REPRESENTATION AND FACTORIZATION THEOREMS FOR ALMOST-L^p-SPACES

J.M. CALABUIG[1], O. GALDAMES BRAVO[2], M.A. JUAN[3], E.A. SÁNCHEZ PÉREZ[1]

Abstract. We extend the notions of p-convexity and p-concavity for Banach ideals of measurable functions following an asymptotic procedure. We prove a representation theorem for the spaces satisfying both properties as the one that works for the classical case: each almost p-convex and almost p-concave space is order isomorphic to an almost-L^p-space. The class of almost-L^p-spaces contains, in particular, direct sums of (infinitely many) L^p-spaces with different norms, that are not in general p-convex —nor p-concave—. We also analyze in this context the extension of the Maurey-Rosenthal factorization theorem that works for p-concave operators acting in p-convex spaces. In this way we provide factorization results that allow to deal with more general factorization spaces than L^p-spaces.

1. Introduction

Representation theorems for Banach function spaces and factorization theorems for operators between these spaces are fundamental tools in the theory of Banach lattices. Regarding representation theorems, the most classic, the so called *Kakutani's Representation Theorem*, states that abstract L^p-spaces are lattice isomorphic to $L^p(\mu)$ for some scalar measure μ. On the other hand, a lattice which is both p-convex and p-concave is lattice isomorphic to an abstract L^p-space and, therefore, lattice isomorphic to an L^p-space. This result can be obtained as a consequence of a general theory that relates these geometric inequalities with factorization of operators through L^p-spaces.

The objective of this work is to extend these lattice geometric notions to include in the class of associated spaces others that are not considered in the classical theory. This is for instance the case of ℓ^∞, c_0 and direct sums with q-norms of L^p-spaces for $q \neq p$. In order to do it, we study the main properties of a class of Banach ideals of measurable functions. This class of spaces, called *almost-L^p-spaces*, were introduced in [9, S.4.1] as a tool for obtaining characterizations of factorization spaces (see [9, Th.4.1]) for p-th power factorable operators (see [14, Ch.4]). These *almost-L^p-spaces* are in some way related to the topic discussed in Chapter IV of [3]. There, complemented subspaces of L^p-spaces associated to direct sums are studied. However, our spaces are not in general L^p-spaces or subspaces of these spaces, but nevertheless inherit many good properties of L^p-spaces. Moreover, they act as factorization spaces for operators "in an L^p-style". That is, in the manner of the classic results that nowadays are known as Maurey-Rosenthal Factorization Theory (see for instance [5, 10, 13, 15]). Recall that the Maurey-Rosenthal Theorems allow

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to factor p-concave operators defined from p-convex Banach function spaces through L^p-spaces.

Due to the relevance of their applications, the so called variable exponent L^p-spaces have also been studied recently (see for example [6, 7, 8] and the references therein). In a sense, they can also be considered as asymptotic versions of the classical Lebesgue spaces, but their definition is not related to ours. Some other attempts of asymptotic generalizations of the main lattice geometric properties of the L^p-spaces —p-convexity and p-concavity— for Banach lattices and operators between them can be found in [4, 16]. However, the notions appearing there — mainly based on interpolation construction— are not related to the ideas developed here.

The paper is divided as follows. After this introductory section, in the next one we present the class of the almost-L^p-spaces, providing also the canonical examples of that kind of spaces. In the third section we study the family of almost-L^p-spaces: first in the general setting (see Theorem 3.2) and secondly by introducing additional lattice properties as the Fatou property (Proposition 3.5) and the order continuity property (Proposition 3.6). In particular, we also show that the classical sequence spaces are almost-L^p-spaces (see Example 3.3). In the fourth and final section we provide the corresponding factorization theorems in this context (see Proposition 4.1 and Theorem 4.4).

2. Notation and preliminaries

Given a complete finite measure space (Ω, Σ, μ), a *Banach function space* $X(\mu)$ (B.f.s. for short) is a space of (classes of μ-a.e.) measurable real valued functions such that if f is measurable, $g \in X(\mu)$ and $|f| \leq |g|$ then $f \in X(\mu)$ and $\|f\|_{X(\mu)} \leq \|g\|_{X(\mu)}$. These spaces are sometimes called *Banach ideals* of the space $L^0(\mu)$, consisting of classes of μ-a.e. equal measurable functions. The closed unit ball of $X(\mu)$ will be denoted by $B_{X(\mu)}$. If f is an element of a B.f.s $X(\mu)$, we write $supp(f)$ for its (μ-a.e. defined) support. If $X(\mu)$ and $Z(\mu)$ are B.f.s., we write $M(X(\mu), Z(\mu))$ for the space of functions that define multiplication operators between $X(\mu)$ and $Z(\mu)$. That is the space of all measurable μ-a.e. equal functions g such that $g \cdot X(\mu) \subseteq Z(\mu)$. Endowed with the natural operator norm, it is a Banach function space over μ. The reader can find more information about these spaces in [2, 17, 18].

For $1 \leq p < \infty$, a linear mapping T from a Banach space into a Banach lattice is said to be *p-convex* if there is a constant M such that

$$\Big\| \Big(\sum_{k=1}^{N} |T(x_k)|^p \Big)^{1/p} \Big\| \leq M \Big(\sum_{k=1}^{N} \|x_k\|^p \Big)^{1/p},$$

for each finite set of vectors $x_, \dots, x_N$ in the domain of T. The smallest such M is denoted by $M^{(p)}(T)$. In a similar way if for a linear map T from a Banach lattice into a Banach space the inequality

$$\Big(\sum_{k=1}^{N} \|T(x_k)\|^p \Big)^{1/p} \leq M \Big\| \Big(\sum_{k=1}^{N} |x_k|^p \Big)^{1/p} \Big\|,$$

holds for each finite set of vectors $x, \dots, x_N$ in the domain of T, the map T is called *p-concave*. Now the smallest such M is denoted by $M_{(p)}(T)$. When the identity map from a Banach lattice into itself is p-convex (resp. p-concave) then the Banach lattice is said to be p-convex (resp. p-concave). The reader is referred to [1, 12] for the unexplained terminology.

