INVOLUTIVENESS OF LINEAR COMBINATIONS OF A QUADRATIC OR TRIPOTENT MATRIX AND AN ARBITRARY MATRIX

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ABSTRACT. In this article, we characterize the involutiveness of the linear combination of the form $a_1A_1 + a_2A_2$ when a_1, a_2 are nonzero complex numbers, A_1 is a quadratic or tripotent matrix, and A_2 is arbitrary, under certain properties imposed on A_1 and A_2.
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1. INTRODUCTION AND PRELIMINARY RESULTS

The symbols $\mathbb{C}$ and $\mathbb{C}^*$ will denote the set of complex numbers and nonzero complex numbers, respectively. Let $\mathbb{C}^{n\times m}$ denote the set of all complex $n\times m$ matrices. If $A \in \mathbb{C}^{n\times m}$, then A^* denotes the conjugate transpose of A. The identity matrix of order n will be denoted by I_n. For a given square complex matrix A, the set of eigenvalues of A will be denoted by $\sigma(A)$. The symbol $\oplus$ will denote the direct sum of matrices.

The inheritance of the idempotency, involutiveness, or tripotency by linear combinations of idempotents, involutive, or tripotents has very useful applications in the theory of distributions of quadratic forms in normal variables (see e.g. [20, 21, 22]). The reader may find in [21] more applications of idempotent and tripotent matrices. The sets of idempotent and involutive matrices can be dealt by a uniform approach: a quadratic matrix.

Let us define the concept of quadratic matrix and review some properties. Following [1], we say that a matrix $A \in \mathbb{C}^{n\times n}$ is said to be *quadratic* if there exists a second degree polynomial $p : \mathbb{C} \to \mathbb{C}$ such that $p(A) = 0$. Thus, quadratic matrices are a wide class of matrices containing idempotent ($A^2 = A$), involutive ($A^2 = I_n$), and several other types of matrices. The reader is referred to [9] to consult deeper properties of quadratic matrices.

In [19, Theorem 2.1], it was established an useful expression for quadratic matrices. Concretely, for $A \in \mathbb{C}^{n\times n}$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$ one has that $(A-\alpha I_n)(A-\beta I_n) = 0$ if and only if exists a nonsingular $S \in \mathbb{C}^{n\times n}$ such that $A = S(\alpha I_p \oplus \beta I_q)S^{-1}$, where $p, q \in \{0, 1, \dots, n\}$. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be an $\{\alpha, \beta\}$-*quadratic matrix* if $(A - \alpha I_n)(A - \beta I_n) = 0$. Observe that, in particular, an idempotent is a $\{0, 1\}$-quadratic matrix and an involutive matrix is a $\{-1, 1\}$-quadratic matrix.

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The linear combination of the form

$$A = a_1 A_1 + a_2 A_2, \qquad A_1, A_2 \in \mathbb{C}^{n\times n},\ a_1, a_2 \in \mathbb{C}^* \tag{1.1}$$

was investigated by many researchers and many useful results was obtained (see e.g. [2, 3, 4, 6, 7, 13, 15, 17, 19, 20, 22] and references therein).

The purpose of this paper is to investigate the necessary and sufficient conditions for $A = a_1A_1 + a_2A_2$ to be involutive matrix, where A_1 is a quadratic or tripotent matrix and A_2 is arbitrary under some certain conditions.

2. Main results

We begin the study of the linear combination (1.1) when A_1 is a quadratic matrix and A_1, A_2 satisfy certain condition.

Theorem 2.1. *Let $A_1, A_2 \in \mathbb{C}^{n\times n} \setminus \{0\}$ and $\alpha, \beta \in \mathbb{C}$ with $0 \neq \alpha \neq \beta$. Moreover, let A be a linear combination of the form (1.1) with $a_1, a_2 \in \mathbb{C}^*$. If A_1 is a $\{\alpha, \beta\}$-quadratic and $A_1A_2A_1 = A_2A_1$, then $A^2 = I_n$ if and only if there is a nonsingular matrix $V \in \mathbb{C}^{n\times n}$ such that*

$$A_1 = V \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} V^{-1} \tag{2.1}$$

and A_2 satisfies one of the following cases.

(i) $\beta \neq 1$

$$A_2 = V \begin{pmatrix} \frac{1-a_1\alpha}{a_2} I_q & 0 & 0 & L \\ 0 & \frac{-1-a_1\alpha}{a_2} I_{p-q} & M & 0 \\ 0 & 0 & \frac{1-a_1\beta}{a_2} I_r & 0 \\ 0 & 0 & 0 & \frac{-1-a_1\beta}{a_2} I_{n-p-r} \end{pmatrix} V^{-1}, \tag{2.2}$$

being $L \in \mathbb{C}^{q\times(n-p-r)}$ and $M \in \mathbb{C}^{(p-q)\times r}$ arbitrary.

(ii) $\beta = 1$, $\alpha a_1 = 1$.

$$A_2 = V \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\alpha-1}{\alpha a_2} I_r & 0 \\ S & 0 & -\frac{\alpha+1}{\alpha a_2} I_{n-r-p} \end{pmatrix} V^{-1}, \tag{2.3}$$

being $S \in \mathbb{C}^{(n-r-p)\times p}$ arbitrary.

(iii) $\beta = 1$, $\alpha a_1 = -1$.

$$A_2 = V \begin{pmatrix} 0 & 0 & 0 \\ R & \frac{\alpha+1}{\alpha a_2} I_r & 0 \\ 0 & 0 & \frac{1-\alpha}{\alpha a_2} I_{n-r-p} \end{pmatrix} V^{-1}, \tag{2.4}$$

being $R \in \mathbb{C}^{r\times p}$ arbitrary.

Proof. Since A_1 is an $\{\alpha, \beta\}$-quadratic matrix, there exist $p \in \{0, 1, \dots, n\}$ and a nonsingular matrix $U \in \mathbb{C}^{n\times n}$ such that $A_1 = U(\alpha I_p \oplus \beta I_{n-p})U^{-1}$. Let us write $A_2 = U \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} U^{-1}$, where $X_1 \in \mathbb{C}^{p\times p}$. Since $A_1A_2A_1 = A_2A_1$ and $\alpha \neq 0$, we conclude that

$$\alpha X_1 = X_1, \qquad \alpha\beta X_2 = \beta X_2, \qquad \beta X_3 = X_3, \qquad \beta^2 X_4 = \beta X_4. \tag{2.5}$$

It is evident that if A_1 is represented as in (2.1) and A_2 is represented as in (2.2), (2.3), or (2.4), and the scalars a_1, α, β satisfy the corresponding conditions, then $A^2 = I_n$.

Let us assume $A^2 = I_n$. We split the proof in several cases.

