Filomat 33:3 (2019), 897–915
https://doi.org/10.2298/FIL1903897D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Exact Solution of a Class of Homogeneous Strongly Coupled Mixed Parabolic Problems

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Abstract. In this paper an exact series solution for homogeneous parabolic coupled systems is constructed using a projection method. An illustrative example is given.

1. Introduction and notation

Coupled partial differential systems with coupled boundary-value conditions are frequent in different areas of science and technology as in scattering problems in quantum mechanics [1, 9, 14], in chemical physics [6, 8, 11], coupled diffusion problems [3, 10, 17], thermo-elastoplastic Modelling [5], etc.

Recently, [15, 16], an exact series solution for the homogeneous initial-value problem

$$u_t(x,t) = Au_{xx}(x,t)\ ,\ 0 < x < 1\ ,\ t > 0 \tag{1}$$
$$A_1u(0,t) + B_1u_x(0,t) = 0\ ,\ t > 0 \tag{2}$$
$$A_2u(1,t) + B_2u_x(1,t) = 0\ ,\ t > 0 \tag{3}$$
$$u(x,0) = f(x)\ ,\ 0 \le x \le 1\ , \tag{4}$$

where $u = (u_1, u_2, \dots, u_m)^T$ and $f(x) = (f_1(x), f_2(x), \dots, f_m(x))^T$ are $m-$dimensional vectors, was constructed under the following hypotheses and notation:

- The matrix coefficient A is a matrix which satisfies the following condition

$$\mathrm{Re}(z) > 0\ ,\ \forall z \in \sigma(A), \tag{5}$$

where $\sigma(C)$ denotes the set of all the eigenvalues of a matrix C in $\mathbb{C}^{m\times m}$ and $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$.

2010 *Mathematics Subject Classification*. 49J20; 15A09
Keywords. Coupled diffusion problems, Coupled boundary conditions, Vector boundary-value differential systems, Sturm-Liouville vector problems, Projections
Received: 13 April 2017; Accepted: 30 July 2019
Communicated by Dragan S. Djordjević
This work has been supported by Spanish *Ministerio de Economía y Competitividad* and the European Regional Development Fund (ERDF) TIN2014-59294-P.
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• Matrices $A_i, B_i, i = 1, 2$, are $m \times m$ complex matrices, and we assume that the block matrix

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \text{ is regular }, \tag{6}$$

and also that the matrix pencil

$$A_1 + \rho B_1 \text{ is regular}, \tag{7}$$

i.e. condition (7) involves the existence of some $\rho_0 \in \mathbb{C}$, so that matrix $A_1 + \rho_0 B_1$ is invertible, see [2].

• Using condition (7) we can define the following matrices $\widetilde{A_1}$ and $\widetilde{B_1}$ by

$$\widetilde{A_1} = (A_1 + \rho_0 B_1)^{-1} A_1 \ , \ \widetilde{B_1} = (A_1 + \rho_0 B_1)^{-1} B_1, \tag{8}$$

which satisfy the condition:

$$\widetilde{A_1} + \rho_0 \widetilde{B_1} = I, \tag{9}$$

where matrix I denotes, as usual, the identity matrix. Under hypothesis (6), is it easy to show that matrix $B_2 - (A_2 + \rho_0 B_2) \widetilde{B_1}$ is regular (see [12] for details) and we can introduce matrices $\widetilde{A_2}$ and $\widetilde{B_2}$ defined by

$$\widetilde{A_2} = \left[B_2 - (A_2 + \rho_0 B_2) \widetilde{B_1}\right]^{-1} A_2 \ , \ \widetilde{B_2} = \left[B_2 - (A_2 + \rho_0 B_2) \widetilde{B_1}\right]^{-1} B_2, \tag{10}$$

which satisfy the following conditions:

$$\widetilde{B_2} - \left(\widetilde{A_2} + \rho_0 \widetilde{B_2}\right) \widetilde{B_1} = I, \widetilde{B_2}\widetilde{A_1} - \widetilde{A_2}\widetilde{B_1} = I. \tag{11}$$