Through this paper we will use —often without an explicit explanation— restrictions and extensions of Banach function spaces when the support set is restricted and extended. Let us explain these notions. Let $A \in \Sigma$. The complement of the set A is denoted as usual by $A^c := \Omega \setminus A$. The restriction to A of the measure space is denoted by $(\Omega \cap A, \Sigma_A, \mu_A)$, where $\Sigma_A := \{B \cap A : B \in \Sigma\}$, which is a σ-algebra over A, and $\mu_A(B) := \mu(B)$, where $B \in \Sigma_A$. Let $X(\mu)$ be a B.f.s. The space $X(\mu_A)$ of functions in $X(\mu)$ restricted to A is still a B.f.s. endowed with the norm $\|f\|_{X(\mu_A)} := \|f\chi_A\|_{X(\mu)}$. Note that if $A \in \Sigma$ with $\mu(A) = 0$, then $X(\mu)$ and $X(\mu_{A^c})$ are clearly order isomorphic and isometric.

Now we are ready for giving the main definition of this paper.

Definition 2.1. Let (Ω, Σ, μ) be a finite measure space. A B.f.s. $X(\mu)$ is said to be an *almost-L^p-space* if for every $\varepsilon > 0$, there exists $A_\varepsilon \in \Sigma$ with $\mu(A_\varepsilon) < \varepsilon$ such that the restriction $X(\mu_{A_\varepsilon^c})$ is order isomorphic to an L^p-space. This means that there is a finite (positive) measure $\tilde{\mu}$ supported on A_ε^c and acting in $\Sigma_{A_\varepsilon^c}$ such that $X(\mu_{A_\varepsilon^c}) = L^p(\tilde{\mu})$ with equivalent norms. Note that in this case μ and $\tilde{\mu}$ are also equivalent.

Following the definition, just taking $A_\varepsilon = \emptyset$ for all $\varepsilon > 0$, we have that an L^p-space is also an almost-L^p-space. Now, let us present a class of spaces that are in a sense the canonical examples of almost-L^p-spaces.

Example 2.2. Consider a Lebesgue measurable disjoint partition (B_n) of $[0, 1]$, μ Lebesgue measure, and the space $X(\mu) = \bigoplus_n L^p(\mu_{B_n})$ that is endowed with the norm

$$\|f\|_{X(\mu)} := \sum_{n=1}^{\infty} \|f\chi_{B_n}\|_{L^p(\mu_{B_n})}, \quad f \in X(\mu).$$

Since $\lim_{r\to\infty} \sum_{n=r}^{\infty} \mu(B_n) = 0$, we have that for each $\varepsilon > 0$ there exists $r_0 \in \mathbb{N}$ —depending on ε— such that $\sum_{n=r_0}^{\infty} \mu(B_n) < \varepsilon$. Therefore let us take $A_\varepsilon = \bigcup_{n=r_0}^{\infty} B_n \in \Sigma$ which satisfies $\mu(A_\varepsilon) < \varepsilon$. Then $X(\mu_{A_\varepsilon^c})$ is a finite sum of disjoint L^p-spaces (with the 1-norm), that is order isomorphic to an L^p-space. Consequently, $X(\mu)$ is an almost-L^p-space.

We finish this section with a useful (and well-known) result regarding the representation theory of Banach lattices by means of L^p-spaces. The reader can find the main original results on this point of view for the representation of p-convex and p-concave Banach lattices in the papers by Krivine [10], Rosenthal [15] and Maurey [13] (see also [11]).

Remark 2.3. The well known representation theory for the L^p-spaces allows to write the abstract ones appearing in Definition 2.1 as concrete spaces as $L^p(gd\mu)$. Here $g^{1/p}$ defines a (norm one) multiplication operator belonging to $M(X(\mu), L^p(\mu))$. Although several classical arguments allow to prove this, we prefer the following direct one: Corollary 5 in [5] gives that if $X(\mu)$ is an order continuous p-convex Banach function space over μ, a p-concave linear operator $T\colon X(\mu) \to E$ can be extended to such a space $L^p(gd\mu)$. Just taking the identity map $\mathrm{id}\colon X(\mu) \to X(\mu)$ as operator T, we get that it factors through $L^p(gd\mu)$ by means of the identity map. This leads to the equality of $X(\mu)$ and $L^p(gd\mu)$ with equivalent norms. That is, there are constants $k, K > 0$ such that

$$k\|f\|_{X(\mu)} \leq \|f\|_{L^p(gd\mu)} \leq K\|f\|_{X(\mu)}, \qquad f \in X(\mu).$$

3. The structure of the almost-L^p-spaces

In this section we will describe the almost-L^p-spaces. Following Example 2.2 we start this section with a result which clarifies the nature of the elements of the spaces as classes of measurable functions on an almost-L^p-space. Observe that given a B.f.s. $X(\mu)$ and given $A \in \Sigma$, the decomposition $f = f\chi_A + f\chi_{A^c}$ for all $f \in X(\mu)$ provides the decomposition $X(\mu) = X(\mu_A) \oplus X(\mu_{A^c})$.

Theorem 3.1. *Let μ be a finite measure and $X(\mu)$ an almost-L^p-space for $1 \leq p < \infty$. Then there exist*

(1) *a disjoint partition $(B_n) \subseteq \Sigma$ of Ω, and*
(2) *a sequence $(g_n) \subseteq L^1(\mu)$ satisfying that $\mathrm{supp}(g_n) \subseteq B_n$ for all $n \in \mathbb{N}$,*

such that each $f \in X(\mu)$ is a μ-a.e. limit of a series defined by a disjoint sequence (f_n), where $f_n \in L^p(g_n d\mu)$ for all $n \in \mathbb{N}$.

Proof. Let $\varepsilon > 0$. Then there exists $A_\varepsilon \in \Sigma$ with $\mu(A_\varepsilon) < \varepsilon$ so that $X(\mu_{A_\varepsilon^c})$ is order isomorphic to an L^p-space. If $\delta > 0$ is any other real number then there exists also a measurable set $A_\delta \in \Sigma$ so that $\mu(A_\delta) < \delta$ and $X(\mu_{A_\delta^c})$ is also order isomorphic to an L^p-space.

