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MANUSCRIPT

Possibilistic compositions and state functions: application to the order promising process for perishables

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In this paper we propose the concepts of composition of possibilistic variables and state functions. While in conventional compositional data analysis, the interdependent components of a deterministic vector must add up to a specific quantity, we consider such components as possibilistic variables. The concept of state function is intended to describe the «*state*» of a dynamic variable over time. If a state function is used to model decay in time, it is called ageing function. We present a practical implementation of our concepts through the development of a model for a supply chain planning problem, specifically the order promising process for perishables. We use the composition of possibilistic variables to model the existence of different non-homogeneous products in a lot (sublots with lack of homogeneity in the product), and the ageing function to establish a shelf life-based pricing policy. To maintain a reasonable complexity and computational efficiency, we propose the procedure to obtain an equivalent interval representation based on α-cuts, allowing to include both concepts by means of linear mathematical programming. Practical experiments were conducted based on data of a Spanish supply chain dedicated to pack and distribute oranges and tangerines. The results validated the functionality of both, the compositions of possibilistic variables and ageing functions, showing also a very good performance in terms of the interpretation of a real problem with a good computational performance.

Keywords: Composition of Possibilistic Variables; State Function; Ageing Function; Order Promising Process; Perishability; Lack of Homogeneity in The Product; Uncertainty.

1. Introduction

There are many situations in mathematical modelling when is necessary to consider interdependent data, i.e., where the variables or the restrictions are not free to behave themselves without having a direct impact on each other. One example of such interdependence is the well-known case where the components should add up to the total amount of a variable, representing for example, a raw material or a finished product. Examples of this situation are the distribution of the components in the chemical (Nureize and Watada 2010) or food (Kilic et al. 2013) industries, or the consistency of the components of a determined material in the metallurgic industry (Slotnick 2011) that generate different *grades* of it. If we analyse this situation in the industry of perishables, it is common to have different "versions" of the same product in the same lot (for instance, different size, different weight, colour, etc.) but all of the subgroups must add up to the total quantity of the lot.

In the area of mathematical modelling of supply chain planning problems (Mula et al. 2010; Peidro et al. 2009), one can find several situations where determined quantities of products and/or raw materials must be subdivided according to their components. We find the case of the order promising process (OPP), which refers to a set of business activities triggered to provide a response to customer order requests (Grillo et al. 2016a). The OPP bases its functionality on the concept of Available-to-Promise (ATP). The ATP represents the uncommitted availability of product computed by subtracting the already committed orders from the quantities in stock and planned supply defined in the Master Production Schedule (MPS) (Ball et al. 2004). Based on the ATP quantities, the OPP answers to each order with an acceptance/rejection decision. There are certain Supply Chains belonging to sectors such as ceramic tile, furniture and agriculture in which products present different characteristics among its units that, in some cases, can vary over time. Furthermore, such characteristics are important and perceptible to the final customers and therefore, the OPP can take these aspects into account when committing their orders. These sectors are affected by the so-called Lack of Homogeneity in the product (LHP) (for more details on LHP, see Alemany et al. (2015, 2013); Grillo et al. (2016a,b)). This is, in fact, the case of fresh fruit and vegetables that are perishables.

Under these circumstances, the computation of the ATP must consider the sub-division of the MPS into homogeneous sub-lots that will not be accurately known until the moment the product is harvested and classified. For this type of supply chain, the lack of homogeneity in the product refers to the existence of different units of the same product (subtype) that differ in characteristics such as colour, weight, size or variety. The subdivision of the MPS lots into different subtypes (in order to compute the ATP as a consequence of LHP) is the first issue addressed in this paper to exemplify a situation in which there is interdependent data with several elements that must add up to a determined total quantity.

The second issue in the modelling of ATP for perishables, is that it involves another source of complexity, namely the product's lifetime (Grillo et al. 2016a). In this case, the ATP is not just affected by the conventional schema of the LHP in terms of the subdivision into subtypes, but also in the way that the shelf life can be considered as another cause of LHP. Since most of the time, customers request certain levels of freshness, it could itself be used to define a subtype. Literature shows that, no models have been developed for this type of problems, specifically in supply chains of perishables (Grillo et al. 2016a).

The distribution of a total quantity into several components is known as compositional data and it has been extensively studied since Aitchison (1982). For reviews on the topic, we refer to Aitchison and J. Egozcue (2005); Pawlowsky-Glahn and Buccianti (2011); Pawlowsky-Glahn and Egozcue (2016). These researchs show that compositional data have mainly focused on the deterministic case, i.e., not considering uncertainty in the elements of the composition. However, there exist several situations in the real world where the composition is not precisely defined and the distribution of the elements is varying in a certain range. As a consequence, the total quantity to which they must add up to is also uncertain. An example of such a situation is the computation of ATP per subtype as mentioned before.

There has been recent interest in the study of uncertainty in the elements of a composition from a probabilistic point of view, under the name of *joint mixability*. A detailed description of joint mixability can be found in the work of Puccetti and Wang (2015); Wang and Wang (2016) and the references therein. However, the main disadvantage of the probabilistic approach is that it normally uses long sets of historical data in order to identify suitable probability distributions. If there is not enough representative data available in order to characterize the uncertainty, then the probabilistic approach is not suggested as is highlighted by authors like Peidro et al. (2010) and Dubois et al. (2003).

Grillo et al. (2016a) makes an extensive literature review where the authors highligh the need of modelling ATP per subtype in the deterministic case (also called "crisp scene"), but they also remarke that the case when there is uncertainty in the subdivision per subtype has not been studied for perishables.

In order to analyse the uncertainty in the compositional data from a new point of view, we introduce possibility theory (Dubois and Prade 2012) as a suitable tool to model it. We introduce the special case when the elements of a composition are considered to be possibilistic variables. Up to the authors' knowledge the latter case has not been studied. In this work we will present the composition of possibilistic variables in a general way, so that it can be applied in a wide range of situations. We will exemplify its practical usefulness through the modelling of ATP under LHP conditions.

As we mentioned in previous paragraphs, the supply chain of perishables has another challenging feature: the subtypes will change some of its characteristics based on the ageing process (Haijema

2013; Entrup et al. 2005). In order to face this issue, we introduce the state function, which is a continuous function whose domain is a determined window of time, and the co-domain represents a characteristic of the product, which we will call *state*, such as quality or value (price). The state function represents the state of a variable at each time. When the state function is used to model the product's decay, it is called ageing function. Considering that the shelf life of a perishable is usually uncertain (there is just a vague idea of it), the ending time in the domain is represented with a possibilistic variable. Hence, the state itself is computed through a possibilistic variable.

The main contributions of this paper are twofold: to introduce the concepts of composition of possibilistic variables and state/ageing function, and to apply the new concepts in a practical example of OPP under LHP for perishables. We model the quantity of homogeneous product through a composition of possibilistic variables and we propose to link the product's price with an ageing function.

For both, compositions and state functions, we propose general procedures based on α-cuts in order to simplify the computations for linear mathematical modelling, a common tool used in applications due to its reasonable complexity and computational efficiency in the solution process.

Numerical experiments have been executed by applying the developed tools to a real case of a Spanish supply chain of the fruit sector, specifically the packing and distribution of oranges and tangerines. We have used a data set based on real information of orders given by the supply chain. The results obtained validated the correct model's functionality with a very good computational performance. We also presented some managerial insights in order to exemplify the usefulness that a tool like this has for decision makers.

The rest of the paper is organized as follows, Section 2 introduces and explains the concepts of composition of possibilistic variables. In Section 3 we describe the state function, meanwhile in Section 4 we describe how to apply both tools in the OPP process for perishables. In Section 5 the numerical experiments are presented. Finally Section 6 presents the main conclusions of this work.

2. Modelling compositions with possibilistic variables

In this section we will describe our main motivations for considering uncertainty in compositional data and our proposal to handle such uncertainty based on possibilistic variables.

2.1 *Imprecision in compositional data*

A wide variety of situations in mathematical modelling needs to take into account variables that are interdependent, for example, when certain values need to add up to another given value. Let us consider the data in Table 1, where the expected composition of a lot of oranges of 100kg coming from a field is described according to the subdivision of the product into its possible quality levels, 1^{st}, 2^{nd}, 3^{rd} and 4^{th}.

Table 1.: Composition of a lot of oranges

Quality	1^{st}	2^{nd}	3^{rd}	4^{th}	Total
(kg)	65	20	10	5	100
(%)	65%	20%	10%	5 %	100%

Clearly, the sum of the components should add up to the total quantity of the lot. Let **x** be a vector in $\mathbb{R}^n$ whose components represent the contribution to the lot of oranges (sublots). If c represents the total quantity of the lot, then we have $\sum_{i=1}^{n} x_i = c$.

With the information in Table 1, it is easy to see that $x_1 = 65$, $x_2 = 20$, $x_3 = 10$, $x_4 = 5$, and $c = 100$, hence, the vector **x** can be considered as a composition of c. The concept of compositional data was originally introduced by Aitchison (1982) and it has been oriented to exact data without considering imprecision. Now, let us consider the case where the values of Table 1 are not precisely known until the moment the analysis of its composition is made. Beforehand, there is just a vague idea regarding the values of the composition, for example *about* 65kg of the 1^{st} quality type, *about* 20kg of the 2^{nd} quality type, *about* 10kg of the 3^{rd} quality type and *about* 5kg of waste. All of them highlighted *"about"* since the exact quantities are not known, but they still have the restriction of adding up to the total quantity of 100kg. Even more complex, the case when they do not have the condition of adding up to 100kg and there is also just a vague idea that the lot will be *about* 100kg with the also vague composition mentioned. This would mean that both things involve imprecision, the values of the composition's components, and the total quantity they must add up to. In the following, we introduce the concepts to model imprecision in compositions, based on possibility distributions.

2.2 *Compositions*

Definition 1. *Let $n \in \mathbb{N}$ be such that $n > 1$. An n-part composition of $c \in \mathbb{R}^+$ is a vector $\mathbf{x} \in (\mathbb{R}^+)^n$ such that $\sum_{i=1}^{n} x_i = x$.*

Definition 2.

(i) A possibilistic variable X on a universe U is described by a possibility distribution π_X, i.e. a mapping $\pi_X : U \to [0,1]$ such that $(\exists u^* \in U)(\pi_X(u^*) = 1)$.

(ii) A possibilistic vector $(X_1, \ldots, X_n)$ on a product universe $U_1 \times \ldots U_n$ consists of n possibilistic variables X_i on U_i and is described by a joint possibility distribution $\pi_{X_1,\ldots,X_n}$, i.e. a mapping $\pi_{X_1,\ldots,X_n} : U_1 \times \ldots U_n \to [0,1]$ such that $(\exists(u_1^*, \ldots, u_n^*) \in U_1 \times \ldots U_n)(\pi_{X_1,\ldots,X_n}(u_1^*, \ldots, u_n^*) = 1)$. The marginal possibility distributions of the variables X_i are given by

$$\pi_{X_i}(u) = \sup\{\pi_{X_1,\ldots,X_n}(u_1, \ldots, u_n) \mid (u_1, \ldots, u_n) \in U_1 \times \ldots \times U_n \wedge u_i = u\}\,. \tag{1}$$

Note that $\pi_{X_i}(u_i^*) = 1$.

(iii) The components of a possibilistic vector are called non-interactive if the joint possibility distribution can be written as

$$\pi_{X_1,\ldots,X_n}(u_1, \ldots, u_n) = \min(\pi_{X_1}(u_1), \ldots, \pi_{X_n}(u_n))\,. \tag{2}$$

Note that in any case, it holds that

$$\pi_{X_1,\ldots,X_n}(u_1, \ldots, u_n) \leqslant \min(\pi_{X_1}(u_1), \ldots, \pi_{X_n}(u_n))\,. \tag{3}$$

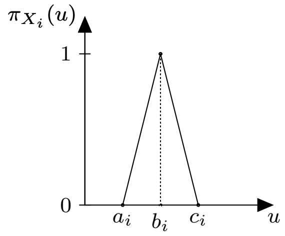
Possibilistic variables can be used to model epistemic uncertainty, the simplest case being that of interval uncertainty. In particular, if a possibilistic variable X on $\mathbb{R}$ is used to model the knowledge that X takes values in an interval $[a, b]$, then its possibility distribution is given by

$$\pi_X(u) = \begin{cases} 1 & \text{, if } u \in [a,b]\,, \\ 0 & \text{, if } u \notin [a,b]\,. \end{cases} \tag{4}$$

Note that π_X is nothing else but the characteristic mapping of the set $[a, b]$.