• Under the assumptions (5), (6), (7), and considering the following essential hypothesis:

$$\begin{array}{c} \text{exist } b_1 \in \sigma\left(\widetilde{B_1}\right) - \{0\}, b_2 \in \sigma\left(\widetilde{B_2}\right), \text{ and } v \in \mathbb{C}^m - \{0\}, \\ \text{such that } \left(\widetilde{B_1} - b_1 I\right) v = \left(\widetilde{B_2} - b_2 I\right) v = 0 \ , \end{array} \tag{12}$$

then, if the vector valued function $f(x)$ satisfies hypotheses

$$\left.\begin{array}{c} f \in C^2([0,1]) \\ (1 - \rho_0 b_1) f(0) + b_1 f'(0) = 0 \\ -\left(\dfrac{1 - b_2 + \rho_0 b_1 b_2}{b_1}\right) f(1) + b_2 f'(1) = 0 \end{array}\right\}, \tag{13}$$

under the additional hypothesis:

$$\begin{array}{c} f(x) \in Ker\left(\widetilde{B_1} - b_1 I\right) \bigcap Ker\left(\widetilde{B_2} - b_2 I\right) \ , \ 0 \le x \le 1 \\ \text{and} \\ Ker\left(\widetilde{B_1} - b_1 I\right) \bigcap Ker\left(\widetilde{B_2} - b_2 I\right) \text{ is an invariant subspace with respect to matrix } A, \end{array} \tag{14}$$

where a subspace E of $\mathbb{C}^m$ is invariant by the matrix $A \in \mathbb{C}^{m \times m}$, if $A(E) \subset E$, then an exact series solution $u(x,t)$ of homogeneous problem (1)–(4) is constructed in Ref. [15].

• Under the above assumptions (5), (6), (7), and replacing the condition (12) by the following hypothesis

$$\begin{array}{c} 0 \in \sigma\left(\widetilde{B}_1\right), a_2 \in \sigma\left(\widetilde{A}_2\right), \textit{ and we have } w \in \mathbb{C}^m - \{0\}, \\ \textit{so that } \widetilde{B}_1 w = \left(\widetilde{A}_2 - a_2 I\right) w = 0 \ . \end{array} \tag{15}$$

if the vector valued function $f(x)$ satisfies the new hypotheses

$$\left.\begin{array}{c} f \in C^2([0,1]) \\ \\ f(0) = 0 \\ \\ a_2 f(1) + f'(1) = 0 \end{array}\right\}, \tag{16}$$

and

$$\begin{array}{c} f(x) \in Ker\left(\widetilde{B}_1\right) \cap Ker\left(\widetilde{A}_2 - a_2 I\right) \ , \ 0 \le x \le 1 \\ \textit{and} \\ Ker\left(\widetilde{B}_1\right) \cap Ker\left(\widetilde{A}_2 - a_2 I\right) \ \textit{ is an invariant subspace respect to matrix } A, \end{array} \tag{17}$$

then an exact series solution $u(x, t)$ of homogeneous problem (1)–(4) is constructed in Ref. [16].

As appear in [15] and [16], the valued vector function $f(x)$ obligatorily have to satisfy, from (14) or (17), one of the following conditions:

$$\begin{array}{lcl} f(x) & \in & Ker\left(\widetilde{B}_1 - b_1 I\right) \cap Ker\left(\widetilde{B}_2 - b_2 I\right) \ , \text{ if } b_1 \neq 0 \\ & \text{or} & \\ f(x) & \in & Ker\left(\widetilde{B}_1\right) \cap Ker\left(\widetilde{A}_2 - a_2 I\right) \ , \text{ if } b_1 = 0. \end{array} \tag{18}$$

This paper deals with the construction of the exact series solution of the problem (1)–(4) with less restrictive conditions on the valued vector function $f(x)$ that given in (18), i.e. for more general vector valued functions $f(x)$. To obtain this objective, we will get adapted the technique given in reference [7, p.281].

