On the Prüfer rank of mutually permutable products of two abelian groups

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Abstract

A group G has finite (or Prüfer or special) rank if every finitely generated subgroup of G can be generated by r elements and r is the least integer with this property. The aim of this paper is to prove the following result: Assume that $G = AB$ is a group which is the product of the mutually permutable abelian subgroups A and B of Prüfer ranks r and s respectively. If G is locally finite, then the Prüfer rank of G is at most $r + s + 3$. If G is an arbitrary group, then the Prüfer rank of G is at most $r + s + 4$.
Mathematics Subject Classification (2010): 20D10, 20D20
Keywords: abelian group, soluble group, polycyclic group, rank, factorisations.

1 Introduction

A group G has finite (or Prüfer or special) rank $r = r(G)$ if every finitely generated subgroup of G can be generated by r elements and r is the least integer with this property. Denote also by $d(G)$ the minimum number of elements required to generate the group G. If the locally soluble group $G = AB$ is the product of two subgroups A and B with finite Prüfer rank, then G is hyperabelian with finite Prüfer rank (see [3, Theorem 1.1]) and, in this case, the Prüfer rank of G is bounded by a function of the Prüfer ranks of A and B (see [1, Theorem 4.3.5]). Unfortunately, this bound is not explicit. If G is a finite p-group for some prime p and the Prüfer ranks of A and B are bounded by r, then the Prüfer rank of G is bounded by a polynomial function of r. Better bounds for factorised finite p-groups were showed in [2]. However, it seems to be difficult to decide if the Prüfer rank of G is bounded by a linear function of the Prüfer ranks of A and B. On the other hand, the class of metabelian groups of finite rank which are products of two abelian subgroups has attracted growing interest recently, particulary in relation to the class of all metabelian groups which are constructible in the sense of Baumslag and Bieri [5].

Our main goal in this paper is to give a linear explicit bound for the Prüfer rank of a mutually permutable product $G = AB$ of two abelian subgroups A and B in terms of the Prüfer ranks of A and B.

Our first theorem gives an upper bound for the Prüfer rank of locally finite mutually permutable products.

Theorem A. *Let the locally finite group $G = AB$ be the product of the mutually permutable abelian subgroups A and B. If A and B have Prüfer ranks r and s respectively, then the Prüfer rank of G is at most $r + s + 3$.*

As a consequence a linear upper bound for the Prüfer rank of an arbitrary mutually permutable product is obtained.

Theorem B. *Let the group $G = AB$ be the product of the mutually permutable abelian subgroups A and B. If A and B have Prüfer ranks r and s respectively, then the Prüfer rank of G is at most $r + s + 4$.*

Recall that two subgroups A and B of a group G *permute* if $AB = BA$ is a subgroup of G. A and B are called *mutually permutable* if every subgroup of A permutes with B and every subgroup of B permutes with A; of course any two normal subgroups are mutually permutable. If every subgroup of A permutes with every subgroup of B we say that A and B are *totally permutable*. Obviously totally permutable subgroups are mutually permutable but the converse does not hold in general. The structure of mutually and totally permutable products has been investigated by several authors in the last twenty five years, especially in the finite case, and received a full discussion in [4]. Mutually permutable products of infinite groups were considered in [6] and [7]. They play an important role in the proof of our main theorems.

Throughout the paper, the word rank will mean Prüfer rank.

2 Preliminary results

We collect in this section some results which are needed in the proof of main theorems. The following known property about ranks can be found in [9, Lemma 1.6.23] and will be used in the sequel without further comment: Let $N \trianglelefteq G$ and $H \leq G$. Then $r(G) \leq r(G/N) + r(N)$ and $r(H) \leq r(G)$.

