LIPSCHITZ OPERATOR IDEALS AND THE APPROXIMATION PROPERTY

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ABSTRACT. We establish the basics of the theory of Lipschitz operator ideals with the aim of recovering several classes of Lipschitz maps related to absolute summability that have been introduced in the literature in the last years. As an application we extend the notion and main results on the approximation property for Banach spaces to the case of metric spaces.

1. INTRODUCTION AND NOTATIONS

Since Farmer and Johnson [20] extended absolute summability to Lipschitz mappings, several works have appeared with the aim of extending different classes of linear operators to the Lipschitz context (see for instance [11, 12, 13, 26] and the references therein).

Although in recent years there is an increasing interest for finding results on approximation properties related to Lipschitz functions and the free spaces $Æ(X)$ of metric spaces X (see [15, 16, 21, 22, 29]), there are no explicit definitions of approximation properties on metric spaces. For example, [5] reproduces the approximation property for Banach spaces considering approximating Lipschitz mappings instead of linear operators.

Recently, new extensions of the approximation property for Banach spaces related to operator ideals have been introduced (see [4, 30]). In [30], the definition is based on a notion of compactness depending on a given operator ideal, that has its roots in the paper by Carl and Stephani [9] and has been developed in the context of polynomials and holomorphy in [2, 3, 23]. The general notion of Lipschitz operator

ideal provides a natural context for giving suitable definitions of such notions.

Our purpose is to develop a basic theory of the Lipschitz operator ideals that unifies all the recently introduced new classes of Lipschitz mappings, and apply it to the establishment of a suitable context for the study of the approximation property intrinsic to metric spaces and Lipschitz mappings.

Throughout the paper X, Y denote pointed metric spaces with a base point denoted by 0, and E, F are (real or complex) Banach spaces. We will write $\mathbb{K}$ for $\mathbb{R}$ or $\mathbb{C}$. If d is the metric on X, B_X stands for the set $\{x \in X : d(x, 0) \leq 1\}$. Given a Banach space Z, Z^* denotes its topological dual, and B_Z is its closed unit ball (which is coherent with the above notation). As usual, $\mathcal{L}(E, F)$ denotes the space of all continuous linear operators from E to F with the operator norm. If $T \in \mathcal{L}(E, F)$, $T^* : F^* \longrightarrow E^*$ is the adjoint operator of T. The Lipschitz space $Lip_0(X, E)$ is the Banach space of all Lipschitz mappings T from X to E that vanish at 0, under the Lipschitz norm $Lip(.)$, where $Lip(T)$ its the infimum of all constants $C \geq 0$ such that $\|T(x) - T(x')\| \leq C d(x, x')$ for all $x, x' \in X$. The notation Lip_0 stands for the class of all Lipschitz mappings from pointed metric spaces to Banach spaces. For $E = \mathbb{K}$, we write $X^{\#} = Lip_0(X) = Lip_0(X, \mathbb{K})$. Along the paper we consider $B_{X^\#}$ endowed with the pointwise topology. It is well know that $Lip_0(X)$ has a predual, namely the space of Arens and Eells Æ(X) [1]. A molecule on X is a scalar valued function m on X with finite support that satisfies $\sum\limits_{x \in X} m(x) = 0$. We denote by $\mathcal{M}(X)$ the linear space of all molecules on X. For $x, x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A. For $m \in \mathcal{M}(X)$ we can write $m = \sum\limits_{j=1}^{n} \lambda_j m_{x_j x_j'}$ for some suitable scalars λ_j, and we write

$$\|m\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{j=1}^{n} |\lambda_j| \, d(x_j, x_j'), \ m = \sum_{j=1}^{n} \lambda_j m_{x_j x_j'} \right\},$$

where the infimum is taken over all representations of the molecule m. Denote by Æ(X) the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$. The map $\delta_X : X \to$ Æ(X) defined by $\delta_X(x) = m_{x0}$ isometrically embeds X in Æ(X). Given $T \in Lip_0(X, E)$, there exists a unique linear map $T_L : $ Æ$(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$. The operator T_L is referred to as the linearization of T (see [35, Theorem 2.2.4 (b)]).

The correspondence $T \longleftrightarrow T_L$ establishes an isomorphism between the vector spaces $Lip_0(X, E)$ and $\mathcal{L}(\text{Æ}(X), E)$. In particular, the spaces $X^{\#}$ and $\text{Æ}(X)^*$ are isometrically isomorphic via the linearization $R(f) := f_L$, where $f_L(m) = \sum_{x \in X} f(x)m(x)$ (see [35, Theorem 2.2.2]).

Sawashima [34] defined the Lipschitz adjoint (or dual) of $T \in Lip_0(X, E)$ as the continuous linear operator

$$
\begin{array}{rlll}
T^{\#}: & Lip_0(E) & \longrightarrow & Lip_0(X) \\
& g & \longmapsto & T^{\#}(g) = g \circ T
\end{array}
$$

The restriction of $T^{\#}$ to E^* is called the Lipschitz transpose map of T and is denoted here by T^t.

Section 2 contains the basics of the theory of Lipschitz operator ideals, that are introduced in a natural way. Especial emphasis is made on examples: we recover the main classes of Lipschitz mappings such as approximable Lipschitz mappings, Lipschitz $p-$summing mappings, Lipschitz $p-$integral mappings, Lipschitz $p-$nuclear mappings, Lipschitz compact and Lipschitz weakly compact mappings. In Section 3 we extend to the Lipschitz mappings setting some linear procedures to construct ideals of operators from a given operator ideal $\mathcal{I}$: by composition with linear operators from $\mathcal{I}$ and, by considering those Lipschitz mappings whose Lipschitz transposes belong to $\mathcal{I}$. It is proved that the Lipschitz dual of a given operator ideal $\mathcal{I}$ coincides with the Lipschitz composition ideal with the dual ideal of $\mathcal{I}$. A direct consequence is the well-known Lipschitz variant of Schauder's (respectively Gantmacher's) theorem: a Lipschitz operator is compact (respectively weakly compact) if, and only if, its transpose is compact (respectively weakly compact). In Section 4 we initiate a study of approximation properties for metric spaces, that come from considering composition ideals of Lipschitz mappings. The main aim is to relate the linear and the Lipschitz theories. To do so, we strongly make use of linearization techniques.

2. Lipschitz operator ideals

The aim of this section is to introduce the concept of Lipschitz operator ideal. We will follow the spirit of the definition of linear operator ideal explained in the excellent monographs [17, 33].

Definition 2.1. A *Lipschitz operator ideal* $\mathcal{I}_{Lip}$ is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E the components

$$
\mathcal{I}_{Lip}(X, E) := Lip_0(X, E) \cap \mathcal{I}_{Lip}
$$

satisfy:

(i) $\mathcal{I}_{Lip}(X, E)$ is a linear subspace of $Lip_0(X, E)$.

(ii) $vg \in \mathcal{I}_{Lip}(X, E)$ for $v \in E$ and $g \in X^{\#}$.

(iii) The ideal property: if $S \in Lip_0(Y, X)$, $T \in \mathcal{I}_{Lip}(X, E)$ and $w \in \mathcal{L}(E, F)$, then the composition wTS is in $\mathcal{I}_{Lip}(Y, F)$.

A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a normed (Banach) Lipschitz operator ideal if there is $\|.\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0, +\infty[$ that satisfies

(i') For every pointed metric space X and every Banach space E, the pair $(\mathcal{I}_{Lip}(X, E), \|.\|_{\mathcal{I}_{Lip}})$ is a normed (Banach) space and $Lip(T) \leq \|T\|_{\mathcal{I}_{Lip}}$ for all $T \in \mathcal{I}_{Lip}(X, E)$.

(ii') $\|Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$.

(iii') If $S \in Lip_0(Y, X)$, $T \in \mathcal{I}_{Lip}(X, E)$ and $w \in \mathcal{L}(E, F)$, then $\|wTS\|_{\mathcal{I}_{Lip}} \leq Lip(S) \|T\|_{\mathcal{I}_{Lip}} \|w\|$.

From conditions (i'), (ii') and (iii') we get for any $v \in E$ and any $g \in X^{\#}$,

$$
\begin{aligned}
\|vg\|_{\mathcal{I}_{Lip}} &= \|(v\ Id_{\mathbb{K}}) \circ Id_{\mathbb{K}} \circ g\|_{\mathcal{I}_{Lip}} \leq \|v\ Id_{\mathbb{K}}\| \|Id_{\mathbb{K}}\|_{\mathcal{I}_{Lip}} Lip(g) \\
&= \|v\|Lip(g) = Lip(vg) \leq \|vg\|_{\mathcal{I}_{Lip}},
\end{aligned}
$$

that is, $\|vg\|_{\mathcal{I}_{Lip}} = \|v\|Lip(g) = Lip(vg)$.

Following [26] a mapping $T \in Lip_0(X, E)$ has Lipschitz finite dimensional rank if the linear hull of the set $\left\{\frac{T(x)-T(x')}{d(x,x')}, x, x' \in X, x \neq x'\right\}$ is a finite dimensional subspace of E. We denote by $Lip_{0\mathcal{F}}(X, E)$ the set of all Lipschitz finite rank mappings from X to E. Clearly, $Lip_{0\mathcal{F}}(X, E)$ is a linear subspace of $Lip_0(X, E)$. It is proved in [26, Proposition 2.4] that having finite dimensional rank is equivalent to having Lipschitz finite dimensional rank. This is also equivalent to saying that the linearization T_L of T has finite rank. Such a mapping admits a finite representation $T = \sum\limits_{k=1}^{n} u_k f_k$, with $(u_k)_{k\leq n} \in E, (f_k)_{k\leq n} \in X^{\#}$. Indeed, given $T \in Lip_{0\mathcal{F}}(X, E)$, according to [26, Proposition 2.4], $T_L \in \mathcal{L}(\text{Æ}(X), E)$ has finite rank. Thus

$$
T_L = \sum_{i=1}^{n} m_i^* \otimes u_i, (u_i)_{i\leq n} \subset E, (m_i^*)_{i\leq n} \subset \text{Æ}(X)^*.
$$

Since $\text{Æ}(X)^*$ and $X^{\#}$ are isometrically isomorphic then, there is $(f_i)_{i\leq n} \subset X^{\#}$ such that $(f_i)_L = m_i^*$.