Claim. *If $A_\varepsilon \cap A_\delta = \emptyset$, then $X(\mu)$ is order isomorphic to an L^p-space.* Let us prove that. Since it is trivial that

$$X(\mu) = X(\mu_{A_\varepsilon}) \oplus X(\mu_{(A_\varepsilon \cup A_\delta)^c}) \oplus X(\mu_{A_\delta}),$$

it will be enough to study the different situations that can occur regarding the values of the measure over the sets involved. Suppose that $\mu(A_\varepsilon) = 0$; then $X(\mu)$ and $X(\mu_{A_\varepsilon^c})$ are order isomorphic and the claim is done. The case for $\mu(A_\delta) = 0$ is analogous. If $\mu(A_\varepsilon) \cdot \mu(A_\delta) \neq 0$, since $X(\mu_{A_\varepsilon}) \subseteq X(\mu_{A_\delta^c})$ and $X(\mu_{A_\delta}) \subseteq X(\mu_{A_\varepsilon^c})$, it is clear that both, $X(\mu_{A_\varepsilon})$ and $X(\mu_{A_\delta})$, are order isomorphic to L^p-spaces, and so the same happens with the direct sum of these spaces. Thus, the study reduces to the behavior of $X(\mu_{(A_\varepsilon \cup A_\delta)^c})$. Now, if $\mu(A_\varepsilon \cup A_\delta)^c = 0$, then $X(\mu)$ is order isomorphic to $X(\mu_{A_\varepsilon}) \oplus X(\mu_{A_\delta})$ and again the proof is done. Finally, if $\mu(A_\varepsilon \cup A_\delta)^c \neq 0$, taking into account that $(A_\varepsilon \cup A_\delta)^c \subseteq A_\varepsilon^c$, we have that $X(\mu_{(A_\varepsilon \cup A_\delta)^c}) \subseteq X(\mu_{A_\varepsilon^c})$. Thus, $X(\mu)$ is order isomorphic to a (finite) direct sum of L^p-spaces, and then order isomorphic to an L^p-space.

Let us continue with the proof. Since the case when $X(\mu)$ is order isomorphic to an L^p-space is trivial, we will assume that $X(\mu)$ is not (order isomorphic to) an L^p-space. Therefore, taking into account the claim we can assume that $A_\varepsilon \cap A_\delta \neq \emptyset$ for every $\varepsilon \neq \delta$ and $\varepsilon, \delta > 0$. Consider the sequence given by $\varepsilon_n = 1/n$ for all $n \in \mathbb{N}$. Then there exists a sequence of measurable subsets $(A_n) \subseteq \Sigma$ such that $\mu(A_n) < \varepsilon_n$

and $X(\mu_{A_n^c})$ is order isomorphic to an L^p-space.

Step 1. Let us define $\mu_1 := \mu_{A_1^c}$ and $B_1 = A_1^c$. Then we have the decomposition

$$X(\mu) = X(\mu_1) \oplus X(\mu_{A_1}),$$

where $X(\mu_1)$ is order isomorphic to an L^p-space, say $L^p(\widetilde{\mu}_1)$.

Step 2. Consider now $\mu_2 := \mu_{A_1 \setminus A_2}$ and $B_2 = A_1 \setminus A_2$. Hence $X(\mu_2)$ is order isomorphic to an L^p-space, say $L^p(\widetilde{\mu}_2)$, since it is easy to check that it is p-convex and p-concave (see Remark 2.3).

Step 3. Using these arguments, we obtain the following decomposition for $X(\mu)$,

$$\begin{aligned} X(\mu) &= X(\mu_{A_2\setminus A_1}) \oplus X(\mu_{(A_1\cup A_2)^c}) \oplus X(\mu_{A_1\setminus A_2}) \oplus X(\mu_{A_1\cap A_2}) \\ &= X(\mu_1) \oplus X(\mu_2) \oplus X(\mu_{A_1\cap A_2}). \end{aligned}$$

Thus, $X(\mu)$ is order isomorphic to $L^p(\widetilde{\mu}_1) \oplus L^p(\widetilde{\mu}_2) \oplus X(\mu_{A_1\cap A_2})$, where $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are disjoint measures supported in the disjoint measurable sets B_1 and B_2.

Step 4. We can apply this procedure inductively and obtain a sequence of spaces $\big(L^p(\widetilde{\mu}_n)\big)$, where the measures $\widetilde{\mu}_n$ are mutually disjoint to each other and supported on B_n. Observe that the set $A = \bigcap_{n=1}^\infty A_n \in \Sigma$ is μ-null —since $\mu(A) \le \mu(A_n) < 1/n$, for all $n \in \mathbb{N}$— and then also $X(\mu_{\bigcap_{n=1}^\infty A_n})$ is so. Therefore we can write each function in $X(\mu)$ as a direct sum of disjoint functions of $\bigoplus_n L^p(\widetilde{\mu}_n)$. But since each measure $\widetilde{\mu}_n$ is absolutely continuous with respect to μ_n (actually they are equivalent) then the *Radon-Nikodym Theorem* gives a sequence of functions (g_n), each g_n being a μ_n integrable function, such that $\widetilde{\mu}_n = g_n\mu_n$. Hence all the elements in $X(\mu)$ are (classes of μ-a.e. equal) functions that can be written as sums of series that converge μ-a.e. Hence they are elements of $\bigoplus_n L^p(g_n d\mu)$, where the direct sum is understood as a μ-a.e. disjoint sum. □

The result above gives a description of the elements of an almost-L^p-space. The natural topology for such a space $X(\mu)$ is given by its norm. But, in general, the closure of the functions having their support in a finite collection of the B_n's does not give the whole space $X(\mu)$. In what follows we will give some concrete results characterizing almost-L^p-spaces under some Banach lattice assumptions. Let us start with a metric result, that relates an almost-L^p-space with a weighted space $L^p(gd\mu)$. In such a case we write that $X(\mu)$ is an almost-L^p-space with respect to the space $L^p(gd\mu)$. In the next result —and in the rest of the paper— we will use the following abuse of notation. Let $g = \sum_{n=1}^\infty g_n$ be a μ-measurable function defined as a sum (pointwise μ-a.e) of disjoint μ-integrable functions. Although g may not be μ-integrable, the set function $\Sigma \ni A \rightsquigarrow \sum_{n=1}^\infty \int_A g_n d\mu$ defines a σ-finite measure. For the sake of simplicity, we will write $\mu_g(\cdot) := \int_\cdot g d\mu$ for this measure.

Theorem 3.2. *Let $X(\mu)$ be a Banach function space over a finite measure μ. The following statements are equivalent.*

(i) *$X(\mu)$ is an almost-L^p-space.*
(ii) *There are sequences $(q_n), (Q_n) \subseteq]0, \infty[$ and a disjoint sequence of integrable functions (g_n) supported on a partition (B_n) of Ω such that for all $f \in X(\mu)$*

and $k \in \mathbb{N}$,

$$\big(\sum_{n=1}^{k} q_n \|f\chi_{B_n}\|^p_{X(\mu)}\big)^{1/p} \leq \big\|f\chi_{\cup_n^k B_n}\big\|_{L^p(gd\mu)} \leq \big(\sum_{n=1}^{k} Q_n \|f\chi_{B_n}\|^p_{X(\mu)}\big)^{1/p},$$

where the measurable function $g = \sum_{n=1}^{\infty} g_n$ *is defined pointwise* μ*-a.e.*

Notice that since the formula $\mu_g(A) := \sum_{n=1}^{\infty} \int_A g_n d\mu$, $A \in \Sigma$, defines a σ-finite measure which is equivalent to μ, we have that $L^0(\mu) = L^0(gd\mu)$.

