite collection of opene $U_1, \ldots, U_k \subset X$.*

Proof. Let $U_1, \ldots, U_k$ be opene sets, and let $m \in \mathbb{N}$. Pick points $x_1, \ldots, x_k$ such that $x_i \in U_i$ and choose $\varepsilon > 0$ such that $B_\varepsilon(x_i) \subset U_i$, where $B_\varepsilon(x_i)$ is the open ball of radius ε centered at x_i, for all $i \in \{1, \ldots, k\}$. By a *hypercyclic* vector we mean that its orbit is dense in the space X. Take a hypercyclic vector $u \in B_\varepsilon(0)$ by [19, Theorem 2.19], and let $y_i = x_i + u$. Then $y_i \in U_i$ by the construction. Since u is hypercyclic there is $n \in \mathbb{N}$, $n > m$, such that $\|T^n u\| > (\|T\| + 1)^m$. Then $\rho(T^{n-j} x_i, T^{n-j} y_i) = \|T^{n-j}(x_i - y_i)\| = \|T^{n-j} u\| > (\|T\| + 1)^{m-j} > 1$ for all $i = 1, \ldots, k$ and $j = 0, \ldots, m-1$. Hence $\{n, n-1, \ldots, n-m+1\} \subset \bigcap_{i=1}^k S_T(U_i, 1)$, and therefore (X, T) is thickly multi-sensitive. $\qquad\square$

Proposition 7.3. *Let (X, T) be an infinite dimensional linear system. Then the following conditions are equivalent:*

1. *For each $\delta > 0$, (X, T) is transitively sensitive with a sensitive constant δ.*
2. *There exists $\delta_0 > 0$ such that (X, T) is transitively sensitive with a sensitive constant δ_0.*
3. *There exists $\delta_0 > 0$ such that $S_T(W_0, \delta_0) \cap N_T(U, V) \neq \varnothing$ for any opene subsets U, V of X and any 0-neighbourhood W_0.*

Proof. We just need to show (3) $\Rightarrow$ (1). Indeed, let $\delta > 0$ be arbitrary, and fix arbitrary opene U, V, W of X. We select $\varepsilon > 0$ and $x \in W$ such that $x + B_\varepsilon(0) \subset W$. Observing $S_T(\lambda W_0, \lambda \delta_0)$ for any scalar $\lambda \neq 0$, and so without loss of generality we assume $\delta > \delta_0$. Let $0 < \varepsilon' < \frac{\delta_0 \varepsilon}{\delta}$, and set $W_0 = B_{\varepsilon'}(0)$. By the hypothesis there are $y, z \in W_0$ and $n \in N_T(U, V)$ such that $\|T^n y - T^n z\| > \delta_0$. Set $y' = x + \frac{\delta}{\delta_0} y$ and $z' = x + \frac{\delta}{\delta_0} z$. We have $y', z' \in W$ and $\|T^n y' - T^n z'\| > \delta$. As opene $U, V, W \subset X$ are arbitrary, (X, T) is transitively sensitive with a sensitive constant δ. $\qquad\square$

In this framework the weak mixing property implies transitive sensitivity too. The following result establishes a very close connection of transitivity with transitive sensitivity. We do not know, however, whether every transitive linear system is transitively sensitive.

Proposition 7.4. *Let (X, T) be an infinite dimensional linear, topologically transitive system. If (X, T) is not transitively sensitive, then there exists a dense open subset $U_0 \subset X$ such that every $x \in U_0$ has a dense orbit.*

Proof. If (X, T) is not transitively sensitive, by Proposition 7.3 we find opene U, V of X and $\delta > 1$ such that $\|T^n x\| \leq \delta$ whenever $n \in N_T(U, V)$ and $\|x\| \leq 1$. We fix an arbitrary opene $V' \subset V$ and select an opene $\widehat{V} \subset V'$ and $\varepsilon > 0$ such that $\widehat{V} + B_\varepsilon(0) \subset V'$. Given $u \in U$, there is $\varepsilon' < \frac{\varepsilon}{\delta}$ such that $U' := u + B_{\varepsilon'}(0) \subset U$. Since T is transitive, there exists $m \in N_T(U', \widehat{V}) \subset N_T(U, V)$. That is, we find $u' = u + w \in U'$ with $\|w\| < \varepsilon'$ and $T^m u' \in \widehat{V}$. By the assumption $\|T^m w\| \leq \delta \varepsilon' < \varepsilon$. Therefore, $T^m u = T^m u' - T^m w \in \widehat{V} + B_\varepsilon(0) \subset V'$. Since $u \in U$ and opene $V' \subset V$ are arbitrary, we obtain that the orbit of every element in U is somewhere dense, thus everywhere dense by transitivity of the system. Finally, the open set $U_0 := \bigcup_{n \in \mathbb{N}} T^{-n}(U)$ is dense, and every element in U_0 has a dense orbit. $\square$

There are (very difficult) examples of linear systems (X, T) such that every non-zero element has a dense orbit [36], but it seems to unknown whether every linear system that admits an open set of elements whose orbit is dense is so that every non-zero element has a dense orbit. It is also worthy to mention that there are (also rare) examples of transitive but not weakly mixing linear systems [10] (see also [7]), but as far as we know there are no examples of transitive non-weakly mixing linear systems such that every non-zero element has a dense orbit.

Concerning weak disjointness, observe that for each separable Banach space the family of all opene subsets admits a countable base, and then it is a routine to show that Theorem 5.2 holds true within linear systems too. Note that the intersection of finitely many thickly syndetically sets is still thickly syndetic, and that an interesting property is that every *topologically ergodic* linear system (X, T) (i.e., each element of $\mathcal{N}_T$ is a syndetic set) satisfies that each element of $\mathcal{N}_T$ is actually a thickly syndetic set (see the exercises in [19, Chapter 2]). Thus any finite family $(X_1, T_1), \ldots, (X_k, T_k)$ of topologically ergodic linear systems is weakly disjoint and, moreover, the product system $(X_1 \times \cdots \times X_k, T_1 \times \cdots \times T_k)$ is topologically ergodic.

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