For $x \in X$ we obtain

$$
\begin{aligned}
T(x) &= T_L \circ \delta_X(x) = T_L(m_{x0}) = \sum_{i=1}^{n} (f_i)_L \otimes u_i(m_{x0}) \\
&= \sum_{k=1}^{n} \langle (f_i)_L, m_{x0} \rangle \, u_i \\
&= \sum_{i=1}^{n} f_i(x) u_i.
\end{aligned}
$$

Thus $T = \sum_{i=1}^{n} u_i f_i$.

The class $Lip_{0\mathcal{F}}$ is the smallest Lipschitz operator ideal.

Definition 2.2. (1) A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is said to be *injective* if for every metric linear injection $I : E \hookrightarrow F$ (that is, I is linear and $\|I(x)\| = \|x\|$ for all $x \in E$) and every $T \in Lip_0(X, E)$, T is in $\mathcal{I}_{Lip}$ whenever $I \circ T \in \mathcal{I}_{Lip}(X, F)$. A normed Lipschitz operator ideal is called injective if moreover $\|T\|_{\mathcal{I}_{Lip}} = \|I \circ T\|_{\mathcal{I}_{Lip}}$. In a straightforward way, the *injective hull* $\mathcal{I}_{Lip}^{inj}$ can be defined as the smallest injective Lipschitz operator ideal that contains $\mathcal{I}_{Lip}$. Note that each Lipschitz operator ideal is contained in an injective Lipschitz operator ideal, and so the notion of injective hull makes sense. Indeed, as in the linear case, $\mathcal{I}_{Lip}^{inj}(X, E)$ is formed by all Lipschitz mappings $T : X \to E$ that satisfy that $J_F \circ T \in \mathcal{I}(X, \ell_\infty(B_{E^*}))$, where J_E is the canonical inclusion $J_E : E \to \ell_\infty(B_{E^*})$. The space $\mathcal{I}_{Lip}^{inj}(X, E)$ is endowed with the norm $\|T\|_{\mathcal{I}_{Lip}}^{inj} := \|J_E \circ T\|_{\mathcal{I}_{Lip}}$, for all $T \in \mathcal{I}_{Lip}^{inj}(X, E)$ (see for example [17, 9.7]).

(2) The *closed hull* $\overline{\mathcal{I}_{Lip}}$ of a Lipschitz operator ideal $\mathcal{I}_{Lip}$ consists of all Lipschitz mappings that can be approximated, with respect to the Lipschitz norm, by a sequence of Lipschitz mappings in $\mathcal{I}_{Lip}$.

2.1. **Approximable Lipschitz operators.** Following [26], a Lipschitz mapping $T \in Lip_0(X, E)$ is said to be *approximable* if it is the limit of a sequence of Lipschitz finite rank operators from X to E in the Lipschitz norm Lip.

The collection of all approximable Lipschitz operators $T \in Lip_0(X, E)$ is denoted by $\overline{Lip_{0\mathcal{F}}(X, E)}$. It is clear that $\overline{Lip_{0\mathcal{F}}(X, E)}$ is an example of Lipschitz ideal. Given E and F Banach spaces, let $\mathcal{L}_f(E, F)$ denote the space of all finite rank linear operators from E to F.

The following is inspired in [6].

Lemma 2.3. *Let $T \in Lip_0(X, E)$. The following statements are equivalent.*

(1) $T \in Lip_{0\mathcal{F}}(X, E)$.
(2) $T_L \in \mathcal{L}_f(\text{Æ}(X), E)$.
(3) $T^t \in \mathcal{L}_f(E^*, X^{\#})$.

Proof. The equivalence between (1) and (2) appears in [26, Proposition 2.4]. The equivalence with (3) follows from the fact that $(T_L)^* = R \circ T^t$ and the ideal property. $\square$

Corollary 2.4. *Let $T \in Lip_0(X, E)$. The following statements are equivalent.*

(1) *T can be approximated by finite rank Lipschitz operators.*
(2) *T_L can be approximated by finite rank linear operators.*
(3) *T^t can be approximated by finite rank linear operators.*

Proof. It is a direct consequence of Lemma 2.3 the fact that the correspondence $T \in Lip_0(X, E) \longmapsto T_L \in \mathcal{L}(\text{Æ}(X), E)$ is an isomorphism, and Hutton's theorem [25, Theorem 2.1] that assures that T_L can be approximated by finite rank operators if, and only if, $(T_L)^*$ can be approximated by finite rank operators. $\square$

2.2. **Lipschitz $p-$summing operators.** Let $1 \leq p < \infty$. In [20] Lipschitz $p-$summing operators defined between metric spaces are introduced. A mapping $T \in Lip_0(X, E)$ is *Lipschitz $p-$summing* if there exists a constant $C \geq 0$ such that for all $(x_i)_{i \leq n}, (x'_i)_{i \leq n}$ in X and all $(a_i)_{i \leq n} \subset \mathbb{R}^+$

$$\sum_{i=1}^{n} a_i \|T(x_i) - T(x'_i)\|^p \leq C^p \sup_{f \in B_{X^{\#}}} \sum_{i=1}^{n} a_i |f(x_i) - f(x'_i)|^p.$$

The infimum of all such constants $C \geq 0$ is denoted by $\pi_p^L(T)$. This class of mappings is denoted by $\Pi_p^L(X, E)$. Thanks to an argument detailed in [20] the scalars $a_1, \ldots, a_n$ can be removed from the definition.

It is well known that $\Pi_p^L(X, E)$ with the norm $\pi_p^L(.)$ is a Banach space that satisfies the ideal property (this is straightforward, see for instance [20]) and that $Lip(T) \leq \pi_p^L(T)$ for every $T \in \Pi_p^L(X, E)$. A direct calculation shows that $\pi_p^L(vg) \leq \|v\|Lip(g)$ for any $v \in E$ and $0 \neq g \in X^{\#}$. Theorem 2 in [20] assures that $\pi_p^L(Id_{\mathbb{K}}) = \pi_p(Id_{\mathbb{K}}) = 1$, where $Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda$. Moreover, consider a metric linear injection $I : E \hookrightarrow F$ and $T \in Lip_0(X, E)$ such that $I \circ T \in \Pi_p^L(X, E)$.

Since I is a metric injection, given $(x_i)_{i \leq n}, (x'_i)_{i \leq n}$ in X we get

$$\left(\sum_{i=1}^{n} \|T(x_i) - T(x'_i)\|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} \|IT(x_i) - IT(x'_i)\|^p \right)^{\frac{1}{p}}$$

$$\leq \pi_p^L(IT) \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^{n} |f(x_i) - f(x'_i)|^p \right)^{\frac{1}{p}}.$$

Hence $T \in \Pi_p^L(X, E)$ and $\pi_p^L(T) \leq \pi_p^L(IT)$. The ideal property gives the reverse inequality and then $\pi_p^L(T) = \pi_p^L(IT)$. That proves that Π_p^L is injective. Summarizing, we have the following.

Proposition 2.5. (Π_p^L, π_p^L) *is an injective Banach Lipschitz operator ideal.*

2.3. **Lipschitz $p-$integral operators.** Let $1 \leq p \leq \infty$ Lipschitz $p-$integral operators from metric spaces to pointed metric spaces are introduced in [20] (see also [13]). A map $T \in Lip_0(X, E)$ is *Lipschitz $p-$integral* if there are a probability measure space (Ω, Σ, μ) and two Lipschitz mappings $A : L_p(\mu) \longrightarrow E^{**}$ and $B : X \longrightarrow L_\infty(\mu)$ such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{\ \ T\ \ } & E & \xrightarrow{\ \ k_E\ \ } & E^{**} \\ \Big\downarrow{\scriptstyle B} & & & & \Big\uparrow{\scriptstyle A} \\ L_\infty(\mu) & \xrightarrow[\ \ i_p^\mu\ \ ]{} & & & L_p(\mu). \end{array}$$

where $i_p^\mu : L_\infty(\mu) \longrightarrow L_p(\mu)$ is the canonical mapping and $k_E : E \longrightarrow E^{**}$ is the canonical isometric embedding. For short, we will say that (A, i_p^μ, B) is a factorization for T. Denote by $\iota_p^L(T)$ the infimum of $Lip(A)Lip(B)$ taken over all factorizations as above. This class of mappings is denoted by $\mathcal{I}_p^L(X, E)$.

Using some standard techniques from [7, Proposition 2.1] (see also [19, Theorem 5.2]) we can show that $\mathcal{I}_p^L(X, E)$ is a Banach Lipschitz operator ideal with the norm $\iota_p^L(.)$. For the convenience of the reader, we include the proof.

Proposition 2.6. $(\mathcal{I}_p^L, \iota_p^L)$ *is a Banach Lipschitz operator ideal.*

Proof. We check all conditions in Definition 2.1.

(i) We start checking that $Lip(T) \leq \iota_p^L(T)$ for all $T \in \mathcal{I}_p^L(X, E)$. Let (A, i_p^μ, B) be a factorization for a $T \in \mathcal{I}_p^L(X, E)$. Clearly $k_E T \in Lip_0(X, E^{**})$ and $Lip(T) = Lip(k_E T) \leq Lip(A)Lip(B)$. Taking the infimum we get $Lip(T) \leq \iota_p^L(T)$.