More refined and very popular possibility distributions are the so-called Triangular Fuzzy Interval (TFI), which allow us to incorporate a possibility gradient (left, central or right), that we denote as $X = (a, b, c)$ (see Figure 1). Note that, for the case of a normal TFI $\pi_X(b) = 1$. The following example gives a basic overview of the application of triangular possibility distributions to the components of the lot in Table 1.

Example 1. *If the composition's components described in Table 1 are considered fuzzy intervals, we have* $X_1 \approx 65$*,* $X_2 \approx 20$*,* $X_3 \approx 10$ *and* $X_4 \approx 5$*.*



$$\pi_{X_i}(u) = \begin{cases} 0 & , \text{ if } u \leqslant a_i, \\ \frac{u-a_i}{b_i-a_i} & , \text{ if } a_i < u \leqslant b_i, \\ \frac{c_i-u}{c_i-b_i} & , \text{ if } b_i < u < c_i, \\ 0 & , \text{ if } u \geqslant c_i. \end{cases}$$

Figure 1.: Triangular fuzzy interval

For each fuzzy interval we consider $X_i = (0.75b_i,\ b_i,\ 1.25b_i)$*, where* b_i *represents the central value. In this case,* $b_1 = 65$*,* $b_2 = 20$*,* $b_3 = 10$ *and* $b_4 = 5$*. Hence, the first TFI is* $X_1 = (48.75, 65, 81.25)$*, the second one is* $X_2 = (15, 20, 25)$*, and so on. In this way, the values* $(60, 23, 9.5, 6)$ *are a specific tuple for the composition and by computing their respective possibility distribution, we obtain* $\pi_{X_i}(u) = (0.69,\ 0.40,\ 0.80,\ 0.20)$ *which can be interpreted as the possibility degree that each value in the tuple belongs to its respective fuzzy interval.*

Computing with possibilistic variables is facilitated by Zadeh's extension principle (Zadeh 1978). Consider a function $f : U_1 \times \ldots \times U_n \to V$ and a possibilistic vector $(X_1, \ldots, X_n)$ on $U_1 \times \ldots \times U_n$, then $Y = f(U_1, \ldots, U_n)$ is the possibilistic variable on V with possibility distribution π_Y defined by

$$\pi_Y(v) = \sup\{\pi_{X_1,\ldots,X_n}(u_1, \ldots, u_n) \mid f(u_1, \ldots, u_n) = v\}\,. \tag{5}$$

Consider a possibilistic vector $(X_1, \ldots, X_n)$ on $\mathbb{R}^n$, then its sum $\sum_{i=1}^n X_i$ is the possibilistic variable Y with possibility distribution

$$\pi_Y(v) = \sup\{\pi_{X_1,\ldots,X_n}(u_1, \ldots, u_n) \mid \sum_{i=1}^{n} u_i = v\}\,. \tag{6}$$

When treating the possibilistic variables as non-interactive, i.e. considering only their marginal possibility distributions, then the corresponding sum will be denoted by $Z = \bigoplus_{i=1}^{n} X_i$, with possibility distribution

$$\pi_Z(v) = \sup\{\min(\pi_{X_1}(u_1), \ldots, \pi_{X_n}(u_n)) \mid \sum_{i=1}^{n} u_i = v\}\,. \tag{7}$$

Based on the concepts above, let us now introduce the compositions built on possibilistic variables.

Definition 3. *Let* $n \in \mathbb{N}$ *be such that* $n > 1$*. A possibilistic vector* $(X_1, \ldots, X_n)$ *on* $(\mathbb{R}^+)^n$ *is called an* n*-part composition of a possibilistic variable* C *on* $\mathbb{R}^+$ *if* $\sum_{i=1}^{n} X_i = C$*.*

In the following discussion, we will develop a practical procedure to construct such n-part compositions. We start from the following situation. Suppose we have an imprecise description of the quantities X_i, $i = 1, \ldots, n$, of certain products expressed in terms of possibility distributions π_{X_i} and an imprecise description of their grand total C expressed in terms of a possibility distribution π_C. An important assumption is

$$C \subseteq \bigoplus_{i=1}^{n} X_i \,, \tag{8}$$

stating that C is included in the sum of the quantities X_i, $i = 1, \ldots, n$, when treated as being non-interactive. This can be seen as a kind of coherence condition, expressing that C is indeed realizable.

The aim now is to define a joint possibility distribution $\pi^*_{X_1,\ldots,X_n}$ of the possibilistic vector $(X_1, X_2, \ldots, X_n)$ satisfying the following conditions:

(i) it holds that $\sum_{i=1}^{n} X_i = C$ (using $\pi^*_{X_1,\ldots,X_n}$), i.e. the possibilistic vector $(X_1, \ldots, X_n)$ is an n-part composition of C;

(ii) $\pi^*_{X_1,\ldots,X_n}$ is the least specific (i.e. the largest) joint possibility distribution realizing (i).

Theorem 1. *The joint possibility distribution of the possibilistic vector $(X_1, \ldots, X_n)$ defined by*

$$\pi^*_{X_1,\ldots,X_n}(u_1, \ldots, u_n) = \min(\pi_C(\sum_{i=1}^{n} u_i), \min_{i=1}^{n} \pi_{X_i}(u_i)) \tag{9}$$

is the largest distribution such that $\sum_{i=1}^{n} X_i = C$. The proof can be found in Appendix A.

Remark 1. *Obviously, it holds that the marginal distribution $\pi^*_{X_i}$ of X_i obtained from $\pi^*_{X_1,\ldots,X_n}$ satisfies $\pi^*_{X_i} \leqslant \pi_{X_i}$.*

Example 2. *Let $n = 2$.*

(i) Consider X_1 and X_2 with triangular distributions $\pi_{X_1} = \pi_{X_2} = \langle 0, 5, 10 \rangle$. Clearly, $X_1 \oplus X_2$ has the triangular distribution $\langle 0, 10, 20 \rangle$ as possibility distribution. Suppose that the grand total C is described by the triangular distribution $\pi_C = \langle 8, 10, 12 \rangle$. Consider the possibility distribution

$$\pi^*_{X_1,X_2}(u_1, u_2) = \min(\pi_C(u_1 + u_2), \min(\pi_{X_1}(u_1), \pi_{X_2}(u_2))) \,,$$

then the marginal distributions of X_1 and X_2 are given by $\pi^*_{X_1} = \pi_{X_1}$ and $\pi^*_{X_2} = \pi_{X_2}$. This joint possibility distribution is shown in Figure 2.

(ii) Consider X_1 and X_2 with triangular distribution $\pi_{X_1} = \pi_{X_2} = \langle 0, 8, 10 \rangle$. Clearly, $X_1 \oplus X_2$ has the triangular distribution $\langle 0, 16, 20 \rangle$ as possibility distribution. Suppose that the grand total C is described by the triangular distribution $\pi_C = \langle 14, 16, 18 \rangle$. Consider the possibility distribution

$$\pi^*_{X_1,X_2}(u_1, u_2) = \min(\pi_C(u_1 + u_2), \min(\pi_{X_1}(u_1), \pi_{X_2}(u_2))) \,,$$

then the marginal distributions of X_1 and X_2 are given by $\pi^*_{X_1} = \pi^*_{X_2} = \langle 4, 8, 10 \rangle$. This joint possibility distribution is shown in Figure 3.

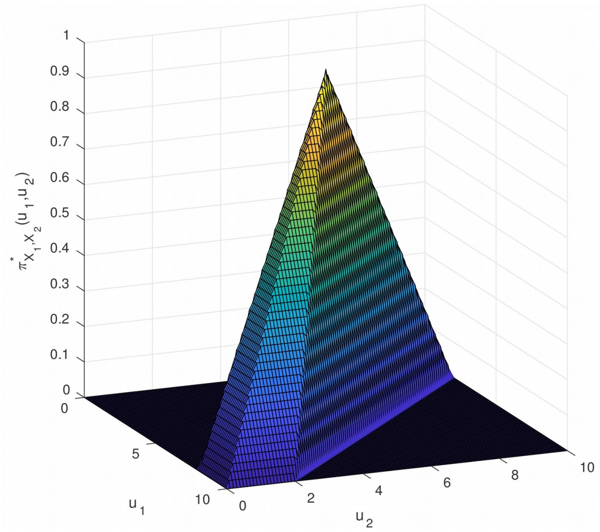


Figure 2.: Joint possibility distribution: case $\pi_{X_1} = \pi_{X_2} = \langle 0,\ 5,\ 10\rangle$.

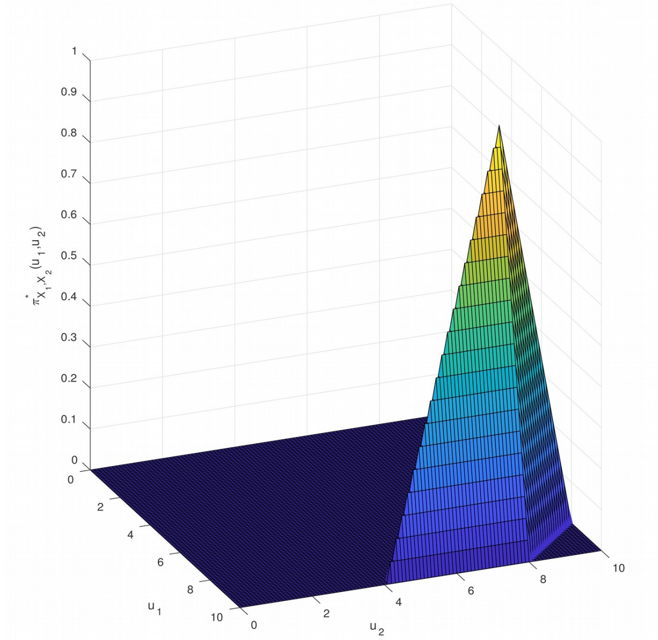


Figure 3.: Joint possibility distribution: case $\pi_{X_1} = \pi_{X_2} = \langle 0,\ 8,\ 10\rangle$.

2.3 *Using compositions in practice*

Suppose we have an imprecise description of the quantities X_i, $i = 1, \ldots, n$, of certain products expressed in terms of possibility distributions π_{X_i} and an imprecise description of their grand total C expressed in terms of a possibility distribution π_C such that $C \subseteq \bigoplus_{i=1}^{n} X_i$. We construct the joint possibility distribution $\pi^*_{X_1,\ldots,X_n}$ as explained in the previous section:

$$\pi^*_{X_1,\ldots,X_n}(u_1, \ldots, u_n) = \min(\pi_C(\sum_{i=1}^{n} u_i), \min_{i=1}^{n} \pi_{X_i}(u_i))\,. \tag{10}$$

In the context of optimization, one usually adopts an α-cut approach. For a given $\alpha \in]0,1]$, we choose values u_i of X_i such that $\pi^*_{X_1,\ldots,X_n}(u_1,\ldots,u_n) \geqslant \alpha$ and $\pi_C(z) \geqslant \alpha$ with $z = \sum_{i=1}^n u_i$, in view of the composition constraint. It is immediate that $\pi^*_{X_1,\ldots,X_n}(u_1,\ldots,u_n) \geqslant \alpha$ if and only $\pi_C(z) \geqslant \alpha$ and $\pi_{X_i}(u_i) \geqslant \alpha$, $i = 1,\ldots,n$.

Hence, we obtain the following extremely simple procedure. For a given $\alpha \in]0,1]$, select u_i, $i = 1,\ldots,n$, such that $\pi_{X_i}(u_i) \geqslant \alpha$ (i.e., select u_i from the α-cut of the original possibility distribution of X_i). Compute the sum $z = \sum_{i=1}^n u_i$; if $\pi_C(z) < \alpha$, then reject the vector of values $(u_1,\ldots,u_n)$, otherwise proceed. There is no need to compute the joint possibility distribution $\pi^*_{X_1,\ldots,X_n}$ explicitly.