We need to use power automorphisms of a group, so perhaps a quick review of facts about these groups would be appropiate. Recall that the *power automorphism group* of a group G, $\mathrm{PAut}(G)$, is the set of all automorphisms of G which leave every subgroup of G invariant. Hence, if $\alpha \in \mathrm{PAut}(G)$, there exists an integer $n_{g,\alpha}$ such that $g^\alpha = g^{n_{g,\alpha}}$, for all $g \in G$. If $n_{g,\alpha} = n_\alpha$ does not depend on the choice of g, then α is called a *universal power automorphism.*

The structures of G and $PAut(G)$ are strictly linked. For instance if G is a finite abelian p-group or more generally a finite regular p-group ([8, Theorem 5.3.1]), then every power automorphism is universal, and via the restriction homomorphism $PAut(G)$ can be embedded in $Aut(\langle g\rangle)$, for every cyclic group $\langle g\rangle$ of G of maximal order. In fact, we have:

Lemma 1 ([8, Theorem 3.4.1])**.** *Every power endomorphism of an abelian group G is locally universal.*

In particular, if G is finite and abelian of exponent p^n, then $PAut(G)$ can be embedded in $Aut(C_{p^n})$.

Another well known fact we shall need is the description of the automorphism group of a cyclic group. We state it here for the sake of completeness.

Lemma 2 ([10, I, Satz 13.19])**.** *Let G be a cyclic group of order p^n, p a prime.*

(i) If p is odd, then $\mathrm{Aut}(G) \simeq C_{p^{n-1}(p-1)}$.

(ii) If $p = 2$, then

- *(a) if $n = 1$, then* $\mathrm{Aut}(G) = 1$;
- *(b) if $n = 2$, then* $\mathrm{Aut}(G) = C_2$;
- *(c) if $n > 2$, then* $\mathrm{Aut}(G) = C_2 \times C_{2^{n-2}}$.

Moreover, consider a cyclic group $C = \langle c\rangle$ of order 2^n, $n > 2$. Then $\mathrm{Aut}(C) = \langle u\rangle \times \langle\alpha\rangle$, *where $c^u = c^{-1}$ and $c^\alpha = c^5$ and the order of α is 2^{n-2}. In particular, the involutions of* $\mathrm{Aut}(C)$ *are exactly u, $\gamma = \alpha^{2n-3}$ and $\eta = u\gamma$. Furthermore, $c^\gamma = c^{2^{n-1}+1}$ and $c^\eta = c^{2^{n-1}-1}$.*

Assume that a group A acts on a group B. Let $a \in A$ and $b \in B$. We say that *a inverts b* if $b^a = b^{-1}$; a inverts B if a inverts every element of B.

The following lemma shows that, in totally permutable products of finite p-groups, the structure of a core-free factor is very restricted.

Lemma 3. *Let the group $G = AB$ be the product of the totally permutable finite abelian p-groups A and B. Assume that* $\mathrm{Core}_G(A) = 1$.

(i) If p is odd, then A is cyclic.

(ii) If $p = 2$, then either A is cyclic or $A = \langle a \rangle \times \langle c \rangle$ for elements $a, c \in A$, such that $o(a) = 2$ and a inverts B.

Proof. First of all note that $A \cap B$ is a normal subgroup of G contained in A. Hence $A \cap B \leq \mathrm{Core}_G(A) = 1$.

Let $a \in A$ be an element of order p and let X be a cyclic subgroup of B. Then X is a maximal subgroup of $X\langle a \rangle$ and so a normalises X. Therefore, a acts as a power automorphism on B. By Lemma 1, a acts as a universal power automorphism on B. Since $\mathrm{Core}_G(A) = 1$, it follows that a acts non-trivially on B. Hence $\Omega_1(A)$ can be embedded in $\mathrm{PAut}(B)$. By Lemma 1, $\Omega_1(A)$ is isomorphic to a subgroup of $\mathrm{Aut}(C_{p^n})$, where $\exp(B) = p^n$.

If p is odd, $\mathrm{PAut}(B)$ is cyclic by Lemma 2(i) and so is $\Omega_1(A)$. Then, being A the direct product of cyclic subgroups, A is cyclic. This establishes (i).

Assume that $p = 2$ and A is not cyclic. Then, by Lemma 2(ii), $\Omega_1(A)$ is a subgroup of the direct product of a cyclic 2-group and a group of order 2, $\exp(B) \geq 8$, and there exists an element $a \in A$ that inverts B by conjugation. Suppose that there exists an element $x \in A$ with $x^2 = a$. Applying [6, Lemma 6], we have that $[x^2, y^2] = 1$ for all $y \in B$. Since conjugation by a inverts y, we must have $y^4 = 1$ for all $y \in B$, so that we are forced to the contradiction $\exp(B) = 4$. Hence $a \notin \Phi(A)$ and $\langle a \rangle$ has a complement in A. By Lemma 2(ii), $A = \langle a \rangle \times \langle c \rangle$ for some $c \in A$. This establishes (ii). □

Our next lemma is essentially a special case of Theorem 6 of [7], with the extra information coming from Theorem 4.2.2 of [8]. According to [8], a group is called *weak* if it is generated by its elements of infinite order. In particular, a nilpotent group is weak if it contains an element of infinite order.