We check now that $\mathcal{I}_p^L(X,E)$ is a linear subspace of $Lip_0(X,E)$. Let $T \in \mathcal{I}_p^L(X,E)$ and let (A, i_p^μ, B) be a decomposition for T. For any $\alpha \in \mathbb{K}$, $k_E(\alpha T) = (\alpha A)i_p^\mu B$. Since $\alpha A \in Lip_0(L_p(\mu), E^{**})$ then αT is Lipschitz $p-$integral and

$$\iota_p^L(\alpha T) \le Lip(\alpha A)Lip(B) = |\alpha|\, Lip(A)Lip(B).$$

It follows that $\iota_p^L(\alpha T) \le |\alpha|\, \iota_p^L(T)$. Then, for $\alpha \ne 0$, we have $\iota_p^L(T) = \iota_p^L(\alpha^{-1}\alpha T) \le |\alpha^{-1}|\, \iota_p^L(\alpha T)$, hence $|\alpha|\, \iota_p^L(T) \le \iota_p^L(\alpha T)$ and so $\iota_p^L(\alpha T) = |\alpha|\, \iota_p^L(T)$.

Let $T_1, T_2 \in \mathcal{I}_p^L(X,E)$ and $\varepsilon > 0$. Consider factorizations $(A_i, i_p^{\mu_i}, B_i)$ for T_i such that $Lip(B_i) = \frac{1}{2}$ and $Lip(A_i) < \iota_p^L(T_i) + \frac{\varepsilon}{2}$, for $i = 1, 2$. We may assume also that $\Omega_1 \cap \Omega_2 = \emptyset$.

Take $\Omega := \Omega_1 \cup \Omega_2$, $\Sigma := \{S \subset \Omega : S \cap \Omega_i \in \Sigma_i, i = 1, 2\}$ and define the probability measure μ on Ω by

$$\mu(S) := \frac{Lip(A_1)\mu_1(S \cap \Omega_1) + Lip(A_2)\mu_2(S \cap \Omega_2)}{Lip(A_1) + Lip(A_2)}.$$

Define $A : L_p(\mu) \longrightarrow E^{**}$ and $B : X \longrightarrow L_\infty(\mu)$ by

$$\begin{aligned} A(f) &= A_1(f\,|_{\Omega_1}) + A_2(f\,|_{\Omega_2}), \\ B(x) &= B_1(x) \cdot \chi_{\Omega_1} + B_2(x) \cdot \chi_{\Omega_2}, \end{aligned}$$

where χ_{Ω_i} is the characteristic function of Ω_i for $i = 1, 2$. We have $A(0) = 0$. If $\frac{1}{p} + \frac{1}{p'} = 1$, using Hölder's inequality we get

$$\begin{aligned} \|A(f) - A(f')\| &\le \sum_{i=1}^{2} \|A_i(f\,|_{\Omega_i}) - A_i(f'\,|_{\Omega_i})\| \\ &\le \sum_{i=1}^{2} Lip(A_i)^{1/p}Lip(A_i)^{1/p'}\,\|(f-f')\,|_{\Omega_i}\|_{L_p(\mu_i)} \\ &\le \left(\sum_{i=1}^{2} Lip(A_1)\right)^{1/p} \left(\sum_{i=1}^{2} Lip(A_i)^{1/p'}\,\|(f-f')\,|_{\Omega_i}\|_{L_p(\mu_i)}\right) \\ &\le \left(\sum_{i=1}^{2} Lip(A_1)\right)^{1/p} \left(\sum_{i=1}^{2} Lip(A_i)\right)^{1/p'} \left(\sum_{i=1}^{2} \|(f-f')\,|_{\Omega_i}\|_{L_p(\mu_i)}^p\right)^{1/p} \\ &= (Lip(A_1) + Lip(A_2))\,\|(f-f')\|_{L_p(\mu)} \end{aligned}$$

for all $f, f' \in L_p(\mu)$. Hence $A \in Lip_0(L_p(\mu), E^{**})$ and $Lip(A) \le Lip(A_1) + Lip(A_2)$. Moreover $B \in Lip_0(X, L_\infty(\mu))$ with $Lip(B) \le 1$

because $B(0) = 0$ and

$$
\begin{aligned}
\|B(x) - B(x')\|_{L_\infty(\mu)} &\leq \|B_1(x) - B_1(x')\|_{L_\infty(\mu_1)} + \|B_2(x) - B_2(x')\|_{L_\infty(\mu_2)} \\
&\leq (Lip(B_1) + Lip(B_2))\, d(x, x') \\
&= d(x, x')
\end{aligned}
$$

for all $x, x' \in X$. For each $x \in X$, we have

$$
\begin{aligned}
Ai_p^\mu B(x) &= Ai_p^\mu(B_1(x)\chi_{\Omega_1} + B_2(x)\chi_{\Omega_2}) \\
&= A_1 i_p^{\mu_1} B_1(x) + A_2 i_p^{\mu_2} B_2(x) \\
&= k_E T_1(x) + k_E T_2(x) \\
&= k_E(T_1 + T_2)(x),
\end{aligned}
$$

and thus $Ai_p^\mu B = k_E(T_1 + T_2)$. Hence $T_1 + T_2 \in \mathcal{I}_p^L(X, E)$ and

$$
\iota_p^L(T_1 + T_2) \leq Lip(A)Lip(B) \leq Lip(A) \leq \iota_p^L(T_1) + \iota_p^L(T_2) + \varepsilon.
$$

Since ε was arbitrary, it follows that $\iota_p^L(T_1 + T_2) \leq \iota_p^L(T_1) + \iota_p^L(T_2)$. To prove the completeness of the space $\mathcal{I}_p^L(X, E)$, take a sequence (T_n) in $\mathcal{I}_p^L(X, E)$ such that $\sum_{n=1}^\infty \iota_p^L(T_n) < \infty$. Since $Lip(.) \leq \iota_p^L(.)$ and $(Lip_0(X, E), Lip(.))$ is a Banach space, there exists $T := \sum_{n=1}^\infty T_n \in Lip_0(X, E)$. We prove that $\sum_{n=1}^\infty T_n = T$ for $\iota_p^L(.)$. Let $\varepsilon > 0$. For each $n \in \mathbb{N}$ we can find a probability space $(\Omega_n, \Sigma_n, \mu_n)$, Lipschitz operators $B_n \in Lip_0(X, L_\infty(\mu_n))$, $A_n \in Lip_0(L_p(\mu_n), E^{**})$ with $Lip(B_n) = 1/2^n$ and $Lip(A_n) \leq \iota_p^L(T_n) + \varepsilon/2^n$ such that $k_E T_n$ factors as

$$
k_E T_n = A_n i_p^{\mu_n} B_n : X \xrightarrow{B_n} L_\infty(\mu_n) \xrightarrow{i_p^{\mu_n}} L_p(\mu_n) \xrightarrow{A_n} E^{**}.
$$

We can assume that the Ω_n's are pairwise disjoint and set $\Omega := \cup_{n \in \mathbb{N}} \Omega_n$ and $\Sigma := \{S \subset \Omega : S \cap \Omega_n \in \Sigma_n, \ \forall n \in \mathbb{N}\}$. Define the probability measure μ on Σ by

$$
\mu(S) := \frac{\sum_{n=1}^\infty \mu_n(S \cap \Omega_n) Lip(A_n)}{\sum_{n=1}^\infty Lip(A_n)},
$$

$S \in \Sigma$. Define $A : L_p(\mu) \longrightarrow E^{**}$ and $B : X \longrightarrow L_\infty(\mu)$ by

$$
\begin{aligned}
A(f) &= \sum_{n=1}^\infty A_n(f \mid_{\Omega_n}), \\
B(x) &= \sum_{n=1}^\infty B_n(x)\chi_{\Omega_n}.
\end{aligned}
$$

Clearly $A(0) = 0$ and $B(0) = 0$. When $p = \infty$ it is clear that $Lip(A) \leq \sum_{n=1}^{\infty} Lip(A_n) < \infty$. For $1 \leq p < \infty$, using a similar argument as above we get

$$\|A(f) - A(f')\| \leq \left(\sum_{n=1}^{\infty} Lip(A_n) \right) \|f - f'\|_{L_p(\mu)}$$

and

$$\|B(x) - B(x')\|_{L_\infty(\mu)} \leq d(x, x'),$$

for all $f, f' \in L_p(\mu)$ and all $x, x' \in X$. Hence, $A \in Lip_0(L_p(\mu), E^{**})$ with $Lip(A) \leq \sum_{n=1}^{\infty} Lip(A_n) \leq \sum_{n=1}^{\infty} \iota_p^L(T_n) + \epsilon$, and $B \in Lip_0(X, L_\infty(\mu))$ with $Lip(B) \leq 1$.

For each $x \in X$, we have

$$\begin{aligned}
Ai_p^\mu B(x) &= A \left(\sum_{n=1}^{\infty} i_p^\mu (B_n(x)\chi_{\Omega_n}) \right) \\
&= \sum_{m=1}^{\infty} A_m \left((\sum_{n=1}^{\infty} i_p^\mu (B_n(x)\chi_{\Omega_n}))|_{\Omega_m} \right) \\
&= \sum_{m=1}^{\infty} A_m i_p^{\mu_m} B_m(x) = \sum_{m=1}^{\infty} k_E T_m(x) = k_E T(x)
\end{aligned}$$

and thus $Ai_p^\mu B = k_E T$. Hence, $T \in \mathcal{I}_p^L(X, E)$ and

$$\iota_p^L(T) \leq Lip(A) Lip(B) \leq Lip(A) \leq \sum_{n=1}^{\infty} \iota_p^L(T_n) + \varepsilon.$$

Since ε was arbitrary, it follows that $\iota_p^L(T) \leq \sum_{n=1}^{\infty} \iota_p^L(T_n)$.

We now show that $T = \sum_{k=1}^{\infty} T_k$ for the ι_p^L norm. For each $n \in \mathbb{N}$, define $t_n : L_p(\mu) \longrightarrow E^{**}$ by $t_n(f) = \sum_{k=n+1}^{\infty} A_k(f|_{\Omega_k})$. By the same argument used above we obtain, $t_n \in Lip_0(L_p(\mu), E^{**})$ with $Lip(t_n) \leq \sum_{k=n+1}^{\infty} Lip(A_k)$ and so $\lim_{n \to \infty} Lip(t_n) = 0$. It is easy to see that $T - \sum_{k=1}^{n} T_k = t_n i_p^\mu B$. Then, $\lim_{n \to \infty} \iota_p^L(T - \sum_{k=1}^{n} T_k) = 0$.