Case I: $\beta \neq 1$. From the third equality of (2.5), we get $X_3 = 0$. So,

$$A_2 = U \begin{pmatrix} X_1 & X_2 \\ 0 & X_4 \end{pmatrix} U^{-1}. \tag{2.6}$$

Hence,

$$A = a_1A_1 + a_2A_2 = U \begin{pmatrix} a_1\alpha I_p + a_2X_1 & a_2X_2 \\ 0 & a_1\beta I_{n-p} + a_2X_4 \end{pmatrix} U^{-1}$$

and

$$A^2 = U \begin{pmatrix} (a_1\alpha I_p + a_2X_1)^2 & a_1a_2(\alpha+\beta)X_2 + a_2^2X_1X_2 + a_2^2X_2X_4 \\ 0 & (a_1\beta I_{n-p} + a_2X_4)^2 \end{pmatrix} U^{-1}.$$

From $A^2 = I_n$, we conclude that

$$(a_1\alpha I_p + a_2X_1)^2 = I_p, \tag{2.7}$$

$$(a_1\beta I_{n-p} + a_2X_4)^2 = I_{n-p}, \tag{2.8}$$

and

$$a_1a_2(\alpha+\beta)X_2 + a_2^2X_1X_2 + a_2^2X_2X_4 = 0. \tag{2.9}$$

By (2.7), there exist $q \in \{0,\ldots,p\}$ and a nonsingular matrix $V_1 \in \mathbb{C}^{p\times p}$ such that

$$a_1\alpha I_p + a_2X_1 = V_1 \begin{pmatrix} I_q & 0 \\ 0 & -I_{p-q} \end{pmatrix} V_1^{-1},$$

which implies

$$X_1 = V_1 \begin{pmatrix} \frac{1-a_1\alpha}{a_2} I_q & 0 \\ 0 & \frac{-1-a_1\alpha}{a_2} I_{p-q} \end{pmatrix} V_1^{-1}. \tag{2.10}$$

From (2.8), there exist $r \in \{0,\ldots,n-p\}$ and a nonsingular matrix $V_2 \in \mathbb{C}^{(n-p)\times(n-p)}$ such that

$$a_1\beta I_{n-p} + a_2X_4 = V_2 \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-p-r} \end{pmatrix} V_2^{-1},$$

that is,

$$X_4 = V_2 \begin{pmatrix} \frac{1-a_1\beta}{a_2} I_r & 0 \\ 0 & \frac{-1-a_1\beta}{a_2} I_{n-p-r} \end{pmatrix} V_2^{-1}. \tag{2.11}$$

Let X_2 be written as $X_2 = V_1 \left(\begin{smallmatrix} K & L \\ M & N \end{smallmatrix}\right) V_2^{-1}$, where $K \in \mathbb{C}^{q\times r}$. By (2.9), (2.10), and (2.11), we get

$$a_1a_2(\alpha+\beta)X_2 + a_2^2(X_1X_2 + X_2X_4) = V_1 \begin{pmatrix} 2a_2K & 0 \\ 0 & -2a_2N \end{pmatrix} V_2^{-1} = 0.$$

Thus, $K = 0$ and $N = 0$. Now, X_2 reduces to

$$X_2 = V_1 \begin{pmatrix} 0 & L \\ M & 0 \end{pmatrix} V_2^{-1}. \tag{2.12}$$

Let us define $V = U(V_1 \oplus V_2)$. We obtain

$$\begin{aligned} A_1 &= U \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_r \end{pmatrix} U^{-1} \\ &= U(V_1\oplus V_2) \begin{pmatrix} V_1^{-1} & 0 \\ 0 & V_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} (V_1^{-1}\oplus V_2^{-1})U^{-1} \\ &= V \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} V^{-1}. \end{aligned}$$

By (2.6), (2.10), (2.11), and (2.12), we obtain

$$\begin{aligned} A_2 &= U \begin{pmatrix} X_1 & X_2 \\ 0 & X_4 \end{pmatrix} U^{-1} \\ &= U \begin{pmatrix} V_1 \begin{pmatrix} \frac{1-a_1\alpha}{a_2} I_q & 0 \\ 0 & \frac{-1-a_1\alpha}{a_2} I_{p-q} \end{pmatrix} V_1^{-1} & V_1 \begin{pmatrix} 0 & L \\ M & 0 \end{pmatrix} V_2^{-1} \\ 0 & V_2 \begin{pmatrix} \frac{1-a_1\beta}{a_2} I_r & 0 \\ 0 & \frac{-1-a_1\beta}{a_2} I_{n-p-r} \end{pmatrix} V_2^{-1} \end{pmatrix} U^{-1} \\ &= U(V_1\oplus V_2) \begin{pmatrix} \frac{1-a_1\alpha}{a_2} I_q & 0 & 0 & L \\ 0 & \frac{-1-a_1\alpha}{a_2} I_{p-q} & M & 0 \\ 0 & 0 & \frac{1-a_1\beta}{a_2} I_r & 0 \\ 0 & 0 & 0 & \frac{-1-a_1\beta}{a_2} I_{n-p-r} \end{pmatrix} (V_1^{-1}\oplus V_2^{-1})U^{-1}. \end{aligned}$$

Thus, A_2 can be written as in (2.2).

Case II: $\beta = 1$. Since $\alpha \neq \beta$, we have $\alpha \neq 1$. Hence from (2.5), we get $X_1 = 0$ and $X_2 = 0$. Then

$$A = a_1A_1 + a_2A_2 = U \begin{pmatrix} \alpha a_1 I_p & 0 \\ a_2X_3 & a_1I_{n-p} + a_2X_4 \end{pmatrix} U^{-1}.$$

Since $A^2 = I_n$, we conclude that

$$(\alpha a_1)^2 = 1, \quad (a_1I_{n-p} + a_2X_4)^2 = I_{n-p}, \quad (1+\alpha)a_1a_2X_3 + a_2^2X_4X_3 = 0. \tag{2.13}$$

By the second equality of (2.13), there exist $r \in \{0,\dots,n-p\}$ and a nonsingular matrix $T \in \mathbb{C}^{(n-p)\times(n-p)}$ such that $a_1I_{n-p} + a_2X_4 = T(I_r \oplus -I_{n-p-r})T^{-1}$, a simple computation shows that

$$X_4 = T \begin{pmatrix} \frac{1-a_1}{a_2} I_r & 0 \\ 0 & \frac{-1-a_1}{a_2} I_{n-r-p} \end{pmatrix} T^{-1}. \tag{2.14}$$