Lemma 4. *Let the group $G = AB$ be the product of the totally permutable abelian subgroups A and B. Assume that $\mathrm{Core}_G(A) = 1$ and B is weak. Then B is normal in G and $|G : B| \leq 2$.*

Proof. We may assume that B is a proper subgroup of G. Note that $A \cap B = 1$. Let $x \in B$ be an element of infinite order and $a \in A$. If a is of infinite order, then $\langle a^2 \rangle$ is normalized by x by [7, Lemma 1(2)] and if a has finite order, $\langle a^2 \rangle$ is normalized by x by [7, Lemma 3]. Since B is weak, it is generated by elements of infinite order. Therefore $\langle a^2 \rangle$ is normal in G. Since $\mathrm{Core}_G(A) = 1$, it follows that $a^2 = 1$.

Therefore it has been proved that A is an elementary abelian 2-group. Consider $a \in A$ and $x \in B$. Then $|\langle a \rangle \langle x \rangle : \langle x \rangle| = 2$. Hence $\langle x \rangle$ is normalized by A. Hence A can be embedded in $\mathrm{PAut}(B)$ since $\mathrm{C}_A(B) \leq \mathrm{Core}_G(A) = 1$.

Applying [8, Corollary 4.2.3], A has order 2. Thus B has index 2 in G and so it is normal in G, as required.

$\square$

3 Ranks of finite p-groups

Some results about the rank of a finite p-group, p a prime, which is a mutually permutable product of two abelian groups will be proved in this section. These results will be crucial to prove our main theorems.

Note that if A is a finite abelian p-group, then $d(A) = r(A)$.

Lemma 5. *Let the finite p-group $G = AB$ be the product of two cyclic subgroups A and B. Then the rank of G is at most 2 if p is odd and at most 3 if $p = 2$.*

Proof. If $p > 2$, then G is metacyclic by [10, III, Satz 11.5]. Thus G is of rank 2. Assume that $p = 2$. If G is a nonmetacyclic 2-group, then G has a unique nonmetacyclic maximal subgroup by [11, Theorem 5.1]. Let M be a metacyclic maximal subgroup of G. Then M is normal in G and G/M is cyclic. Hence $r(G) \leq r(M) + r(G/M) \leq 2 + 1 = 3$, as required. $\square$

Lemma 6. *Let $G = \langle a\rangle\langle b\rangle$ be the product of two cyclic groups $\langle a\rangle$ and $\langle b\rangle$ such that $|\langle a\rangle| = |\langle b\rangle| = 2^2$ and $\langle a\rangle \cap \langle b\rangle = 1$. Then $\langle a^2, b^2\rangle$ is contained in $\mathrm{Z}(G)$. Furthermore, if G is nonmetacyclic, then $[a, b] = a^2b^2$.*

Proof. By [4, Corollary 3.1.9], G is the totally permutable product of $\langle a\rangle$ and $\langle b\rangle$. Then $\langle a^2\rangle\langle b\rangle$ is a subgroup of G and $\langle b\rangle$ is a normal subgroup of $\langle a^2\rangle\langle b\rangle$. If a^2 does not centralise $\langle b\rangle$, we have that a^2 inverts $\langle b\rangle$. In this case $\langle b\rangle\langle a^2\rangle$ is isomorphic to the dihedral group of order 8. Hence $\langle b\rangle$ is a characteristic subgroup of $\langle b\rangle\langle a^2\rangle$ which is normal in G. Hence $\langle b\rangle \trianglelefteq G$ and then $b^a \in \langle b\rangle$. It follows that $b^a = b$ or $b^a = b^{-1}$. In both cases, we have $b^{a^2} = b$, against supposition. Hence $[a^2, b] = 1$ and $a^2 \in \mathrm{Z}(G)$. By using the same arguments with b^2, we get $\langle a^2, b^2\rangle \leq \mathrm{Z}(G)$. Assume now G is a nonmetacyclic group. By [11, Proposition 2.12], $\Phi(G) = \langle a^2, b^2\rangle$ and $G/\langle a^2, b^2\rangle$ is abelian. Hence $[a, b] \in \langle a^2\rangle\langle b^2\rangle$. Since $[a, b] \neq 1$, we deduce that $[a, b] = a^2, b^2$ or a^2b^2. If $[a, b] = a^2$ or $[a, b] = b^2$, then either $\langle a\rangle$ or $\langle b\rangle$ is a normal subgroup of G and G is metacyclic. By this contradiction $[a, b] = a^2b^2$, as required. $\square$