3. Modelling state functions

Let us consider the lot of oranges as described in Example 1. In this case, each composition's component represents a sub-lot of oranges depending on the quality conditions. In the previous section, we explained how to model such composition using possibilistic variables. Now, what happens if the quality conditions are not static and could change with time, as it is normal for perishable products? Or even more, what happens if another characteristic like the orange's price is linked with the ageing process? We now aim to analyse how a situation like this can be modelled.

3.1 *State functions*

The state of a product with a finite life span $[t_i, t_e]$ is modelled by a state function $h : [t_i, t_e] \to \mathbb{R}$ representing the state $h(t)$ of the product at time t. State functions can, for instance, be used to model the decaying quality of a perishable product over time; such state functions will be called ageing functions. An ageing function then is a decreasing function $h : [t_i, t_e] \to [s_i, s_e]$ with $h(t_i) = s_i$ and $h(t_e) = s_e$.

In particular, for a perishable product, the ending time t_e is usually not precisely known and one has only a vague idea about it. Similarly as before, such an imprecise ending time will be modelled as a possibilistic variable T_e with possibility distribution π_{T_e} satisfying $\pi_{T_e}(t_e) = 1$. We suppose that the (closure of the) support of T_e (i.e. the closure of the set where π_{T_e} is not zero) is the interval $[\ell_e, u_e]$, with a length that is relatively small compared to that of the life span $[t_i, t_e]$. The latter assumption implies that if the real ending time would be a value t^* in $[\ell_e, u_e]$, then the given ageing function could be adapted by linear rescaling to obtain an ageing function $h^{(t^*)}$ for a life span $[t_i, t^*]$. For any $t \in [t_i, t^*]$, it then holds that

$$h^{(t^*)}(t) = h\left(t_i + \frac{t - t_i}{t^* - t_i}(t_e - t_i)\right). \tag{11}$$

Obviously, it holds that $h^{(t^*)}(t_i) = s_i$ and $h^{(t^*)}(t^*) = s_e$.

In case the ending time t_e is described by a possibilistic variable T_e, it is obvious that the state time t is not known precisely either and is described by a possibilistic variable S_t with possibility distribution π_{S_t} on $[t_i, u_e]$ defined by

$$\pi_{S_t}(s) = \sup\{\pi_{T_e}(t^*) \mid h^{(t^*)}(t) = s\}. \tag{12}$$

In case no t^* exists such that $h^{(t^*)}(t) = s$, then it obviously holds that $\pi_{S_t}(s) = 0$. The underlying principle is again Zadeh's extension principle, this time not used to extend a function allowing for fuzzy inputs, but to extend a function allowing for a fuzzy parameter (the ending time). In words, the Eq.(12) states that the degree of possibility of s being the state at time t is determined by the most possible ending time t^* for which it holds that the state at time t, according to the ageing function corresponding to t^*, equals s.

In case the ageing function is strictly decreasing, there exists at most one t^* and Eq.(12) reduces to $\pi_{S_t}(s) = \pi_{T_e}(t^*)$ if $h^{(t^*)}(t) = s$, and $\pi_{S_t}(s) = 0$ if no such t^* exists. In case the ageing function is constant on some subinterval, then this uniqueness is not guaranteed.

Example 3. *Let us illustrate the above procedure on a simple example. Let the ageing function be linearly decreasing and π_{T_e} be a TFI with parameters (l_e, t_e, u_e). Then π_{S_t} is a TFI as well with parameters (a_t, b_t, c_t) given by $a_t = h^{(\ell_e)}(t)$, $b_t = h^{(t_e)}(t) = h(t)$ and $c_t = h^{(u_e)}(t)$.*

Obviously, the α-cuts of π_{S_t} can be easily found. Figure 4 graphically illustrates this procedure.

Example 4. *As a second example, we consider an ageing function that remains constant on part of its domain, and illustrate the impact of an imprecise ending time. Consider the piecewise linear ageing function $h : [0, 10] \to [0, 2]$ with $h(0) = 2$, $h(2) = 1$, $h(8) = 1$ and $h(10) = 0$.*

Let π_{T_e} be the TFI with parameters $(8, 10, 12)$. We give some examples of S_t for different values of t:

(i) the state at $t = 1.6$ is the TFI with parameters $(1, 1.2, 1.33)$;
(ii) the state at $t = 2$ is the TFI with parameters $(1, 1, 1.17)$;
(iii) the state at any time $t \in [2.4, 6.4]$ is the crisp value 1 (TFI with parameters $(1, 1, 1)$);
(iv) the state at time $t = 10$ is the TFI with parameters $(0, 0, 0.83)$.

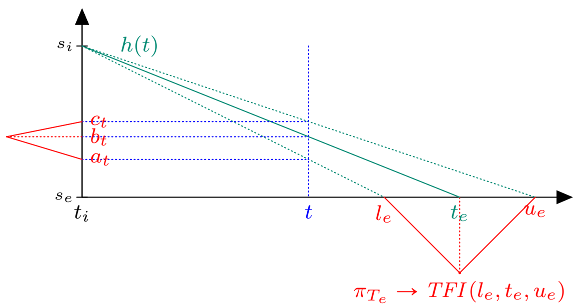


Figure 4.: Example of linear ageing function.

4. Application to the order promising process for perishables

In this section the application of the previous concepts of compositions and state functions to the OPP modelling for perishables, such as fresh fruit and vegetables, is presented.

4.1 *Problem description*

The problem consists on deciding the acceptance/rejection of the customer order proposals and the assignation of the ATP per subtype while maximizing profits. The model presented here is a synthesized version of the one presented in Grillo et al. (2017a). In the previous paper, the authors developed a complete model for the OPP for fruit supply chains. They considered the LHP effect when the lots are classified and separated into several subtypes, which are defined by characteristics of the product like size, colour, quality, calibre, etc. They also presented an extended multi-objective crisp MILP model with which they modelled the LHP's effect in the ATP quantities for fruit supply chains. Furthermore, they expressed the price of the product as a function of its remaining shelf life at the moment of delivery to the customers. The price values were considered to be decreasing over time, due to the shelf life consumption. Both topics, the LHP's effect and the pricing policy related to freshness, were considered deterministic. The model included other novel aspects, such as allowing for partial orders, i.e., orders of which not all of its lines (each line in an order asks for a specific quantity of a subtype) could be served as requested. In such case, they assumed that the order could be partially served with a penalization cost, and that the lines that could not be served can be renegotiated. They also considered transportation, holding, and expired product costs. Another considered aspect was the possibility of directly sending the remaining product to produce by-products in the case that such product does not belong to any of the subtypes allowed to fulfill orders.

As previously mentioned, the main objective of this section is to exemplify how we can implement compositions of possibilistic variables in a real problem of a supply chain. Specifically, we model the LHP's effect by using the compositions. We also exemplify how to use state functions in the specific case where they can be applied jointly with compositions. The state/ageing functions replace the original deterministic pricing policy based on the ranges of freshness and the associated price. The main novelty is that we consider the uncertainty propagated to the price from the original uncertainty in the product's lifetime.

In order to do this, we simplify the model of Grillo et al. (2017a), to focus on the implementation of the already mentioned tools. To perform such simplification, a set of considerations and assumptions must be taken into account based on the original model.

The following assumptions are made about the order proposals of the customer:

(1) Each order is composed by one or several lines. Each line specifies the subtype and the quantity requested of it.
(2) The order's due date (asked date to deliver the order) is assumed to be the same for all its lines.
(3) Customers require homogeneous units of the same subtype (each line must be exactly fulfilled with the asked subtype). It is also not possible to accumulate different subtypes in order to fulfil future orders.
(4) Customers need to be served on time. Any delay is allowed for the delivery of the product.
(5) The orders are promised in batching mode (Grillo et al. 2017a), i.e., they are accumulated for a given window of time known as batching interval. The orders are promised at the beginning of each batching interval with the expected quantities of ATP.

The following assumptions are made about the product:

(1) Units of the same subtype have the same characteristics. Different subtypes of the same product can differ in attributes such as quality, color, size and weight, among others.
(2) The total MPS per product and time period are uncertain in supply chains of perishables due to multiple uncontrollable factors, such as weather, temperature, humidity, etc, that impact in the maturation process.
(3) Planned lots in the MPS are assumed to be composed by different sublots of homogeneous subtypes as an effect of the LHP.
(4) The exact amount of each subtype in the MPS is also uncertain until production and classification activities have been performed. Beforehand, it is only possible to have an estimation.
(5) The ATP per subtype is calculated based on the previous division of the MPS in homogeneous sublots of the same subtype, and subtracting the orders already committed (see details of this definition in Ball et al. (2004)).
(6) The product is perishable. It deteriorates over time and becomes waste when it expires.
(7) For perishables, the product's shelf life can be used as another homogeneity criteria, because customers normally require minimum levels of freshness (which is not necessarily of the same quality).
(8) Since deterioration depends not only on the harvesting time but also on additional factors, the maximum product's shelf life involves uncertainty. If another product's attribute is linked

with ageing process, e.g., the selling price, it will also inherit such uncertainty.

From the characteristics mentioned before, in Subsection 4.2 we will explain how to model homogeneous ATP through the concepts of compositions with possibilistic variables and an ageing-dependent pricing policy for products with ageing functions.

4.2 *Notations*

The notation follows the convention: i) indices are represented as single italic letters; ii) sets are represented with single italic capital letters with the indices they refer to; iii) input data (given parameters and parameters computed from other inputs) are represented as single boldface letters with their respective indices; iv) the model's decision variables are represented as single boldface, non-italic capital letters; and finally, v) the computed variables (calculated from the model's decision variables through equations) are represented as single-capital letters.

Indices

i	Products, $i \in \{1, \cdots, I\}$.
h	Harvesting time, $h \in \{-SL, \cdots, T-1\}$.
s	Subtypes, $s \in \{1, \cdots, S\}$.
t	Time buckets, $t \in \{0, \cdots, T\}$.
o	Customer order proposals, $o \in \{1, \cdots, O\}$.

Sets

O_{is}	Set of subtypes s of product i requested in the customer order proposal o in the current model's execution.
I_s	Set of subtypes s in which each product i can be classified.

Input data

$\mathbf{d}^o$	Due date of the order proposal o.
$\mathbf{c}_{is}^{ht}$	Total committed quantity from previous execution of subtype s of product i, in period t, and harvested in h.
$\mathbf{f}_{is}^o$	Fending unitary cost if $\mathbf{r}_{is}^o$ is rejected.
$\mathbf{h}_{is}$	Holding unitary cost of subtype s of product i per time period.
$\mathbf{k}_{is}^h$	Initial stock of subtype s of product i harvested in h, available at the beginning of each execution.
$\mathbf{n}^o$	Total number of lines in each order o.
$\mathbf{r}_{is}^o$	Requested quantity of the subtype s of product i, in order o.
$\boldsymbol{\rho}^o$	Unitary transport cost of order o.
$\mathbf{t}^o$	Transporting time to the delivery place of order o.
$\mathbf{w}_{is}$	Waste unitary cost per unit of the subtype s of product i.
$\boldsymbol{\tau}$	Length of the batching interval in time periods.

Computed intervals

A_{is}^{ht}	The homogeneous Available-To-Promise of subtype s of product i, harvested in h and available in t, after classification activities and after taking into account the committed quantities once the model has been executed.
G_{is}^{ht}	Quantity of A_{is}^{ht} becoming waste (garbage) due to expiration.

Possibilistic variables

$\mathbf{E}_{is}^{o}$	Possibilistic variable representing the price (earning) assigned to the subtype s of product i in order o, considering its harvesting time and the due date $\mathbf{d}^o$ when is asked to be delivered. It is computed from an ageing function whose maximum time (representing the product's shelf life) is described by a possibilistic variable.
$\mathbf{L}_{is}$	Possibilistic variable representing the maximum lifetime (shelf-life) for subtype s of product i.
$\mathbf{M}_{is}^{ht}$	Possibilistic variable representing the MPS per subtype s of product i, harvested in period h and available in t.
$\mathbf{P}_{i}^{ht}$	Possibilistic variable representing the MPS of product i, harvested in period h and available t.

Decision variables

$\mathbf{Y}_{is}^{oht}$	Binary variable that takes the value of 1 if the requested quantity of subtype s of product i in order o is completely served by the corresponding A_{is}^{ht} and the value of 0 otherwise.
$\mathbf{U}^{o}$	Binary variable that takes the value of 1 if all subtypes s in order o are finally fulfilled and consequently the order is promised.