(ii) Fix a point $x_0 \in X$ and take $\Omega = \{x_0\}$, $\Sigma = \{\Omega, \emptyset\}$ and $\mu : \Sigma \longrightarrow \mathbb{R}$ defined by $\mu(\Omega) = 1, \mu(\emptyset) = 0$. Then (Σ, Ω, μ) is a probability space. Clearly, $L_\infty(\mu)$ and $L_p(\mu)$ contain only constant functions.
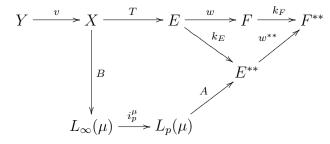
Let $g \in X^\#$ and $v \in E$. Define $B \in Lip_0(X, L_\infty(\mu))$ and $A \in Lip_0(L_p(\mu_i), E^{**})$ by $A(t\mathbf{1}) = t k_E(v)$ for all $t \in \mathbb{K}$ and $B(x) = g(x)\mathbf{1}$

for all $x \in X$, where $\mathbf{1}$ is the constant function equal to 1 on Ω. It is clear that

$$(k_E(vg))(x) = g(x)k_E(v) = g(x)A(\mathbf{1}) = A(g(x)\mathbf{1}) = Ai_p^\mu(g(x)\mathbf{1}) = Ai_p^\mu B(x)$$

for all $x \in X$. Then $vg \in \mathcal{I}_p^L(X, E)$ and $\iota_p^L(vg) \leq Lip(A)Lip(B) = \|v\| \, Lip(g)$.

Let $Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda$. By [13, p. 5275] $\iota_p^L(Id_{\mathbb{K}}) = \iota_p(Id_{\mathbb{K}}) = 1$.

(iii) Let $v \in Lip_0(Y, X)$, $T \in \mathcal{I}_p^L(X, E)$ and $w \in \mathcal{L}(E, F)$. Since T is Lipschitz $p-$integral then we have the following factorization

Since the mappings $B \circ v$ and $w^{**} \circ A$ are Lipschitz then $w \circ T \circ v$ is Lipschitz $p-$integral and

$$\iota_p^L(w \circ T \circ v) = \iota_p^L(w^{**} \circ A \circ i_p^\mu \circ B \circ v) \leq \|w\| \, Lip(A)Lip(B)Lip(v).$$

Taking the infimum, we obtain

$$\iota_p^L(w \circ T \circ v) \leq \|w\| \, \iota_p^L(T)Lip(v).$$

$\square$

2.4. **Lipschitz $p-$nuclear operators.** Let $1 \leq p \leq \infty$. Lipschitz $p-$nuclear operators from metric spaces to Banach spaces are introduced in [13]. A map $T \in Lip_0(X, E)$ is *Lipschitz $p-$nuclear* if there are two Lipschitz mappings $A : \ell_p \longrightarrow E$ and $B : X \longrightarrow \ell_\infty$ and $\lambda = (\lambda_n)_n \in \ell_p$ such that the following diagram commutes:

where $M_\lambda((x_n)_n) := (\lambda_n x_n)_n$, for all $(x_n)_n \in \ell_\infty$. Let $\nu_p^L(T)$ denote the infimum of $Lip(A) \, \|M_\lambda\| \, Lip(B)$ over all factorizations as above; note that $\|M_\lambda\| = \|\lambda\|_{\ell_p}$. The set of all such T is denoted by $\mathcal{N}_p^L(X, E)$.

Proposition 2.7. $(\mathcal{N}_p^L, \nu_p^L)$ *is a Banach Lipschitz operator ideal.*

Proof. We start proving that $Lip(T) \leq \nu_p^L(T)$ for any $T \in \mathcal{N}_p^L(X, E)$. Assume that $T \in \mathcal{N}_p^L(X, E)$. Then, there exist $A \in Lip_0(X, \ell_\infty)$, $B \in Lip_0(\ell_p, E)$ and a diagonal operator $M_\lambda \in \mathcal{L}(\ell_\infty, \ell_p)$ with $\lambda \in \ell_p$ such that T factors as

$$T : X \xrightarrow{A} \ell_\infty \xrightarrow{M_\lambda} \ell_p \xrightarrow{B} E.$$

Clearly $T \in Lip_0(X, E)$ and

$$Lip(T) = Lip(BM_\lambda A) \leq Lip(B) \, \|M_\lambda\| \, Lip(A).$$

Taking the infimum we get $Lip(T) \leq \nu_p^L(T)$.

The spaces $L_p(\mu)$ and $L_\infty(\mu)$ are the standard sequence spaces ℓ_p and ℓ_∞ respectively, whenever μ is the counting measure on $\mathbb{N}$. Then, an easy adaptation of the proof of Proposition 2.6 proves that $\mathcal{N}_p^L$ is a Banach Lipschitz operator ideal. $\square$

2.5. **Lipschitz compact and Lipschitz weakly compact operators.** Following [26] a Lipschitz map $T \in Lip_0(X, E)$ is *Lipschitz compact* (*Lipschitz weakly compact*) if the set

$$\left\{ \frac{T(x) - T(x')}{d(x, x')}, x, x' \in X, x \neq x' \right\}$$

is relatively compact (respectively, relatively weakly compact) in E. Denote by $Lip_{0\mathcal{K}}(X, E)$ and $Lip_{0\mathcal{W}}(X, E)$ the sets of Lipschitz compact operators and Lipschitz weakly compact operators from X to E, respectively. The ideal property has been proved in [26, Proposition 2.3]. Now we prove that $Lip_{0\mathcal{K}}(X, E)$ (respectively $Lip_{0\mathcal{W}}(X, E)$) contains the finite dimensional rank Lipschitz operators. Let T be a finite dimensional rank Lipschitz operator, then by [26, Proposition 2.4] the linearization T_L has finite dimensional rank and so is compact (respectively weakly compact) and by [26, Proposition 2.1 and Proposition 2.2] T is Lipschitz compact (respectively Lipschitz weakly compact). By [26, Corollary 2.6] the spaces $Lip_{0\mathcal{K}}(X, E)$ and $Lip_{0\mathcal{W}}(X, E)$ are closed subspaces of $Lip_0(X, E)$). Then $Lip_{0\mathcal{K}}$ and $Lip_{0\mathcal{W}}$ are Banach Lipschitz operator ideals with the norm Lip.

We can get the next result regarding the classical approximation property; we will adapt this notion to the Lipschitz setting in a more general way in the last section of the paper.

Proposition 2.8. *If E has the approximation property and $T \in Lip_0(X, E)$, then T is Lipschitz compact if, and only if, T can be approximated by Lipschitz finite rank operators.*

Proof. If T is Lipschitz compact then T_L is compact and by the approximation property of E we conclude that T_L can be approximated

by finite rank operators. Then, by Corollary 2.4 T can be approximated by Lipschitz finite rank operators. For the converse see [26, Propositions 2.1 and 2.4]. $\qquad\square$

Remark 2.9. If $\mathcal{I}_{Lip}$ is a Banach Lipschitz operator ideal and we define $\mathcal{I}_{Lip}\cap\mathcal{L}$ as the class of all linear operators between Banach spaces that belong to $\mathcal{I}_{Lip}$, then $\mathcal{I}_{Lip}\cap\mathcal{L}$ is a linear operator ideal. It is shown in [20, Theorem 2] that for a linear operator T, $\pi_p^L(T)$ coincides with the absolutely $p-$summing norm and then, $\Pi_p^L\cap\mathcal{L}$ is the ideal of absolutely $p-$summing operators. The same thing occurs for Lipschitz $p-$integral operators: for a linear operator T, $\iota_p^L(T)$ coincides with the $p-$integral norm and then, $\mathcal{I}_p^L\cap\mathcal{L}$ is the ideal of $p-$integral operators. Also, from [13, Theorem 2.1] it follows that $\mathcal{N}_p^L(X, E^*)\cap\mathcal{L}(X, E^*) = \mathcal{N}_p(X, E^*)$ whenever X is a separable Banach space and E^* is a dual Banach space.

3. Methods to produce Lipschitz operator ideals

We undertake here some methods to produce Lipschitz operator ideals.

3.1. **Composition ideal of Lipschitz mappings.**

Definition 3.1. (Composition Ideals) Given an operator ideal $\mathcal{I}$, a Lipschitz mapping $T \in Lip_0(X, E)$ belongs to the *composition Lipschitz operator ideal* $\mathcal{I}\circ Lip_0$, denoted $T \in \mathcal{I}\circ Lip_0(X, E)$, if there are a Banach space F, a Lipschitz operator $S \in Lip_0(X, F)$ and an operator $u \in \mathcal{I}(F, E)$ such that $T = u \circ S$. If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a normed operator ideal we write $\|T\|_{\mathcal{I}\circ Lip_0} = \inf \|u\|_{\mathcal{I}} Lip(S)$, where the infimum is taken over all u, S as above.

Proposition 3.2. *Let $\mathcal{I}$ be an operator ideal. The following are equivalent for $T \in Lip_0(X, E)$:*
    (1) $T \in \mathcal{I}\circ Lip_0(X, E)$.
    (2) $T_L \in \mathcal{I}(\text{Æ}(X), E)$.
    If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a normed operator ideal, then

$$\|T\|_{\mathcal{I}\circ Lip_0} = \|T_L\|_{\mathcal{I}}.$$

Proof. $(1) \Rightarrow (2)$ Assume that $T \in \mathcal{I}\circ Lip_0(X, E)$. Then there is a Banach space F, a Lipschitz operator $S \in Lip_0(X, F)$ and an operator $u \in \mathcal{I}(F, E)$ such that $T = u \circ S$. Since $T_L = u \circ S_L$, the ideal property ensures that $T_L \in \mathcal{I}(\text{Æ}(X), E)$. If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is normed then,

$$\|T_L\|_{\mathcal{I}} = \|u \circ S_L\|_{\mathcal{I}} \leq \|u\|_{\mathcal{I}} \|S_L\| = \|u\|_{\mathcal{I}} Lip(S).$$

Taking the infimum over all such factorizations we obtain $\|T_L\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I} \circ Lip_0}$.