Throughout this paper we will assume the results and nomenclature given in [15, 16]. We denote by Θ the null matrix, and by I or $I_{n\times n}$ the identity matrix of dimension n. The kernel and the image of a matrix B are denoted by $Ker(B)$ and $Im(B)$, respectively. If B is a matrix in $\mathbb{C}^{n\times m}$, we denote by $B^\dagger$ its Moore-Penrose pseudoinverse [2]. A collection of examples, properties and applications of this concept may be found in [13], and $B^\dagger$ can be efficiently computed with the *Matlab* and *Mathematica* computer algebra systems. We will need to use two well known properties of the Moore-Penrose pseudoinverse:

Lemma 1.1 ([2]). *Let be B a matrix in $\mathbb{C}^{s\times s}$, then,*

$$Ker(B) = Im\left(I - B^\dagger B\right) \ , \ Im(B) = Ker\left(I - BB^\dagger\right), \tag{19}$$

Lemma 1.2 ([7]). *Let be P, Q and R matrices in $\mathbb{C}^{s\times s}$ so that $R = Q(I - P^\dagger P)$, then,*

$$Ker(P) \cap Ker(Q) = Im\left[(I - P^\dagger P)(I - R^\dagger R)\right] \ .$$

and a new property which we will demonstrate:

Lemma 1.3. *Let be A, B two matrices in $\mathbb{C}^{s\times s}$, then:*

(*i*) *If $\left(I - BB^\dagger\right) AB = \Theta$, then $Im(B)$ is an invariant subspace for matrix A.*

(*ii*) *If* $BA\left(I - B^{\dagger}B\right) = \Theta$ *then* $Ker(B)$ *is an invariant subspace for matrix* A.

Proof: (*i*) Let be $x \in Im(B)$, then using lemma 1.1 one gets that $x \in Ker\left(I - BB^{\dagger}\right)$, then $\left(I - BB^{\dagger}\right)x = 0 \Longrightarrow BB^{\dagger}x = x$. Furthermore, taking into account the hypothesis $\left(I - BB^{\dagger}\right)AB = \Theta$, then $AB = BB^{\dagger}AB$, thus by right multiplying the latter equality by $B^{\dagger}x$ one gets

$$A(BB^{\dagger}x) = BB^{\dagger}A(BB^{\dagger}x) \Longrightarrow (I - BB^{\dagger})Ax = 0 \,,$$

then

$$Ax \in Ker(I - BB^{\dagger}),$$

and using lemma 1.1 again one gets $Ax \in Im(B)$, which indicates that $Im(B)$ is an invariant subspace for matrix A.

(*ii*) The second implication is shown analogously to (*i*), taking into account relation $Ker(B) = Im(I - B^{\dagger}B)$ given in lemma 1.1. □

This paper is organized as follows: In section 2 the solution of (1)–(4) is obtained using a projection method. In section 3 we will introduce an algorithm and give an significative example.

2. Projections

As conditions (14) and (17) involving kernel intersections, to become to be more operational, we will express these intersections in terms of images, taking into account the property given in lemma 1.2:

$$Ker(M) \cap Ker(N) = Im(H), \tag{20}$$

where matrix H is given by

$$H = \left(I - M^{\dagger}M\right)\left(I - \left[N\left(I - M^{\dagger}M\right)\right]^{\dagger}\left(N\left(I - M^{\dagger}M\right)\right)\right). \tag{21}$$

We introduce the following sets:

$$\left.\begin{array}{rcl} G_1 & = & \{b_1(1), \ldots, b_1(s_1)\} \\ G_2 & = & \{b_2(1), \ldots, b_2(s_2)\} \\ G_3 & = & \{a_2(1), \ldots, a_2(s_3)\} \end{array}\right\} \tag{22}$$

with the different eigenvalues of matrices $\widetilde{B_1}$, $\widetilde{B_2}$ and $\widetilde{A_2}$ respectively. We denote by

$$H\left(b_1(i), b_2(j)\right) \text{ , if } b_1(i) \neq 0, \;\; 1 \leq i \leq k \text{ , } i \leq s_1 \text{ , } j \leq s_2 \tag{23}$$

the matrix defined by (21) where matrices M and N take the values

$$M = \left(\widetilde{B_1} - b_1(i)I\right) \text{ and } N = \left(\widetilde{B_2} - b_2(j)I\right) \,, \tag{24}$$

and we denote by

$$H\left(0, a_2(k)\right) \text{ , } if \; 0 \in G_1 \text{ , } 1 \leq k \leq s_3, \tag{25}$$

the matrix defined by (21) where matrices M and N take the values

$$M = \widetilde{B_1} \text{ and } N = \left(\widetilde{A_2} - a_2(k)I\right) \,. \tag{26}$$

