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Weak proximal normal structure and coincidence quasi-best proximity points

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Abstract

We introduce the notion of pointwise cyclic-noncyclic relatively nonexpansive pairs involving orbits. We study the best proximity point problem for this class of mappings. We also study the same problem for the class of pointwise noncyclic-noncyclic relatively nonexpansive pairs involving orbits. Finally, under the assumption of weak proximal normal structure, we prove a coincidence quasi-best proximity point theorem for pointwise cyclic-noncyclic relatively nonexpansive pairs involving orbits. Examples are provided to illustrate the observed results.

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1. Introduction

Let A, B be nonempty subsets of Banach space X. A mapping $T : A \cup B \to A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. On the other hand, a mapping $S : A \cup B \to A \cup B$ is said to be *noncyclic* if $S(A) \subseteq A$ and $S(B) \subseteq B$.

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For a cyclic mapping $T : A \cup B \to A \cup B$, a point $p \in A \cup B$ is said to be a best proximity point provided that

$$d(p, Tp) = \operatorname{dist}(A, B).$$

Furthermore, we say that a pair (A, B) of subsets in a Banach space satisfies a property if each of the sets A and B has that property. Similarly, the pair (A, B) is called convex if both A and B are convex; moreover we write

$$(A, B) \subseteq (E, F) \Leftrightarrow A \subseteq E, B \subseteq F.$$

In addition, we will use the following notations:

$$\delta(A, B) = \sup\{\|x - y\| : x \in A, y \in B\};$$
$$\delta(x, B) = \sup\{\|x - y\| : y \in B\}.$$

For a nonempty, bounded and convex subset F of a Banach space X, we write

$$r_x(F) = \sup\{\|x - y\| : y \in F\};$$
$$r(F) = \inf\{r_x(F) : x \in F\};$$
$$F_c = \{x \in F : r_x(F) = r(F)\}.$$

In 2017, M. Gabeleh introduced the notion of a pointwise cyclic relatively nonexpansive mapping involving orbits, and proved a theorem on the existence of best proximity points.

Definition 1.1 ([11]). Let (A, B) be a nonempty pair of subsets of a Banach space X. A mapping $T : A \cup B \to A \cup B$ is said to be pointwise cyclic relatively nonexpansive involving orbits if T is cyclic and for any $(x, y) \in A \times B$, if $\|x - y\| = \operatorname{dist}(A, B)$, then

$$\|Tx - Ty\| = \operatorname{dist}(A, B),$$

and otherwise, there exists a function $\alpha : A \times B \to [0, 1]$ such that

$$\|Tx - Ty\| \leq \alpha(x, y)\|x - y\| + (1 - \alpha(x, y)) \min\{\delta_x[\mathcal{O}^2(y; \infty)], \delta_y[\mathcal{O}^2(x; \infty)]\},$$

where, for any $(x, y) \in A \times B$

$$\delta_x[\mathcal{O}^2(y; \infty)] = \sup_{n \in \mathbb{N}} \|x - T^{2n}y\|, \quad \delta_y[\mathcal{O}^2(x; \infty)] = \sup_{n \in \mathbb{N}} \|T^{2n}x - y\|.$$

Note that, if $A = B$, then we say that T is a pointwise nonexpansive mapping involving orbits. In [12], M. Gabeleh, O. Olela Otafudu, and N. Shahzad considered a pair of mappings T and S. According to [12], for a nonempty pair of subsets (A, B) in a metric space (X, d), and a cyclic-noncyclic pair $(T; S)$ on $A \cup B$ (that is, $T : A \cup B \to A \cup B$ is cyclic and $S : A \cup B \to A \cup B$ is noncyclic); they called a point $p \in A \cup B$ a *coincidence best proximity point* for $(T; S)$ if

$$d(Sp, Tp) = \operatorname{dist}(A, B).$$

Note that if $S = I$, the identity map on $A \cup B$, then $p \in A \cup B$ is a best proximity point for T.

In 2019, A. Abkar and M. Norouzian introduced the concept of coincidence quasi-best proximity point and proved the existence of such points for quasi-cyclic-noncyclic contraction pairs. We remark that the coincidence quasi-best proximity point theory is more general and includes both the best proximity point theory and the coincidence best proximity point theory.

Definition 1.2 ([2])**.** Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : X \to X$ be a quasi-cyclic-noncyclic pair on $A \cup B$; that is, $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. A point $p \in A \cup B$ is said to be a coincidence quasi-best proximity point for $(T; S)$ if

$$d(Sp, Tp) = \mathrm{dist}(S(A), S(B)).$$

In case that $S = I$, the point p reduces to a best proximity point for T.

In this article, we will focus on the coincidence quasi-best proximity point problem for pointwise cyclic-noncyclic and noncyclic-noncyclic relatively non-expansive pairs. To do this, we need to recall some definitions and theorems. We begin with the following definition which is a modification of a concept in [8].

Definition 1.3. Let (A, B) be a nonempty pair of subsets of a Banach space X and $S : A \cup B \to A \cup B$ be a noncyclic mapping on $A \cup B$. A convex pair $(S(A), S(B))$ is called a proximal pair if for each $(a_1, b_1) \in A \times B$, there exists $(a_2, b_2) \in A \times B$ such that for each $i, j \in \{1, 2\}$ with $i \neq j$ we have

$$\|Sa_i - Sb_j\| = \mathrm{dist}(S(A), S(B)).$$

Given (A, B) a pair of nonempty subsets of a Banach space X, the associated proximal pair of $(S(A), S(B))$ is the pair $(S(A_0^s), S(B_0^s))$ given by

$$A_0^s := \{a \in A : \|Sa - Sb\| = \mathrm{dist}(S(A), S(B)) \text{ for some } b \in B\},$$

$$B_0^s := \{b \in B : \|Sa - Sb\| = \mathrm{dist}(S(A), S(B)) \text{ for some } a \in A\},$$

In fact, if the pair $(S(A), S(B))$ is nonempty, weakly compact and convex, then its associated pair $(S(A_0^s), S(B_0^s))$ is also nonempty, weakly compact and convex. Furthermore, we have

$$\mathrm{dist}(S(A_0^s), S(B_0^s)) = \mathrm{dist}(S(A), S(B)).$$

The proof of the above statements goes in the same lines as in the case for the pair (A, B); see for instance [21]. Here's a definition we derive from [8] and we've made some changes to meet our needs.