4.3 *Mathematical modelling*

Objective function: Profits generated as the difference between the incomes of the promised orders and the total cost of inventory holding, rejecting orders, wasted product due to expiration and transport cost. Eq. (13) computes the objective.

$$
\begin{aligned}
&\textit{SI:} && \sum_{(o,i,s)\in O_{is}} \mathbf{r}_{is}^{o}\mathbf{E}_{is}^{o} && \rightarrow \text{Selling income.}\\
&\textit{HCO:} && \sum_{(o,i,s)\in O_{is}} \mathbf{h}_{is}\mathbf{r}_{is}^{o}(\mathbf{d}^{o}-\mathbf{t}^{o}-\sum_{h,t} t\mathbf{Y}_{is}^{oht}) && \rightarrow \text{Holding cost of committed orders.}\\
&\textit{HCA:} && \sum_{(i,s)\in I_s}\sum_{h,t\leqslant\tau} (\boldsymbol{\tau}-t)\mathbf{h}_{is}A_{is}^{ht} && \rightarrow \text{Holding cost of the remaining ATP.}\\
&\textit{RC:} && \sum_{(o,i,s)\in O_{is}}\sum_{h,t} \mathbf{f}_{is}^{o}(1-\mathbf{Y}_{is}^{oht}) && \rightarrow \text{Rejected order's cost.}\\
&\textit{WC:} && \sum_{(i,s)\in I_s}\sum_{h,t} \mathbf{w}_{is}\mathrm{G}_{is}^{ht} && \rightarrow \text{Wasting cost due to expiration.}\\
&\textit{TC:} && \sum_{(o,i,s)\in O_{is}}\sum_{h,t} \boldsymbol{\rho}^{o}\mathbf{r}_{is}^{o}\mathbf{Y}_{is}^{oht} && \rightarrow \text{Transporting cost.}
\end{aligned}
$$

$$\textit{Maximize Profits: } Z = \textit{SI} - \textit{HCO} - \textit{HCA} - \textit{RC} - \textit{WC} - \textit{TC}. \tag{13}$$

Crisp constraints: The set of constraints in Eq. (14) guarantees that an order will be served only if all its lines are served. On the other hand, the order will not be served and any of its order lines will be reserved.

$$\sum_{(i,s)\in I_s} \sum_{h,t} \mathbf{Y}_{is}^{oht} = \mathbf{n}^o \mathbf{U}^o \qquad \forall o \in O_{is}. \tag{14}$$

The set of constraints in Eq. (15) guarantees that each subtype requested in each order line can only be fulfilled with ATP harvested in the same h and reserved in only one t. It is not possible to accumulate subtypes to fulfil order lines.

$$\sum_{h,t} \mathbf{Y}_{is}^{oht} \leqslant 1 \qquad \forall (o, i, s) \in O_{is}. \tag{15}$$

The set of constraints in Eq. (16) states that no tardiness is allowed for any promised order. The reservation time must take place before the order's due date less the required transporting time.

$$\sum_{h,t} t\mathbf{Y}_{is}^{oht} \leqslant \mathbf{d}^o - \mathbf{t}^o \qquad \forall (o, i, s) \in O_{is}. \tag{16}$$

Possibilistic constraints: The set of constraints in Eq. (17) represents the balance between the total MPS and the production schedule per subtype. This constraint is modelled through a composition of possibilistic variables.

$$\mathbf{P}_i^{ht} = \sum_{s\in I_s} \mathbf{M}_{is}^{ht} \qquad \forall i \in I_s,\, h,\, t. \tag{17}$$

The set of constraints in Eq. (18) establishes that the A_{is}^{ht} can not be negative. This ensures that promised quantities cannot be higher than the existing MPS.

$$\mathrm{A}_{is}^{ht} \geqslant 0 \qquad \forall (i, s) \in I_s, h, t. \tag{18}$$

Where: The Available To Promise A_{is}^{ht} is calculated in Eq. (19). By definition (Alemany et al. 2015), the ATP is obtained as the difference between the MPS in each period of time, less the quantities already promised in previous executions and the quantities promised during the current execution.

$$A_{is}^{ht} = \begin{cases} \mathbf{k}_{is}^{h} - \mathbf{c}_{is}^{ht} - \sum\limits_{o\in O_{is}} r_{is}^{o} \mathbf{Y}_{is}^{oht} & , \ \forall (i, s) \in I_s, h, t = 0, \\ \mathbf{M}_{is}^{ht} - \mathbf{c}_{is}^{ht} - \sum\limits_{o\in O_{is}} r_{is}^{o} \mathbf{Y}_{is}^{oht} & , \ \forall (i, s) \in I_s, h, t > 0. \end{cases} \tag{19}$$

Furthermore, Eq. (20) calculates the part of the ATP becoming waste in each period, because of ageing effect. It is important to remark that the latter quantity depends on the condition of $t - h$

to be higher than the maximum shelf life, which would imply the expiring.

$$G_{is}^{ht} = \begin{cases} A_{is}^{ht} & , \quad if\ t-h > \mathbf{L}_{is}, \qquad \forall (i,s) \in I_s, h, t, \\ 0 & , \quad otherwise. \end{cases} \tag{20}$$

4.4 *Equivalent MILP model*

In this subsection we describe the required computations on possibilistic variables and computed intervals of the model, in order to represent it with an equivalent MILP model. We consider all the possibilistic variables as TFI; for example $\mathbf{P}_i^{ht} = \left(p_i^{1ht}, p_i^{2ht}\, p_i^{3ht}\right)$. A possible value in such TFI is represented as a single italic letter with the respective indices, for example, p_i^{ht} is a possible value of $\mathbf{P}_i^{ht}$. This same logic also applies for all the computed intervals.

4.4.1 Interval representation based on an α-cut

First, we exemplify how the possibilistic approach works when compared to the crisp one. We analyse the example of Grillo et al. (2016b). They stated that a determined perishable product can be subdivided into different subtypes (that could be based on quality, size, colour, etc). If the composition of such lot is not certainly known until the classification activities have been performed, the planification of the orders to be accepted can only be based on an approximation. They used a parameter representing the proportion to approximate the subdivision. For example they consider a lot of 350kg that is subdivided in the following way: 50% of first subtype, 30% of the second one and 20% of the last one. Then, the final quantities are 175kg (350*0.5), 105kg (350*0.3) and 70kg (350*0.2); the parameter of proportion for each subtype is 0.5, 0.3 and 0.2 respectively. Figure 5 illustrates the general idea of the LHP: the subdivision of lots into subtypes.

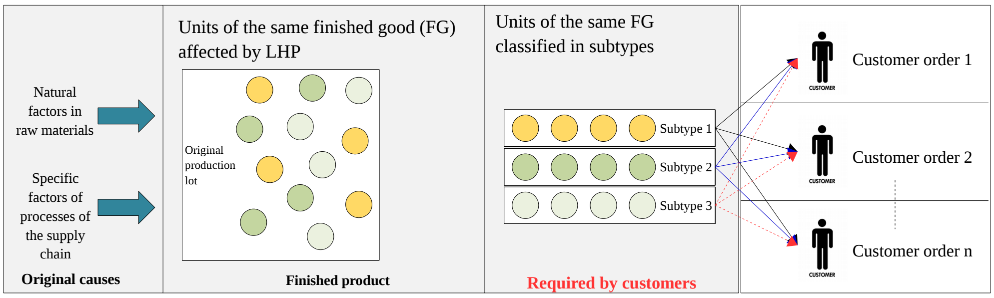


Figure 5.: LHP working logic. Adapted from Grillo et al. (2016a)

In the cases when customers do not care about a specific subtype, and only require an homogeneous product, an order of 150kg could only be served with product of subtype 1; an order of 100kg could be served with subtypes 2 and 3, and an order of less than 70kg could be served with any of the subtypes in the example. On the other hand, if the customer specifies the subtype, an order of 200kg of subtype 1 could not be served. The risk of shortage (Grillo et al. 2015) is high if the resulting quantity of each subtype is not enough to fulfill all the customer orders. This example decribes how the crisp case of the model works. For more details on how to handle this problem using a crisp approach, readers are referred to Grillo et al. (2017a).

It can be noted that the values 175, 105 and 70 add up to 350, then the vector $(175, 105, 70)$ is a composition of 350. In our case, we consider the values of the vector to be possibilistic variables given the inherent epistemic uncertainty as a consequence of the LHP. We use possibilistic variables because there is not much available information prior to the classification activities of the final quantities of the subtypes.

Now, for simplicity, we only analyse the first subtype for which the crisp case has the value of 175kg. We consider the quantity of the first subtype to be represented by a possibilistic variable described by the TFI (150, 175, 200). In this case a possible value is $u = 160$; this value evaluated in the possibility distribution of the TFI (as explained in Example 1) gives a possibility degree of 0.4. Then, we say that the possibiltity degree of having 160kg of subtype 1 in the lot is 0.4. It can be noted how the value of 190 would also have the same possibility degree of 0.4 according to the formula given in Example 1. Figure 6-(A) illustrates this idea.

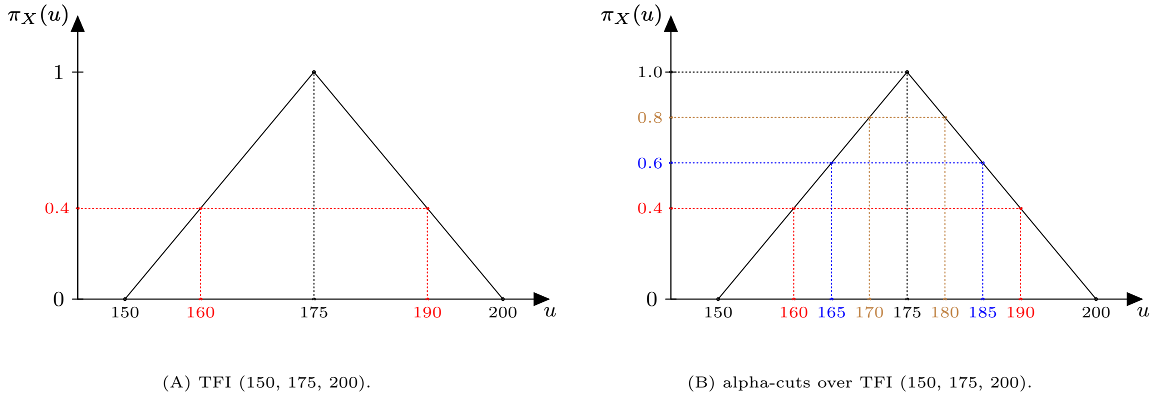


Figure 6.: The possibilistic variables and α-cuts

The idea illustrated in Figures 6-(A) and 6-(B) is a key element of the representation theorem (Zeng et al. 2006): if we consider an α-cut in a possibility distribution, we obtain an interval of values in its support that ensures a level of the possibility degree (the α-cut) as a minimum value. In the example given in Figure 6-(A), if $\alpha = 0.4$ for the possibility distribution of the TFI (150, 175, 200), we can ensure that $\pi_X(u) \geqslant 0.4 \ \forall u \in [160, 190]$. The α-cut approach allows us to compute the equivalent interval representations of the possibilistic variables of the model described in Subsection 4.3.

It is important to highlight that the greater the value of α is, the higher the possibility degree. Consequently, the amplitude of the resulting interval of such α in the support of the possibility distribution will be smaller. In the previous example, if $\alpha = 0.6$, the resulting interval will be [165, 185], if $\alpha = 0.8$ the interval will be [170, 180] and so on. Finally, if $\alpha = 1$, we obtain the case when the α-cut considers the highest possibility degree in the possibility distribution. In this case we recover the original crisp counterpart, since the resulting interval is [175, 175] which only contains one possible value. The latter statement is shown in Figure 6-(B).

What we need to do to obtain equivalent interval representations and executing MILP is to apply α-cuts as described in the previous paragraphs while ensuring that the rules explained in Subsection 2.3 hold. With the aim of obtaining a suitable formula for the resulting intervals from possibilistic variables, we consider the same example of TFI (a, b, c) and the possibiliy distribution given in Example 1. Clearly $\pi_X(u) \geqslant \alpha$ if and only if

$$u \in [a + \alpha(b - a), c - \alpha(c - b)], \tag{21}$$

The calculations to obtain Eq. (21) also work in a similar way for other type of possibility distri-butions, e.g., trapezoidal.