By lemma 1.2, condition

$$Ker\left(\widetilde{B}_1 - b_1(i)I\right) \cap Ker\left(\widetilde{B}_2 - b_2(j)I\right) \neq \{0\}\ ,\ b_1(i) \neq 0$$

is equivalent to condition

$$H\left(b_1(i), b_2(j)\right) \neq \Theta\ ,$$

and also condition

$$Ker\left(\widetilde{B}_1\right) \cap Ker\left(\widetilde{A}_2 - a_2(k)I\right) \neq \{0\} \text{ if } 0 \in G_1$$

is equivalent to condition

$$H\left(0, a_2(k)\right) \neq \Theta\ .$$

We consider the subsets of $G_1 \times G_2$ or $G_1 \times G_3$ given by

$$S_2 = \{(b_1(i_l), b_2(j_l)) \in G_1 \times G_2\ :\ b_1(i_l) \neq 0\ ,\ H\left(b_1(i_l), b_2(j_l)\right) \neq \Theta\ ,\ 1 \leq l \leq q \leq s_1 s_2\} \tag{27}$$

or

$$S_3 = \{(0, a_2(k_{l'})) \in G_1 \times G_3\ :\ 0 \in G_1\ ,\ H\left(0, a_2(k_{l'})\right) \neq \Theta\ ,\ 1 \leq l' \leq r \leq s_3\} \tag{28}$$

and the block matrix defined by

$$\mathcal{H} = \left[H\left(b_1(i_1), b_2(j_1)\right) \ldots H\left(b_1(i_q), b_2(j_q)\right)\ \right] \quad \text{if } 0 \notin G_1 \tag{29}$$

or

$$\mathcal{H} = \left[H\left(b_1(i_1), b_2(j_1)\right) \ldots H\left(b_1(i_q), b_2(j_q)\right)\ H\left(0, a_2(k_1)\right) \ldots H\left(0, a_2(k_r)\right)\right] \quad \text{if } 0 \in G_1\ . \tag{30}$$

Suppose that the vector valued function $f(x)$ satisfies $f \in C^2\left([0,1]\right)$ and $f(x) \in Im\left(\mathcal{H}\right)$, i.e.:

$$\left(I - \mathcal{H}\mathcal{H}^{\dagger}\right) f(x) = 0\ ,\ 0 \leq x \leq 1 \tag{31}$$

and also satisfies that

$$\left.\begin{array}{l}
[0 \ldots 0\ H\left(b_1(i_l), b_2(j_l)\right)\ 0 \ldots 0]\,\mathcal{H}^{\dagger}\left(\left(1 - \rho_0 b_1\left(i_l\right)\right) f(0) + b_1\left(i_l\right) f'(0)\right) = 0 \\
[0 \ldots 0\ H\left(b_1(i_l), b_2(j_l)\right)\ 0 \ldots 0]\,\mathcal{H}^{\dagger}\left(-\left(\dfrac{1 - b_2(j_l) + \rho_0 b_1\left(i_l\right) b_2(j_l)}{b_1\left(i_l\right)}\right) f(1) + b_2(j_l) f'(1)\right) = 0 \\
1 \leq l \leq q\ \ ,\ \text{if } 0 \notin G_1
\end{array}\right\} \tag{32}$$

and also satisfies that

$$\left.\begin{array}{l}
[0 \ldots 0\ H\left(0, a_2(k_{l'})\right) 0 \ldots 0]\,\mathcal{H}^{\dagger} f(0) = 0 \\
[0 \ldots 0\ H\left(0, a_2(k_{l'})\right)\ 0 \ldots 0]\,\mathcal{H}^{\dagger}\left(a_2(k_{l'}) f(1) + f'(1)\right) = 0 \\
1 \leq l' \leq r\ \ ,\ \text{if } 0 \in G_1
\end{array}\right\} \tag{33}$$

From (23) and (25) and lemma 1.2 one gets that

$$ImH\left(b_1(i_1), b_2(j_1)\right) = Ker\left(\widetilde{B}_1 - b_1(i_l)I\right) \cap Ker\left(\widetilde{B}_2 - b_2(j_l)I\right)$$

and

$$ImH\left(0, a_2(k_{l'})\right) = Ker\left(\widetilde{B}_1\right) \cap Ker\left(\widetilde{A}_2 - a_2(k_{l'})I\right). \tag{34}$$