Proof. (i) $\Rightarrow$ (ii). Assume that $X(\mu)$ is an almost-L^p-space and let us take $f \in X(\mu)$. By using Theorem 3.1 we can write $f = \sum_{n=1}^{\infty} f\chi_{B_n}$, μ-a.e. where $f_n = f\chi_{B_n} \in L^p(g_n d\mu)$ and being g_n measurable functions with support equal to B_n —(B_n) a disjoint partition of Ω—. Moreover, by the description given in the proof of Theorem 3.1 we have that $X(\mu_{B_n})$ is order isomorphic to $L^p(g_n d\mu)$. Then for each $n \in \mathbb{N}$ there are positive constants q_n and Q_n such that

$$q_n\big\|f\chi_{B_n}\big\|^p_{X(\mu)} \leq \big\|f\chi_{B_n}\big\|^p_{L^p(g_n d\mu)} \leq Q_n\big\|f\chi_{B_n}\big\|^p_{X(\mu)}.$$

Consider the finite sum $\sum_{n=1}^{k} f_n$, $k \in \mathbb{N}$. Then

$$\sum_{n=1}^{k} q_n\|f\chi_{B_n}\|^p_{X(\mu)} \leq \sum_{n=1}^{k} \|f\chi_{B_n}\|^p_{L^p(g_n d\mu)} \leq \sum_{n=1}^{k} Q_n\|f\chi_{B_n}\|^p_{X(\mu)}.$$

Consequently, since for every $f \in X(\mu)$ we have that

$$\begin{aligned} \|f\chi_{\cup_n^k B_n}\|^p_{L^p(gd\mu)} &= \int_\Omega \big|f\chi_{\cup_{n=1}^k B_n}\big|^p \big(\sum_{m=1}^{\infty} g_m\big) d\mu = \sum_{n=1}^{k} \int_\Omega \big|f\chi_{B_n}\big|^p g_n d\mu \\ &= \sum_{n=1}^{k} \|f\chi_{B_n}\|^p_{L^p(g_n d\mu)}, \end{aligned}$$

we obtain the result.
(ii) $\Rightarrow$ (i). Since μ is finite, for each $\varepsilon > 0$ we find a finite number n_ε such that $\mu(C_{n_\varepsilon}) < \varepsilon$ for $C_{n_\varepsilon} := \bigcup_{n=n_\varepsilon+1}^{\infty} B_n$. The inequalities in (ii) together with *Holder's inequality* give that

$$\begin{aligned} \min\{q_n : n = 1, \ldots, n_\varepsilon\} \cdot \|f\chi_{C^c_{n_\varepsilon}}\|_{X(\mu)} &\leq \sum_{n=1}^{n_\varepsilon} q_n\|f\chi_{B_n}\|_{X(\mu)} \\ = \sum_{n=1}^{n_\varepsilon} q_n^{1/p} q_n^{1/p'} \|f\chi_{B_n}\|_{X(\mu)} &\leq \big(\sum_{n=1}^{n_\varepsilon} q_n\big)^{1/p'} \cdot \big(\sum_{n=1}^{n_\varepsilon} q_n\|f\chi_{B_n}\|^p_{X(\mu)}\big)^{1/p} \\ &\leq \big(\sum_{n=1}^{n_\varepsilon} q_n\big)^{1/p'} \cdot \|f\chi_{C^c_{n_\varepsilon}}\|_{L^p(gd\mu)}, \end{aligned}$$

where p' is the conjugate exponent of p which is defined by $1/p + 1/p' = 1$.

On the other hand,

$$\begin{aligned}\|f\chi_{C^c_{n_\varepsilon}}\|_{L^p(gd\mu)} &\le \big(\sum_{n=1}^{n_\varepsilon} Q_n \|f\chi_{B_n}\|^p_{X(\mu)}\big)^{1/p} \\ &\le (\sum_{n=1}^{n_\varepsilon} Q_n)^{1/p}\cdot \max\big\{\|f\chi_{B_n}\|_{X(\mu)} : n=1,\dots,n_\varepsilon\big\} \\ &\le (\sum_{n=1}^{n_\varepsilon} Q_n)^{1/p}\cdot \|f\chi_{C^c_{n_\varepsilon}}\|_{X(\mu)}.\end{aligned}$$

Since this holds for every $\varepsilon > 0$ this proves the isomorphism, and therefore $X(\mu)$ is an almost-L^p-space. □

Example 3.3. Let $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ be a finite measure space. Then $\ell^p(\mu)$ is an almost-L^q-space, with respect to the space $\ell^q(\mu)$, for every $1 \le q, p < \infty$. Observe that $\chi_{\{k\}}(j)$ is 1 if $j = k$ and 0 in other case, hence $|x_k\chi_{\{n\}}|^p\mu(\{k\}) = 0$ whenever $k \ne n$. Therefore for each $x = (x_n) \in \ell^p(\mu)$ and each $n = 1, 2, \dots$ we have that

$$\begin{aligned}|x_n|^q\mu(\{n\}) &= \big(|x_n|^p\big)^{q/p}\mu(\{n\}) = \mu(\{n\})^{1-q/p}\,\big(|x_n\chi_{\{n\}}|^p\mu(\{n\})\big)^{q/p} \\ &= \mu(\{n\})^{1-q/p}\,\big(|x_1\chi_{\{n\}}|^p\mu(\{1\}) + \cdots + |x_n\chi_{\{n\}}|^p\mu(\{n\}) + \cdots\big)^{q/p} \\ &= \mu(\{n\})^{1-q/p}\,\|x\chi_{\{n\}}\|^q_{\ell^p(\mu)}.\end{aligned}$$

This implies that

$$\|x\|_{\ell^q(\mu)} = \left(\sum_{n=1}^{\infty} |x_n|^q\mu(\{n\})\right)^{1/q} = \left(\sum_{n=1}^{\infty} \mu(\{n\})^{1-q/p}\,\|x\chi_{\{n\}}\|^q_{\ell^p(\mu)}\right)^{1/q}.$$

In this case taking $B_n := \{n\}$ and $Q_n = q_n := \mu(B_n)^{1-q/p}$ in the previous theorem, for $x \in \ell^p(\mu)$ we obtain

$$\left(\sum_{n=1}^{\infty} q_n\,\|x\chi_{B_n}\|^q_{\ell^p(\mu)}\right)^{1/q} \le \|x\|_{\ell^q(\mu)} \le \left(\sum_{n=1}^{\infty} Q_n\,\|x\chi_{B_n}\|^q_{\ell^p(\mu)}\right)^{1/q}.$$

Note that the inequalities above are actually equalities.