$(2) \Rightarrow (1)$ Consider the factorization of T given by $T = T_L \circ \delta_X$. Since δ_X is Lipschitz and $T_L \in \mathcal{I}(Æ(X), E)$ then $T \in \mathcal{I} \circ Lip_0(X, E)$ and, if $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is normed we have

$$\|T\|_{\mathcal{I} \circ Lip_0} = \|T_L \circ \delta_X\|_{\mathcal{I} \circ Lip_0} \leq \|T_L\|_{\mathcal{I}} Lip(\delta_X) = \|T_L\|_{\mathcal{I}}.$$

$\square$

Corollary 3.3. *If $\mathcal{I}$ is a (normed, closed, Banach) operator ideal then, $\mathcal{I} \circ Lip_0$ is a (respectively normed, closed, Banach) Lipschitz operator ideal.*

Proof. Let us check that a composition Lipschitz operator ideal is a Lipschitz operator ideal. Using Proposition 3.2,

$$T, S \in \mathcal{I} \circ Lip_0(X, E) \iff T_L, S_L \in \mathcal{I}(Æ(X), E)$$
$$\iff (\alpha T + \beta S)_L = \alpha T_L + \beta S_L \in \mathcal{I}(Æ(X), E), \forall \alpha, \beta \in \mathbb{K}$$
$$\iff \alpha T + \beta S \in \mathcal{I} \circ Lip_0(X, E), \forall \alpha, \beta \in \mathbb{K}.$$

As a consequence of Proposition 3.2, $\mathcal{I} \circ Lip_0(X, E)$ contains the finite dimensional rank Lipschitz operators. To prove the ideal property consider $v \in Lip_0(Y, X)$, $T \in \mathcal{I} \circ Lip_0(X, E)$ and $w \in \mathcal{L}(E, F)$. There are a Banach space G, a Lipschitz operator $S \in Lip_0(X, G)$ and an operator $u \in \mathcal{I}(G, E)$ such that $T = u \circ S$. From the ideal property, $w \circ u \in \mathcal{I}(G, E)$. It is clear that $S \circ v \in Lip_0(Y, E)$ and we conclude that $w \circ T \circ v \in \mathcal{I} \circ Lip_0(Y, F)$. Therefore $\mathcal{I} \circ Lip_0$ is a Lipschitz operator ideal.

Now we show that $\mathcal{I} \circ Lip_0$ is closed whenever $\mathcal{I}$ is. Consider a sequence $(T_i)_i \in \mathcal{I} \circ Lip_0(X, E)$ converging to T in $Lip_0(X, E)$. From

$$\lim_{i \to \infty} \|(T_i)_L - T_L\| = \lim_{i \to \infty} \|(T_i - T)_L\| = \lim_{i \to \infty} Lip(T_i - T) = 0.$$

it follows that $T_L \in \mathcal{I}(Æ(X), E)$ and so, $T \in \mathcal{I} \circ Lip_0(X, E)$.

Clearly, $(\mathcal{I} \circ Lip_0, \|\cdot\|_{\mathcal{I} \circ Lip_0})$ is a normed Lipschitz ideal whenever $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is normed. Just for the sake of completeness note that if $v \in Lip_0(Y, X)$, $T \in \mathcal{I} \circ Lip_0(X, E)$ and $w \in \mathcal{L}(E, F)$ then, by [27, Lemma 3.1]) the linearization of $w \circ T \circ v$ is $w \circ T_L \circ \hat{v}$ for a suitable linear operator $\hat{v} \in \mathcal{L}(Æ(Y), Æ(X))$, with $\|\tilde{v}\| = Lip(v)$.

$$\begin{aligned} \|w \circ T \circ v\|_{\mathcal{I} \circ Lip_0} &= \|w \circ T_L \circ \hat{v}\|_{\mathcal{I} \circ Lip_0} \\ &\leq \|w\| \|T_L\|_{\mathcal{I}} \|\hat{v}\| \\ &= \|w\| \|T\|_{\mathcal{I} \circ Lip_0} Lip(v). \end{aligned}$$

To finish the proof, it easily follows from Proposition 3.2 that if $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a Banach operator ideal then, $(\mathcal{I} \circ Lip_0, \|.\|_{\mathcal{I} \circ Lip_0})$ is a Banach Lipschitz operator ideal. $\qquad\square$

Example 3.4. Let $1 < p \leq \infty$. Cohen [14] introduces the class of *strongly $p-$summing* linear operators as those $T \in \mathcal{L}(E, F)$ fulfilling that there exists $C \geq 0$ such that for all $n \in \mathbb{N}$, $x_1, ..., x_n \in E$ and $y_1^*, ..., y_n^* \in F^*$

$$\sum_{i=1}^n |\langle T(x_i), y_i^* \rangle| \leq C \Big(\sum_{i=1}^n \|x_i\|^p \Big)^{\frac{1}{p}} \sup_{\varphi \in B_{F^{**}}} \Big(\sum_{i=1}^n |\varphi(y_i^*)|^{p'} \Big)^{\frac{1}{p'}}.$$

The space of all strongly $p-$summing linear operators is denoted by $\mathcal{D}_p(E, F)$ and the infimum of all C by $d_p(T)$. In [14, Theorem 2.2.2] it is shown that $T \in \mathcal{D}_p(E, F)$ if, and only if, $T^* \in \Pi_{p'}(F^*, E^*)$ whenever $1 < p \leq \infty$. In [36] the strongly Lipschitz $p-$summing mappings are introduced. A map $T \in Lip_0(X, E)$ is *strongly Lipschitz $p-$summing* $(1 < p \leq \infty)$, if there are a Banach space F and an operator $u \in D_p(F, E)$ such that

$$|\langle y^*, T(x) - T(x') \rangle| \leq d(x, x') \|u^*(y^*)\| \text{ for all } x, x' \in X, y^* \in E^*.$$

The infimum of all constants $d_p(u)$ is denoted $d_{st,p}^L(T)$. This class of mappings is denoted by $\mathcal{D}_{st,p}^L(X, E)$ and with the norm $d_{st,p}^L(T)$ it is a Banach space. Moreover, it is proved that T is strongly Lipschitz $p-$summing if, and only if, T_L is strongly $p-$summing. Then by Proposition 3.2 every strongly Lipschitz $p-$summing mapping T can be considered the composition of a Lipschitz mapping and a strongly $p-$summing mapping, that is, $\mathcal{D}_{st,p}^L = D_p \circ Lip_0$.

Example 3.5. The following variant of Lipschitz integral mapping is introduced in [7]. A mapping $T \in Lip_0(X, E)$ is *Lipschitz Grothendieck-integral* if there are a probability measure space (Ω, Σ, μ), a linear operator $A \in \mathcal{L}(L_1(\mu), E^{**})$ and a Lipschitz mapping $B \in Lip_0(X, L_\infty(\mu))$ such that $k_E \circ T = A \circ i_1^\mu \circ B$. The Lipschitz G-integral norm of T is defined by $Lip_{GI}(T) = \inf \|A\| Lip(B)$, where the infimum is taken over all factorizations as above. This class of mappings is denoted by $Lip_{0GI}(X, E)$. If we denote by $(\mathcal{I}_1, \iota_1)$ the Banach ideal of integral linear operators (we refer to [19, 17] for integral linear operators) then, $\mathcal{I}_1(F, E) \subset Lip_{0GI}(F, E) \subset \mathcal{I}_1^L(F, E)$, for all Banach spaces E and F. The maximality of $\mathcal{I}_1$ implies that $\iota_1(T) = Lip_{GI}(T) = \iota_1^L(T)$, for all $T \in \mathcal{I}_1(F, E)$. In [7, Proposition 2.1] it is proved that Lip_{0GI} is a Banach Lipschitz operator ideal (a similar notion of Lipschitz operator ideal is considered in [7], without going into a general study).

Proposition 2.4 in [7] shows that $T \in Lip_{0GI}(X, E)$ if, and only if, $T_L \in \mathcal{I}_1(\text{Æ}(X), E)$. Hence, by Proposition 3.2, $Lip_{0GI} = \mathcal{I}_1 \circ Lip_0$.

Proposition 3.6. *Let X be a pointed metric space and E be a Banach space. We have*

(1) $Lip_{0\mathcal{K}}(X, E) = \mathcal{K} \circ Lip_0(X, E)$ *isometrically.*
(2) $Lip_{0\mathcal{W}}(X, E) = \mathcal{W} \circ Lip_0(X, E)$ *isometrically.*

Proof. It is a consequence of Propositions 2.1 and 2.2 in [26] and Proposition 3.2. $\square$

Remark 3.7. In general, a Lipschitz operator ideal $\mathcal{I}_{Lip}$ does not coincide with $\mathcal{I} \circ Lip_0$. For example, the ideal of Lipschitz p-summing operators Π_p^L and the corresponding composition $\Pi_p \circ Lip_0$ do not coincide. Indeed, $\delta_{\mathbb{K}}$ is Lipschitz p-summing, but its linearization is the identity map on the infinite dimensional space $\text{Æ}(\mathbb{K})$ and so, it cannot be absolutely p-summing. Proposition 3.2 ensures now that $\delta_{\mathbb{K}}$ does not belong to $\Pi_p \circ Lip_0$. The same situation occurs for Lipschitz p-nuclear and Lipschitz p-integral operators.

3.2. **The dual Lipschitz ideal.** The *dual* of an operator ideal $\mathcal{I}$ is defined as follows: for Banach spaces E and F,

$$\mathcal{I}^{dual}(E, F) = \{u \in \mathcal{L}(E, F) : u^* \in \mathcal{I}(F^*, E^*)\},$$

where $u^* : F^* \longrightarrow E^*$ is the adjoint of u. It is well known that $\mathcal{I}^{dual}$ is an operator ideal. We write $\|u\|_{\mathcal{I}^{dual}} = \|u^*\|_{\mathcal{I}}$.