Finally we need to extrapolate this same idea to an interval implementation for the state functions. To do so, we base our explanation on Figure 4 used in Examples 3 and 4 of Subsection 3.1. We now include in Figure 7 the illustration of the α-cut for a linear state function.

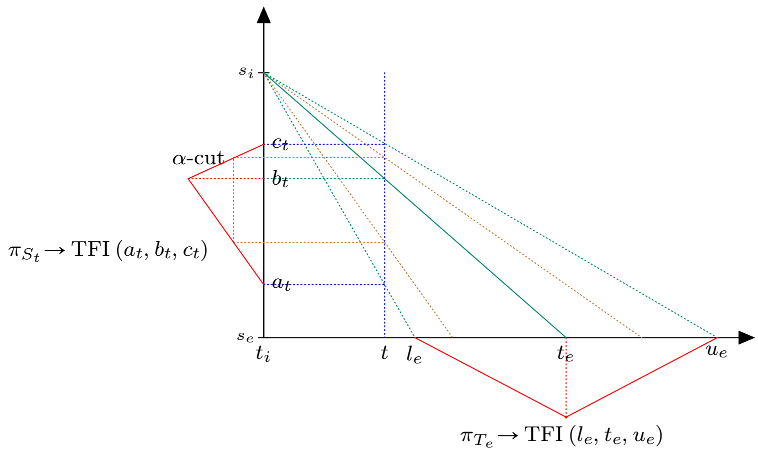


Figure 7.: Possibilistic variables and and α-cut for state functions

In Figure 7, we consider that the price of a product (green linear ageing function) is linked to its imprecise lifetime. The product's lifetime is modelled by using a possibilistic variable T_e whose possibility distribution π_{T_e} is described by the TFI (l_e, t_e, u_e). As explained in Subsection 3.1, the rescaling of the ageing function with ending time t_e to the points l_e and u_e (green dotted linear functions) allows us to project another possibilistic variable π_{S_t} (see y-axis in Figure 7). An α-cut approach can easily be used for the possibility distributions π_{T_e} and π_{S_t} by applying the same idea described in Eq. (21). The resulting interval for one α value in the support of the respective possibility distribution can be projected to the other possibility distribution as shown in the brown dotted lines in Figure 7. In our example, we consider an α-cut for the possibility distribution of the price, i.e., π_{S_t}. Considering a low value of α (near to 0) means that we consider a wider interval in the support of π_{S_t} (more possible values for the price). It also means that we consider a wider interval in the support of the possibility distribution modelling the lifetime. For example, if the maximum lifetime is the TFI (8, 12, 14), making the interval of the final price narrower (through a greater α value) means that we take into account less possible values of the lifetime, for example [10, 13]. This also means that for $\alpha = 1$ we recover the crisp counterpart, because the lifetime will be exactly 12 and only one price is possible for such case.

In the following subsections we will describe the required computations in order to obtain interval representations for the possibilistic price and compositions in the OPP model, using the approach explained in the previous paragraphs.

4.4.2 Computations with the possibilistic product price

We use a piecewise linear ageing function $f(t)$ describing the price state in the time. It is based on the information of the pricing policy given in Grillo et al. (2017a) where deterministic *price ranges* were applied. Based on the information given in the latter study, we consider a decreasing linear change from one price to another. Hence, the resulting curve has a piecewise linear form. It is important to highlight that other types of functions (all the ones satisfying the conditions explained in Subsection 3.1) can also be applied in a similar way. The form of the ageing function depends on the behaviour of the variable described by it. In our specific case, the piecewise linear function was a suitable option. The maximum time value of such ageing function is described by $\mathrm{L}_{is} = \left(l_{is}^1, l_{is}^2, l_{is}^3\right)$. It considers 4 ranges of price, as shown in Figure 8, with a linear decreasing transition time between consecutive ranges. This function is used for each subtype s of product i. The number of constant intervals of the function and the parameters of L_{is} should be given, according to the normal behaviour of the product and the decision maker's need.

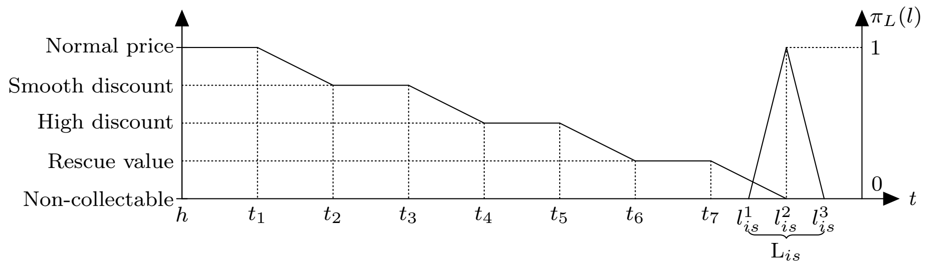


Figure 8.: Price-ageing function

The price's state over time is represented by $\mathbf{E}_{is}^o = \left(e_{is}^{1o}, e_{is}^{2o}, e_{is}^{3o}\right)$ where $e_{is}^{1o} = f^{(l_{is}^1)}(t)$, $e_{is}^{2o} = f^{(l_{is}^2)}(t)$ and $e_{is}^{3o} = f^{(l_{is}^3)}(t)$.

Now we can easily compute a conventional interval for the product's price by applying an α-cut on $\mathbf{E}_{is}^o$, as exemplified in Subsection 4.4.1. We obtain $(\mathbf{E}_{is}^o)_\alpha = [e_{is}^{o\,(\mathrm{a})}, e_{is}^{o\,(\mathrm{b})}]$, where:

$$e_{is}^{o\,(\mathrm{a})} = e_{is}^{1o} + \alpha(e_{is}^{2o} - e_{is}^{1o}),\ e_{is}^{o\,(\mathrm{b})} = e_{is}^{3o} - \alpha(e_{is}^{3o} - e_{is}^{2o}). \tag{22}$$

Note that in order to compute the price of each subtype requested in $\mathbf{r}_{is}^o$, it is required to evaluate $e_{is}^{o\,(\mathrm{a})}$ and $e_{is}^{o\,(\mathrm{b})}$ in $t = \mathbf{d}^o \sum_{h,t} \mathbf{Y}_{is}^{oht}$ if it is promised to be delivered in its due date $\mathbf{d}^o$.

Now, the model's objective function becomes a matter of limits, lower and upper profit as if each subtype in each order earns the minimum and maximum values of the interval of shelf life-based price respectively.

SI(Low):	$\sum_{(o,i,s)\in O_{is}} \mathbf{r}_{is}^o e_{is}^{o\,(\mathrm{a})}$	→ Lower bound of selling income.
SI(Up):	$\sum_{(o,i,s)\in O_{is}} \mathbf{r}_{is}^o e_{is}^{o\,(\mathrm{b})}$	→ Upper bound of selling income.
HCO:	$\sum_{(o,i,s)\in O_{is}} \mathbf{h}_{is}\mathbf{r}_{is}^o(\mathbf{d}^o - \mathbf{t}^o - \sum_{h,t} t\mathbf{Y}_{is}^{oht})$	→ Holding cost of committed orders.
HCA:	$\sum_{(i,s)\in I_s} \sum_{h,t\leqslant\tau} (\boldsymbol{\tau} - t)\mathbf{h}_{is}a_{is}^{ht}$	→ Holding cost of the remaining ATP.
RC:	$\sum_{(o,i,s)\in O_{is}} \sum_{h,t} \mathbf{f}_{is}^o(1 - \mathbf{Y}_{is}^{oht})$	→ Rejected order's cost.
WC:	$\sum_{(i,s)\in I_s} \sum_{h,t} \mathbf{w}_{is}g_{is}^{ht}$	→ Wasting cost due to expiration.
TC:	$\sum_{(o,i,s)\in O_{is}} \sum_{h,t} \boldsymbol{\rho}^o\mathbf{r}_{is}^o\mathbf{Y}_{is}^{oht}$	→ Transporting cost.

Lower profit:

$$Z_a = \mathit{SI(Low)} - \mathit{HCO} - \mathit{HCA} - \mathit{RC} - \mathit{WC} - \mathit{TC}. \tag{23}$$

Upper profit:

$$Z_b = \mathit{SI(Up)} - \mathit{HCO} - \mathit{HCA} - \mathit{RC} - \mathit{WC} - \mathit{TC}. \tag{24}$$

Note that *HCA* and *WC* now depend on possible values of their possibilistic variables respectively.

The optimization strategy will rely on the decision maker's needs, for example, maximize the lower profit, maximize the upper profit or maximize the average of both (balanced).

4.4.3 *Computations with the possibilistic constraints*

We start by describing the computations on the constraint in Eq. (17) modelled as a composition of possibilistic variables. We now follow the α-cut approach explained in Subsection 2.3 and exemplified in Subsection 4.4.1. Since we have already used an α-cut for the computations in the product's price, in this part we use a γ-cut to differentiate them. We must include the following equations according to the procedure; Eq.(25) ensures the key condition for the joint possibility distribution given in Eq. (10):

$$p_i^{ht} = \sum_{s \in I_s} m_{is}^{ht} \quad \forall i \in I_s,\, h,\, t. \tag{25}$$

Equations (26) and (27) implement the logic explained in Eq. (21) in order to define equivalent intervals for each composition's component, that in our case represents the MPS per each subtype:

$$m_{is}^{ht} \geqslant m_{is}^{1ht} + \gamma(m_{is}^{2ht} - m_{is}^{1ht}) \quad \forall (i,s) \in I_s,\, h,\, t. \tag{26}$$

$$m_{is}^{ht} \leqslant m_{is}^{3ht} - \gamma(m_{is}^{3ht} - m_{is}^{2ht}) \quad \forall (i,s) \in I_s,\, h,\, t. \tag{27}$$

Equations (28) and (29) have a similar interpretation but for the equivalent intervals of compositions adding up to the grand total, that in our case represents the total MPS per product (the sum of the quantities of the subtypes):

$$p_i^{ht} \geqslant p_i^{1ht} + \gamma(p_i^{2ht} - p_i^{1ht}) \quad \forall i,\, h,\, t. \tag{28}$$

$$p_i^{ht} \leqslant p_i^{3ht} - \gamma(p_i^{3ht} - p_{is}^{2ht}) \quad \forall i,\, h,\, t. \tag{29}$$

And the remaining constraints will now turn now into:

$$a_{is}^{ht} \geqslant 0 \quad \forall (i,s) \in I_s, h, t. \tag{30}$$

Where:

$$a_{is}^{ht} = \begin{cases} \mathbf{k}_{is}^{h} - \mathbf{c}_{is}^{ht} - \sum\limits_{o \in O_{is}} r_{is}^{o} \mathbf{Y}_{is}^{oht} &, \quad \forall (i,s) \in I_s, h, t = 0, \\ m_{is}^{ht} - \mathbf{c}_{is}^{ht} - \sum\limits_{o \in O_{is}} r_{is}^{o} \mathbf{Y}_{is}^{oht} &, \quad \forall (i,s) \in I_s, h, t > 0. \end{cases} \tag{31}$$

and

$$g_{is}^{ht} = \begin{cases} a_{is}^{ht} & , \quad \textit{if } t-h > l_{is}^{3}, \quad \forall (i,s) \in I_s, h, t, \\ 0 & , \quad \textit{otherwise}. \end{cases} \tag{32}$$

The crisp constraints remain the same.

4.4.4 Equivalent MILP model

Summarizing, the MILP equivalent model will be

Maximize:

(1) Eq. (23) for the profit's lower bound maximization,
(2) Eq. (24) for the profit's upper bound maximization,
(3) $\frac{Z_a+Z_b}{2}$ for a balanced optimization strategy.