In the case when the space $X(\mu)$ has particular lattice properties —as order continuity or the Fatou property—, more convenient characterizations are suitable. Recall that a B.f.s. $X(\mu)$ has the *Fatou property* if for every increasing sequence of positive functions (f_n) in $X(\mu)$ with $\sup\|f_n\| < \infty$ it follows that there exists $f = \sup f_n$ in $X(\mu)$ and $\|f\| = \sup\|f_n\| < \infty$. Also, the space $X(\mu)$ is said to be *order continuous* if for every decreasing sequence of positive functions (f_n) in $X(\mu)$ that converges to 0 μ-a.e., it follows that $\|f_n\| \downarrow 0$.

Let us remark first that almost-L^p-spaces are, in general, neither order continuous nor Fatou.

Remark 3.4. (1) Consider the finite measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is defined as

$$\mu(A) := \sum_{n\in A} 1/2^n, \quad A \in 2^{\mathbb{N}}.$$

Note that ℓ^∞ and c_0 are Banach function spaces over μ with the usual supremum norm. Moreover, for $1 \le p < \infty$, let us see that both of them are almost-L^p-spaces with respect to the space $\ell^p(\mu)$. Indeed, consider the partition given by the atoms, i.e. $B_n = \{n\}$, $n \in \mathbb{N}$. Clearly, if $k \in \mathbb{N}$ and $f \in \ell^\infty$, we have

$$\begin{aligned}\big\|f\chi_{\{1,\ldots,k\}}\big\|_{\ell^\infty} &\le 2^{k/p}\cdot\big\|f\chi_{\{1,\ldots,k\}}\big\|_{\ell^p(\mu)}\\ &\le 2^{k/p}\cdot\Big(\sum_{n=1}^{k} 1/2^n\Big)^{1/p}\cdot\max\big\{|f(n)| : n=1,...,k\big\}\\ &= 2^{k/p}\cdot\Big(\sum_{n=1}^{k} 1/2^n\Big)^{1/p}\cdot\|f\chi_{\{1,\ldots,k\}}\|_{\ell^\infty}.\end{aligned}$$

Since for every $\varepsilon > 0$ we find a natural number n_ε such that $\mu(\{n \ge n_\varepsilon\}) < \varepsilon$, we obtain that ℓ^∞ is an almost-$L^p(\mu)$ space. Therefore, we have a Banach function space that is an almost-L^p-space and has the Fatou property, but is not order continuous. Exactly the same computations show that c_0 is also an almost-L^p-space; in this case, it is order continuous, but it does not satisfy the Fatou property.

(2) The situations given in Example 3.3 and in (1) can be extended to the class of all sequence spaces. Let $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ be a finite measure space and take *any* sequence space $\ell(\mu)$ over μ. Then $\ell(\mu)$ is an almost-L^q-space with respect to the space $\ell^q(\mu)$ for each $1 \le q < \infty$. To see this, let $\varepsilon > 0$. Since μ is finite there exists $n_\varepsilon \in \mathbb{N}$ such that $\mu(A_\varepsilon) < \varepsilon$ where $A_\varepsilon = \{n_\varepsilon+1, n_\varepsilon+2, \dots\}$. Observe that $\ell(A_\varepsilon^c) = \ell(\{1, \dots, n_\varepsilon\})$ is finite dimensional. In consequence it is isomorphic to the finite dimensional space $\ell^q(A_\varepsilon^c)$ which is an L^q-space, that is, $\ell(\mu)$ is an almost-L^q-space.

Part (1) of Remark 3.4 motivates the following two results. They describe the elements of the almost-L^p- spaces under the assumption of specific lattice properties of the spaces involved. We assume in them the description provided by Theorem 3.2 of the function g as a sum of a disjoint sequence (g_n).

Proposition 3.5. *Let $X(\mu)$ be an almost-L^p-space with respect to $L^p(gd\mu)$ with the Fatou property. Then the following assertions are equivalent for a measurable function f.*

(i) *The function f belongs to $X(\mu)$.*
(ii) *There is a disjoint sequence $(f_n) \subseteq L^p(g_n d\mu)$ such that $f = \sum_{n=1}^\infty f_n$ μ-a.e. and $\sup_k \big\|\sum_{n=1}^k f_n\big\|_{X(\mu)} < \infty$.*

Proof. (i) $\Rightarrow$ (ii) is a consequence of Theorem 3.1 and the fact that $\|\cdot\|_{X(\mu)}$ is a lattice norm. Indeed, this theorem gives the sequence (f_n) of disjoint functions for which we have for all $k \in \mathbb{N}$

$$\big\|\sum_{n=1}^{k} f_n\big\|_{X(\mu)} = \big\||\sum_{n=1}^{k} f_n|\big\|_{X(\mu)} = \big\|\sum_{n=1}^{k} |f_n|\big\|_{X(\mu)} \le \||f|\|_{X(\mu)} = \|f\|_{X(\mu)}.$$

(ii) $\Rightarrow$ (i) is a direct consequence of the Fatou property; since $X(\mu)$ is an almost-L^p-space, each function $h_k = \sum_{n=1}^k f_n$ belongs to $X(\mu)$. The sequence $(|h_k|)$ is then order bounded by the measurable function $|f|$ and converges to it pointwise; taking into account that $\sup_k \|h_k\|_{X(\mu)} < \infty$, we obtain that $f \in X(\mu)$. □

Proposition 3.6. *Let $X(\mu)$ be an order continuous almost-L^p-space with respect to $L^p(gd\mu)$. The following assertions are equivalent for a measurable function f.*

(i) *The function f belongs to $X(\mu)$.*

(ii) *There is a disjoint sequence* $(f_n) \subseteq L^p(g_n d\mu)$ *such that* $f = \sum_{n=1}^{\infty} f_n$ μ*-a.e. and* $\lim_{k,m} \| \sum_{n=k}^{m} f_n \|_{X(\mu)} = 0$.