Thus, $Im\mathcal{H}$ is the direct sum of subspaces H_l and $H_{l'}$ given by

$$H_l = ImH\left(b_1(i_l), b_2(j_l)\right) \ , \ \ 1 \le l \le q \ , \ H_{l'} = ImH\left(0, a_2(k_{l'})\right) \ , \ 1 \le l' \le r \tag{35}$$

and the projection $f_l(x)$ of vector valued function $f(x)$ on subspace H_l is given by

$$f_l(x) = [0 \dots 0 \, H\left(b_1(i_l), b_2(j_l)\right) \ 0 \dots 0] \, \mathcal{H}^{\dagger} f(x) \ , \ 1 \le l \le q \ , \ 0 \le x \le 1 \tag{36}$$

or

$$f_{l'}(x) = [0 \dots 0 \, H\left(0, a_2(k_{l'}\right) 0 \dots 0] \, \mathcal{H}^{\dagger} f(x) \ , \ 1 \le l' \le r \ , \ 0 \le x \le 1. \tag{37}$$

If we assume that projection $f_{l'}(x) = 0$ when $0 \notin G_1$, from (31) one gets

$$f(x) = \sum_{l'=1}^{r} f_{l'(x)} + \sum_{l=1}^{q} f_l(x) = \mathcal{H}\mathcal{H}^{\dagger} f(x) \ \ , \ \ 0 \le x \le 1 \ .$$

Under these hypotheses on $f(x)$, one gets that $f_l(x)$ and $f_{l'}(x)$ are twice continuously differentiable on $[0, 1]$, and by (32) and (33) it follows that

$$\left.\begin{array}{rcl} (1 - \rho_0 b_1\,(i_l)) f_l(0) + b_1\,(i_l)\, f_l'(0) & = & 0 \\ -\left(\dfrac{1 - b_2(j_l) + \rho_0 b_1\,(i_l)\, b_2(j_l)}{b_1\,(i_l)}\right) f_l(1) + b_2(j_l) f_l'(1) & = & 0 \end{array}\right\} \tag{38}$$

and

$$\left.\begin{array}{rcl} f_{l'}(0) & = & 0 \\ a_2(k_0) f_{l'}(1) + f_{l'}'(1) & = & 0 \end{array}\right\} \tag{39}$$

If subspaces $ImH\left(b_1(i_l), b_2(j_l)\right)$ with $1 \le l \le q$ and $ImH\left(0, a_2(k_{l'})\right)$ with $1 \le l' \le r$, are invariant with respect the matrix A, i.e., that is, if

$$\left[I - H\left(b_1(i_1), b_2(j_1)\right)\left(H\left(b_1(i_1), b_2(j_1)\right)\right)^{\dagger}\right] AH\left(b_1(i_1), b_2(j_1)\right) = 0 \ , \ 1 \le l \le q \tag{40}$$

and

$$\left[I - H\left(0, a_2(k_{l'})\right)\left(H\left(0, a_2(k_{l'})\right)\right)^{\dagger}\right] AH\left(0, a_2(k_{l'})\right) = 0 \ , \ 0 \in G_1 \ \ 1 \le l' \le r \tag{41}$$

from (29), (30), (40) and (41), taking into account Theorem 2 of Ref. [15] and Theorem 4 of [16], one gets that the series

$$u(x, t, l) =$$

$$= \begin{cases} \displaystyle\sum_{\lambda_{mn} \in \mathcal{F}(l)} e^{-\lambda_{mn}^2(l)At} \left((1 - \rho_0 b_1(i_l) \sin\left(\lambda_n(l)x\right) - b_1(i_l)\lambda_n(i_l) \cos\left(\lambda_n(l)x\right)\right) C_{\lambda_{mn}(l)}, \text{if } 0 \notin \mathcal{F}(l) \\ \\ (1 - \rho_0 b_1(i_l))\, x - b_1(i_l) C_{0(l)} + \\ \displaystyle + \sum_{\lambda_{mn} \in \mathcal{F}(l')} e^{-\lambda_{mn}^2(l)At} \left((1 - \rho_0 b_1(i_l) \sin\left(\lambda_n(l)x\right) - b_1(i_l)\lambda_n(i_l) \cos\left(\lambda_n(l)x\right)\right) C_{\lambda_{mn}(l)}, \text{if } 0 \in \mathcal{F}(l) \end{cases} \tag{42}$$