Definition 3.8. The *Lipschitz dual* of a given operator ideal $\mathcal{I}$ is defined by

$$\mathcal{I}^{Lip_0-dual}(X, E) = \left\{T \in Lip_0(X, E) : T^t \in \mathcal{I}(E^*, X^{\#})\right\}$$

If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a quasinormed operator ideal, define

$$\|T\|_{\mathcal{I}^{Lip_0-dual}} = \left\|T^t\right\|_{\mathcal{I}}.$$

Theorem 3.9. *If $\mathcal{I}$ is an operator ideal then,*

$$\mathcal{I}^{Lip_0-dual} = \mathcal{I}^{dual} \circ Lip_0$$

Moreover, if $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is normed then,

$$\|\cdot\|_{\mathcal{I}^{Lip_0-dual}} = \|\cdot\|_{\mathcal{I}^{dual} \circ Lip_0}.$$

Proof. Assume that $T \in \mathcal{I}^{Lip_0-dual}$, that is, $T^t \in \mathcal{I}(E^*, X^{\#})$. Consider the linearization operator $R : X^{\#} \to \text{Æ}(X)^*$ defined in Section 1. As we said there, it is linear, continuous and $\|R\| \leq 1$. Since $(T_L)^* =$

$R \circ T^t \in \mathcal{I}(E^*, \!\!\!\!Æ(X)^*)$ then, $T_L \in \mathcal{I}^{dual}(Æ(X), E)$. By Proposition 3.2 we conclude $T \in \mathcal{I}^{dual} \circ Lip_0$. Besides,

$$
\begin{aligned}
\|T\|_{\mathcal{I}^{dual} \circ Lip_0} &= \|T_L\|_{\mathcal{I}^{dual}} = \|(T_L)^*\|_{\mathcal{I}} = \|R \circ T^t\|_{\mathcal{I}} \\
&\leq \|R\| \, \|T^t\|_{\mathcal{I}} = \|T\|_{\mathcal{I}^{Lip_0-dual}} .
\end{aligned}
$$

Assume that $T \in \mathcal{I}^{dual} \circ Lip_0(X, E)$. Then $T = u \circ S$ with $u^* \in \mathcal{I}(E^*, F^*)$ and $S \in Lip_0(X, F)$ for some Banach space F. We have $T^t = S^t \circ u^*$. By the ideal property we conclude that $T^t \in \mathcal{I}(E^*, X^{\#})$, that is $T \in \mathcal{I}^{Lip_0-dual}$. To finish the proof, we show the equality of the norms. Let $\epsilon > 0$, choosing F, S and u such that $\|u\|_{\mathcal{I}^{dual}} \cdot Lip(S) \leq (1 + \epsilon) \|T\|_{\mathcal{I}^{dual} \circ Lip_0}$ we get

$$
\begin{aligned}
\|T\|_{\mathcal{I}^{Lip_0-dual}} &= \|T^t\|_{\mathcal{I}} = \|S^t \circ u^*\|_{\mathcal{I}} \leq \|u^*\|_{\mathcal{I}} \, \|S^t\| \\
&= \|u\|_{\mathcal{I}^{dual}} \, Lip(S) \\
&\leq (1 + \epsilon) \|T\|_{\mathcal{I}^{dual} \circ Lip_0} .
\end{aligned}
$$

Letting $\epsilon \longrightarrow 0$ we obtain $\|T\|_{\mathcal{I}^{Lip_0-dual}} \leq \|T\|_{\mathcal{I}^{dual} \circ Lip_0}$. $\qquad \square$

Propositions 3.4 and 3.5 in [26] can be proved directly from our previous results.

Corollary 3.10. *A Lipschitz operator is compact (weakly compact) if, and only if, its transpose is a compact (weakly compact) linear operator.*

Proof. We get the results directly from Proposition 3.6, Schauder's (Gantmacher's) theorem and Theorem 3.9; indeed, they give the equalities $Lip_{0\mathcal{K}} = \mathcal{K} \circ Lip_0 = \mathcal{K}^{dual} \circ Lip_0 = \mathcal{K}^{Lip_0-dual}$ and $Lip_{0\mathcal{W}} = \mathcal{W} \circ Lip_0 = \mathcal{W}^{dual} \circ Lip_0 = \mathcal{W}^{Lip_0-dual}$. $\qquad \square$

4. Applications: the approximation property for Lipschitz operator ideals

Recall first the classical approximation property: a Banach space E has the *approximation property* if for every Banach spaces F (or just for E) every linear operator $T : E \to F$ can be approximated uniformly on compact sets by finite rank operators or, equivalently, for every Banach space F every compact linear operator $T : F \to E$ can be approximated uniformly on bounded sets by finite rank operators [24, Proposition 35]. The approximation property has been widely studied and the reader can find several books where it has been treated (for example in [17, Ch.I,§5] and [10]). Roughly speaking, three are the main ingredients that play an important role in the approximation property:

the approximating operators (the finite rank operators), the approximated operators (the linear or compact operators) and the bornology where the convergence is considered (compact or bounded sets). In the last decades, some variants of the approximation property related to an operator ideal $\mathcal{I}$ have been considered. These variants concern to what extent the approximation property can spread whenever its ingredients are replaced by some others related to $\mathcal{I}$. The main situations that have been considered are:

• To replace finite rank operators by operators belonging to an operator ideal. The main purpose in [4] is the study of Banach spaces E for which every linear operator $T \in \mathcal{L}(E, E)$ can be approximated uniformly on compact subsets of E by operators in $\mathcal{I}(E, E)$.

• To replace bounded linear or compact operators by elements in a general ideal $\mathcal{I}$. This variant of approximation property on a Banach space E related to an operator ideal $\mathcal{I}$ studies when the space of finite rank operators $\mathcal{F}(F, E)$ is dense in $\mathcal{I}(F, E)$ for every Banach space F (see for example [32] and the references therein).

• To replace compact sets by another class of sets with some kind of compactness related to $\mathcal{I}$. The new class of sets is formed by $\mathcal{I}$-compact sets. This notion was introduced by Carl and Stephani [9] and the related approximation property has been studied in e.g. [18, 30, 31].
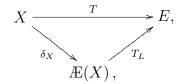
The space of all finite rank linear operators between two Banach spaces E and F is denoted by $\mathcal{F}(E, F)$. Let $\mathcal{I}$ be an operator ideal. A subset B of a Banach space E is *relatively $\mathcal{I}$-compact* if there is a Banach space G and an operator $S \in \mathcal{I}(G, E)$ such that $B \subseteq S(M)$, where M is a compact subset of G. The associated operator definition of this set-theoretic notion is the following. Let T be a linear operator between Banach spaces E and F and consider an operator ideal $\mathcal{I}$. It is said that T is $\mathcal{I}$-*compact* if $T(B_E)$ is a relatively $\mathcal{I}$-compact subset of F. The operator ideal formed by all linear $\mathcal{I}$-compact operators is denoted by $\mathcal{K}_{\mathcal{I}}$ (see [30, Sec.2]).

Let $(\mathcal{I}, \| \cdot \|_{\mathcal{I}})$ be a Banach operator ideal. Following [32], a Banach space E has the $\mathcal{I}$-*approximation property* if for every Banach space F, the space of finite rank linear operators $\mathcal{F}(F, E)$ is $\| \cdot \|_{\mathcal{I}}$-dense in $\mathcal{I}(F, E)$. In [30] it is also considered the situation of replacing the norm $\| \cdot \|_{\mathcal{I}}$ by the operator norm $\| \cdot \|$ in $\mathcal{L}$: a Banach space E has the $\mathcal{I}$-*uniform approximation property* if for every Banach space F, the space of finite rank linear operators $\mathcal{F}(F, E)$ is $\| \cdot \|$-dense in $\mathcal{I}(F, E)$. Note that, whenever the ideal of compact operators $\mathcal{K}$ is considered, the $\mathcal{K}$-(uniform) approximation property is just the approximation property.

In particular, a Banach space E has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property if for every Banach space F, $\mathcal{F}(F, E)$ is norm dense in

$\mathcal{K}_{\mathcal{I}}(F, E)$, that is, every $\mathcal{I}$-compact operator from F into E can be uniformly approximated by finite rank operators. In [30, Proposition 3.2] it is proved that a Banach space E has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property if, and only if, the identity $Id_E : E \to E$ can be approximated uniformly on relatively $\mathcal{I}$-compact sets by finite rank operators.

This is the starting point of our analysis of the "Lipschitz version" of these notions, that is explained in what follows. We will consider Lipschitz ideals $\mathcal{I}_{Lip}$ that are composition ideals, that is, $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ for some linear operator ideal $\mathcal{I}$. Taking into account that each Lipschitz operator can be factored as

$$
\begin{array}{ccc}
X & \xrightarrow{\quad T \quad} & E, \\
& \searrow{\delta_X} \qquad \nearrow{T_L} & \\
& \textit{Æ}(X), &
\end{array}
$$

$\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ satisfies that a Lipschitz map T belongs to $\mathcal{I}_{Lip}(X, E)$ if, and only if, $T_L \in \mathcal{I}(\textit{Æ}(X), X)$ (see Proposition 3.2).

We consider the following natural definitions.

Definition 4.1. Let X and Y be pointed metric spaces, and $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ a composition Lipschitz operator ideal.

A set $K \subseteq X$ is *relatively $\mathcal{I}$-Lipschitz compact* if there is a pointed metric space Y and a Lipschitz operator $S : Y \to \textit{Æ}(X)$ in $\mathcal{I}_{Lip}$ such that $\delta_X(K) \subseteq S(M)$, where M is a compact subset of Y.

Let Y and X be pointed metric spaces. We say that a Lipschitz operator $\phi : Y \to X$ is *$\mathcal{I}$-Lipschitz compact* if $\phi(B_Y) = \phi(\{x \in Y : d(x, 0) \leq 1\})$ is relatively $\mathcal{I}$-Lipschitz compact.

Note that, if E is a Banach space then, a Lipschitz operator $T : X \to E$ is $\mathcal{I}$-Lipschitz compact if $T(\{x \in X : d(x, 0) \leq 1\})$ is a relatively $\mathcal{I}$-Lipschitz compact subset of E.