By the first equality of (2.13), we get $a_1 = 1/\alpha$ or $a_1 = -1/\alpha$. (we can use the first equality of (2.13) because if $p = 0$, then (2.1) would yield $A_1 = \beta I_n$, which is not possible in view that A_1 is $\{\alpha, \beta\}$-quadratic matrix). Let us write X_3 as $X_3 = T\left(\begin{smallmatrix} R \\ S \end{smallmatrix}\right)$, where $R \in \mathbb{C}^{r\times p}$. Then

$$(1+\alpha)a_1a_2X_3 + a_2^2X_4X_3 = T\begin{pmatrix} (\alpha a_1a_2 + a_2)R \\ (\alpha a_1a_2 - a_2)S \end{pmatrix}.$$

Thus, from the last equality of (2.13) we have that

$$(2.15)\qquad (\alpha a_1a_2 + a_2)R = 0 \qquad \text{and} \qquad (\alpha a_1a_2 - a_2)S = 0.$$

Case II.a: $\alpha a_1 = 1$. Equalities (2.14) and (2.15) reduce to

$$(2.16)\qquad X_4 = T\begin{pmatrix} \frac{\alpha-1}{\alpha a_2}I_r & 0 \\ 0 & -\frac{\alpha+1}{\alpha a_2}I_{n-r-p} \end{pmatrix}T^{-1} \qquad \text{and} \qquad R = 0.$$

We have

$$A_1 = U(\alpha I_p \oplus I_{n-p})U^{-1} = U(I_p \oplus T)(\alpha I_p \oplus I_{n-p})(I_p \oplus T^{-1})U^{-1}$$

and

$$\begin{aligned} A_2 &= U\begin{pmatrix} 0 & 0 \\ X_3 & X_4 \end{pmatrix}U^{-1} \\ &= U\begin{pmatrix} 0 & 0 \\ T\begin{pmatrix} 0 \\ S \end{pmatrix} & T\begin{pmatrix} \frac{\alpha-1}{\alpha a_2}I_r & 0 \\ 0 & -\frac{\alpha+1}{\alpha a_2}I_{n-r-p} \end{pmatrix}T^{-1} \end{pmatrix}U^{-1} \\ &= U(I_p \oplus T)\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\alpha-1}{\alpha a_2}I_r & 0 \\ S & 0 & -\frac{\alpha+1}{\alpha a_2}I_{n-r-p} \end{pmatrix}(I_p \oplus T^{-1})U^{-1}. \end{aligned}$$

It is enough to define $V = U(I_p \oplus T)$ to get the expression of this case.

Case II.b: $\alpha a_1 = -1$. The proof of this case is quite similar to the previous one. $\square$

Example 2.2. Let us solve in this example the following problem. Let

$$A_1 = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Find all numbers $a_1, a_2 \in \mathbb{C}^*$ such that $a_1A_1 + a_2A_2$ is involutive.

Observe that A_1 is a $\{2, 1\}$-quadratic matrix and $A_1A_2A_1 = A_2A_1$. Furthermore, a simple computation shows that $\sigma(A_2) = \{0, -1\}$. Following the notation of the previous result, we can assume $\alpha = 2$ and $\beta = 1$. By the previous result, we must consider two alternatives: $a_1 = 1/2$ or $a_1 = -1/2$. For the first alternative, by (2.3), we obtain $\sigma(A_2) \subset \{0, 1/(2a_2), -3/(2a_2)\}$, and thus, $-1 \in \{1/(2a_2), -3/(2a_2)\}$, so we have two possibilities for a_2, namely $a_2 = -1/2$ or $a_2 = 3/2$. For the second alternative, by (2.4), we obtain $\sigma(A_2) \subset \{0, 3/(2a_2), -1/(2a_2)\}$, and thus, $-1 \in \{3/(2a_2), -1/(2a_2)\}$, so we have now two possibilities for a_2, namely $a_2 = -3/2$ or $a_2 = 1/2$.

Observe that we have four possibilities:

$$(a_1, a_2) \in \{(1/2, -1/2), (1/2, 3/2), (-1/2, 1/2), (-1/2, -3/2)\}.$$

It is enough now to check if $a_1 A_1 + a_2 A_2$ is involutive. None of the above four possibilites yield to the involutiveness of $a_1 A_1 + a_2 A_2$. Thus, we do not find $a_1, a_2 \in \mathbb{C}^*$ such that $a_1 A_1 + a_2 A_2$ is involutive. ■

Example 2.3. Let A_1 be as in the previous example. We shall find all matrices $A_2 \in \mathbb{C}^{3\times 3}$ and $a_1 \in \mathbb{C}^*$ that $A_1 A_2 A_1 = A_2 A_1$ and $a_1 A_1 + A_2$ is involutive.

By following the notation of the previous result, we can assume $\alpha = 2$, $\beta = 1$. Obviously, we have $a_2 = 1$. Only cases (ii) and (iii) of the previous result can be satisfied, and thus, $a_1 = \pm 1/\alpha = \pm 1/2$. By a diagonalization of A_1, the expression (2.1) in this example is

$$A_1 = V \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^{-1}, \quad V = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad p = 1.$$

If case (ii) of the previous result holds, then $a_1 = 1/2$. Now, A_2 must be of the form (2.3). Since $r \in \{0, \ldots, n-p\} = \{0, 1, 2\}$, then depending on the value of r, some blocks of A_2 dissapear, yielding to the following possibilities for $V^{-1} A_2 V$ (respectively for $r = 0, 1, 2$):

$$\begin{pmatrix} 0 & 0 & 0 \\ x & -3/2 & 0 \\ y & 0 & -3/2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ z & 0 & -3/2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix},$$

being $x, y, z \in \mathbb{C}$ arbitrary. The case (iii) can be dealt by a similar way. We omit the details. ■

Remark 2.4. Observe that under the hypothesis of Theorem 2.1, $\sigma(A_2)$ is localized with no effort, because (2.2), (2.3), and (2.4) prove that A_2 is similar to certain triangular matrices.

Remark 2.5. We shall show how to manage the condition $A_1 A_2 A_1 = A_1 A_2$ with no effort. Let M be an $\{\alpha, \beta\}$-quadratic matrix. It is evident that M^* is a $\{\overline{\alpha}, \overline{\beta}\}$-quadratic matrix. Thus, if $A_1, A_2 \in \mathbb{C}^{n\times n}$ are such that $a_1 A_1 + a_2 A_2$ is involutive, A_1 is $\{\alpha, \beta\}$-quadratic matrix, and $A_1 A_2 A_1 = A_1 A_2$, we can apply Theorem 2.1 to A_1^* and A_2^*.

As an example of the wide applicability of this result we shall prove two corollaries.