Lemma 7. *Let $G = AB$ be the product of the mutually permutable finite abelian 2-groups A and B with $A \cap B = 1$. Assume that $s = r(B) = 1$ or 2. If A is either cyclic or $A = \langle a\rangle \times \langle y\rangle$ such that $o(y) = 2$ and y inverts B, then $r(G)$ is at most $s + 3$.*

Proof. Since $A \cap B = 1$, we have that A and B are totally permutable subgroups of G. Let $D = \mathrm{Core}_G(B)$. By [4, Lemma 4.1.10], G/D is the product of the totally permutable subgroups AD/D and B/D. Moreover, B/D is core-free in G/D. By Lemma 3, either B/D is cyclic or $B/D = \langle bD \rangle \times \langle xD \rangle$, where $o(bD) = 2^m$ and $o(xD) = 2$. Note that if $g \in D$, then $g^z \in \langle g \rangle \langle z \rangle \cap D = \langle g \rangle (\langle z \rangle \cap D) = \langle z \rangle$ for all $z \in A$. Therefore $\langle g \rangle$ is a normal subgroup of G.

We distinguish two cases:

(i) $r(B) = 1$. Let C be a cyclic subgroup of A such that $|A : C| \leq 2$. Then CB is a normal subgroup of G such that $r(CB) \leq 3$ by Lemma 5, and G/CB is cyclic. Therefore $r(G) \leq 4$ and the lemma holds in this case.

(ii) Assume that $r(B) = 2$. If B/D is cyclic, then D is not contained in $\Phi(B)$, the Frattini subgroup of B. Let $u \in D \setminus \Phi(B)$. Then $U = \langle u \rangle$ is a normal subgroup of G. Since $B/\Phi(B)$ is an elementary group of order 4 and $U\Phi(B)$ is a proper subgroup of B, it follows that $G/U\Phi(B)$ is cyclic of order 2. In addition $U\Phi(B)/U$ is contained in $\Phi(B/U)$, so that B/U is cyclic. Since G/U is the product of the totally permutable subgroups AU/U and B/U by [4, Lemma 4.1.10], G/U satisfies the hypotheses of the theorem. Since $r(B/U) = 1$, we have that $r(G/U) \leq 4$ by Case (i). Therefore $r(G) \leq 5$.

Suppose that B/D is not cyclic. Then $B/D = \langle bD \rangle \times \langle xD \rangle$, where $o(bD) = 2^m$ and $o(xD) = 2$. In this case, $x^2 \in D$ and so $\langle x^2 \rangle$ is a normal subgroup of G. The lemma will therefore follow should we succeed in proving that $r(G/\langle x^2 \rangle) \leq 4$. Without loss of generality, we may assume that $x^2 = 1$. Since $xD \notin \Phi(B/D)$, we have that x is an element of order 2 in $B \setminus \Phi(B)$. Let M be a complement of $\langle x \rangle$ in B. Now B cannot be cyclic. Hence $M = \langle b \rangle$ is cyclic for some $b \in B$, and $B = \langle b \rangle \times \langle x \rangle$.

Assume that A is cyclic. Then $r(A\langle x \rangle) \leq 3$ by Lemma 5, and $A\langle x \rangle$ is a normal subgroup of G such that $G/A\langle x \rangle$ is cyclic. Therefore $r(G) \leq 4$.