Definition 1.4. Let (K_1, K_2) be a nonempty pair of subsets of a Banach space X and $S : K_1 \cup K_2 \to K_1 \cup K_2$ be a noncyclic mapping on $K_1 \cup K_2$. We say that a convex pair $(S(K_1), S(K_2))$ has proximal normal structure (**PNS**) if for any closed, bounded, convex and proximal pair $(S(H_1), S(H_2)) \subseteq (S(K_1), S(K_2))$ which

$$\mathrm{dist}(S(H_1), S(H_2)) = \mathrm{dist}(S(K_1), S(K_2)), \quad \delta(S(H_1), S(H_2)) > \mathrm{dist}(S(H_1), S(H_2)),$$

there exists $(x,y) \in H_1 \times H_2$ such that

$$\delta(Sx, S(H_2)) < \delta(S(H_1), S(H_2)), \quad \delta(Sy, S(H_1)) < \delta(S(H_1), S(H_2)).$$

Note that the pair (K, K) has proximal normal structure if and only if K has normal structure in the sense of Brodskii and Milman (see [4] and [20]).

Theorem 1.5 ([8])**.** *Every bounded, closed and convex pair in a uniformly convex Banach space X has proximal normal structure.*

The following definition is a modification of what already appeared in [11].

Definition 1.6. Let (K_1, K_2) be a nonempty pair of subsets of a Banach space X and $S : K_1 \cup K_2 \to K_1 \cup K_2$ be a noncyclic mapping on $K_1 \cup K_2$. We say that a convex pair $(S(K_1), S(K_2))$ has weak proximal normal structure (**WPNS**) if for each nonempty, weakly compact and convex proximal pair $(S(H_1), S(H_2)) \subseteq (S(K_1), S(K_2))$ for which

$$\operatorname{dist}(S(H_1), S(H_2)) = \operatorname{dist}(S(K_1), S(K_2)), \quad \delta(S(H_1), S(H_2)) > \operatorname{dist}(S(H_1), S(H_2)),$$

there exists $(x,y) \in H_1 \times H_2$ such that

$$\delta(Sx, S(H_2)) < \delta(S(H_1), S(H_2)), \quad \delta(Sy, S(H_1)) < \delta(S(H_1), S(H_2)).$$

In this article, we intend to generalize some results of [8] and [11]. Our results have the following advantages: First, we introduce the class of the pointwise cyclic-noncyclic and noncyclic-noncyclic relatively nonexpansive pairs involving orbits, that in particular, includes the class of pointwise cyclic-noncyclic and noncyclic-noncyclic relatively nonexpansive mappings involving orbits. Second, we consider a pair of mappings while the previous articles are concerned with one single mapping, and finally, we study the coincidence quasi-best proximity point problem, which in particular, includes the best proximity point problem as a special case.

2. Cyclic-noncyclic pairs

We begin this section by introducing the new concept of a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits.

Definition 2.1. Assume that (A, B) is a nonempty pair of subsets of a Banach space X and $T, S : A \cup B \to A \cup B$ are two mappings. A pair $(T; S)$ is said to be a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits if $(T; S)$ is a cyclic-noncyclic pair and for any $(x,y) \in A \times B$, if $\|x - y\| = \operatorname{dist}(S(A), S(B))$, then

$$\|Tx - Ty\| = \operatorname{dist}(S(A), S(B)), \quad \|Sx - Sy\| = \operatorname{dist}(S(A), S(B))$$

and otherwise, there exists a function $\alpha : A \times B \to [0,1]$ such that

$$\|Tx - Ty\| \leq \alpha(x,y)\|Sx - Sy\| + (1 - \alpha(x,y)) \max\{\delta_x[\mathcal{O}^2(y;\infty)], \delta_y[\mathcal{O}^2(x;\infty)]\},$$

where, for any $(x,y) \in A \times B$

$$\delta_x[\mathcal{O}^2(y;\infty)] = \sup_{n \in \mathbb{N}} \|x - T^{2n}y\|, \quad \delta_y[\mathcal{O}^2(x;\infty)] = \sup_{n \in \mathbb{N}} \|T^{2n}x - y\|.$$

We note that if $S = I$, then the class of pointwise cyclic-noncyclic relatively nonexpansive pairs involving orbits reduces to the class of pointwise cyclic relatively nonexpansive mappings involving orbits introduced in [11].

Definition 2.2 ([20])**.** We say that a Banach space X has the property (C) if every bounded decreasing sequence of nonempty, closed and convex subsets of X have a nonempty intersection.

For $C \subseteq X$, we denote the diameter of C by $\delta(C)$. A point $x \in C$ is a diametral point of C provided that $\sup\{\|x - y\| : y \in C\} = \delta(C)$. A convex set $K \subseteq X$ is said to have normal structure if for each bounded convex subset H of K which contains at least two points, there is some point $x \in H$ which is not a diametral point of H.

Lemma 2.3 ([20])**.** *Assume that X is a Banach space with the property (C), then F_c is nonempty, closed and convex.*

Lemma 2.4 ([20])**.** *Assume that F is a closed and convex subset of a Banach space X which contains at least two points. If F has normal structure, then $\delta(F_c) < \delta(F)$.*

Theorem 2.5. *Assume that K is a nonempty, bounded, closed and convex subset of a Banach space X with property (C). Suppose that K has normal structure. Let (T, S) be a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits on K. Then there exists a point $p \in K$ such that $\|Tp - Sp\| = 0$.*

Proof. Suppose Γ denotes the collection of all nonempty, closed and convex subsets of K such that (T, S) is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits on K. By Zorn's Lemma, Γ has a minimal member, say F. We complete the proof by verifying that F consists of a single point. Assume that $x \in F_c$. In this case, for any $y \in F_c$ we have

$$\begin{aligned} \|Sx - y\| &\leq \sup\{\|z - y\| : z \in F\} \\ &= r_y(F) = r(F), \end{aligned}$$

therefore,

$$\sup\{\|Sx - y\| : x \in F_c\} \leq r(F).$$

Then,

$$\begin{aligned} r_{Sx}(F) &= \sup\{\|Sx - y\| : y \in F\} \\ &\leq \sup\{\|Sx - y\| : x \in F_c, y \in F\} \\ &\leq \sup\{r(F), y \in F\} \\ &= r(F). \end{aligned}$$