Corollary 2.6. *Let $A_1, A_2 \in \mathbb{C}^{n\times n}$ be two nonzero linearly independent idempotent matrices such that $A_1 A_2 A_1 = A_2 A_1$. Moreover, let A be a linear combination of the form (1.1) with $a_1, a_2 \in \mathbb{C}^*$. Then $A^2 = I_n$ if and only if one of the following conditions holds.*

(i) $(a_1, a_2) \in \{(-1, 2), (1, -2)\}$ *and* $A_1 = I_n$.
(ii) $(a_1, a_2) \in \{(2, -1), (-2, 1)\}$ *and* $A_2 = I_n$.

(iii) $(a_1, a_2) \in \{(1,1), (-1,-1)\}$ *and* $A_1A_2 = A_2A_1 = 0$, $A_1 + A_2 = I_n$.
(iv) $(a_1, a_2) \in \{(1,-1), (-1,1)\}$ *and* $A_1 + A_2 = A_1A_2 + I_n$, $A_2A_1 = 0$.

Proof. It is straightforward that any characteristic (i)–(iv) leads to $A^2 = I_n$.

Assume that $A^2 = I_n$. Since A_1 is a nonzero idempotent, we have two possibilities: $A_1 = I_n$ or A_1 is a $\{1,0\}$-quadratic matrix. For the first of the above possibilites, by writting $A_2 = W(I_x \oplus 0)W^{-1}$, being $x = \operatorname{rank}(A_2)$ with $0 \neq x \neq n$ we easily obtain the characteristic (i) of the Theorem.

Now, we assume that A_1 is a $\{1,0\}$-quadratic matrix. From Theorem 2.1, there exist a nonsingular matrix $V \in \mathbb{C}^{n\times n}$, $p \in \{1,\ldots,n-1\}$, $q \in \{0,1,\ldots,p\}$, and $r \in \{0,1,\ldots,n-p\}$ such that $A_1 = V(I_p \oplus 0)V^{-1}$ and

$$A_2 = V \begin{pmatrix} \frac{1-a_1}{a_2}I_q & 0 & 0 & L \\ 0 & -\frac{1+a_1}{a_2}I_{p-q} & M & 0 \\ 0 & 0 & \frac{1}{a_2}I_r & 0 \\ 0 & 0 & 0 & -\frac{1}{a_2}I_{n-r-p} \end{pmatrix} V^{-1}. \tag{2.17}$$

We shall denote $\lambda = \frac{1-a_1}{a_2}$, $\mu = -\frac{1+a_1}{a_2}$, $\rho = 1/a_2$ and $\sigma = -1/a_2$.

Since $A_2^2 = A_2$, we get

$$\begin{pmatrix} \lambda I_q & 0 \\ 0 & \mu I_{p-q} \end{pmatrix}^2 = \begin{pmatrix} \lambda I_q & 0 \\ 0 & \mu I_{p-q} \end{pmatrix},$$

$$\begin{pmatrix} \rho I_r & 0 \\ 0 & \sigma I_{n-r-p} \end{pmatrix}^2 = \begin{pmatrix} \rho I_r & 0 \\ 0 & \sigma I_{n-r-p} \end{pmatrix}.$$

We shall split the proof to the following cases according to the values of q.

(I) If $q = 0$, then $\mu \in \{0,1\}$. Hence $a_1 = -1$ or $a_1 + a_2 = -1$.
(II) If $q = p$, then $\lambda \in \{0,1\}$. Hence $a_1 = 1$ or $a_1 + a_2 = 1$.
(III) If $0 < q < p$, then $(a_1, a_2) = (-1, 2)$ or $(a_1, a_2) = (1, -2)$.

Again we split the proof to the following cases according the value of r.

(A) If $r = 0$, then $\sigma^2 = \sigma$. Since $\sigma \neq 0$, then we obtain $a_2 = -1$.
(B) If $r = n - p$, then $\rho^2 = \rho$. Thus, $a_2 = 1$.
(C) If $0 < r < n - p$ we arrive at a contradiction.

We now combine cases (I), (II), (III) with (A), (B). The combinations (III-A) and (III-B) are clearly unfeasible.

(I-A): We have $q = r = 0$ and $a_1 = a_2 = -1$. From (2.17), we have $A_2 = V(0 \oplus I_{n-p})V^{-1}$. This situation leads to the part of the characteristic (iii) of the theorem.

(II-A): We have $q = p$, $r = 0$, and $(a_1, a_2) \in \{(1,-1), (2,-1)\}$. From (2.17), we have

$$A_2 = V \begin{pmatrix} \lambda I_p & L \\ 0 & \sigma I_{n-p} \end{pmatrix} V^{-1}.$$

Since $A_2^2 = A_2$, we get $(\lambda + \sigma - 1)L = 0$. If $(a_1, a_2) = (1, -1)$, then $A_2 = V \left(\begin{smallmatrix} 0 & L \\ 0 & I_{n-p} \end{smallmatrix}\right) V^{-1}$ and this situation leads to the part of the characteristic (iv). If $(a_1, a_2) = (2, -1)$, then $\lambda = \sigma = 1$, and therefore, $(\lambda + \sigma - 1)L = 0$ leads to $L = 0$. So, we have $A_2 = I_n$. This is the part of the characteristic (ii).

(I-B): We have $q = 0$, $r = n-p$, and $(a_1, a_2) \in \{(-1, 1), (-2, 1)\}$. From (2.17), we have $A_2 = V \begin{pmatrix} \mu I_p & M \\ 0 & \rho I_{n-p} \end{pmatrix} V^{-1}$. Since $A_2^2 = A_2$, we get $(\mu + \rho - 1)M = 0$. If $(a_1, a_2) = (-1, 1)$, then $A_2 = V \begin{pmatrix} 0 & M \\ 0 & I_{n-p} \end{pmatrix} V^{-1}$ and this situation leads to the part of the characteristic (iv). If $(a_1, a_2) = (-2, 1)$, then $(\mu + \rho - 1)M = 0$ leads to $M = 0$ and $A_2 = I_n$, and this is the part of the characteristic (ii).

(II-B): We have $q = p$, $r = n - p$, and $a_1 = a_2 = 1$. From (2.17), we get $A_2 = V(0 \oplus I_{n-p})V^{-1}$. This situation leads to a part of the characteristic (iii).

The proof is finished. □

Corollary 2.7. *Let $A_1, A_2 \in \mathbb{C}^{n\times n} \setminus \{0\}$ be two linearly independent matrices such that $A_1^2 = I_n$, $A_2^2 = A_2$, $A_1A_2A_1 = A_2A_1$ and let A be a linear combination of the form (1.1) with $a_1, a_2 \in \mathbb{C}^*$. Then $A^2 = I_n$ if and only if $(a_1, a_2) \in \{(-1, 2), (1, -2)\}$ and $A_1A_2 = A_2A_1 = A_2$.*

Proof. It is evident that if $A_1A_2 = A_2A_1 = A_2$, then $(A_1 - 2A_2)^2 = (-A_1 + 2A_2)^2 = I_n$.