Subject to:

- Eq.(14) to Eq.(16)
- Eq.(25) to Eq.(30)

Considering:

- Eq.(31) and Eq.(32)

4.4.5 Implementing dynamic batching mode

In order to implement the model in a batching mode, it is required to update the input data between consecutive executions. Specifically, the parameters $\mathbf{k}_{is}^{h}$ and $\mathbf{c}_{is}^{ht}$ should be updated according to the procedure explained in Grillo et al. (2017a). The main difference in this case is that we consider the orders are served at the beginning of each period. Then, it is expected that any quantity of the initial stock is previously committed because it is the remaining product after serving orders. Considering e as the current execution in the batching mode, the update is computed in the following Eqs. (33) and (34)

$$\mathbf{c}(e+1)_{is}^{ht} = \mathbf{c}(e)_{is}^{ht} + \sum_{o \in O_{is}} \mathbf{r}_{is}^{o} \mathbf{Y}_{is}^{oht}, \quad \forall (i,s) \in I_s, h, t > \boldsymbol{\tau}. \tag{33}$$

$$\mathbf{k}(e+1)_{is}^{h} = \sum_{t \leqslant \boldsymbol{\tau}} \left(a_{is}^{ht} - g_{is}^{ht} \right), \quad \forall (i,s) \in I_s, h. \tag{34}$$

Finally, once the parameters $\mathbf{c}(e+1)_{is}^{ht}$ and $\mathbf{k}(e+1)_{is}^{h}$ have been computed, it is required to execute for both of them an update of the indices t and h. This is because, from one execution to the next one, the product aged $\boldsymbol{\tau}$ periods. This is achieved with the following Eq. (35)

$$ind(e+1) = ind(e) - \boldsymbol{\tau}, \quad \forall ind = h \vee ind = t. \tag{35}$$

Note that for the case of the parameter $\mathbf{c}(e+1)_{is}^{ht}$ with its indices h and t updated, it should coincide with enough supply in the possibilistic variables $\mathbf{P}_{i}^{ht}$ and $\mathbf{M}_{is}^{ht}$ in the execution $e+1$; otherwise the model would go infeasible.

5. Experimental design: application to an orange and tangerine supply chain

In this section we will validate the model with an application to a real case of a Spanish supply chain of the fruit sector, specifically the packing and distribution of orange and tangerine. The implementation was carried out using the CMPL mathematical programming language (Steglich and Schleiff 2010) to code the model, and an algorithm of execution to implement the batching mode and different instances of evaluation developed in Java. CMPL supports different commercial and non commercial optimizers available in the market. In our case, the Gurobi 7.2 solver was used. Experiments were executed by an Intel (R) Core (TM) i7-4510 CPU 2.60 GHz processor, with 8GB RAM under a Ubuntu 16.04 - Linux operative system.

5.1 *Input data overview*

In order to test the model, we have based our data set in the one presented in Grillo et al. (2017a). It comes from a Spanish supply chain dedicated to pack and distribute oranges and tangerines. The scenario includes one packing plant with two products (oranges and tangerines) that can be subdivided in 8 and 7 subtypes respectively. Transporting costs and times are considered as well as inventory holding cost and wasting cost due to the product's decay. The orders from customers have between one to ten lines. A total of eighty-eight incoming orders are considered as in Grillo et al. (2017a). The orders are promised twice a week by considering a 3-day batching interval and a 6-day horizon length. The global horizon for the experiments includes seventeen periods subdivided into four executions. The complete data set for the model can be consulted in Applendix B.

5.2 *Definition of evaluation instances*

Here we define the cases that will be evaluated in the numerical experiments. We focus on the elements that need to be given by the decision maker beforehand the execution.

Optimization strategy: Given the three possibilities mentioned in Subsection 4.4.4, for simplicity, we use the profit's average maximization as the objective function (which is one of the most common strategies used in this type of optimization).

The α-cut and γ-cut: As we have previously described in Subsections 4.4.2 and 4.4.3, the α-cut and γ-cut are used in order to find equivalent crisp intervals in the computations of the possibilistic price and the possibilistic composition modelling the master plans. Both parameters must be in the interval [0, 1]. We will discretise both parameters starting in 0.2 with steps of 0.2 until 1. Thus we will evaluate the values {0.2, 0.4, 0.6, 0.8, 1.0}. We will use such values because they are representative in order to explore the trend of the model as both parameters increase their values. In the case under study, the α-cut regulates the uncertainty degree in the product's shelf life; the closer α is to 1, the less uncertainty in the extension of the shelf life, and as a consequence, the price of the product resembles more to the one determined by the piecewise linear ageing function used in our case. Meanwhile, the γ-cut has the same role, but in the total MPS and the MPS per subtype. It is very important to highlight that the case $\alpha = 1$ and $\gamma = 1$ is the equivalent to the crisp counterpart of the model (as explained and exemplified in Subsection 4.4.1). We consider this case in order to compare it to all the other cases (i.e. $\alpha, \gamma \in \{0.2, 0.4, 0.6, 0.8\}$).

Summarizing, we will execute the model 5 times per each case of the α-cut and 5 times per each case of the γ-cut. Since the model will be executed in batching mode, a full run is composed by

4 executions given the batching window considered and the planning horizon. This brings a total amount of $5 * 5 * 4 = 100$ executions.

5.3 *Results*

5.3.1 Committed orders and generated profits

In terms of practical application, the decision maker's main interest of our model is the resulting values of the binary variables $\mathbf{Y}_{is}^{oht}$ and $\mathbf{U}^o$ (as well as the profit generated by them) since they answer to the questions of what orders to accept, and what ATP to use in order to fulfil the orders. The model returns the values of $\mathbf{Y}_{is}^{oht}$ and $\mathbf{U}^o$, considering that the supply remains within the intervals established by the γ-cut in the TFI describing the possibilistic variables $\mathbf{P}_i^{ht}$ and $\mathbf{M}_{is}^{ht}$. Additionally, the limits of profit in the objective function should be seen as the lower and upper *possible* values that can be earned if such configuration of the variables $\mathbf{Y}_{is}^{oht}$ and $\mathbf{U}^o$ was executed in reality. This is, the selling incomes achieved if the set of accepted orders are paid based on the lower and upper price respectively resulting from the α-cut in $\mathbf{E}_{is}^o$. Note that the parameters of the TFI used to describe $\mathbf{E}_{is}^o$ depend on the product's maximum shelf life, i.e., the TFI describing $\mathbf{L}_{is}$. This is the key point of our shelf-life based pricing policy.

Regarding the expected behaviour of the results in relation to the parameters α and γ, we can say that the lower and upper limits of the profit should get closer in the way that both parameters increase their value. This is because both the master plan and the shelf life get closer to the central values of their respective TFI. The greater the value of γ, the smaller the crisp interval for the MPS. Regarding to the shelf life, since the product's price is linked with its ageing process, the greater α, the smaller the difference between bounds of the product's price in the same expected delivery date. The results of the profit obtained are shown graphically in Figure 9.

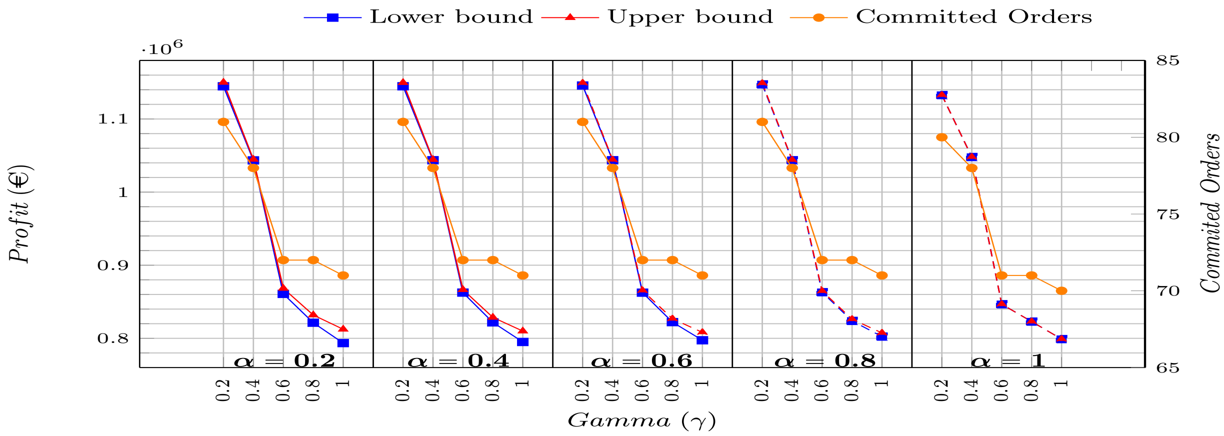


Figure 9.: Experimental results

It can be observed how the expected trend is properly achieved. In the cases when the same value of alpha is maintained, the limits of the profit get farther away when the value of gamma is near to 1. This behaviour is expected, since for one same value of alpha, the lower and upper product's price remains the same, while the supply varies within the intervals defined by the γ-cut. Hence, if more or less supply are available according to gamma's variation, then more or less orders are accepted (see orange curve in Figure 5). Those orders will be paid within the same upper and lower price if alpha stays constant. Otherwise they will be paid with a smaller upper price and with a bigger lower price if alpha gets near to 1. This is why, in the way that alpha gets near to one, the lower and upper limits of profit (curves blue and red in Figure 5) get closer independently of the value of gamma.

It is important to remark that when gamma is near to 0, the supply has a very open range of variation, and the model has more solution spaces to find better combinations. This fact can also be confirmed in Figure 5; by the way that the value of gamma increases, both lower and upper limits of profit, and the total quantity of committed orders lower. As we mentioned before, this is because the possible values of supply, m_{is}^{ht} and p_i^{ht}, have less possible combinations, and they get closer to the crisp values m_{is}^{2ht} and p_i^{2ht} respectively. The negative part of gamma near to 0 is that such open solution space implies a huge level of uncertainty. Then, there is a proportional risk level in the final solution, i.e., the resulting accepted orders cannot be served when the real quantities of supply are available due to shortages. If the orders finally served with real supply are considerably different of those accepted with planned supply, the real profit could go out of the interval established by the respective lower and upper planned bounds. Take a special look at Figure 5 for the gamma values 0.2 and 0.4. The orders accepted and the profits achieved are considerably higher than the rest of the tested cases. This is because of the already explained opened solution space in the possibilistic composition applied. See how variations in the supply limits can critically affect the possibility to accept or reject an order. But in term of practical implementations, such solutions will involve high uncertainty and the orders could not finally be served. On the other hand, for combinations of gamma near to 1, it is expected that the planned supply has less uncertainty involved and will be near to the real quantities. Then, the accepted orders have considerably more probability to finally be served. The decision of what case is better or not will be briefly described in Subsection 5.3.2.

These results validate the proper functionality of the model. The composition of possibilistic variables applied to model the LHP's effect has worked in a very good way in terms of the interpretation of the situation and the complexity in the modelling approach achieved. For its part, the state functions and ageing functions have worked adequately and have a very good performance in terms of the usefulness they have when linking related variables to the product's ageing process.

5.3.2 Managerial insights

From the managerial point of view, it is required to analyse in detail the objective function, in order to evaluate the suitability of the different instances of α and γ tested. Table 2 shows the different results of the profit's components.