Then, for any $x \in F_c$ we have $r_{Sx}(F) = r(F)$; that is, $S : F_c \to F_c$. Moreover, for any $x, y \in F_c$ we have $\|Sx - Sy\| \leq r(F)$. On other hand, for any $x, y \in F_c$,

$$\begin{aligned}\delta_x[\mathcal{O}^2(y;\infty)] &= \sup_{n\in\mathbb{N}} \|x - T^{2n}y\| \\ &\leq \sup\{\|x - z\| : z \in F\} \\ &= r_x(F) = r(F).\end{aligned}$$

Similarly, for any $x, y \in F_c$ we have $\delta_y[\mathcal{O}^2(x;\infty)] \leq r(F)$. In particular, for each $x, y \in F_c$,

$$\begin{aligned}\|Tx - Ty\| &\leq \alpha(x,y)\|Sx - Sy\| + (1-\alpha(x,y))\max\{\delta_x[\mathcal{O}^2(y;\infty)], \delta_y[\mathcal{O}^2(x;\infty)]\} \\ &\leq \alpha(x,y)r(F) + (1-\alpha(x,y))r(F) \\ &= r(F);\end{aligned}$$

that is, $r_{Tx}(F) = r(F)$. Then, $T : F_c \to F_c$. By Lemma 2.3, we have $F_c \in \Gamma$. If $\delta(F) > 0$, then by Lemma 2.4, F_c is properly contained in F which contradicts the minimality of F. Hence $\delta(F) = 0$ and F consists of a single point; this is, there exists a point $p \in K$ such that $Tp = p$ and $Sp = p$. So, there exists a $p \in K$ such that $\|Tp - p\| = 0$. □

Theorem 2.6. *Assume that (A, B) is a nonempty pair of subsets in a Banach space X with PNS. Let $T, S : A \cup B \to A \cup B$ be a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits, and such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Suppose that $(S(A), S(B))$ is a weakly compact and convex pair of subsets in X. Then there exists $(x, y) \in A \times B$ such that for $p \in \{x, y\}$ we have*

$$\|Tp - Sp\| = \mathrm{dist}(S(A), S(B)).$$

Proof. The result follows from Theorem 2.5 if $\mathrm{dist}(S(A), S(B)) = 0$, so we assume that $\mathrm{dist}(S(A), S(B)) > 0$. Let $(S(A_0^s), S(B_0^s))$ be the associated proximal pair of $(S(A), S(B))$. We have already observed that $S(A_0^s)$ and $S(B_0^s)$ are nonempty, weakly compact and convex, moreover

$$\mathrm{dist}(S(A_0^s), S(B_0^s)) = \mathrm{dist}(S(A), S(B)).$$

Assume that $x \in A_0^s$, then there exists $y \in B_0^s$ such that $\|Sx - Sy\| = \mathrm{dist}(S(A), S(B))$. On other hand, $(T; S)$ is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits. Thus,

$$\|T(Sx) - T(Sy)\| = \mathrm{dist}(S(A), S(B)), \quad \|S(Sx) - S(Sy)\| = \mathrm{dist}(S(A), S(B)).$$

This implies that

$$\|S(Sx) - S(Sy)\| = \mathrm{dist}(S(A_0^s), S(B_0^s)),$$

and

$$\|T(Sx) - T(Sy)\| = \mathrm{dist}(S(A_0^s), S(B_0^s)).$$

Therefore, we have

$$T(Sx) \in S(B_0^s), \quad T(Sy) \in S(A_0^s);$$

that is,

$$T(S(A_0^s)) \subseteq S(B_0^s), \quad T(S(B_0^s)) \subseteq S(A_0^s).$$

Similarly,

$$S(S(A_0^s)) \subseteq S(A_0^s), \quad S(S(B_0^s)) \subseteq S(B_0^s).$$

So, for each $x \in A_0^s$ and $y \in B_0^s$ we have

$$\|T(Sx) - T(Sy)\| = \operatorname{dist}(S(A_0^s), S(B_0^s)),$$

and

$$\|S(Sx) - S(Sy)\| = \operatorname{dist}(S(A_0^s), S(B_0^s)).$$

Clearly $(S(A_0^s), S(B_0^s))$ also has proximal normal structure. Now, assume that Ω denotes the collection of all nonempty subsets $S(F)$ of $S(A_0^s) \cup S(B_0^s)$ for which $S(F) \cap S(A_0^s)$ and $S(F) \cap S(B_0^s)$ are nonempty, closed, convex, and such that

$$T(S(F) \cap S(A_0^s)) \subseteq S(F) \cap S(B_0^s), \quad T(S(F) \cap S(B_0^s)) \subseteq S(F) \cap S(A_0^s),$$

and

$$S(S(F) \cap S(A_0^s)) \subseteq S(F) \cap S(A_0^s), \quad S(S(F) \cap S(B_0^s)) \subseteq S(F) \cap S(B_0^s),$$

and so

$$\operatorname{dist}(S(F) \cap S(A_0^s), S(F) \cap S(B_0^s)) = \operatorname{dist}(S(A), S(B)).$$

Since, $S(A_0^s) \cup S(B_0^s) \in \Omega$ and Ω is nonempty, we may assume that $\{S(F_\alpha)\}_{\alpha \in \Omega}$ is a decreasing chain in Ω such that $S(F_0) = \cap_{\alpha \in \Omega} S(F_\alpha)$. Then $S(F_0) \cap S(A_0^s) = \cap_{\alpha \in \Omega}(S(F_\alpha) \cap S(A_0^s))$, so $S(F_0) \cap S(A_0^s)$ is nonempty, closed and convex. Similarly, $S(F_0) \cap S(B_0^s)$ is nonempty, closed and convex. Also,

$$T(S(F_0) \cap S(A_0^s)) \subseteq S(F_0) \cap S(B_0^s), \quad T(S(F_0) \cap S(B_0^s)) \subseteq S(F_0) \cap S(A_0^s)$$

and

$$S(S(F_0) \cap S(A_0^s)) \subseteq S(F_0) \cap S(A_0^s), \quad S(S(F_0) \cap S(B_0^s)) \subseteq S(F_0) \cap S(B_0^s).$$