Now, assume that $A^2 = I_n$. Since $A_1^2 = I_n$, we have three possibilities for A_1: $A_1 = I_n$ or $A_1 = -I_n$, or A_1 is a $\{-1, 1\}$-quadratic matrix. If $A_1 = I_n$, then by writing $A_2 = R(I_x \oplus 0)R^{-1}$, where $x = \operatorname{rank}(A_2) \in \{1, \ldots, n-1\}$, we get $(a_1, a_2) \in \{(1, -2), (-1, 2)\}$, which is the part (i) of Corollary 2.1. If $A_1 = -I_n$, then by $A_1A_2A_1 = A_2A_1$, we obtain $A_2 = 0$ which is not possible . If A_1 is a $\{-1, 1\}$-quadratic matrix, then by Theorem 2.1, there exist a nonsingular matrix V and $p \in \{1, \ldots, n-1\}$ such that $A_1 = V(-I_p \oplus I_{n-p})V^{-1}$ and A_2 is written as in (2.3) or (2.4). Case (ii) of Theorem 2.1 and $A_2^2 = A_2$ lead to $(a_1, a_2) = (-1, 2)$ and $A_2 = V(0 \oplus I_r \oplus 0)V^{-1}$, whereas case (iii) and $A_2^2 = A_2$ lead to $(a_1, a_2) = (1, -2)$ and $A_2 = V(0 \oplus 0 \oplus I_{n-r-p})V^{-1}$. □

A square matrix A is called *group invertible* if there exists a matrix X such that $AXA = A$, $XAX = X$, and $AX = XA$. It can be proved that this matrix X is unique (if it exists) and it is customarily written as $A^{\#}$ (see [5, Section 4.4]). It is easy to see that any diagonalizable matrix is group invertible. This generalized inverse is necessary to define the *sharp ordering*: Let $A, B \in \mathbb{C}^{n\times n}$ be two group invertible matrices. We write $A \overset{\#}{\leq} B$ when $AA^{\#} = BA^{\#}$ and $A^{\#}A = A^{\#}B$ (see [18, Chapter 4]). If A is nonsingular and $A \overset{\#}{\leq} B$, then obviously $A = B$. Thus, if we assume in addition that A is an $\{\alpha, \beta\}$-quadratic matrix and we want to deal with non-trivial linear combinations $aA + bB$, where $a, b \in \mathbb{C}^*$, we can assume that α or β are zero.

Theorem 2.8. *Let $A_1 \in \mathbb{C}^{n\times n}$ be an $\{\alpha, 0\}$-quadratic matrix, $A_2 \in \mathbb{C}^{n\times n}$, $\alpha \in \mathbb{C}^*$, and A be a linear combination of the form (1.1) with $a_1, a_2 \in \mathbb{C}^*$. Assume that A_1 and A_2 are linearly independent matrices and $A_1 \overset{\#}{\leq} A_2$. Then*

$$A^2 = I_n \qquad \Leftrightarrow \qquad a_2^2(A_2 - A_1)^2 = I_n - \alpha^{-1}A_1, \quad 1 = [\alpha(a_1 + a_2)]^2 .$$

Proof. Since A_1 is an $\{\alpha, 0\}$-quadratic matrix, then there exists a nonsingular matrix U such that $A_1 = U(\alpha I_p \oplus 0)U^{-1}$, where $p \in \{1, \dots, n-1\}$. Let us write $A_2 = U \left(\begin{smallmatrix} X & Y \\ Z & T \end{smallmatrix}\right) U^{-1}$, where $X \in \mathbb{C}^{p\times p}$. By employing the two conditions of the sharp ordering, we get that $A_2 = U(\alpha I_p \oplus T)U^{-1}$.

$\Rightarrow$: Since $A^2 = I_n$, we get that $1 = [\alpha(a_1 + a_2)]^2$ and $(a_2T)^2 = I_{n-p}$. Therefore,

$$a_2^2(A_2 - A_1)^2 = a_2^2U(0 \oplus T^2)U^{-1} = U(0 \oplus I_{n-p})U^{-1} = I_n - \alpha^{-1}A_1.$$

$\Leftarrow$: The condition $a_2^2(A_2 - A_1)^2 = I_n - \alpha^{-1}A_1$ leads to $a_2^2T^2 = I_{n-p}$. Hence $a_1A_1 + a_2A_2 = U((a_1 + a_2)\alpha I_p \oplus a_2T)U^{-1}$ clearly is involutive. □

Remark 2.9. Let $A_1, A_2 \in \mathbb{C}^{n\times n}$ satisfy the hypotheses of the former theorem. If we want check the existence of $a_1, a_2 \in \mathbb{C}^*$ such that $a_1A_1 + a_2A_2$ is involutive (and in this case, find such a_1, a_2), then the above result gives us a procedure. First, find the spectrum of A_1, or equivalenty, find $\alpha \in \sigma(A_1)\backslash\{0\}$. Second, check if $(A_2-A_1)^2$ is a scalar multiple of $I_n-\alpha^{-1}A_1$. If not, the problem has not solution. If yes, from $a_2^2(A_2-A_1)^2 = I_n-\alpha^{-1}A_1$, we can find the feasible values of a_2 and from $1 = [\alpha(a_1 + a_2)]^2$, we can find the feasible values of a_1.

Let us deal now with another condition that was appeared in [16], namely $A_2A_1^{\#}A_1 = A_1$. We will assume that A_1 is singular (if otherwise, then $A_2A_1^{\#}A_1 = A_1$ reduces to $A_1 = A_2$). Observe that $A_2A_1^{\#}A_1 = A_1 \Leftrightarrow A_2A_1^{\#} = A_1A_1^{\#}$. Hence $A_2A_1^{\#}A_1 = A_1$ implies $A_1 \overset{\#}{\leq} A_2$.

Let us observe that if A_1, A_2 satisfy $A_2A_1^{\#} = A_1A_1^{\#}$ and $A_1A_2 = A_2A_1$, then by writing $A_1 = U(K \oplus 0)U^{-1}$, where U and K are nonsingular (this is possible because A_1 is group invertible), we have that A_2 can be writen as $A_2 = U(K \oplus T)U^{-1}$ for some matrix T. Hence $A_1^{\#}A_1 = A_1^{\#}A_2$; which leads to $A_1 \overset{\#}{\leq} A_2$. Therefore, in next theorem we will assume the condition $A_1A_2 \neq A_2A_1$, since otherwise, we can apply Theorem 2.8.