Assume that $A = \langle a \rangle \times \langle y \rangle$ such that $o(y) = 2$ and y inverts B. Write $N = \langle a \rangle \langle b \rangle$. Then N is a normal subgroup of G and $G/N \simeq C_2 \times C_2$. By Lemma 5, we have that $r(N) \leq 3$ and then that $r(G) \leq 5$. What we must prove is that $d(H) \leq 4$ for all subgroups H of G. Assuming this to be false, let us choose a subgroup H of G such that $d(H) \geq 5$.

Since $d(H) \leq r(G) \leq 5$, we have that $d(H) = 5$. If HN/N is cyclic, we have $d(H) \leq r(H) \leq r(H/H \cap N) + r(N) \leq 1 + 3 = 4$, a contradiction which shows that $G = HN$. Denote $H_1 = H \cap N$. Then we have $H/H_1 \simeq G/N \simeq C_2 \times C_2$ and so $d(H_1) \geq 3$. Since $d(H_1) \leq r(N) \leq 3$, we have that $d(H_1) = 3$ and so H_1 is not metacyclic. Note that H_1 cannot equal to N because $d(N) = 2$. Therefore H_1 lies inside a nonmetacyclic maximal X

subgroup of N. Applying [11, Theorem 5.1], it follows that $X = \langle ab, a^2, b^2\rangle$, $d(X) = 3$, and $o(a), o(b) \geq 4$. By [11, Proposition 2.12], we have that $\Phi(\Phi(N)) = \langle a^4, b^4\rangle$ and $|N : \Phi(\Phi(N))| = 2^4$. Clearly $\langle a^4, b^4\rangle \leq \Phi(X)$. In addition, $|N : \Phi(X)| = |N : X||X : \Phi(X)| = 2 \cdot 2^3 = 2^4$, therefore we can conclude that $\Phi(X) = \langle a^4, b^4\rangle = \Phi(\Phi(N))$. Note that $\langle a^2, b^2\rangle$ is a metacyclic maximal subgroup of X and H_1 is not contained in $\langle a^2, b^2\rangle$ since we agreed that $d(H_1) = 3$. Therefore $X = H_1\langle a^2, b^2\rangle$.

We write $\bar{g}$ to denote the image of g in $\bar{G} = G/\Phi(X)$. Also if K is any subgroup of G, then $\bar{K}$ is the image of K in G. Applying Lemma 6 to the group $\bar{N} = \langle\bar{a}\rangle\langle\bar{b}\rangle$, $o(\bar{a}) = o(\bar{b}) = 2^2$, we obtain that $\langle\bar{a}^2, \bar{b}^2\rangle \leq \mathrm{Z}(\bar{N})$. Since $\bar{X}$ is an elementary abelian group of order 8 contained in $\bar{N}$, it follows that $\bar{N}$ is nonmetacyclic. Hence $[\bar{a}, \bar{b}] = \bar{a}^2\bar{b}^2$ by Lemma 6. Moreover since $\bar{x}$ centralises $\bar{a}^2$ and $\bar{y}$ centralises $\bar{b}^2$, we have that $\langle\bar{a}^2, \bar{b}^2\rangle \leq \mathrm{Z}(\bar{G})$.

Let us now prove that $\bar{H}$ is non-abelian. Since $y \in G = HN$, there exists $n \in N$ such that $ny \in H$. In addition, there exists $t \in \langle a^2, b^2\rangle$ such that $abt \in H_1$.

Bearing in mind that $\bar{n}$ is a product of a power of $\bar{a}$ and a power of $\bar{b}$, we can conclude that $(\bar{a}\bar{b})^{\bar{n}} \in \{\bar{a}\bar{b}, \bar{b}\bar{a}\}$. Therefore

$$[\overline{abt}, \overline{ny}] = [\overline{ab}, \overline{ny}] = (\bar{a}\bar{b})^{-1}(\bar{a}\bar{b})^{\bar{n}\bar{y}} = \bar{b}^{-1}\bar{a}^{-1}(\bar{a}\bar{b})^{\bar{y}} \textit{ or } \bar{b}^{-1}\bar{a}^{-1}(\bar{b}\bar{a})^{\bar{y}},$$

which is equal to $\bar{b}^{-2}$ or $[\bar{a}, \bar{b}]\bar{b}^{-2}$. Thus $[\overline{abt}, \overline{ny}] = \bar{a}^2$ or $\bar{b}^2$. Consequently $[\overline{abt}, \overline{ny}] \neq 1$ and $\bar{H}$ is non-abelian.