To show that $S(F_0) \in \Omega$ we only need to verify that

$$\operatorname{dist}(S(F_0) \cap S(A_0^s), S(F_0) \cap S(B_0^s)) = \operatorname{dist}(S(A), S(B)).$$

Note that for each $\alpha \in J$ it is possible to select

$$Sx_\alpha \in S(F_\alpha) \cap S(A_0^s), \quad Sy_\alpha \in S(F_\alpha) \cap S(B_0^s)$$

such that

$$\|Sx_\alpha - Sy_\alpha\| = \operatorname{dist}(S(A), S(B)).$$

It is also possible to choose convergent subnets $\{Sx_{\alpha'}\}$ and $\{Sy_{\alpha'}\}$ (with the same indices), say

$$\lim_{\alpha'} Sx_{\alpha'} = Sx, \quad \lim_{\alpha'} Sy_{\alpha'} = Sy.$$

Then clearly $Sx \in S(F_0) \cap S(A_0^s)$ and $Sy \in S(F_0) \cap S(B_0^s)$. By weak lower semicontinuity of the norm, we have $\|Sx - Sy\| \leq \mathrm{dist}(S(A), S(B))$; hence,

$$\mathrm{dist}(S(A), S(B)) \leq \mathrm{dist}(S(F_0) \cap S(A_0^s), S(F_0) \cap S(B_0^s)) \leq \|Sx - Sy\| \leq \mathrm{dist}(S(A), S(B)).$$

Therefore,

$$\mathrm{dist}(S(F_0) \cap S(A_0^s), S(F_0) \cap S(B_0^s)) = \mathrm{dist}(S(A), S(B)).$$

Since, every chain in Ω is bounded below by a member of Ω, Zorn's Lemma implies that Ω has a minimal element, say $S(K)$. Assume that $S(K_1) = S(K) \cap S(A_0^s)$ and $S(K_2) = S(K) \cap S(B_0^s)$. Observe that if

$$\delta(S(K_1), S(K_2)) = \mathrm{dist}(S(K_1), S(K_2)),$$

then for any $x \in S(K_1)$, we have

$$\|Tx - Sx\| = \mathrm{dist}(S(K_1), S(K_2)) = \mathrm{dist}(S(A), S(B)).$$

Similarly, for any $y \in S(K_2)$, we have

$$\|Ty - Sy\| = \mathrm{dist}(S(K_1), S(K_2)) = \mathrm{dist}(S(A), S(B)).$$

Now, we assume that

$$\delta(S(K_1), S(K_2)) > \mathrm{dist}(S(K_1), S(K_2)).$$

We complete the proof by showing that this leads to a contradiction. Since $S(K)$ is minimal, it follows that $(S(K_1), S(K_2))$ is a proximal pair in $(S(A_0^s), S(B_0^s))$. By the PNS property of X, there exist $(x_1, y_1) \in K_1 \times K_2$ and $\beta \in (0, 1)$ such that

$$\delta(Sx_1, S(K_2)) \leq \beta\delta(S(K_1), S(K_2)) \quad \text{and} \quad \delta(Sy_1, S(K_1)) \leq \beta\delta(S(K_1), S(K_2)).$$

Since, $(S(K_1), S(K_2))$ is a proximal pair, there exists $(x_2, y_2) \in K_1 \times K_2$ such that for each distinct $i, j \in \{1, 2\}$, we have

$$\|Sx_i - Sy_j\| = \mathrm{dist}(S(K_1), S(K_2)).$$

So, for each $u \in S(K_2)$ we have

$$\begin{aligned}\|\frac{Sx_1 + Sx_2}{2} - u\| &\leq \|\frac{Sx_1 - u}{2}\| + \|\frac{Sx_2 - u}{2}\| \\ &\leq \frac{\beta\delta(S(K_1), S(K_2))}{2} + \frac{\delta(S(K_1), S(K_2))}{2} \\ &= \alpha\delta(S(K_1), S(K_2)),\end{aligned}$$

where $\alpha = \frac{1+\beta}{2} \in (0, 1)$. Assume that $Sw_1 = \frac{(Sx_1+Sx_2)}{2}$ and $Sw_2 = \frac{(Sy_1+Sy_2)}{2}$. Then

$$\delta(Sw_1, S(K_2)) \leq \alpha\delta(S(K_1), S(K_2)) \quad \text{and} \quad \delta(Sw_2, S(K_1)) \leq \alpha\delta(S(K_1), S(K_2)).$$

Since,

$$\begin{aligned}\operatorname{dist}(S(K_1), S(K_2)) &\le \|Sw_1 - Sw_2\| \\ &= \|\frac{(Sx_1 + Sx_2)}{2} - \frac{(Sy_1 + Sy_2)}{2}\| \\ &\le \frac{1}{2}[\|Sx_1 - Sy_2\| + \|Sx_2 - Sy_1\|] \\ &= \operatorname{dist}(S(K_1), S(K_2)),\end{aligned}$$

we have $\|Sw_1 - Sw_2\| = \operatorname{dist}(S(K_1), S(K_2))$. Put

$$\begin{aligned}S(L_1) &= \{Sx \in S(K_1) : \delta(Sx, S(K_2)) \le \alpha\delta(S(K_1), S(K_2))\}, \\ S(L_2) &= \{Sy \in S(K_2) : \delta(Sy, S(K_1)) \le \alpha\delta(S(K_1), S(K_2))\}.\end{aligned}$$

Then for each $i = 1, 2$, $S(L_i)$ is a nonempty, closed and convex subset of $S(K_i)$ and since $Sw_1 \in S(L_1)$ and $Sw_2 \in S(L_2)$, we have

$$\operatorname{dist}(S(K_1), S(K_2)) \le \operatorname{dist}(S(L_1), S(L_2)) \le \|Sw_1 - Sw_2\| = \operatorname{dist}(S(K_1), S(K_2)).$$