Proof. (ii) $\Rightarrow$ (i). The sequence of the functions $h_k := \sum_{n=1}^{k} f_n$ converges in $X(\mu)$, since it defines a Cauchy sequence by the condition given in (ii). It converges also μ-a.e. to its limit, which belongs to $X(\mu)$ due to the fact that $X(\mu)$ is a Banach function space and so there is a subsequence converging almost everywhere to f. In particular, this implies that the limit of the sequence (h_k) is f, and then it belongs to $X(\mu)$.
(i) $\Rightarrow$ (ii) is a consequence of Theorem 3.1 and the order continuity of $X(\mu)$. Since $f \in X(\mu)$, this theorem gives the sequence of disjoint functions (f_n) such that $f = \sum_{n=1}^{\infty} f_n$ μ-a.e. Consider the sequence of the functions $r_k := \sum_{n=k}^{\infty} f_n \in X(\mu)$. The decreasing sequence formed by the positive functions $|r_k| := \sum_{n=k}^{\infty} |f_n|$ converges to 0 μ-a.e. Therefore, by the order continuity of $X(\mu)$, $\lim_k \|r_k\|_{X(\mu)} = \lim_k \||r_k|\|_{X(\mu)} = 0$, and so the sequence is Cauchy, which gives (ii). □

4. Factorization of operators through almost-L^p-spaces

In this section we will consider a B.f.s. $X(\mu)$, a Banach space E and $1 \le p < \infty$. The lattice $X(\mu)$ is *almost-p-convex* if for every $\varepsilon > 0$ there exist $A_\varepsilon \in \Sigma$ and $K_\varepsilon > 0$ such that $\mu(A_\varepsilon) < \varepsilon$ and for all every finite choice of functions $\{f_1, \dots, f_n\} \subseteq X(\mu)$,

$$\left\| \left(\sum_{j=1}^{n} |f_j \chi_{A_\varepsilon^c}|^p \right)^{1/p} \right\|_{X(\mu)} \le K_\varepsilon \left(\sum_{j=1}^{n} \|f_j \chi_{A_\varepsilon^c}\|_{X(\mu)}^p \right)^{1/p}.$$

For a given $\varepsilon > 0$ and an associated measurable set A_ε, the infimum of all K_ε satisfying the previous inequality will be denoted by $M_{(p)}(A_\varepsilon^c, X(\mu))$.

Let $T\colon X(\mu) \to E$ be an operator. Suppose that for a given $\varepsilon > 0$ there exist $A_\varepsilon \in \Sigma$ and $Q_\varepsilon > 0$ such that $\mu(A_\varepsilon) < \varepsilon$ and the inequality

$$\left(\sum_{j=1}^{n} \|T(f_j \chi_{A_\varepsilon^c})\|_E^p \right)^{1/p} \le Q_\varepsilon \left\| \left(\sum_{j=1}^{n} |f_j|^p \right)^{1/p} \chi_{A_\varepsilon^c} \right\|_{X(\mu)},$$

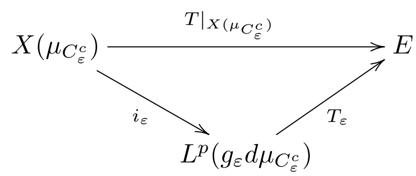
holds for every finite choice of functions $\{f_1, \dots, f_n\} \subseteq X(\mu)$. Then T is said to be *almost-p-concave*. For $\varepsilon > 0$ and an associated A_ε, the smallest possible value of the constants Q_ε above is denoted by $M^{(p)}(A_\varepsilon^c, T)$. As usual for concavity type properties, a B.f.s. $X(\mu)$ is said to be *almost-p-concave* if the identity map on $X(\mu)$ is almost-p-concave.

As in the classical case of p-concave operators defined on order continuous p-convex B.f.s. (see for instance [10, 13, 15]), the following characterization occurs in our context.

Proposition 4.1. *Let* $X(\mu)$ *be an almost-p-convex order continuous Banach function space over the finite measure* μ*, where* $1 \le p < \infty$*. For a Banach space valued operator* $T\colon X(\mu) \to E$*, the following statements are equivalent.*

(i) *The operator* T *is almost-p-concave.*

(ii) *For every $\varepsilon > 0$ there is a measurable set C_ε such that $\mu(C_\varepsilon) < \varepsilon$ and the restriction of T to $X(\mu_{C_\varepsilon^c})$ can be extended as*



for some function $0 \le g_\varepsilon$ such that $g_\varepsilon^{1/p} \in M(X(\mu_{C_\varepsilon^c}), L^p(\mu_{C_\varepsilon^c}))$ with norm in this space less or equal to 1.

Proof. (i) $\Rightarrow$ (ii). Fix an $\varepsilon > 0$. Then by definition there are measurable sets A_ε and B_ε such that $\mu(A_\varepsilon) < \varepsilon/2$ and $\mu(B_\varepsilon) < \varepsilon/2$, and out of A_ε the space is p-convex —i.e. $X(\mu_{A_\varepsilon^c})$ is p-convex— and out of B_ε, the operator T is p-concave. Take $C_\varepsilon := A_\varepsilon \cup B_\varepsilon \in \Sigma$. Therefore, $\mu(C_\varepsilon) < \varepsilon$ and in $C_\varepsilon^c := (A_\varepsilon \cup B_\varepsilon)^c$ we have a p-concave operator $T|_{X(\mu_{C_\varepsilon^c})}$ on the p-convex order continuous Banach function space $X(\mu_{C_\varepsilon^c})$. The arguments given in Remark 2.3 provide the desired extension. For the converse, take $\varepsilon > 0$. Then there is $C_\varepsilon \in \Sigma$ such that $\mu(C_\varepsilon) < \varepsilon$ and satisfies the properties given in (ii). Therefore, for each finite family $f_1, \dots, f_n \in X(\mu)$, we have

$$\begin{aligned}
\left(\sum_{j=1}^n \|T(f_j \chi_{C_\varepsilon^c})\|_E^p \right)^{1/p} &\le \|T_\varepsilon\| \left(\sum_{j=1}^n \|f_j \chi_{C_\varepsilon^c}\|^p_{L^p(g_\varepsilon d\mu_{C_\varepsilon^c})} \right)^{1/p} \\
&= \|T_\varepsilon\| \left(\sum_{j=1}^n \int_{C_\varepsilon^c} |f_j|^p \chi_{C_\varepsilon^c} g_\varepsilon d\mu_{C_\varepsilon^c} \right)^{1/p} \\
&= \|T_\varepsilon\| \left(\int_{C_\varepsilon^c} \Big(\sum_{j=1}^n |f_j|^p \Big) g_\varepsilon \chi_{C_\varepsilon^c} d\mu_{C_\varepsilon^c} \right)^{1/p} \\
&= \|T_\varepsilon\| \left\| \left(\sum_{j=1}^n |f_j|^p \right)^{1/p} g_\varepsilon^{1/p} \chi_{C_\varepsilon^c} \right\|_{L^p(\mu_{C_\varepsilon^c})} \\
&\le \|T_\varepsilon\| \left\| \left(\sum_{j=1}^n |f_j|^p \right)^{1/p} \chi_{C_\varepsilon^c} \right\|_{X(\mu)},
\end{aligned}$$

and the result follows. □

Corollary 4.2. *An order continuous Banach function space is an almost-L^p-space if and only if it is almost-p-convex and almost-p-concave.*