Theorem 2.10. *Let $A_1 \in \mathbb{C}^{n\times n}$ be $\{\alpha, 0\}$-quadratic, $A_2 \in \mathbb{C}^{n\times n}$, $\alpha \in \mathbb{C}^*$, and A be a linear combination of the form (1.1) with $a_1, a_2 \in \mathbb{C}^*$. Assume that A_1 and A_2 are linearly independent matrices, $A_1A_2 \neq A_2A_1$, and $A_2A_1^{\#}A_1 = A_1$. Then $A^2 = I_n$ if and only if there exist a nonsingular matrix $V \in \mathbb{C}^{n\times n}$, $p \in \{1, \dots, n-1\}$, and $q \in \{0, \dots, n-p\}$ such that $A_1 = V(\alpha I_p \oplus 0)V^{-1}$ and it is satisfied one of the following cases:*

(i) $\alpha(a_1 + a_2) = 1$.

(i.1) $A_2 = V \left(\begin{smallmatrix} \alpha I_p & Y \\ 0 & -\frac{1}{a_2} I_{n-p} \end{smallmatrix}\right) V^{-1}$ *being* $Y \neq 0$ *an arbitrary matrix in* $\mathbb{C}^{p\times(n-p)}$.

(i.2) $A_2 = V \begin{pmatrix} \alpha I_p & 0 & Y \\ 0 & \frac{1}{a_2} I_q & 0 \\ 0 & 0 & -\frac{1}{a_2} I_{n-p-q} \end{pmatrix} V^{-1}$ *being* $Y \neq 0$ *an arbitrary matrix in* $\mathbb{C}^{p\times q}$.

(ii) $\alpha(a_1 + a_2) = -1$.

(ii.1) $A_2 = V \left(\begin{smallmatrix} \alpha I_p & Y \\ 0 & \frac{1}{a_2} I_{n-p} \end{smallmatrix}\right) V^{-1}$ *being* $Y \neq 0$ *an arbitrary matrix in* $\mathbb{C}^{p\times(n-p)}$.

(ii.2) $A_2 = V \begin{pmatrix} \alpha I_p & Y & 0 \\ 0 & \frac{1}{a_2} I_q & 0 \\ 0 & 0 & -\frac{1}{a_2} I_{n-p-q} \end{pmatrix} V^{-1}$ *being* $Y \neq 0$ *an arbitrary matrix in* $\mathbb{C}^{p\times(n-p-q)}$.

Proof. The 'if' part is evident. We will prove the reciprocal: There exist a nonsingular matrix $U \in \mathbb{C}^{n\times n}$ and $p \in \{1, \dots, n-1\}$ such that

$$A_1 = U(\alpha I_p \oplus 0)U^{-1}, \quad A_2 = U \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} U^{-1}, \quad X \in \mathbb{C}^{p\times p}. \tag{2.18}$$

By employing $A_2 A_1^{\#} A_1 = A_1$, we get $X = \alpha I_p$ and $Z = 0$. Let us observe that since $0 \neq p \neq n$, then all blocks in (2.18) for A_1 and A_2 appear. Since $A_1 A_2 \neq A_2 A_1$, then $Y \neq 0$. By using $A^2 = I_n$, we get

$$\alpha(a_1+a_2) \in \{-1, 1\}, \qquad \alpha(a_1+a_2)Y + a_2 YT = 0, \qquad a_2^2 T^2 = I_{n-p}. \tag{2.19}$$

The last equality of (2.19) implies the existence of a nonsingular matrix $U_1 \in \mathbb{C}^{(n-p)\times(n-p)}$ such that $a_2 T = U_1(I_q \oplus -I_{n-p-q})U_1^{-1}$ for some $q \in \{0, \dots, n-p\}$. Let us write $Y = \left(\begin{smallmatrix} Y_1 & Y_2 \end{smallmatrix}\right) U_1^{-1}$ for $Y_1 \in \mathbb{C}^{p\times q}$. From the second equality of (2.19), we get

$$[\alpha(a_1 + a_2) + 1]\, Y_1 = 0 \qquad \text{and} \qquad [\alpha(a_1 + a_2) - 1]\, Y_2 = 0. \tag{2.20}$$

We have the following possibilities:

(a) If $q = 0$, in view of the decompositions of T and Y, we get $T = -\frac{1}{a_2} I_{n-p}$ and $Y = Y_2 U_1^{-1}$. Since $Y \neq 0$, then $Y_2 \neq 0$, hence (2.20) leads to $\alpha(a_1 + a_2) = 1$. Setting $V = U$ allows us to prove the case (i.1).

(b) If $q = n - p$, in view of the decompositions of T and Y, we get $T = \frac{1}{a_2} I_{n-p}$ and $Y = Y_1 U_1^{-1}$. Since $Y \neq 0$, then $Y_1 \neq 0$, hence (2.20) leads to $\alpha(a_1 + a_2) = -1$. Setting $V = U$ allows us to prove the case (ii.1).

(c) If $0 \neq q \neq n - p$, since $Y \neq 0$, then $Y_1 \neq 0$ or $Y_2 \neq 0$.

(c.i) If $Y_1 \neq 0$, then (2.20) implies $\alpha(a_1+a_2) = -1$ and $Y_2 = 0$. Also we have

$$A_2 = U \begin{pmatrix} \alpha I_p & (Y_1 \ \ 0)U_1^{-1} \\ 0 & a_2^{-1}U_1(I_q \oplus -I_{n-p-q})U_1^{-1} \end{pmatrix} U^{-1}$$
$$= U \begin{pmatrix} I_p & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} \alpha I_p & (Y_1 \ \ 0) \\ 0 & a_2^{-1}(I_q \oplus -I_{n-p-q}) \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & U_1^{-1} \end{pmatrix} U^{-1}.$$

Setting $V = U(I_p \oplus U_1)$ and renaming $Y = Y_1$ permit obtain the case (ii.2).

(c.ii) If $Y_2 \neq 0$, then again (2.20) yields $\alpha(a_1+a_2) = 1$ and $Y_1 = 0$. As before, we have

$$A_2 = U \begin{pmatrix} \alpha I_p & (0 \ \ Y_2)U_1^{-1} \\ 0 & a_2^{-1}U_1(I_q \oplus -I_{n-p-q})U_1^{-1} \end{pmatrix} U^{-1}$$
$$= U \begin{pmatrix} I_p & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} \alpha I_p & (0 \ \ Y_2) \\ 0 & a_2^{-1}(I_q \oplus -I_{n-p-q}) \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & U_1^{-1} \end{pmatrix} U^{-1}.$$

Setting $V = U(I_p \oplus U_1)$ and renaming $Y = Y_2$ permit obtain the case (i.2).

The proof is finished. □

Remark 2.11. Observe that under the hypothesis of the above theorem, finding the spectrum of A_2 is simple since A_2 is a triangular matrix.