Therefore,

$$\operatorname{dist}(S(L_1), S(L_2)) = \operatorname{dist}(S(K_1), S(K_2)) = \operatorname{dist}(S(A), S(B)).$$

Now, assume that $Sx \in S(L_1)$ and $Sy \in S(K_2)$. Then $Sx \in S(A_0^s)$ and $Sy \in S(B_0^s)$; that is, $x \in A_0^s$ and $y \in B_0^s$. Thus,

$$\|T(Sx) - T(Sy)\| = \operatorname{dist}(S(A), S(B)) \le \delta(Sx, S(K_2)) \le \alpha\delta(S(K_1), S(K_2)).$$

So, $T(Sy) \in B(T(Sx); \alpha\delta(S(K_1), S(K_2))) \cap S(K_1)$; that is,

$$T(S(K_2)) \subseteq B(T(Sx); \alpha\delta(S(K_1), S(K_2))) \cap S(K_1) := S(K_1').$$

Clearly $S(K_1')$ is closed and convex. Also, if $Sy \in S(K_2)$ satisfies $\|Sx - Sy\| = \operatorname{dist}(S(A), S(B))$, then

$$\|T(Sx) - T(Sy)\| = \operatorname{dist}(S(K_1), S(K_2)).$$

Since, $T(Sy) \in S(K_1')$, we conclude that $\operatorname{dist}(S(K_1'), S(K_2)) = \operatorname{dist}(S(A), S(B))$. Therefore, $S(K_1') \cup S(K_2) \in \Omega$ and by the minimality of $S(K)$ we must have $S(K_1') = S(K_1)$. Hence,

$$S(K_1) \subseteq B(T(Sx); \alpha\delta(S(K_1), S(K_2)));$$

that is, $\delta(T(Sx), S(K_1)) \le \alpha\delta(S(K_1), S(K_2))$ and since $Sx \in S(L_1)$ was arbitrary, we obtain $T(S(L_1)) \subseteq S(L_2)$. Similarly, $T(S(L_2)) \subseteq S(L_1)$, $S(S(L_1)) \subseteq S(L_1)$ and $S(S(L_2)) \subseteq S(L_2)$. Thus, $S(L_1) \cup S(L_2) \in \Omega$, but $\delta(S(L_1), S(L_2)) \le \alpha\delta(S(K_1), S(K_2))$, contradicting the minimality of $S(K)$. □

Corollary 2.7. *Assume that (A, B) is a nonempty pair of subsets in a uniformly convex Banach space X. Let $T, S : A \cup B \to A \cup B$ be a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits, and such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Suppose that $(S(A), S(B))$ is a bounded,* *closed and convex pair of subsets in X. Then there exists $(x, y) \in A \times B$ such that for $p \in \{x, y\}$ we have*

$$\|Tp - Sp\| = \operatorname{dist}(S(A), S(B)).$$

3. Noncyclic-noncyclic pairs

In this section we study the case in which both mappings are noncyclic. Indeed, we first introduce a pointwise noncyclic-noncyclic relatively nonexpansive pair involving orbits, and proceed to study its best proximity points.

Definition 3.1. Assume that (A, B) is a nonempty pair of subsets of a Banach space X and $T, S : A \cup B \to A \cup B$ are two mappings. A pair $(T; S)$ is said to be a pointwise noncyclic-noncyclic relatively nonexpansive pair involving orbits if $(T; S)$ is a noncyclic-noncyclic pair and for any $(x, y) \in A \times B$, if $\|x - y\| = \operatorname{dist}(S(A), S(B))$, then

$$\|Tx - Ty\| = \operatorname{dist}(S(A), S(B)), \quad \|Sx - Sy\| = \operatorname{dist}(S(A), S(B))$$

and otherwise, there exists a function $\alpha : A \times B \to [0, 1]$ such that

$$\|Tx - Ty\| \leq \alpha(x, y)\|Sx - Sy\| + (1 - \alpha(x, y)) \max\{\delta_x[\mathcal{O}(y; \infty)], \delta_y[\mathcal{O}(x; \infty)]\},$$

where, for any $(x, y) \in A \times B$

$$\delta_x[\mathcal{O}(y; \infty)] = \sup_{n \in \mathbb{N}} \|x - T^n y\|, \quad \delta_y[\mathcal{O}(x; \infty)] = \sup_{n \in \mathbb{N}} \|T^n x - y\|.$$

Theorem 3.2. *Assume that (A, B) is a nonempty pair of subsets in a strictly convex Banach space X with PNS, and $T, S : A \cup B \to A \cup B$ is a pointwise noncyclic-noncyclic relatively nonexpansive pair involving orbits such that $T(A) \subseteq S(A)$ and $T(B) \subseteq S(B)$. Suppose that $(S(A), S(B))$ is a weakly compact and convex pair of subsets in X. Then, there exists $x_0 \in A$ and $y_0 \in B$ such that*

$$Tx_0 = x_0, \quad Ty_0 = y_0$$

and

$$\|x_0 - y_0\| = \operatorname{dist}(S(A), S(B)).$$

Proof. Suppose that $(S(A_0^s), S(B_0^s))$ is the associated proximal pair of $(S(A), S(B))$, and choose $x \in A_0^s$. Then there exists $y \in B_0^s$ such that $\|Sx - Sy\| = \operatorname{dist}(S(A), S(B))$, and furthermore

$$\|T(Sx) - T(Sy)\| = \operatorname{dist}(S(A), S(B)) = \operatorname{dist}(S(A_0^s), S(B_0^s)).$$