Proof. Assume that the Banach ideal $X(\mu)$ is almost-p-convex and almost-p-concave and consider the identity map id: $X(\mu) \to X(\mu)$. By using Proposition 4.1 for every $\varepsilon > 0$ we find a measurable set A_ε such that $\mu(A_\varepsilon) < \varepsilon$ and satisfying the factorization diagram —for the operator id—. This gives the order isomorphism between $X(\mu_{A_\varepsilon^c})$ and $L^p(g_\varepsilon d\mu_{A_\varepsilon^c})$ —where g_ε is the function provided by the quoted proposition—. Since the converse is obvious, this gives the result. □

Definition 4.3. Let $Z(\mu)$ be a Banach ideal of μ-measurable functions. We say that an operator $T\colon X(\mu) \to E$ *almost extends to* $Z(\mu)$, if for each $\varepsilon > 0$ there is a measurable set A_ε such that

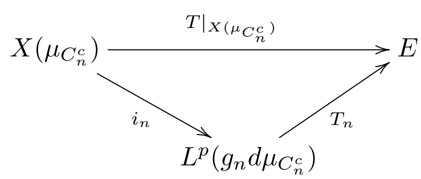
- $\mu(A_\varepsilon) < \varepsilon$,
- $X(\mu_{A_\varepsilon^c})$ is included in $Z(\mu)$ with inclusion i_ε, and
- $T|_{X(\mu_{A_\varepsilon^c})} = T_0 \circ i_\varepsilon$ for an operator $T_0 : Z(\mu) \to E$.

Note that this definition is equivalent to the one that is obtained when the requirements are imposed only for the constants $\varepsilon_n > 0$ of a null sequence (ε_n).

Theorem 4.4. *Let $X(\mu)$ be an order continuous almost-p-convex Banach function space over the finite measure μ for $1 \le p < \infty$. For a Banach space valued operator $T\colon X(\mu) \to E$ the following statements are equivalent.*

(i) *The operator T is almost-p-concave.*
(ii) *There is an almost-L^p-space, $Z(\mu)$, such that T almost extends to $Z(\mu)$.*

Proof. (i) $\Rightarrow$ (ii). Let T be an almost-p-concave operator. Consider the sequence $(1/n)$ and use Proposition 4.1 to find a sequence of measurable sets C_n such that $\mu(C_n) < 1/n$ and provides a sequence of extensions of $X(\mu_{C_n^c})$ through $L^p(g_n d\mu_{C_n^c})$ of $T|_{X(\mu_{C_n^c})}$ as



for functions $0 \le g_n$ such that $g_n^{1/p} \in M(X(\mu_{C_n^c}), L^p(\mu_{C_n^c}))$. It can be easily seen that the sequence of sets (C_n) can be chosen to be decreasing, just taking in each step n the next corrected C_n as $C_1 \cap \ldots \cap C_{n-1}$. Note that this intersection could be empty for some n; but in such a case $X(\mu)$ is actually an L^p-space (see the *Claim* in the proof of Theorem 3.2) and the result is trivial. For the aim of clarity, let us denote $M_{(p)}(C_n^c, X(\mu))$ by $M_{(n)}$ and $M^{(p)}(C_n^c, T)$ by $M^{(n)}$. It can be found as an application of Corollary 5 of [5] that each extension satisfies that

$$\|T(f\chi_{C_n^c})\|_E \le M_{(n)} M^{(n)} \Big(\int_{C_n^c} |f|^p g_n d\mu \Big)^{1/p}, \tag{4.1}$$

where g_n, $n \in \mathbb{N}$, are the functions appearing in Proposition 4.1; recall that $g_n^{1/p}$ define multiplication operators of norm less or equal to one. Define now the measurable sets $B_n = C_n^c \setminus C_{n-1}^c$, $n \in \mathbb{N}$, where $C_0^c = \emptyset$. Since (C_n) is supposed to be decreasing, (B_n) is a well defined sequence of disjoint measurable sets satisfying $B_n \downarrow \emptyset$. Let us consider now the set

$$Z(\mu) = \left\{ f \in L^0(\mu) : \sum_{k=1}^\infty M_{(k)} M^{(k)} \Big(\int_{B_k} |f|^p g_k d\mu \Big)^{1/p} < \infty \right\}.$$

Note that $Z(\mu)$ is actually $\bigoplus_{k=1}^\infty L^p(\widetilde{g}_k d\mu)$ for $\widetilde{g}_k = \big(M_{(k)} M^{(k)}\big)^p g_k$, which is a Banach ideal in $L^0(\mu)$, where the direct sum is endowed with the 1-norm,

$$\|f\|_{Z(\mu)} = \sum_{k=1}^\infty \|f\|_{L^p(\widetilde{g}_k d\mu)} = \sum_{k=1}^\infty M_{(k)} M^{(k)} \Big(\int_{B_k} |f|^p g_k d\mu \Big)^{1/p}.$$

Moreover, $(Z(\mu), \|\cdot\|_{Z(\mu)})$ is an almost-L^p-space. Write $g = \sum_{n=1}^{\infty} g_n \chi_{B_n}$. Indeed, take $f \in Z(\mu)$ and put $q_n = Q_n = (M_{(n)} M^{(n)})^{-p}$ for each $n \in \mathbb{N}$. Hence for all $k \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{n=1}^{k} q_n \|f\chi_{B_n}\|_{Z(\mu)}^p &= \sum_{n=1}^{k} Q_n \|f\chi_{B_n}\|_{Z(\mu)}^p = \sum_{n=1}^{k} Q_n (M_{(n)} M^{(n)})^p \int_{B_n} |f|^p g_n d\mu \\ &= \sum_{n=1}^{k} \int_{B_n} |f|^p g_n d\mu = \big\| f\chi_{\cup_{n=1}^k B_n} \big\|_{L^p(gd\mu)}^p.\end{aligned}$$