and

$$u(x,t,l') = \begin{cases} \displaystyle\sum_{\lambda_{mn}\in\mathcal{F}(l')} e^{-\lambda_{mn}^2(l')At}\sin(\lambda_n(l')x)\,C_{0(l')} & , \quad \text{if } a_2(k_{l'}) \neq -1 \\ x\,C_{0(l')} + \displaystyle\sum_{\lambda_{mn}\in\mathcal{F}(l')} e^{-\lambda_{mn}^2(l')At}\sin(\lambda_n(l')x)C_{\lambda_{mn}(l)} & , \quad \text{if } a_2(k_{l'}) = -1 \end{cases} \tag{43}$$

if $0 \in G_1$, where $\mathcal{F}(l)$, $\lambda_n(l)$ and $C_{\lambda_{mn}(l)}$, are given by Theorem 2 of Ref. [15], are solutions of problem (1)–(3) with the initial conditions

$$\left.\begin{array}{rcl} u(x,0,l) & = & f_l(x) \\ u(x,0,l') & = & f_{l'}(x) \end{array}\right\} , \quad 0 \le x \le 1. \tag{44}$$

Thus

$$u(x,t) = \sum_{l=1}^{q} u(x,t,l) + \sum_{l'=1}^{r} u(x,t,l') \tag{45}$$

considering $u(x,t,l') = 0$ if $0 \notin G_1$, is an exact solution of problem (1)–(4).

Summarizing, the following result has been established:

Theorem 2.1. *Let be $A \in \mathbb{C}^{m\times m}$, satisfying condition (5). Let $\widetilde{B_1}$, $\widetilde{B_2}$ and $\widetilde{A_2}$ be matrices defined by (8) and (11). We assume that conditions (12) and (15) holds. Let be S_2, S_3 and matrix $\mathcal{H}$ defined by (27), (28) and (29)–(30) respectively, and let be $f(x)$ a vector valued function twice continuously differentiable on $[0,1]$ which satisfies conditions (31)-(32) and (38)-(39). Under these hypotheses, vector valued function $u(x,t)$ defined by (45), where $u(x,t,l)$ and $u(x,t,l')$ are defined by (42) and (43) respectively, is an exact solution of problem (1)–(4).*

3. Algorithm and Example

We can summarize the process to calculate the solution of the homogeneous problem with homogeneous conditions (1)–(4) in Algorithm 1.

Algorithm 1 Solution of homogeneous problem (1)–(4)

1: Check that matrix A satisfies (5).

2: Check that the block matrix $\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}$ is regular.

3: Determine a number $\rho_0 \in \mathbb{R}$ so that the matrix pencil $A_1 + \rho_0 B_1$ is regular.

4: Determine matrices $\widetilde{A_1}$, $\widetilde{B_1}$, defined by (8), and determine matrices $\widetilde{A_2}$, $\widetilde{B_2}$ defined by (11).

5: Determine $\sigma(\widetilde{B_1}), \sigma(\widetilde{B_1}), \sigma(\widetilde{B_1})$ and $G_i, i = 1, 2, 3$, defined by (22).

6: Build matrices $H(b_1(i), b_2(j))$ defined by (23), and if $0 \in G_1$, build matrices $H(0, a_2(k))$ defined by (25). Discard the null matrices.

7: Select matrices $H(b_1(i), b_2(j))$ which are invariant by matrix A, i.e., satisfy condition (40). Let q be the number of these matrices. If $0 \in G_1$, select matrices $H(0, a_2(k))$ which are invariant by matrix A, i.e., satisfy condition (41). Let r be the number of these matrices.

8: Build matrix $\mathcal{H}$ defined by (29) if $0 \notin G_1$ and by (30) if $0 \in G_1$.

9: Check that $f \in C^2([0, 1])$ and satisfies condition (31).

10: Check that for each non null matrices $H(b_1(i), b_2(j))$ invariant by matrix A, conditions (32) holds, and if $0 \in G_1$, that matrices $H(0, a_2(k))$ satisfy conditions (33).