Thus, $T : S(A_0^s) \to S(A_0^s)$ and similarly, $T : S(B_0^s) \to S(B_0^s)$. Now let Ω denote the collection of nonempty subsets $S(F)$ of $S(A_0^s) \cup S(B_0^s)$ for which $S(F) \cap S(A_0^s)$ and $S(F) \cap S(B_0^s)$ are nonempty, closed and convex,

$$T(S(F) \cap S(A_0^s)) \subseteq S(F) \cap S(A_0^s), \quad T(S(F) \cap S(B_0^s)) \subseteq S(F) \cap S(B_0^s),$$

$$S(S(F) \cap S(A_0^s)) \subseteq S(F) \cap S(A_0^s), \quad S(S(F) \cap S(B_0^s)) \subseteq S(F) \cap S(B_0^s)$$

and

$$\operatorname{dist}(S(F)\cap S(A_0^s), S(F)\cap S(B_0^s)) = \operatorname{dist}(S(A), S(B)).$$

Since, $S(A_0^s)\cup S(B_0^s)\in\Omega$, Ω is nonempty. We proceed as in the proof of Theorem 2.6 to show that Ω has a minimal element $S(K)$. Assume that $S(K_1) = S(K)\cap S(A_0^s)$, and $S(K_2) = S(K)\cap S(B_0^s)$. First, assume that one of the sets is a singleton, say $S(K_1) = \{x\}$. Then $Tx = x$ and if y is the unique point of $S(K_2)$ for which $\|x-y\| = \operatorname{dist}(S(K_1), S(K_2))$, it must be the case that $Ty = y$. Since, $\|y-x\| = \operatorname{dist}(S(A), S(B))$, we are finished. So, we may assume that $S(K_1)$ and $S(K_2)$ have positive diameter and because the space is strictly convex, this in turn implies that

$$\delta(S(K_1), S(K_2)) > \operatorname{dist}(S(K_1), S(K_2)).$$

We shall see that this leads to a contradiction. Since $(S(A_0^s), S(B_0^s))$ has proximal normal structure, we may define $S(L_1)$ and $S(L_2)$ as in the proof of Theorem 2.6. Choose $Sx\in S(L_1)$. For any $Sy\in S(K_2)$, we have $Sx\in S(A_0^s)$ and $Sy\in S(B_0^s)$; that is, $x\in A_0^s$ and $y\in B_0^s$. Thus, $\|Sx-Sy\| = \operatorname{dist}(S(A), S(B))$ and so,

$$\|T(Sx)-T(Sy)\| = \operatorname{dist}(S(A), S(B)) \le \delta(Sx, S(K_2)) \le \alpha\delta(S(K_1), S(K_2)).$$

This implies that

$$T(Sy)\in B(T(Sx);\alpha\delta(S(K_1), S(K_2)))\cap S(K_2),$$

thus,

$$T(S(K_2))\subseteq B(T(Sx);\alpha\delta(S(K_1), S(K_2)))\cap S(K_2).$$

It follows from the minimality of $S(K)$ that $S(K_2)\subseteq B(T(Sx);\alpha\delta(S(K_1), S(K_2)))$ and this in turn implies that

$$\delta(T(Sx), S(K_2)) \le \alpha\delta(S(K_1), S(K_2)).$$

Therefore, $T(Sx)\in S(L_1)$; in fact $T(S(L_1))\subseteq S(L_1)$. Similarly, $T(S(L_2))\subseteq S(L_2)$, $S(S(L_1))\subseteq S(L_1)$ and $S(S(L_2))\subseteq S(L_2)$. Since, $S(L_1)$ and $S(L_2)$ are, respectively, nonempty, closed and convex subsets of $S(K_1)$ and $S(K_2)$ and since for $\alpha < 1$ we have

$$\delta(S(L_1), S(L_2)) \le \alpha\delta(S(K_1), S(K_2)),$$

which contradicts the minimality of $S(K)$. □

Corollary 3.3. *Assume that (A, B) is a nonempty pair of subsets in a uniformly convex Banach space X and $T, S : A\cup B\to A\cup B$ is a pointwise noncyclic-noncyclic relatively nonexpansive pair involving orbits such that $T(A)\subseteq S(A)$ and $T(B)\subseteq S(B)$. Suppose that $(S(A), S(B))$ is a bounded, closed and convex pair of subsets in X. Then, there exists $x_0\in A$ and $y_0\in B$ such that*

$$Tx_0 = x_0, \quad Ty_0 = y_0$$

and

$$\|x_0-y_0\| = \operatorname{dist}(S(A), S(B)).$$

4. WPNS and cyclic-noncyclic pairs

In this section, and under weak proximal normal structure, we discuss the coincidence quasi-best proximity point problem for pointwise cyclic-noncyclic relatively nonexpansive pairs involving orbits.

Lemma 4.1. *Assume that (A, B) is a nonempty pair of subsets in a Banach space X, and $T, S : A \cup B \to A \cup B$ is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Suppose that $(S(A), S(B))$ is a weakly compact and convex pair of subsets in X. Then, there exists $(S(K_1), S(K_2)) \subseteq (S(A_0^s), S(B_0^s)) \subseteq (S(A), S(B))$ which is minimal with respect to being nonempty, closed, convex and T and S-invariant pair of subsets of $(S(A), S(B))$, such that*

$$\operatorname{dist}(S(K_1), S(K_2)) = \operatorname{dist}(S(A), S(B)).$$

Moreover, the pair $(S(K_1), S(K_2))$ is proximal.

Proof. The proof essentially goes in the same lines as in the proof of Theorem 2.6. We omit the details. □

Theorem 4.2. *Assume that (A, B) is a nonempty pair of subsets in a Banach space X with $WPNS$, and $T, S : A \cup B \to A \cup B$ is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Suppose that $(S(A), S(B))$ is a weakly compact and convex pair of subsets in X. Then $(T; S)$ has a coincidence quasi-best proximity point.*

Proof. By Lemma 4.1, assume that $(S(K_1), S(K_2))$ is a minimal, weakly compact, convex and proximal pair which is T and S-invariant, and such that $\operatorname{dist}(S(K_1), S(K_2)) = \operatorname{dist}(S(A), S(B))$. Notice that

$$\overline{con}(T(S(K_1))) \subseteq S(K_2)$$

and so,

$$T(\overline{con}(T(S(K_1)))) \subseteq T(S(K_2)) \subseteq \overline{con}(T(S(K_2))).$$

Similarly,

$$T(\overline{con}(T(S(K_2)))) \subseteq \overline{con}(T(S(K_1)));$$

that is, T is cyclic on $\overline{con}(T(S(K_1))) \cup \overline{con}(T(S(K_2)))$.