Table 2.: Objective function components

α	γ	SI(Low)	SI(Up)	HCO	HCA	RC	WC	TC
0.2	0.2	1,444,992	1,439,689	21,771	21,155	56,553	19,787	175,700
	0.4	1,370,322	1,368,980	24,003	23,647	88,269	22,584	167,087
	0.6	1,235,201	1,227,720	28,632	28,362	132,532	23,672	153,747
	0.8	1,209,038	1,198,515	30,226	29,248	146,114	22,928	148,652
	1	1,191,265	1,172,360	31,577	29,933	151,126	19,795	146,453
0.4	0.2	1,444,686	1,439,725	21,771	21,155	56,553	19,787	175,700
	0.4	1,370,051	1,369,189	24,003	23,647	88,269	22,584	167,087
	0.6	1,233,460	1,229,472	28,810	28,184	132,532	23,672	153,747
	0.8	1,205,819	1,198,973	30,273	29,201	146,114	22,928	148,652
	1	1,189,421	1,174,529	31,383	30,061	151,348	20,274	146,453
0.6	0.2	1,444,187	1,440,738	21,808	21,118	56,553	19,787	175,700
	0.4	1,369,729	1,369,189	24,003	23,647	88,269	22,584	167,087
	0.6	1,232,774	1,229,871	28,691	28,303	132,532	24,151	153,747
	0.8	1,204,386	1,199,804	30,194	29,280	146,114	23,408	148,652
	1	1,187,666	1,176,863	31,383	30,061	151,348	20,274	146,453
0.8	0.2	1,443,730	1,442,174	21,835	20,999	56,553	19,787	175,700
	0.4	1,369,505	1,369,189	24,003	23,647	88,269	22,584	167,087
	0.6	1,232,024	1,230,515	28,731	28,263	132,532	24,151	153,747
	0.8	1,203,795	1,201,493	30,194	29,280	146,114	23,408	148,652
	1	1,186,864	1,181,896	31,372	30,072	151,348	20,274	146,453
1	0.2	1,430,974	1,430,974	22,718	20,002	63,891	17,641	174,014
	0.4	1,367,529	1,367,529	24,543	22,292	88,269	17,350	167,087
	0.6	1,215,756	1,215,756	30,161	27,915	140,546	19,408	151,108
	0.8	1,199,473	1,199,473	31,094	28,707	150,070	19,372	147,333
	1	1,180,333	1,180,333	32,199	30,354	156,221	17,198	145,427

For supply chain planning problems, in this type of optimization based on possibility distributions and α-cuts, it appears as a common practice to discard values of alpha less than 0.5 because they are considered as a very high level of uncertainty (Alemany et al. 2015). Then, in order to analyse the data of Table 2, let us consider the values of α and γ larger than 0.5 (0.6 and 0.8 in the evaluated cases) compared to the case when $\alpha = \gamma = 1$ (the crisp case). Table 3 shows the resulting variation percentages of such comparison.

Table 3.: Variation percentage (%) vs crisp case

α	γ	SI(Low)	SI(Up)	HCO	HCA	RC	WC	TC	Z_a	Z_b
0.6	0.6	4.20	4.44	-10.89	-6.76	-15.16	40.44	5.72	7.95	8.31
	0.8	1.65	2.04	-6.23	-3.54	-6.47	36.11	2.22	2.91	3.48
0.8	0.6	4.25	4.38	-10.77	-6.89	-15.16	40.44	5.72	8.03	8.22
	0.8	1.79	1.99	-6.23	-3.54	-6.47	36.11	2.22	3.12	3.41

From the pricing policy side, it is clear that better results are achieved when $\alpha = 0.8$ for both, the selling incomes and cost performance. One can think that better results should be achieved for $\alpha = 0.6$ because this means more variability in the products' shelf life. This is not necessarily true because of the conflicting situation with wasting costs at expense of selling more fresh product, and the interaction with the uncertainty in the supply. Hence, for this specific case, with this data set in the evaluated instances, it would be a better option to consider $\alpha = 0.8$ given the results in the profit for both cases of gamma.

In regard to the uncertainty in supply, based on the configuration of the data set for the possibilistic variable $\mathbf{M}_{is}^{ht}$, a value of γ equal to 0.6 would imply a range of variation from 0 to 0.9 truck (one truck includes around 30 pallets, 750 Kg) of the product at most, depending on the subtype. Meanwhile, a value of γ equal to 0.8 would imply a range from 0 to 0.4 truck. Based on this, both cases are relatively reasonable, and it will depend on the decision makers to choose the option. If they are able to risk more, $\gamma = 0.6$ would be the option with the aim to earn about 5% more than if they consider $\gamma = 0.8$. Otherwise if they prefer to handle a less risky solution, with less profit (about 3% more than the crisp case), but with a relatively small risk, $\gamma = 0.8$ would be the option. There are other approaches used to evaluate the suitability of the α-cuts, for example in Alemany et al. (2015) where an interactive procedure is applied to compute a fuzzy decision vector based on the decision maker's requirements, or in Grillo et al. (2017b) where another interactive procedure based on fuzzy TOPSIS is applied. These types of analysis are out of the scope of this work, but readers are refereed to them in order to see practical examples. The work of Grillo et al. (2017b) provides an example of OPP modelling for non perishables. Readers are referred to this work for more details on how non-perishable products behave in the presence of LHP. The authors conducted a study for a ceramic tile supply chain. In general terms, the perishability exposes a considerably higher level of complexity since the subtypes change through time. The perishability itself can also be used to define the subtype. By comparing both studies, the case of perishables increases the risk of shortages, since the causes of the uncertainty are more complex to describe and control, for example the weather. The case of perishables requires greater attention on the LHP management, not just for the risk of shortage itself, but also for the profitability of the supply chain; it could be highly affected due to the wasting cost and the variation of the product's price.

5.3.3 Computational efficiency

Finally, Table 4 shows the computational efficiency data of the model executions. It can be seen how the model has a very good performance in terms of resolution time with the solutions matrix's size considered in this case.

Table 4.: Computational efficiency

Execution	Constraints	Binary variables	Continuous variables	Non-zeros	Aver. sol. time (s)	% Gap
e_1	12,194	13,246	7,438	130,718	2.08	0.001
e_2	12,188	12,790	7,438	126,923	2.09	0.001
e_3	12,192	13,094	7,438	129,453	2.09	0.001
e_4	12,174	11,726	7,438	118,106	1.96	0.001

6. Conclusions and future research lines

Compositions of possibilistic variables are based on the concept of compositional data where different elements of a vector should add up to a specific resulting quantity. We considered not only the case when the elements are possibilistic variables, but also the case when the resulting quantity is another possibilistic variable. This type of compositions can be applied to a wide open variety of situations when the conventional compositional data is not recommended due to the existence of epistemic uncertainty in the composition's components. The α-cut approach is applied in order to simplify the application of compositions into linear mathematical programming, obtaining more computational efficient models.

Furthermore, we introduced the concepts of state functions, which describe the "state" of a variable over time. If a state function is used to model product's decay, it is called ageing function. This type of functions are defined from an initial time until an uncertain ending time represented by a possibilistic variable. Ageing functions allow to link related shelf life-based variables, for example price, and also describe them as possibilistic variables. This modelling approach has the advantage of being an easy and good performing option to be applied in several types of situations when dealing with perishables.

In order to exemplify the application of compositions with possibilistic variables and ageing functions into linear mathematical modelling, we have developed a model of a supply chain planning problem, specifically the order promising process, where both concepts can be applied simultaneously. The compositions are used to represent the effect of the so called Lack of Homogeneity in Product, LHP, in the master production schedule when the handled product is perishable and as a consequence it must be classified into subtypes. The total master production schedule and its corresponding quantities per subtype are represented as a composition of possibilistic variables. Moreover, since the product is perishable, some of its characteristics can change within time. We use an ageing function to link the product's price. Hence, this application includes both concepts, the compositions of possibilistic variables to model the master production schedule from which the orders must be promised, and the ageing functions to model product's price at the delivery time.

Practical experiments have been executed by applying the model to a real case of a Spanish supply chain of the fruit sector, specifically the packing and distribution of oranges and tangerines. We have used a data set based on real information given by the supply chain, and we have executed different instances in a batching ordering mode in rolling horizon. The results obtained validated the model's correct functionality with a very good computational performance. We also presented some managerial insights in order to exemplify the usefulness that a tool like this has for decision makers.

Finally, the following future reseach lines are identified: regarding to the compositions of possibilistic variables, it is required to generalise the approach of using possibilistic variables from the case where the variables add up to a determined total, to the case where other operations (such as multiplication, subtraction, etc.) are considered. Such study will allow us to compute the epistemic uncertainty in diffent types of problems where the uncertainty has an interdependent nature.

For state functions, it is required to investigate the performance of other types of curves, not only linear functions or decreasing piecewise linear functions. Also, special attention should be given to extend these notions to the case when the curve is not necessarily a decreasing function. Such case would apply to situations when the state increases within time, e.g. the price of wine.

Compositions of possibilistic variables and state functions can be applied to other type of problems, i.e., not just supply chain problems, but to other areas like technology, mathematics and other problems in the engineering field. Regarding to the order-promising processes, we recommend to apply them in an extended model, considering additional features like different manufacturing strategies, renegotiation processes for the rejected orders, advance and delays, etc. We also recommend the application of these tools to problems of more operative level for perishables, where the changes of state can occur in very short periods of time, for example, operations of transformation of a row material, freezing or holding of finished products, transportation under controlled conditions, etc. Another interesting case would be the application to the handling of by-products.

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Appendix A. Proof of theorem 1.

Note that the assumption $C \subseteq \bigoplus_{i=1}^{n} X_i$ implies that

$$\pi_C(v) \leqslant \sup\{\min(\pi_{X_1}(u_1), \ldots, \pi_{X_n}(u_n)) \mid \sum_{i=1}^{n} u_i = v\}\,.$$

Let $Y = \sum_{i=1}^{n} X_i$, then

$$\begin{aligned}
\pi_Y(v) &= \sup\{\min(\pi_C(\sum_{i=1}^{n} u_i), \min(\pi_{X_1}(u_1), \ldots, \pi_{X_n}(u_n))) \mid \sum_{i=1}^{n} u_i = v\} \\
&= \sup\{\min(\pi_C(v), \min(\pi_{X_1}(u_1), \ldots, \pi_{X_n}(u_n))) \mid \sum_{i=1}^{n} u_i = v\} \\
&= \min(\pi_C(v), \sup\{\min(\pi_{X_1}(u_1), \ldots, \pi_{X_n}(u_n)) \mid \sum_{i=1}^{n} u_i = v\}) \\
&= \pi_C(v)\,.
\end{aligned}$$

Moreover, consider another possibility distribution $\pi^{\#}_{X_1,\ldots,X_n}$ such that

$$\pi^{\#}_{X_1,\ldots,X_n}(u_1, \ldots, u_n) > \pi^{*}_{X_1,\ldots,X_n}(u_1, \ldots, u_n)$$

for some point $(u_1, \ldots, u_n)$, while still $\sum_{i=1}^{n} X_i = C$ (using $\pi^{\#}_{X_1,\ldots,X_n}$). Since $\min(\pi_{X_1}, \ldots, \pi_{X_n})$ is an upper bound to the joint possibility distribution, it must hold that $\pi^{\#}_{X_1,\ldots,X_n}(u_1, \ldots, u_n) >$

$\pi_C(\sum_{i=1}^{n} u_i)$. However, this implies that for $v = \sum_{i=1}^{n} u_i$, it holds that $\pi_Y^{\#}(v) > \pi_C(v)$, a contradiction.

Appendix B. Input data

We have based our data set in the one presented in Grillo et al. (2017a) with the required modifications in order to implement the compositions of possibilistic variables and the state functions. The global horizon length includes 17 periods (in order to consider two weeks of incoming orders at least), subdivided into four OPP executions with a planning horizon of six periods each. The orders are promised twice a week, considering a 3-day batching interval. The executions occur in periods 1, 4, 8 and 11 simulating real dynamics.

Two products (i) are contemplated (oranges and tangerines) with subtypes (s) defined according to the quality level, the calibre and the packaging type. Table B1 shows the products with their respective subtypes and the parameters of the possibilistic variable $\mathbf{L}_{is}$.