Consequently H' is not contained in $\Phi(X)$. Therefore H' is not contained in $\Phi(H_1)$. In particular, $H/\Phi(H_1)$ is non-abelian and so $\Phi(H_1) < \Phi(H)$. Since $d(H_1) = 3$ and $\mid H : H_1 \mid = 2^2$, we obtain that $\mid H : \Phi(H_1) \mid = \mid H : H_1 \mid\mid H_1 : \Phi(H_1) \mid = 2^5$. Hence $\mid H : \Phi(H) \mid \leq 2^4$ and then $d(H) \leq 4$, contrary to the choice of H, which is our final contradiction.

Therefore $d(H) \leq 4$ for all subgroups H of G, and so $r(G) \leq 4$. The proof is of the lemma is complete. □

Lemma 8. *Let a finite p-group $G = AB$ be the mutually permutable product of two abelian p-groups A and B with* $\mathrm{Core}_G(A) = 1$. *Suppose that B has rank s. Then the rank of G is at most $s + 2$ if p is odd and at most $s + 3$ if $p = 2$.*

Proof. Since $A \cap B \leq \mathrm{Core}_G(A) = 1$, it follows that the product is totally permutable. Let $D = \mathrm{Core}_G(B)$. Then $G/D = (AD/D)(B/D)$ is the totally permutable product of AD/D and B/D such that B/D is core-free in G/D. Assume that p is odd. It follows from Lemma 3(i) that A and B/D are both cyclic. By Lemma 5 we obtain $r(G/D) \leq 2$. Hence $r(G) \leq s + 2$.

Now we assume that G is a 2-group. It follows from Lemma 3(ii) that A is cyclic or $A = \langle a \rangle \times \langle y \rangle$, where y inverts B and $o(y) = 2$. If $r(B) = s \leq 2$, the result directly follows from Lemma 7. If $B \trianglelefteq G$, the result is trivial.

Now we only consider the case $s \geq 3$ and $B/D \neq 1$. It follows from Lemma 3(ii) that $d(B/D) \leq 2$. Thus $D \not\leq \Phi(B)$ otherwise $r(B) = d(B/D) \leq 2$, contrary to $s \geq 3$. Write $d = d(B/D)$ and note that $d = 1$ or 2. Let $|D : D \cap \Phi(B)| = 2^t$, where $t \geq 1$. It is easy to see that

$$2^t = |D\Phi(B) : \Phi(B)| = \frac{|B : \Phi(B)|}{|B : D\Phi(B)|} \leq \frac{|B : \Phi(B)|}{|B/D : \Phi(B/D)|} = 2^{s-d},$$

which implies that $t + d \leq s$.

Note that $D/D \cap \Phi(B)$ is an elementary abelian 2-group of order 2^t. Let $x_1(D \cap \Phi(B)), ..., x_t(D \cap \Phi(B))$ be generators of $D/D \cap \Phi(B)$ and denote $K = \langle x_1, x_2, \ldots, x_t \rangle \leq D$. We see that K is a normal subgroup of G. Actually $K \leq B$ is normal in B. Let $x \in K$ and $a \in A$. As $\langle a \rangle \langle x \rangle$ is a subgroup, Then $x^a \in \langle a \rangle \langle x \rangle \cap D = \langle x \rangle$, as desired. Since $D = K(D \cap \Phi(B))$, we have that $d(B/K) = d(B/D) = d$.

Note that G/K is the totally product of AK/K and B/K with $r(B/K) = d = 1$ or 2, and AK/K is cyclic or $AK/K = \langle aK \rangle \times \langle yK \rangle$ such that yK inverts B/K and $o(yK) = 2$. It follows from Lemma 7 that $r(G/K) \leq r(B/K) + 3 = d + 3$. Hence $r(G) \leq r(K) + d + 3 = t + d + 3 \leq s + 3$. Now the proof is complete. □

4 Proof of Theorem A

Before we prove Theorem A, we have to deal with the case that G is finite. Our next lemma is a result of Lucchini concerning the number of generators of a finite group. It will be essential in the proof of Theorem A.