On other hand, S is noncyclic on $\overline{con}(S(S(K_1))) \cup \overline{con}(S(S(K_2)))$. The minimality of $(S(K_1), S(K_2))$ implies that

$$\overline{con}(T(S(K_1))) = S(K_2) \quad \text{and} \quad \overline{con}(T(S(K_2))) = S(K_1).$$

Besides,

$$\overline{con}(S(S(K_1))) = S(K_1) \quad \text{and} \quad \overline{con}(S(S(K_2))) = S(K_2).$$

We note that if $\delta(S(K_1), S(K_2)) = \operatorname{dist}(S(K_1), S(K_2)) = \operatorname{dist}(S(A), S(B))$, then every point of $S(K_1) \cup S(K_2)$ is a coincidence quasi-best proximity point of $(T;S)$ and we are finished. Otherwise, since $(S(A),S(B))$ has WPNS, there exists a point $(x_1,y_1)\in K_1\times K_2$ and $c\in(0,1)$, so that

$$\delta(Sx_1,S(K_2))\leq c\,\delta(S(K_1),S(K_2)),\quad \delta(Sy_1,S(K_1))\leq c\,\delta(S(K_1),S(K_2)).$$

Since $(S(K_1),S(K_2))$ is a proximal pair, there exists $(x_2,y_2)\in K_1\times K_2$ such that

$$\|Sx_1-Sy_2\|=\|Sx_2-Sy_1\|=\operatorname{dist}(S(A),S(B)).$$

Put $Su:=\frac{Sx_1+Sx_2}{2}$ and $Sv:=\frac{Sy_1+Sy_2}{2}$. Then, $(Su,Sv)\in S(K_1)\times S(K_2)$ and

$$\|Su-Sv\|=\operatorname{dist}(S(K_1),S(K_2)).$$

Moreover, for each $z\in K_2$, we have

$$\begin{aligned}\|Su-Sz\|&=\|\frac{Sx_1+Sx_2}{2}-Sz\|\\&\leq\frac{1}{2}[\|Sx_1-Sz\|+\|Sx_2-Sz\|]\\&\leq\frac{c+1}{2}\delta(S(K_1),S(K_2)).\end{aligned}$$

Now, if $r:=\frac{c+1}{2}$, then $r\in(0,1)$ and $\delta(Su,(S(K_2))\leq r\delta(S(K_1),S(K_2))$. Similarly, we can see that $\delta(Sv,(S(K_1))\leq r\delta(S(K_1),S(K_2))$. Assume that

$$\begin{aligned}S(L_1)&=\{Sx\in S(K_1):\delta(Sx,S(K_2))\leq r\delta(S(K_1),S(K_2))\},\\S(L_2)&=\{Sy\in S(K_2):\delta(Sy,S(K_1))\leq r\delta(S(K_1),S(K_2))\}.\end{aligned}$$

Thus, $(Su,Sv)\in S(L_1)\times S(L_2)$ and so, $\operatorname{dist}(S(L_1),S(L_2))=\operatorname{dist}(S(K_1),S(K_2))$. Moreover, $(S(L_1),S(L_2))$ is a weakly compact and convex pair in X. We show that T is cyclic on $S(L_1)\cup S(L_2)$. Suppose $Sx\in S(L_1)$ and $Sy\in S(K_2)$. Then, similar to proof of Theorem 2.6, $Sx\in S(A_0^s)$ and $Sy\in S(B_0^s)$; that is, $x\in A_0^s$ and $y\in B_0^s$. Thus,

$$\|T(Sx)-T(Sy)\|=\operatorname{dist}(S(A),S(B))\leq\delta(Sx,S(K_2))\leq r\delta(S(K_1),S(K_2)).$$

So, $T(Sy)\in B(T(Sx);r\delta(S(K_1),S(K_2)))$; that is,

$$T(S(K_2))\subseteq B(T(Sx);r\delta(S(K_1),S(K_2)))$$

and

$$S(K_1)=\overline{con}T(S(K_2))\subseteq B(T(Sx);r\delta(S(K_1),S(K_2))).$$

Therefore, $\delta(T(Sx),S(K_1))\leq r\delta(S(K_1),S(K_2))$; that is, $T(Sx)\in S(L_2)$. Thus, $T(S(L_1))\subseteq S(L_2)$. Similarly, $T(S(L_2))\subseteq S(L_1)$, $S(S(L_1))\subseteq S(L_1)$ and $S(S(L_2))\subseteq S(L_2)$. Hence, T is cyclic and S is noncyclic on $S(L_1)\cup S(L_2)$. The minimality of $(S(K_1),S(K_2))$ now implies that

$$S(L_1)=S(K_1)\quad\text{and}\quad S(L_2)=S(K_2).$$

Now, we have

$$\delta(S(K_1),S(K_2))=\sup_{x\in K_1}\delta(Sx,S(K_2))\leq r\delta(S(K_1),S(K_2)),$$

which is a contradiction. □

5. Examples

We clarify the above results with some examples.

Example 5.1. Let $A = [-4, 0]$ and $B = [0, 4]$ be subsets of the uniformly convex Banach space $(\mathbb{R}, |.|)$. For any $x \in A \cup B$ we define

$$Tx = -\frac{1}{4}x, \quad Sx = \frac{1}{2}x.$$

Then,

$$T(A) = [0, 1] \subseteq [0, 2] = S(B), \quad T(B) = [-1, 0] \subseteq [-2, 0] = S(A).$$

Moreover, for any $(x, y) \in A \times B$, we define

$$\alpha(x, y) = \begin{cases} 0, & if \quad x = y \\ 1, & if \quad x \neq y. \end{cases}$$

If $(x, y) \in A \times B$ such that $\|x - y\| = \text{dist}(S(A), S(B)) = 0$, then $x = y$ and

$$\|Tx - Ty\| = \text{dist}(S(A), S(B)), \quad \|Sx - Sy\| = \text{dist}(S(A), S(B)).$$