Example 2.12. Let

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

We will find all $a_1, a_2 \in \mathbb{C}^*$ such that $a_1A_1 + a_2A_2$ is involutive.

First of all, we check that A_1 and A_2 satisfy the hypotheses of Theorem 2.10. It is simple to see that $A_1^2 = 2A_1$; and therefore A_1 is a $\{2,0\}$-quadratic matrix and $A_1^{\#} = \frac{1}{4}A_1$. Now, checking $A_2A_1^{\#}A_1 = A_1$ and $A_1A_2 \neq A_2A_1$ is straightforward.

A_1 is a $\{2,0\}$-quadratic matrix. Hence $a_1 + a_2 \in \{-1/2, 1/2\}$. The matrix A_2 is a triangular matrix; so, it is clear that $\sigma(A_2) = \{1,2\}$. Then characteristics (i.2) and (ii.2) of Theorem 2.10 are impossible. Characteristic (i.1) leads to $-1/a_2 = 1$ and $a_1+a_2 = 1/2$; which yields $(a_1,a_2) = (3/2,-1)$. Characteristic (ii.1) implies $(a_1,a_2) = (-3/2,1)$. Both of these lead to $A^2 = I_2$. Observe that this reasoning is idependent on the size of the involved matrices (we have choosen 2×2 matrices for the sake of the readability). ■

Example 2.13. Let A_1 be as in the previous example. We will find all matrices $A_2 \in \mathbb{C}^{2\times 2}$ and $a_1 \in \mathbb{C}^*$ such that $A_2A_1^{\#}A_1 = A_1$ and $a_1A_1 + A_2$ is involutive.

We have

$$A_1 = V\mathrm{diag}(2,0)V^{-1}, \qquad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

If $A_2A_1 = A_1A_2$, then by the proof of Theorem 2.8, we obtain $A_2 = V\mathrm{diag}(2,t)V^{-1}$ with $t \in \{-1,1\}$. Also, Theorem 2.8 implies $1 = [2(a_1+1)]^2$, that is to say $a_1 \in \{-1/2, -3/2\}$.

If $A_2A_1 \neq A_1A_2$, then by Theorem 2.10 there exists $y \in \mathbb{C}^*$ such that $2(a_1+1) = 1$, $A_2 = V\left(\begin{smallmatrix} 2 & y \\ 0 & -1\end{smallmatrix}\right)V^{-1}$ or $2(a_1+1) = -1$, $A_2 = V\left(\begin{smallmatrix} 2 & y \\ 0 & 1\end{smallmatrix}\right)V^{-1}$. I.e.,

$$a_1 = -\frac{1}{2},\ A_2 = \frac{1}{2}\begin{pmatrix} 1-y & 3+y \\ 3-y & 1+y \end{pmatrix},$$

or

$$a_1 = -\frac{3}{2},\ A_2 = \frac{1}{2}\begin{pmatrix} 3-y & 1+y \\ 1-y & 3+y \end{pmatrix}. \qquad \blacksquare$$

Next results concern with linear combinations of the form (1.1), when A_1 is tripotent (i.e. $A_1^3 = A_1$). If A_1 is tripotent, then Theorem 2.1 of [6] implies the existence of a nonsingular $U \in \mathbb{C}^{n\times n}$ such that

$$A_1 = U(I_s \oplus -I_t \oplus 0)U^{-1}, \tag{2.21}$$

where $s, t \in \{0, 1, \dots, n\}$ and $s + t = \mathrm{rank}(A_1)$. It is evident that if $t = 0$, then A_1 is idempotent. Also, it would be clear that if $s = 0$, then $-A_1$ is idempotent. In next result we impose the hypothesis $A_1^2 \neq \pm A_1$ since $A_1^2 \neq \pm A_1$ were covered in Theorem 2.1.

Theorem 2.14. *Let A_1, $A_2 \in \mathbb{C}^{n\times n}$ be two linearly independent matrices such that $A_1^3 = A_1$, $A_1^2 \neq \pm A_1$, $A_1A_2A_1 = A_2A_1$ and let A be a linear combination of the form (1.1) with $a_1, a_2 \in \mathbb{C}^*$. Then $A^2 = I_n$ if and only if there exists a nonsingular matrix $V \in \mathbb{C}^{n\times n}$ such that*

$$A_1 = V(I_k \oplus I_l \oplus -I_m \oplus -I_j \oplus 0 \oplus 0)V^{-1} \tag{2.22}$$

and one of the following conditions holds.

(i) $a_1 = 1$ *and*

$$A_2 = V\begin{pmatrix} 0 & 0 & Y_1 & Y_2 & \frac{-a_2}{2}(Y_1Z_1 + Y_2Z_2) & W_2 \\ 0 & -\frac{2}{a_2}I_l & 0 & 0 & W_3 & 0 \\ 0 & 0 & 0 & 0 & Z_1 & 0 \\ 0 & 0 & 0 & 0 & Z_2 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a_2}I_e & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{a_2}I_f \end{pmatrix}V^{-1}, \tag{2.23}$$

where $Y_1 \in \mathbb{C}^{k\times m}$, $Y_2 \in \mathbb{C}^{k\times j}$, $Z_1 \in \mathbb{C}^{m\times e}$, $Z_3 \in \mathbb{C}^{j\times e}$, $W_2 \in \mathbb{C}^{k\times f}$, $W_3 \in \mathbb{C}^{l\times e}$ and k, l, m, j, e, f are nonnegative integers;

(ii) $a_1 = -1$ *and*

$$A_2 = V\begin{pmatrix} \frac{2}{a_2}I_k & 0 & 0 & 0 & 0 & W_2 \\ 0 & 0 & Y_1 & Y_2 & W_3 & \frac{a_2}{2}(Y_1Z_1 + Y_2Z_2) \\ 0 & 0 & 0 & 0 & 0 & Z_1 \\ 0 & 0 & 0 & 0 & 0 & Z_2 \\ 0 & 0 & 0 & 0 & \frac{1}{a_2}I_e & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{a_2}I_f \end{pmatrix}V^{-1}, \tag{2.24}$$

where $Y_1 \in \mathbb{C}^{l\times m}$, $Y_2 \in \mathbb{C}^{l\times j}$, $Z_1 \in \mathbb{C}^{m\times f}$, $Z_2 \in \mathbb{C}^{j\times f}$, $W_2 \in \mathbb{C}^{k\times f}$, $W_3 \in \mathbb{C}^{l\times e}$ *and* k, l, m, j, e, f *are nonnegative integers.*