Lemma 9 ([13, Lemma 1]). *Let G be a finite group with a normal p-subgroup N. Assume that the Sylow p-subgroups of G can be generated by r elements:*

(a) If G/N can be generated by d elements, where $d \geq r + 1$, then G can be generated by d elements.

(b) If $p = 2$ and G/N can be generated by d elements, where $d \geq r$, then G can be generated by d elements.

Lemma 10. *Let G be a finite soluble group. Suppose that all its Sylow 2-subgroups can be generated by $d + 1$ elements and the other Sylow subgroups can be generated by d elements. Then G can be generated by $d + 1$ elements.*

Proof. We work by induction on $|G|$. As G is soluble, there is a normal p-subgroup N of G for some prime p. By induction, G/N can be generated by $d+1$ elements. Let P be a Sylow p-subgroup of G. If $p=2$, then P can be generated by $d+1$ elements by hypothesis. Applying Lemma 9 (b), we have that G can be generated by $d+1$ elements. If $p \neq 2$, then P can be generated by d elements by hypothesis. It follows from Lemma 9 (a) that G can be generated by $d+1$ elements. □

We are ready to prove Theorem A.

Proof of Theorem A. Now consider $H = \langle a_1b_1, ..., a_nb_n \rangle$ a finitely generated subgroup of G, where $a_i \in A$ and $b_i \in B$. As G is locally finite, H is finite. Moreover, H is soluble since G is metabelia by Itô's Lemma. Let $C = \mathrm{Core}_G(A)$ and G/C is the totally permutable product of A/C and BC/C as $A \cap B \leq C$.

Write $L = \langle a_1, ..., a_n \rangle$,$F = \langle b_1, ..., b_n \rangle$, $X = \langle L, F \rangle$ and $D = C \cap X$. Note that all of them are finite and $H \leq X$. Note that XC/C is the totally permutable product of LC/C and FC/C. Since X/D is isomorphic to XC/C, it follows that X/D is a finite group that is the product of the totally permutable subgroups LD/D and FD/D and $LD/D \cap FD/D = 1$. Let L_p and F_p be the Sylow p-subgroups of L and F, respectively. Then $U/D = (L_pD/D)(F_pD/D)$ is a Sylow p-subgroup of X/D. Also, U/D is the product of the totally permutable subgroups L_pD/D and F_pD/D.

Let $S/D = \mathrm{Core}_{U/D}(L_pD/D)$. It is easy to see that U/S is the totally permutable product of L_pS/S and F_pS/S, moreover, L_pS/S is core-free in U/S. By Lemma 8, we get that $r(U/S) \leq r(F_pS/S) + 2$(or $r(F_pS/S) + 3$ if $p = 2$). Since $S \leq L_pD \leq A$ and $F_pS/S \leq BS/S$, it implies that $r(U) \leq r + s + 2$(or $r + s + 3$ if $p = 2$). Clearly the rank of the Sylow p-subgroup of U is at most $r + s + 2$(or $r + s + 3$ if $p = 2$). Note that the Sylow p-subgroup of U is also the Sylow p-subgroup of X and a Sylow p-subgroup of H is contained in a Sylow p-subgroup of X, we obtain that the rank of a Sylow p-subgroup of H is at most $r+s+2$(or $r+s+3$ if $p = 2$). Recall that H is a finite soluble group, by Lemma 10, H can be generated by $r + s + 3$ elements. Consequently, the rank of G is at most $r + s + 3$. □

5 Proof of Theorem B

Recall that $d(G)$ is the minimum number of elements required to generate the group G. Now define $f(G)$ to be the maximum of $\{d(G/H)\}$ for every normal subgroup H of finite index in G. From [12, Theorem], we have the following:

Lemma 11. *If G is a polycyclic group, then $d(G) \leq f(G) + 1$.*

Now we are ready to prove Theorem A.

Proof of Theorem B. If A or B is normal in G, $r(G) \leq r + s$. Thus we may assume that neither A nor B are normal in G. Let $C = \mathrm{Core}_G(A)$. Since $A \cap B \leq C$, we have that G/C is the totally permutable product of A/C and BC/C and A/C is core-free in G/C.