We will write $\mathcal{K}_{\mathcal{I}}^L(X, E)$ for the class of all $\mathcal{I}$-Lipschitz compact operators from X to E, considered as a topological subspace of the Lipschitz operators. This class satisfies some good composition properties. For example, taking into account that the Lipschitz image of a relatively $\mathcal{I}$-Lipschitz compact set is again relatively $\mathcal{I}$-Lipschitz compact, we obtain the following fact: if $T : X \to Y$ is $\mathcal{I}$-Lipschitz compact and $R \in Lip_0(Y, Z)$, then $R \circ T : X \to Z$ is also $\mathcal{I}$-Lipschitz compact.

Proposition 4.2. *Let X be a pointed metric space and let $A \subset X$. Then A is relatively $\mathcal{I}$-Lipschitz compact if, and only if, $\delta_X(A)$ is relatively $\mathcal{I}$-compact.*

Moreover, if E is a Banach space, every relatively $\mathcal{I}$-Lipschitz compact subset of E is relatively $\mathcal{I}$-compact.

Proof. Assume that A is relatively $\mathcal{I}$-Lipschitz compact. There is a pointed metric space Y, $M \subset Y$ compact and $S \in \mathcal{I}_{Lip}(Y, Æ(X))$ such that $\delta_X(A) \subseteq S(M)$. Since we can write $S = S_L \circ \delta_Y$ with $S_L \in \mathcal{I}(Æ(Y), Æ(X))$, and $\delta_Y(M)$ is compact, we conclude that $\delta_X(A)$ is a relatively $\mathcal{I}$-compact set.

Now assume that $\delta_X(A)$ is relatively $\mathcal{I}$-compact. Let Z be a Banach space, K a compact subset of Z and $T \in \mathcal{I}(Z, Æ(X))$ such that $\delta_X(A) \subseteq T(K)$. This implies that $T \in \mathcal{I}_{Lip}(Z, Æ(X))$, which now does give the result.

Finally, assume that X is a Banach space E. Let us see that every relatively $\mathcal{I}$-Lipschitz compact subset of E is relatively $\mathcal{I}$-compact. Take a relatively $\mathcal{I}$-Lipschitz compact subset K of E. This means that there is a Lipschitz map $S : Y \to Æ(E)$ in $\mathcal{I}_{Lip}$ from a pointed metric space Y and a compact subset $M \subseteq Y$ such that $\delta_E(K) \subseteq S(M)$. We have the canonical factorization for S given by $S_L \circ \delta_Y$, and since $\mathcal{I}_{Lip}$ is a composition Lipschitz ideal, we have that S_L belongs to the linear ideal $\mathcal{I}$. Then the compact subset $M_1 := \delta_Y(M)$ of $Æ(Y)$ satisfies that there is a linear operator S_L in $\mathcal{I}$ such that $\delta_E(K) \subseteq S_L(M_1)$. Consider $(Id_E)_L : Æ(E) \to E$. Then we have that $K = (Id_E)_L(\delta_E(K)) \subseteq (Id_E)_L \circ S_L(M_1)$, where $(Id_E)_L \circ S_L$ is a linear map in $\mathcal{I}$, and so K is a relatively $\mathcal{I}$-compact subset of E.

$\square$

From Proposition 4.2, we obtain the next

Corollary 4.3. *A linear map $T : F \to E$ between Banach spaces is $\mathcal{I}$-Lipschitz compact if, and only if, it is linear $\mathcal{I}$-compact.*

In what follows we will show more concrete information about the relation between the linear and the Lipschitz $\mathcal{I}$-compactness for operators. We start with a lemma that is already known. The linearization $\beta_E : Æ(E) \to E$ of the identity map in E is known as the barycentric map, and it is in fact a quotient map (see for example [28, p.25] and [21, Lemma 2.4], where this map is denoted by β). This gives a proof of the lemma; however, we write a direct proof for the aim of completeness.

Lemma 4.4. *Let E be a Banach space and let $\beta_E : Æ(E) \to E$ be the linearization of the identity map $Id^E : E \to E$. Then, $\beta_E(B_{Æ(E)}) = B_E$.*

Proof. Since $\beta_E \circ \delta_E(x) = Id^E(x) = x$ and $\delta_E(B_E) \subset B_{Æ(E)}$, the inclusion $B_E \subset \beta_E(B_{Æ(E)})$ is trivial. Let us prove the other inclusion.

It is enough to prove that $\beta_E(B_{\mathcal{M}(E)}) = B_E$. Assume that there is $m \in B_{\mathcal{M}(E)}$ such that $\beta_E(m)$ does not belong to B_E. The Hahn-Banach theorem gives a linear functional $\phi \in E^*$ of norm $\|\phi\| = 1$ such that $|\phi(\beta_E(m))| > 1$. Write $f := \phi \circ \beta_E \in \mathcal{M}(E)^*$ and take a real number a so that $1 < a < |f(m)|$. Let $\epsilon > 0$. Consider a representation of m, $m = \sum_{i=1}^n \lambda_i m_{x_i, x_i'}$ so that $\sum_{i=1}^n |\lambda_i| \|x_i - x_i'\| < 1 + \epsilon$. Then,

$$a < |f(m)| = |\sum_{i=1}^n \lambda_i \phi(\beta_E(\delta_E(x_i) - \delta_E(x_i')))| = |\phi(\sum_{i=1}^n \lambda_i(x_i - x_i'))| < 1 + \epsilon.$$

Letting $\epsilon \to 0$, we get a contradiction. $\qquad\square$

The map $(\delta_E \circ T)_L$ appearing in (iii) of the next result is already known (see [21, Lemma 2.2]): it is the unique linear map $\hat{T} : Æ(F) \to Æ(E)$ such that $\hat{T} \circ \delta_F = \delta_E \circ T$.

Proposition 4.5. *Let F and E be Banach spaces, and $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ a composition Lipschitz operator ideal. Consider a Lipschitz map $T : F \to E$.*

(i) *If T is linear and $\mathcal{I}$-Lipschitz compact, then $T_L : Æ(F) \to E$ is (linear) $\mathcal{I}$-compact.*

(ii) *If $\delta_E \circ T_L : Æ(F) \to Æ(E)$ sends the unit ball to a relatively $\mathcal{I}$-compact set, then T is $\mathcal{I}$-Lipschitz compact.*

(iii) *If $(\delta_E \circ T)_L : Æ(F) \to Æ(E)$ is (linear) $\mathcal{I}$-compact, then T is $\mathcal{I}$-Lipschitz compact.*

(iv) *If T is linear, then T is $\mathcal{I}$-Lipschitz compact if, and only if, $T_L : Æ(F) \to E$ is (linear) $\mathcal{I}$-compact.*

Proof. (i) Assume that T is linear and $\mathcal{I}$-Lipschitz compact. By definition, $T(B_F)$ is relatively Lipschitz $\mathcal{I}$-compact. By Proposition 4.2, $T(B_F)$ is relatively $\mathcal{I}$-compact. Note that since T is linear so is $T \circ \beta_F : Æ(F) \to E$ (where β_F is the barycentric map defined above), and

$$(T \circ \beta_F) \circ \delta_F = T \circ (\beta_F \circ \delta_F) = T.$$

But T_L is by definition the unique linear map $Æ(F) \to E$ such that $T = T_L \circ \delta_F$, so we obtain that $T_L = T \circ \beta_F$. Now, using Lemma 4.4,

$$T_L(B_{Æ(F)}) = T \circ \beta_F(B_{Æ(F)}) = T(B_F),$$

so $T_L(B_{Æ(F)})$ is a relatively $\mathcal{I}$-compact set and thus T_L is linear $\mathcal{I}$-compact.

(ii) This is an immediate consequence of Proposition 4.2. By hypothesis we have that $\delta_E \circ T_L(B_{Æ(F)})$ is relatively $\mathcal{I}$-compact. By Proposition 4.2, we have that $T_L(B_{Æ(F)})$ is a relatively $\mathcal{I}$-Lipschitz compact

set. Since $\delta_F(B_F) \subset B_{Æ(F)}$ and $T = T_L \circ \delta_F$, it follows that $T(B_F)$ is relatively $\mathcal{I}$-Lipschitz compact and thus T is $\mathcal{I}$-Lipschitz compact.

(iii) If $(\delta_E \circ T)_L$ is $\mathcal{I}$-compact, then $(\delta_E \circ T)_L(B_{Æ(F)})$ is relatively $\mathcal{I}$-compact. Since $(\delta_E \circ T)(B_F) = ((\delta_E \circ T)_L) \circ \delta_F)(B_F) \subseteq (\delta_E \circ T)_L(B_{Æ(F)})$ it follows that $(\delta_E \circ T)(B_F)$ is relatively $\mathcal{I}$-compact and thus by Proposition 4.2 we conclude that $T(B_F)$ is relatively $\mathcal{I}$-Lipschitz compact.

(iv) We only need to prove ($\Longleftarrow$), since (i) gives the other implication.

Suppose that T_L is $\mathcal{I}$-compact. Then $T_L(B_{Æ(F)})$ is relatively $\mathcal{I}$-compact and thus it is relatively $\mathcal{I}$-Lipschitz compact by Proposition 4.2. Since $\delta_F(B_F) \subset B_{Æ(F)}$,

$$T(B_F) = (T_L \circ \delta_F)(B_F) \subset T_L(B_{Æ(F)}),$$

so $T(B_F)$ is relatively $\mathcal{I}$-Lipschitz compact, and thus T is $\mathcal{I}$-Lipschitz compact.

$\square$

Proposition 4.5 gives that for linear operators, being (Lipschitz) compact and being $\mathcal{L}$-Lipschitz compact coincide. The first three statements of the following result have been proved in [26, Proposition 2.1].