Table B1.: Products, subtypes and shelf-life values

		$\mathbf{L}_{is}$ (days)		
Product (i)	Subtypes (s)	l_{is}^1	l_{is}^2	l_{is}^3
i_1 (Orange)	s_1 (First quality, calibre 1, Box-paperboard 10kg) s_2 (First quality, calibre 1, Net 1.2kg) s_3 (First quality, calibre 1, Bulk) s_4 (First quality, calibre 2, Box-paperboard 10kg) s_5 (First quality, calibre 3, Box-paperboard 10kg) s_6 (First quality, calibre 4, Box-paperboard 10kg)	8	10	12
	s_7 (Second quality, calibre 1, Bulk) s_8 (Second quality, calibre 4, Box-wood 15kg)	7	9	11
i_2 (Tangerine)	s_9 (First quality, calibre 5, Box-paperboard 10kg) s_{10} (First quality, calibre 5, Net 750g) s_{11} (First quality, calibre 5, Net 1.2kg) s_{12} (First quality, calibre 6, Box-paperboard 10kg) s_{13} (First quality, calibre 7, Box-paperboard 10kg) s_{14} (First quality, calibre 8, Box-paperboard 10kg)	8	10	12
	s_{15} (Second quality, calibre 8, Box-paperboard 10kg)	7	9	11

Table B2 shows the price data required to build the piecewise linear ageing function of each subtype as presented in Figure 8. Regarding the different time points presented in Figure 8, where there are changes in the price value, the following procedure is considered (depending on the harvesting time h of the ATP reserved for the orders of each subtype):

- $t_1 = h + l_{is}^2 * 0.40$
- $t_2 = h + l_{is}^2 * 0.45$
- $t_3 = h + l_{is}^2 * 0.60$
- $t_4 = h + l_{is}^2 * 0.65$
- $t_5 = h + l_{is}^2 * 0.80$
- $t_6 = h + l_{is}^2 * 0.85$
- $t_7 = h + l_{is}^2 * 0.90$

Table B3 shows the inventory holding and wasting costs for each subtype. The fending unitary

Table B2.: Price data

i	s	Normal price	Smooth discount	High discount	Rescue value
i_1	s_1	1.19	0.833	0.595	0.357
	s_2	1.17	0.819	0.585	0.351
	s_3	1.15	0.805	0.575	0.345
	s_4	1.13	0.791	0.565	0.339
	s_5	1.07	0.749	0.535	0.321
	s_6	1.01	0.707	0.505	0.303
	s_7	0.92	0.644	0.46	0.276
	s_8	0.95	0.665	0.475	0.285
i_2	s_9	1.21	0.847	0.605	0.363
	s_{10}	1.18	0.826	0.59	0.354
	s_{11}	1.24	0.868	0.62	0.372
	s_{12}	1.12	0.784	0.56	0.336
	s_{13}	1.06	0.742	0.53	0.318
	s_{14}	1.01	0.707	0.505	0.303
	s_{15}	0.81	0.567	0.405	0.243

(*) Data in €

cost $\mathbf{f}^o_{is}$ was obtained as 50% of the maximum income generated for the order line if it was served with the maximum possible price.

Table B3.: Inventory holding and waste costs

i	s	$\mathbf{h}_{is}$ $\left(\frac{€}{kg \cdot day}\right)$	$\mathbf{w}_{is}$ $\left(\frac{€}{kg}\right)$
i_1	s_1	0.018	0.3221
	s_2	0.018	0.322
	s_3	0.017	0.322
	s_4	0.017	0.322
	s_5	0.016	0.322
	s_6	0.015	0.322
	s_7	0.014	0.322
	s_8	0.014	0.322
i_2	s_9	0.018	0.327
	s_{10}	0.018	0.327
	s_{11}	0.019	0.327
	s_{12}	0.017	0.327
	s_{13}	0.016	0.327
	s_{14}	0.015	0.327
	s_{15}	0.012	0.327

Regarding the configuration of the incoming orders, we also follow the description given in Grillo et al. (2017a). Each customer usually places four orders per week. The due date ($\mathbf{d}^o$) of each order usually occurs 4-5 days after the arrival date. Customer orders are composed by several order lines randomly varying between 1 to 10. The authors presented a total of 88 orders following the rule that the sum of the subtypes requested in all the lines in the same order usually equals to 30 pallets (realistic quantity which is the equivalent to one truck). They used such rule based on samples of real orders. Since in this case we have considered the existence of just one packing plant, and the authors originally considered two plants, we have made corrections in the transporting time and due date of the orders, maintaining the general rules they presented. Table B4 shows the configuration of each order with the due date, number of order lines, transporting time and transporting cost.

Concerning to the supply data required to implement the compositions of possibilistic variables, it was necessary to make some additional considerations. First, in Grillo et al. (2017a) they considered the possibility to reserve lines even when the entire order could not be served. They also considered

Table B4.: Order's data

Exec.	o	$\mathbf{d}^o$	$\mathbf{n}^o$	$\mathbf{t}^o$	$\boldsymbol{\rho}^o$	Exec.	o	$\mathbf{d}^o$	$\mathbf{n}^o$	$\mathbf{t}^o$	$\boldsymbol{\rho}^o$
e_1	o_1	3	9	1	0.10	e_3	o_{45}	3	9	1	0.10
	o_2	4	2	2	0.10		o_{46}	2	2	1	0.10
	o_3	3	4	1	0.10		o_{47}	4	4	1	0.10
	o_4	4	1	2	0.15		o_{48}	3	1	1	0.15
	o_5	3	1	1	0.10		o_{49}	4	1	1	0.10
	o_6	4	7	2	0.15		o_{50}	5	7	2	0.15
	o_7	5	3	2	0.15		o_{51}	5	3	2	0.15
	o_8	5	1	2	0.15		o_{52}	5	1	2	0.15
	o_9	2	1	1	0.10		o_{53}	3	1	1	0.10
	o_{10}	6	7	2	0.15		o_{54}	4	7	2	0.15
	o_{11}	6	6	2	0.15		o_{55}	6	6	2	0.15
	o_{12}	3	4	1	0.10		o_{56}	2	4	1	0.1
	o_{13}	5	6	1	0.15		o_{57}	6	6	1	0.15
	o_{14}	4	4	2	0.15		o_{58}	3	4	1	0.15
	o_{15}	6	1	2	0.15		o_{59}	6	1	2	0.15
	o_{16}	5	5	2	0.15		o_{60}	4	5	2	0.15
	o_{17}	4	3	2	0.15		o_{61}	5	3	2	0.15
	o_{18}	6	2	2	0.15		o_{62}	6	2	2	0.15
	o_{19}	6	5	2	0.15		o_{63}	6	5	2	0.15
	o_{20}	6	1	2	0.15		o_{64}	6	1	2	0.15
	o_{21}	6	10	1	0.15		o_{65}	6	9	1	0.15
	o_{22}	6	4	2	0.15		o_{66}	6	4	2	0.15
e_2	o_{23}	4	9	1	0.10	e_4	o_{67}	5	7	1	0.1
	o_{24}	2	2	1	0.10		o_{68}	3	2	1	0.10
	o_{25}	2	1	1	0.10		o_{69}	3	1	1	0.10
	o_{26}	5	1	2	0.15		o_{70}	5	1	2	0.15
	o_{27}	3	4	1	0.10		o_{71}	2	4	1	0.10
	o_{28}	3	1	1	0.15		o_{72}	4	1	2	0.15
	o_{29}	5	1	2	0.15		o_{73}	6	1	2	0.15
	o_{30}	5	6	2	0.15		o_{74}	5	5	2	0.15
	o_{31}	2	3	1	0.10		o_{75}	4	3	1	0.10
	o_{32}	5	6	2	0.15		o_{76}	4	5	2	0.15
	o_{33}	6	4	2	0.15		o_{77}	6	4	2	0.15
	o_{34}	2	5	1	0.10		o_{78}	2	5	1	0.10
	o_{35}	6	7	2	0.15		o_{79}	6	7	2	0.15
	o_{36}	4	5	1	0.15		o_{80}	3	5	1	0.15
	o_{37}	6	4	2	0.15		o_{81}	6	4	2	0.15
	o_{38}	3	1	1	0.15		o_{82}	4	1	2	0.15
	o_{39}	3	2	1	0.15		o_{83}	4	2	1	0.15
	o_{40}	6	10	1	0.15		o_{84}	6	8	1	0.15
	o_{41}	6	3	2	0.15		o_{85}	6	3	2	0.15
	o_{42}	6	4	2	0.15		o_{86}	6	3	2	0.15
	o_{43}	6	1	2	0.15		o_{87}	6	1	2	0.15
	o_{44}	6	4	2	0.15		o_{88}	6	4	2	0.15

(*) $\boldsymbol{\rho}^o$ is expressed in $\left(\frac{€}{\text{unit}\cdot\text{day}}\right)$, $\mathbf{d}^o$ and $\mathbf{f}^o$ are expressed in days

a percentage of about 8% to 10% of the incoming production lots as non-usable to fulfil orders. That product was intended to be used for the production of by-products or directly wasted. It is also required to consider that it is worth to use compositions of posibilistic variables in presence of epistemic uncertainty, i.e., when there is just a vague idea of the real value. This means a considerable high level of uncertainty that in our case should be reflected in the parameters of the possibilistic variables $\mathbf{P}_i^{ht}$ and $\mathbf{M}_{is}^{ht}$. Due to all the previous reasons, in order to define our supply data, we have taken the master production schedule presented in Grillo et al. (2017a) and we have cut the quantities in 30%. The resulting quantities will be the central value of the triangular fuzzy set representing $\mathbf{P}_i^{ht}$. In order to define the limits of such triangular fuzzy set, we consider a variation of $\pm 50\%$ from the central value, with the aim to reflect a very high level of uncertainty. The resulting inputs are shown in Table B5. The harvesting time (h) can take negative values because it is assumed that the product can be harvested before the first period in the planning horizon.

The parameters of the possibilistic variable $\mathbf{M}_{is}^{ht}$ are obtained based on the values of $\mathbf{P}_{i}^{ht}$ given in Table B5 and considering the proportional subdivision originally used in (Grillo et al. 2017a). According to them, the proportional subdivision was obtained based on the sum of the subtypes requested in the order lines with the same due date during each period. The total sum was set as MPS 2-3 periods randomly before the due date in order to confer the model the possibility of taking a wide open window of time in transportation once the product is packed.

Table B5.: Master Production Schedule

				$\mathbf{P}_i^{ht}$ (kg)		
Exec.	i	h	t	p_i^{1ht}	p_i^{2ht}	p_i^{3ht}
e_1	i_1	-1	1	14,013	28,026	42,039
		1	2	22,129	44,258	66,387
		2	3	54,829	109,659	164,488
		2	4	20,619	41,239	61,858
		3	5	23,249	46,499	69,748
		5	6	21,896	43,792	65,688
	i_2	-1	1	10,650	21,301	31,951
		1	2	9,788	19,576	29,363
		2	3	28,506	57,012	85,517
		3	4	23,887	47,775	71,662
		3	5	34,917	69,834	104,750
		4	6	30,226	60,453	90,679
e_2	i_1	-1	1	20,619	41,239	61,858
		0	2	23,249	46,499	69,748
		2	3	21,896	43,792	65,688
		3	5	32,106	64,211	96,317
		5	6	36,846	73,692	110,538
	i_2	0	1	23,887	47,775	71,662
		0	2	34,085	68,170	102,255
		1	3	30,982	61,965	92,947
		3	5	27,608	55,216	82,824
		4	6	16,983	33,966	50,949
e_3	i_1	0	2	32,106	64,211	96,317
		2	3	36,846	73,692	110,538
		2	4	36,284	72,568	108,852
		3	5	28,881	57,763	86,644
		5	6	14,392	28,785	43,177
	i_2	0	2	27,608	55,216	82,824
		1	3	16,983	33,966	50,949
		2	4	26,482	52,964	79,446
		4	5	23,604	47,207	70,811
		4	6	14,968	29,936	44,904
e_4	i_1	-1	1	36,284	72,568	108,852
		0	2	28,881	57,763	86,644
		2	3	14,392	28,785	43,177
		2	4	14,105	28,209	42,314
	i_2	-1	1	26,482	52,964	79,446
		1	2	23,604	47,207	70,811
		1	3	14,968	29,936	44,904
		3	4	7,772	15,544	23,316

Finally, the initial stock $\mathbf{k}_{is}^{h}$ is presented in Table B6 and it coincides with the one given in Grillo et al. (2017a).

If readers are interested in reproducing the numerical experiments, they can easily follow the description given here. Otherwise they can ask us the exact data we have used by e-mail.

Table B6.: Initial stock

i	s	h	$\mathbf{k}_{is}^{h}$ (kg)
i_1	s_1	-3	1300
	s_1	-2	2500
	s_2	-2	8500
	s_2	-1	7500
	s_3	-3	902
	s_3	-2	7500
	s_4	-3	1600
	s_4	-1	9000
	s_5	-3	6000
	s_5	-2	2500
	s_6	-2	733
	s_6	-1	4000
	s_8	-2	2300
	s_8	-1	7000
i_2	s_9	-2	4000
	s_9	-1	5000
	s_{10}	-2	33000
	s_{10}	-1	15000
	s_{11}	-2	4000
	s_{11}	-1	3800
	s_{12}	-2	1500
	s_{12}	-1	1500
	s_{13}	-2	7000
	s_{13}	-1	3000
	s_{14}	-2	7000
	s_{14}	-1	6800
	s_{15}	-2	3000
	s_{15}	-1	2300