Therefore, thanks to Theorem 3.2, $Z(\mu)$ is an almost-L^p-space. In order to finish this first part of the proof let us see that T almost extends to $Z(\mu)$. Consider the sequence (C_n). Then if $\varepsilon > 0$ we find a natural number n such that

• $\mu(C_n) < \varepsilon$,

• $X(\mu_{C_n^c})$ is included in $Z(\mu)$. Indeed, take $f \in X(\mu_{C_n^c})$ and recall that $L^p(g_n d\mu)$ contains $X(\mu_{C_n^c})$ —see the diagram above—, so we obtain

$$\begin{aligned}\|f\|_{Z(\mu)} &= \sum_{k=1}^{n} M_{(k)} M^{(k)} \Big(\int_{B_k} |f|^p g_k d\mu \Big)^{1/p} = \sum_{k=1}^{n} M_{(k)} M^{(k)} \big\| f\chi_{B_k} g_k^{1/p} \big\|_{L^p(\mu_{C_n^c})} \\ &\le \sum_{k=1}^{n} \Big(M_{(k)}\, M^{(k)}\, \|g_k^{1/p}\|_{M(X(\mu_{C_k^c}), L^p(\mu_{C_k^c}))} \|f\chi_{B_k}\|_{X(\mu_{C_n^c})} \Big) \\ &\le \Big(\sum_{k=1}^{n} M_{(k)} M^{(k)} \Big) \|f\|_{X(\mu_{C_n^c})}.\end{aligned}$$

• Finally, by using (4.1) for each $f \in Z(\mu)$ one has

$$\begin{aligned}\|T(f)\|_E &\le \sum_{n=1}^{\infty} \|T(f\chi_{B_n})\|_E = \sum_{n=1}^{\infty} \|T\big((f\chi_{B_n})\chi_{C_n^c}\big)\|_E \\ &\le \sum_{n=1}^{\infty} M_{(n)} M^{(n)} \Big(\int_{C_n^c} |f\chi_{B_n}|^p g_n d\mu \Big)^{1/p} = \|f\|_{Z(\mu)}.\end{aligned}$$

Consequently, the operator T can be defined and is continuous in the domain $Z(\mu)$.

(ii) $\Rightarrow$ (i). Fix $\varepsilon > 0$ and assume that T almost extends to an almost-L^p-space, $Z(\mu)$. Hence we can find two measurable sets A_ε and B_ε, positive constants $\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon$ and a positive measurable function $0 \le g_\varepsilon$ such that

• $\mu(A_\varepsilon) < \varepsilon/2$ and $\mu(B_\varepsilon) < \varepsilon/2$,

• $g_\varepsilon^{1/p} \in M(Z(\mu_{A_\varepsilon^c}), L^p(\mu_{A_\varepsilon^c}))$ and

$$\alpha_\varepsilon \|f\|_{Z(\mu_{A_\varepsilon^c})} \le \|f\|_{L^p(g_\varepsilon d\mu)} \le \beta_\varepsilon \|f\|_{Z(\mu_{A_\varepsilon^c})}, \qquad f \in Z(\mu_{A_\varepsilon^c}), \tag{4.2}$$

• $T|_{X(B_\varepsilon^c)} = T_0 \circ i_\varepsilon$ and

$$\|f\|_{Z(\mu)} \le \gamma_\varepsilon \|f\|_{X(\mu_{B_\varepsilon^c})}, \qquad f \in X(\mu_{B_\varepsilon^c}). \tag{4.3}$$

We define now $C_\varepsilon = A_\varepsilon \cup B_\varepsilon$, which satisfies $\mu(C_\varepsilon) < \varepsilon$. Take $f_1, \dots, f_n$ in $X(\mu)$. Using (4.2) and (4.3) we get

$$\begin{aligned}
\Big(\sum_{j=1}^n \|T(f_j\chi_{C_\varepsilon^c})\|_E^p\Big)^{1/p} &= \Big(\sum_{j=1}^n \|T_0\circ i_\varepsilon(f_j\chi_{C_\varepsilon^c})\|_E^p\Big)^{1/p} \le \|T_0\|\Big(\sum_{j=1}^n \|f_j\chi_{C_\varepsilon^c}\|_{Z(\mu)}^p\Big)^{1/p} \\
&\le \|T_0\|\alpha_\varepsilon^{-1}\Big(\sum_{j=1}^n \|f_j\chi_{C_\varepsilon^c}\|_{L^p(g_\varepsilon d\mu)}^p\Big)^{1/p} = \|T_0\|\alpha_\varepsilon^{-1}\Big(\int_{C_\varepsilon^c}\Big(\sum_{j=1}^n |f_j|^p\Big) g_\varepsilon d\mu\Big)^{1/p} \\
&= \|T_0\|\alpha_\varepsilon^{-1}\left\|\Big(\sum_{j=1}^n |f_j|^p\Big)^{1/p}\chi_{C_\varepsilon^c}\right\|_{L^p(g_\varepsilon d\mu)} \le \|T_0\|\frac{\beta_\varepsilon}{\alpha_\varepsilon}\left\|\Big(\sum_{j=1}^n |f_j|^p\Big)^{1/p}\chi_{C_\varepsilon^c}\right\|_{Z(\mu)} \\
&\le \|T_0\|\frac{\beta_\varepsilon\gamma_\varepsilon}{\alpha_\varepsilon}\left\|\Big(\sum_{j=1}^n |f_j|^p\Big)^{1/p}\chi_{C_\varepsilon^c}\right\|_{X(\mu)}.
\end{aligned}$$

Consequently, T is almost-p-concave as can be seen just taking $Q_\varepsilon = \|T_0\|\beta_\varepsilon\gamma_\varepsilon/\alpha_\varepsilon$. This finishes the proof. □

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[1] Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València. Camino de Vera s/n, 46022 Valencia, Spain.
Email address: jmcalabu@mat.upv.es
Email address: easancpe@mat.upv.es

[2] Facultat de Ciències Matemàtiques, Universitat de València. Dr. Moliner 50, 46100 Burjassot (Valencia), Spain.
Email address: orlando.galdames@uv.es

[3] Departamento de Contabilidad, Finanzas y Control de Gestión, Facultad de Administración y Dirección de Empresas, Universidad Católica de Valencia San Vicente Mártir. C/Corona 34, 46003 Valencia, Spain.
Email address: ma.juan@ucv.es