Otherwise,

$$\begin{aligned} \|Tx - Ty\| = \|\frac{1}{4}y - \frac{1}{4}x\| &= \frac{1}{2}\|\frac{1}{2}y - \frac{1}{2}x\| \\ &= \frac{1}{2}\|Sy - Sx\| = \frac{1}{2}\|Sx - Sy\| \\ &\leq \|Sx - Sy\| \\ &= \alpha(x, y)\|Sx - Sy\| + (1 - \alpha(x, y)) \max\{\delta_x[\mathcal{O}^2(y; \infty)], \delta_y[\mathcal{O}^2(x; \infty)]\}. \end{aligned}$$

Thus, $(T; S)$ is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits, and by Corollary 2.7, there exists $(x, y) \in A \times B$ such that

$$\|Tx - Sx\| = \text{dist}(S(A), S(B)), \quad \|Ty - Sy\| = \text{dist}(S(A), S(B)).$$

Example 5.2. Let $A = [-4, -1]$ and $B = [1, 4]$ be subsets in $(\mathbb{R}, |.|)$. Let $K_1 = [-4, -2]$, $K_2 = [2, 4]$ and

$$Sx = \begin{cases} -\sqrt{-x} - 2, & if \quad x \in A \setminus K_1 \\ \sqrt{x} + 2, & if \quad x \in B \setminus K_2 \\ -3, & if \quad x \in K_1 \\ 3, & if \quad x \in K_2. \end{cases}$$

Therefore, S is a noncyclic mapping. Moreover,

$$S(A) = [-4, -3] \subseteq A, \quad S(B) = [3, 4] \subseteq B.$$

So, $(S(A), S(B))$ is a closed, convex and bounded pair and we have

$$\text{dist}(S(A), S(B)) = 6.$$

Suppose that

$$Tx = \begin{cases} \sqrt{-x}+2, & if \quad x \in A \setminus K_1 \\ -\sqrt{x}-2, & if \quad x \in B \setminus K_2 \\ 3, & if \quad x \in K_1 \\ -3, & if \quad x \in K_2. \end{cases}$$

Therefore, T is a cyclic mapping. Besides,

$$T(A) = [3,4] = S(B) \subseteq B, \quad T(B) = [-4,-3] = S(A) \subseteq A.$$

Moreover, we suppose that for any $(x, y) \in A \times B$,

$$\alpha(x,y) = \begin{cases} 1, & if \quad (x,y) \in (A \setminus K_1) \times (B \setminus K_2) \\ 0, & otherwise. \end{cases}$$

If $\|x-y\| = \text{dist}(S(A), S(B))$, then $(x, y) \in K_1 \times K_2$ and we have

$$\|Sx - Sy\| = \| -3-3\| = 6 = \text{dist}(S(A), S(B))$$

and

$$\|Tx - Ty\| = \|3-(-3)\| = 6 = \text{dist}(S(A), S(B)).$$

Onherwise, for any $(x, y) \in (A \setminus K_1) \times (B \setminus K_2)$, we have

$$\begin{aligned} \|Tx - Ty\| &= \|\sqrt{-x}+2-(-\sqrt{y}-2)\| \\ &= \|\sqrt{-x}+\sqrt{y}+4\| = \|\sqrt{y}+2-(-\sqrt{-x}-2)\| \\ &= \|Sy - Sx\| = \|Sx - Sy\| \\ &\leq \alpha(x,y)\|Sx-Sy\| + (1-\alpha(x,y))\max\{\delta_x[\mathcal{O}(y;\infty)], \delta_y[\mathcal{O}(x;\infty)]\}. \end{aligned}$$

Thus, $(T;S)$ is a pointwise cyclic-noncyclic relatively nonexpansive pair involving orbits, and by Corollary 2.7, there exists $(x, y) \in A \times B$ such that

$$\|Tx - Sx\| = \text{dist}(S(A), S(B)), \quad \|Ty - Sy\| = \text{dist}(S(A), S(B)).$$

In fact, for any $(x, y) \in K_1 \times K_2$, we have

$$\|Tx - Sx\| = 6 = \text{dist}(S(A), S(B)), \quad \|Ty - Sy\| = 6 = \text{dist}(S(A), S(B)).$$

We clarify the above result with an example.

Example 5.3. Assume that $A = [-4, 0]$ and $B = [0, 4]$ are subsets of $(\mathbb{R}, |.|)$. For any $x \in A \cup B$, we set

$$Tx = \frac{1}{4}x, \quad Sx = \frac{1}{2}x.$$

Then,

$$T(A) = [-1,0] \subseteq [-2,0] = S(A), \quad T(B) = [0,1] \subseteq [0,2] = S(B).$$

Moreover, we suppose that for any $(x, y) \in A \times B$,

$$\alpha(x,y) = \begin{cases} 0, & if \quad x = y \\ 1, & if \quad x \neq y. \end{cases}$$

If $(x, y) \in A \times B$ such that $\|x - y\| = \text{dist}(S(A), S(B)) = 0$, then $x = y$ and

$$\|Tx - Ty\| = \text{dist}(S(A), S(B)), \quad \|Sx - Sy\| = \text{dist}(S(A), S(B)).$$

Otherwise,

$$\begin{aligned}\|Tx - Ty\| &= \|\frac{1}{4}x - \frac{1}{4}y\| = \frac{1}{2}\|\frac{1}{2}x - \frac{1}{2}y\| \\ &= \frac{1}{2}\|Sx - Sy\| \\ &\leq \|Sx - Sy\| \\ &= \alpha(x, y)\|Sx - Sy\| + (1 - \alpha(x, y)) \max\{\delta_x[\mathcal{O}(y; \infty)], \delta_y[\mathcal{O}(x; \infty)]\}.\end{aligned}$$

Thus, $(T; S)$ is a pointwise noncyclic-noncyclic relatively nonexpansive pair involving orbits, and by Corollary 3.3, there exists $(x_0, y_0) \in A \times B$ such that

$$\|x_0 - y_0\| = \text{dist}(S(A), S(B)).$$

In fact, for $x_0 = 0$ and $y_0 = 0$, we have $Tx_0 = x_0$, $Ty_0 = y_0$ and

$$\|x_0 - y_0\| = \text{dist}(S(A), S(B)).$$

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