Suppose BC/C is weak. Then BC/C is a normal subgroup of G/C with $|G/C : BC/C| = 2$ by Lemma 4. Therefore we have the normal series $1 \leq C \leq BC \leq G$ with the rank of C at most $r = r(A)$, the rank of BC/C at most $s = r(B)$ and G/BC cyclic. Hence the rank of G is at most $r+s+1$. If $D = \mathrm{Core}_G(B)$, a similar argument applies if AD/D is weak.

Thus we may assume that both AD/D and BC/C are periodic groups. On the other hand, since $A \cap B$ is a normal subgroup of G, we have that $A \cap D = A \cap B = C \cap B$. This implies that $A/(A \cap B)$ and $B/(A \cap B)$ are both periodic groups. Denote $Z = A \cap B$. Since Z is abelian of finite rank, we can consider a free abelian subgroup E of Z of maximal rank, k say. Then Z/E is a periodic group of finite rank, and E/E^m is a finite group of order m^k for every positive integer m, where E^m is the subgroup generated by $\{x^m : x \in E\}$. Since $E \leq \mathrm{Z}(G)$ we have that G/E^m is the product of A/E^m and B/E^m both periodic groups of finite rank. It then follows that G/E^m is locally finite (and of finite rank) by a theorem of Černikov [1, Theorem 3.2.12].

If $E = 1$, then G is locally finite. It follows from Theorem A that $r(G) \leq r + s + 3$, as desired. Assume now that $E \neq 1$. Let H be a finitely generated subgroup of G. As G/E^m is locally finite for all positive integers m and $HE^m/E^m \leq G/E^m$, by Theorem A, we can argue that $HE^m/E^m \simeq H/(H \cap E^m)$ has rank at most $r + s + 3$.

Let X be a subgroup of H. Then $X/(X \cap E) \simeq XE/E$ is finite. Moreover E is finitely generated because it has finite rank. This implies that $X \cap E$ is finitely generated since subgroups of finitely generated abelian groups are finitely generated. Consequently X is finitely generated, which implies that every subgroup of H is finitely generated. Note that H is soluble. Hence H is polycyclic.

We next claim that for every normal subgroup N of H of finite index, there exists a positive integer m such that $H \cap E^m \leq N$. Since $H/(N \cap E)$ is finite, we may assume without loss of generality that $N \leq E$. Then $E/N = U/N \times W/N$, where U/N is free abelian and W/N is finite. Assume the exponent of W/N is m. Then $(E/N)^m = (U/N)^m$ is free abelian. Now $(H \cap E^m)N/N \leq H/N$ which is finite. Also $(H \cap E^m)N/N \leq E^mN/N \leq$

$(E/N)^m$. This implies that $(H \cap E^m)N/N$ is a finite subgroup of a free abelian group. Consequently $(H \cap E^m)N/N = 1$ and $H \cap E^m \leq N$, as claimed.

Recall that $H/(H \cap E^m)$ has rank at most $r+s+3$ for all positive integers m. It follows that H/N has rank at most $r+s+3$ for every normal subgroup N of H of finite index. Now we can apply Lemma 11 to conclude that $d(H) \leq r+s+4$. Consequently, the rank of G is at most $r+s+4$. The proof of the theorem is now complete. □

We bring the paper to a close with an example of a group $G = AB$ of rank 3, which is the totally permutable product of the abelian subgroups A and B with $A \cap B = 1$.

Example Let the group H be

$$H = \langle a, b, c \mid a^2 = b^2 = c^4 = 1, [a, b] = 1, a^c = ab, b^c = b \rangle$$

Then H is the totally permutable product of $\langle ac \rangle$ and $\langle c \rangle$. The rank of G is 3 since $K = \langle a \rangle \times \langle b \rangle \times \langle c^2 \rangle$ is a subgroup of rank 3.

Acknowledgements

The first and third authors are supported by the grant MTM2014-54707-C3-1-P from the Ministerio de Economía y Competitividad, Spain, and FEDER, European Union. The first and fourth authors are supported by Prometeo/2017/057 of Generalitat (Valencian Community, Spain). The third author is also supported by the predoctoral grant 201606890006 from the China Scholarship Council.

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