Proof. Since $A_1^3 = A_1$, and $A_1^2 \neq \pm A_1$, there exist a nonsingular matrix $U \in \mathbb{C}^{n\times n}$ and $s, t \in \{1, \dots, n-1\}$ such that A_1 is written as in (2.21). Let us write A_2 as $A_2 = U \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} U^{-1}$, where $X_{11} \in \mathbb{C}^{s\times s}$ and $X_{22} \in \mathbb{C}^{t\times t}$. Since $A_1A_2A_1 = A_2A_1$, we get $X_{21} = 0$, $X_{22} = 0$, $X_{31} = 0$, $X_{32} = 0$ which imply $A_2 = U \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ 0 & 0 & X_{23} \\ 0 & 0 & X_{33} \end{pmatrix} U^{-1}$. Hence,

$$A = a_1A_1 + a_2A_2 = U \begin{pmatrix} a_1I_s + a_2X_{11} & a_2X_{12} & a_2X_{13} \\ 0 & -a_1I_t & a_2X_{23} \\ 0 & 0 & a_2X_{33} \end{pmatrix} U^{-1}.$$

Since $A^2 = I_n$, we conclude that

$$(a_1I_s + a_2X_{11})^2 = I_s, \tag{2.25}$$

$$a_1^2 I_t = I_t, \tag{2.26}$$

$$(a_2X_{33})^2 = I_{n-s-t}, \tag{2.27}$$

$$X_{11}X_{12} = 0, \tag{2.28}$$

$$-a_1X_{23} + a_2X_{23}X_{33} = 0, \tag{2.29}$$

$$a_1X_{13} + a_2X_{11}X_{13} + a_2X_{12}X_{23} + a_2X_{13}X_{33} = 0. \tag{2.30}$$

Since $t > 0$, by (2.26), we have $a_1 = 1$ or $a_1 = -1$.

If $a_1 = 1$, by (2.25), there exists a nonsingular matrix $V_1 \in \mathbb{C}^{s\times s}$ such that $I_s + a_2X_{11} = V_1(I_k \oplus -I_l)V_1^{-1}$, hence,

$$X_{11} = V_1 \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2}{a_2}I_l \end{pmatrix} V_1^{-1}. \tag{2.31}$$

By (2.27), there exists a nonsingular $V_3 \in \mathbb{C}^{(n-s-t)\times(n-s-t)}$ such that $a_2X_{33} = V_3(I_e \oplus -I_f)V_3^{-1}$, hence,

$$X_{33} = V_3 \begin{pmatrix} \frac{1}{a_2}I_e & 0 \\ 0 & -\frac{1}{a_2}I_f \end{pmatrix} V_3^{-1}. \tag{2.32}$$

Let us write X_{12} and X_{23} as follows

$$X_{12} = V_1 \begin{pmatrix} P \\ Q \end{pmatrix} \quad \text{and} \quad X_{23} = \begin{pmatrix} R & S \end{pmatrix} V_3^{-1}, \tag{2.33}$$

where $P \in \mathbb{C}^{k\times t}$ and $R \in \mathbb{C}^{t\times e}$. By (2.28), (2.31), and the first equality of (2.33), we get $Q = 0$. By (2.29), (2.32), and the second equality of (2.33), we get $S = 0$. Hence X_{12} and X_{23} can be rewritten as

$$X_{12} = V_1 \begin{pmatrix} Y_1 & Y_2 \\ 0 & 0 \end{pmatrix} V_2^{-1} \quad \text{and} \quad X_{23} = V_2 \begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \end{pmatrix} V_3^{-1}, \tag{2.34}$$

where $Y_1 \in \mathbb{C}^{k\times m}$, $Y_2 \in \mathbb{C}^{k\times j}$, $Z_1 \in \mathbb{C}^{m\times e}$, $Z_2 \in \mathbb{C}^{j\times e}$. Let us write $X_{13} = V_1 \left(\begin{smallmatrix} W_1 & W_2 \\ W_3 & W_4 \end{smallmatrix}\right) V_3^{-1}$, where $W_1 \in \mathbb{C}^{k\times e}$. By (2.30), we have $W_1 = \frac{-a_2}{2}(Y_1Z_1 + Y_2Z_2)$ and $W_4 = 0$, so

$$X_{13} = V_1 \begin{pmatrix} \frac{-a_2}{2}(Y_1Z_1 + Y_2Z_2) & W_2 \\ W_3 & 0 \end{pmatrix} V_3^{-1}. \tag{2.35}$$

From (2.31), (2.32), (2.34), and (2.35), it follows that A_1 can be writen as in (2.22) and A_2 as in (2.23) where $V = U(V_1 \oplus I_t \oplus V_3)$.

If $a_1 = -1$, using the same method, it is easy to verify that A_1 can be written as in (2.22) and A_2 as in (2.24). $\square$

Next result deal with another condition.

Theorem 2.15. *Let $A_1, A_2 \in \mathbb{C}^{n\times n}$ be linearly independent matrices. Moreover, let A be a linear combination of the form (1.1) with $a_1, a_2 \in \mathbb{C}^*$. If $A_1^3 = A_1$ and $A_1 \overset{\#}{\leq} A_2$, then $A^2 = I_n$ if and only if there exist a nonsingular matrix $V \in \mathbb{C}^{n\times n}$, $p, q \in \{0, \dots, n\}$, and $r \in \{0, \dots, n-p-q\}$ such that $a_1 + a_2 \in \{-1, 1\}$,*

$$A_1 = V(I_p \oplus -I_q \oplus 0 \oplus 0)V^{-1},$$

and

$$A_2 = V\left(I_p \oplus -I_q \oplus \frac{1}{a_2}I_r \oplus -\frac{1}{a_2}I_{n-r-p-q}\right)V^{-1}.$$

Proof. The 'if' part is evident. We shall prove the 'only' part: Since $A_1 \overset{\#}{\leq} A_2$, then $A_1A_1^\# = A_2A_1^\#$ and $A_1^\#A_1 = A_1^\#A_2$, which by pre and postmultiplying by A_1^2, we get $A_1A_2 = A_2A_1 = A_1^2$. Since $A_1^3 = A_1$, there exists a unitary matrix U such that $A_1 = U(P \oplus 0)U^{-1}$, where $P = I_p \oplus -I_q$ and $p, q \in \{0, \dots, n\}$. Observe that $p + q \neq 0$, since otherwise $A_1 = 0$. By using $A_1A_2 = A_2A_1 = A_1^2$, we deduce the existence of $T \in \mathbb{C}^{(n-p-q)\times(n-p-q)}$ such that $A_2 = U(P \oplus T)U^{-1}$. Since $A^2 = I_n$, we get that $1 = (a_1 + a_2)^2$ and $(a_2T)^2 = I_{n-p-q}$. The proof finishes as in the proof of Theorem 2.8. $\square$

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