Corollary 4.6. *Let F and E be Banach spaces and consider a linear operator $T : F \to E$. The following assertions are equivalent.*

(i) *T is compact.*
(ii) *$T_L : Æ(F) \to E$ is compact.*
(iii) *T is Lipschitz compact as a Lipschitz map.*
(iv) *T is $\mathcal{L}$-Lipschitz compact as a Lipschitz map.*

4.1. **The $\mathcal{I}$-approximation property for Lipschitz operators.** Consider a linear operator ideal $\mathcal{I}$ and let $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ be the associated composition Lipschitz operator ideal. On $Lip_0(X, E)$, we define the topology *Lipschitz-$\tau_{\mathcal{I}}$* of uniform convergence on $\mathcal{I}$-Lipschitz compact sets in the space of operators $Lip_0(X, E)$ as the one generated by the seminorms

$$q_K(T) := \sup_{x \in K} \|T(x)\| = \sup_{m \in \delta_K(K)} \|T_L(m)\|_E,$$

where K is a relatively $\mathcal{I}$-Lipschitz compact set of X.

Note that this topology induces on the space $\mathcal{L}(F, E)$, of linear operators between Banach spaces F and E, the topology $\tau_{\mathcal{I}}$ of uniform convergence on $\mathcal{I}$-compact sets.

Definition 4.7. Let X be a pointed metric space. Consider a class of operators $\mathcal{O}(X, Æ(X)) \subseteq Lip_0(X, Æ(X))$ with the operations inherited

from this linear space. We say that X has the $\mathcal{I}$-*Lipschitz approxima-tion property with respect to* $\mathcal{O}(X, \text{Æ}(X))$ if $\delta_X : X \to \text{Æ}(X)$ belongs to the Lipschitz-$\tau_{\mathcal{I}}$-closure of $\mathcal{O}(X, \text{Æ}(X))$.

Of course, when looking for a genuine version of the approximation property for metric spaces, the elements of $\mathcal{O}$ must have some sort of finite-range-type property. In fact, the case $\mathcal{O} = Lip_{0\mathcal{F}}$ provides the main classical characterization of an approximation type property, as we will show in what follows. There are also two more interesting cases of sets of operators $\mathcal{O}$ that will be analyzed in the next section.

The first result is a natural extension of Proposition 3.6 of [31] for the Lipschitz case that can be obtained as a consequence of the factor-ization of the Lipschitz operators through $\text{Æ}(X)$.

Proposition 4.8. *Let X be a pointed metric space. The following assertions are equivalent.*

(i) *X has the $\mathcal{I}$-Lipschitz approximation property with respect to $Lip_{0\mathcal{F}}(X, \text{Æ}(X))$.*

(ii) *For every Banach space E, $Lip_{0\mathcal{F}}(X, E)$ is Lipschitz-$\tau_{\mathcal{I}}$ dense in $Lip_0(X, E)$.*

Proof. (i) $\Rightarrow$ (ii). Consider a Banach space E and $\phi \in Lip_0(X, E)$. Take $\varepsilon > 0$ and an $\mathcal{I}$-Lipschitz compact subset K of X. Let $g_\varepsilon \in Lip_0(X, \text{Æ}(X))$ satisfying that $\sup_{x \in K} \|\delta_X(x) - g_\varepsilon(x)\|_{\text{Æ}(X)} < \varepsilon$. Since there is a factorization for ϕ given by $\phi = \phi_L \circ \delta_X$, for each $x \in K$ we get that

$$\|\phi(x) - \phi_L \circ g_\varepsilon(x)\| \leq \|\phi_L\| \|\delta_X(x) - g_\varepsilon(x)\|_{\text{Æ}(X)} \leq \varepsilon \, Lip(\phi).$$

This gives the proof. (ii) $\Rightarrow$ (i) is obvious.

$\square$

The second main property related to the approximation property concerns the approximation of compact operators by finite rank ones. Let us show the Lipschitz version. If A is a subset of a Banach space E then $\overline{co}(A)$ denotes the closed convex hull of A.

Proposition 4.9. *Let X be a pointed metric space and $\mathcal{I}$ be an operator ideal. If X has the Lipschitz $\mathcal{I}$-approximation property with respect to $Lip_{0\mathcal{F}}(X, \text{Æ}(X))$ then, for any pointed metric space Z and any $\mathcal{I}$-Lipschitz compact mapping $\phi : Z \to X$, the mapping $\delta_X \circ \phi$ can be approximated by finite rank operators of $Lip_{0\mathcal{F}}(Z, \text{Æ}(X))$ uniformly on B_Z.*

Proof. Assume that ϕ is Lipschitz $\mathcal{I}$-compact. There is a relatively $\mathcal{I}$-Lipschitz compact subset K of X such that $\phi(B_Z) \subseteq K$. Fix $n \in \mathbb{N}$.

Then by the approximation property of X there is a finite rank Lipschitz map g_n such that $\sup_{x \in K} \|\delta_X(x) - g_n(x)\|_{Æ(X)} < 1/n$. Consequently,

$$\sup_{z \in B_Z} \|\delta_X \circ \phi(z) - g_n \circ \phi(z)\|_{Æ(X)} \leq \sup_{x \in K} \|\delta_X(x) - g_n(x)\|_{Æ(X)} < \frac{1}{n}.$$

This gives a sequence $(g_n \circ \phi)_n$ of finite rank Lipschitz maps converging to $\delta_X \circ \phi$ uniformly on B_Z, and the result follows.

$\square$

4.2. **The relation between Lipschitz and linear approximation properties.** The purpose of this section is to show that the new concepts and results we have stated for the Lipschitz setting fit with the definitions and properties of the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property. To establish the connection, we consider the $\mathcal{I}$-Lipschitz approximation property with respect to the sets $\mathcal{O}_1(E, Æ(E)) = \delta_E \circ Lip_{0\mathcal{F}}(E, E)$ and $\mathcal{O}_2(X, Æ(X)) = Lip_{0\mathcal{F}}(Æ(X), Æ(X)) \circ \delta_X$. We will show that the choice of $\mathcal{O}_1$ or $\mathcal{O}_2$ depends on which version of the two canonical cases we want to get: either when X is a Banach space and has the approximation property or when $Æ(X)$ has the approximation property.

Our first aim now is to prove that the $\mathcal{I}$-Lipschitz approximation property is weaker than the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property when they can be compared, that is, if a Banach space E has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property, then it has the $\mathcal{I}$-Lipschitz approximation property with respect to the set $\delta_E \circ Lip_{0\mathcal{F}}(E, E)$ too. This clearly provides a lot of examples of our new approximation property for pointed metric spaces.

Proposition 4.10. *Let $\mathcal{I}$ be an operator ideal. Let E be a Banach space with the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property. Then it has the $\mathcal{I}$-Lipschitz approximation property as a metric space with respect to the set $\delta_E \circ Lip_{0\mathcal{F}}(E, E)$.*

Proof. Suppose that E has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property. Then there is a sequence of finite rank operators $(T_n)_n$ that converges to Id_E in the $\tau_{\mathcal{I}}$ topology. Let us show that the sequence $(\hat{T}_n)_n$ of Lipschitz maps defined as $\hat{T}_n(x) = \delta_E(T_n(x)) \in Æ(E)$, $x \in E$, converges to δ_E in the Lipschitz-$\tau_{\mathcal{I}}$ topology. For a fixed relatively $\mathcal{I}$-Lipschitz compact subset K of E, consider the seminorm

$$q_K(R) := \sup_{x \in K} \|R(x)\|_{Æ(E)}, \quad R \in Lip_0(E, Æ(E)).$$

Recall that, by Proposition 4.2, every relatively $\mathcal{I}$-Lipschitz compact subset of E is relatively $\mathcal{I}$-compact. Thus, K is also a relatively $\mathcal{I}$-compact subset of E, and then this formula defines also a seminorm of the topology $\tau_{\mathcal{I}}$. We get

$$q_K(\delta_E - \hat{T}_n) = \sup_{x \in K} \|m_{x,0} - \delta_E(T_n(x))\|_{\text{Æ}(E)}$$

$$= \sup_{x \in K} \|m_{x,0} - m_{T_n(x),0}\|_{\text{Æ}(E)} \leq \sup_{x \in K} \|x - T_n(x)\|_E.$$

This proves the result. $\qquad\square$

Some results are known about the standard approximation property for the free spaces $\text{Æ}(X)$ (see [15, 16, 22] and the references therein). In this direction, we can also show that under the hypothesis that $\text{Æ}(X)$ has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property, we have that X has the $\mathcal{I}$-Lipschitz approximation property too with respect to a finite-range-type class of operators, showing that our definition can also be applied in these cases. Recall that $\mathcal{F}(E, E)$ denotes the space of all finite rank operators on the Banach space E.

Proposition 4.11. *Let $\mathcal{I}$ be an operator ideal and consider the associated composition Lipschitz operator ideal $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$. Let X be a pointed metric space. If $\text{Æ}(X)$ has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property, then X has the $\mathcal{I}$-Lipschitz approximation property with respect to the class $\mathcal{F}(\text{Æ}(X), \text{Æ}(X)) \circ \delta_X$.*

Proof. Suppose that $\text{Æ}(X)$ has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property as a Banach space. Fix a relatively $\mathcal{I}$-Lipschitz compact subset K of X. By Proposition 4.2 $\delta_X(K)$ is a relatively $\mathcal{I}$-compact set.

By [30, Proposition 3.2], there is a sequence of linear finite rank operators $T_n : \text{Æ}(X) \to \text{Æ}(X)$ such that T_n converges to $Id_{\text{Æ}(X)}$ uniformly on $\delta_X(K)$. Consider the finite rank Lipschitz maps $\tilde{T}_n := T_n \circ \delta_X$, that define Lipschitz operators from X to $\text{Æ}(X)$. It follows that

$$\sup_{x \in K} \|\delta_X(x) - \tilde{T}_n(x))\|_{\text{Æ}(X)} = \sup_{x \in K} \|\delta_X(x) - T_n \circ \delta_X(x)\|_{\text{Æ}(X)}$$

$$= \sup_{w \in \delta_X(K)} \|w - T_n(w)\|_{\text{Æ}(X)}$$

and this finishes the proof. $\qquad\square$

lot with his/her contributions, that have given new light for a better understanding of the ideas that it contains.

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