11: For each non null matrices $H(b_1(i), b_2(j))$ invariant by matrix A, build projection $f_l(x)$ of $f(x)$ on subspace $ImH(b_1(i), b_2(j))$ defined by (36). Check that these projections satisfy condicions (38).

12: If $0 \in G_1$, for each non null matrices $H(0, a_2(k))$ invariant by matrix A, build projection $f_{l'}(x)$ of $f(x)$ on subspace $ImH(0, a_2(k))$ defined by (37). Check that these projections satisfy condicions (39).

If these conditions are not satisfied, algorithm stops because it is not possible find the solution of problem (1)–(4) for the given data. Otherwise, we have two possibilities:

Option A: If $0 \notin G_1$, go to algorithm 2.

Option B: If $0 \in G_1$, go to algorithm 3.

Algorithm 2 Algorithm for the option A of algorithm 1. For each non null matrices $H\left(b_1(i), b_2(j)\right)$ invariant by matrix A, we obtain the solution of the problem

$u_t(x,t,l) = Au_{xx}(x,t,l)\ ,\ 0 < x < 1\ ,\ t > 0$

$A_1 u(0,t,l) + B_1 u_x(0,t,l) = 0\ ,\ t > 0$

$A_2 u(1,t,l) + B_2 u_x(1,t,l) = 0\ ,\ t > 0$

$u(x,0) = f_l(x)\ ,\ 0 \leq x \leq 1$

1: For each non null matrices $H\left(b_1(i), b_2(j)\right)$ invariant by matrix A, take $b_1 = b_1(i), b_2 = b_2(j)$.
2: Determine the positive solutions of the equation (16) of Ref. [15] and determine the set $\mathcal{F}$ given by (27) of Ref. [15]
3: Determine degree p of minimal polynomial of matrix A.
4: Build block matrix $G_\lambda(\rho_0)$ defined by (31) of Ref. [15].
5: Determine $\lambda \in \mathcal{F}$ so that $Rank\left(G_\lambda(\rho_0)\right) < m$.
6: Include the eigenvalue $\lambda = 0$ if $1 \in \sigma\left(-\widetilde{A_2}\widetilde{A_1}\right)$.
7: Determine α given by (44) of Ref. [15].
8: Determine vectors C_{λ_n} defined by (47) of Ref. [15].
9: Determine functions X_{λ_n} defined by (41) of Ref. [15].
10: Determine solution $u(x,t,l)$ given by (42). Return to start the procedure from step 1 taking a different matrix $H\left(b_1(i), b_2(j)\right)$. This cycle will be repeated until maximum of q times.
11: Determine solution of (1)–(4) in the form given by $u(x,t) = \sum_{l=1}^{q} u(x,t,l)$ defined by (45).

Algorithm 3 Algorithm for the option B of algorithm 1 ($0 \in G_1$). Furthermore the solutions $u(x,t,l)$ obtained by the algorithm 2, for each non null matrices $H\left(0, a_2(k)\right)$ invariant by matrix A, we obtain the solution of the problem

$u_t(x,t,l') = Au_{xx}(x,t,l')\ ,\ 0 < x < 1\ ,\ t > 0$

$A_1 u(0,t,l') + B_1 u_x(0,t,l') = 0\ ,\ t > 0$

$A_2 u(1,t,l') + B_2 u_x(1,t,l') = 0\ ,\ t > 0$

$u(x,0) = f_{l'}(x)\ ,\ 0 \leq x \leq 1$

1: Determine, using algorithm 2, each solution $u(x,t,l)$ asociated to the non null matrices $H\left(b_1(i), b_2(j)\right)$ invariant by matrix A. Avoid step 11 in this algorithm 2, and continue the algorithm 3 in step 2.
2: For each non null matrices $H\left(0, a_2(k)\right)$ invariant by matrix A, take $a_2 = a_2(k)$.
3: Determine the positive solutions of (2.5) of Ref. [16] and determine set $\mathcal{F}$, denoted $\mathcal{F}_\star$, given by (2.8) of Ref. [16].
4: Build the block matrix $G_\lambda(\rho_0)$ defined by (2.12) of Ref. [16].
5: Determine $\lambda \in \mathcal{F}_\star$ so that $Rank\left(G_\lambda(\rho_0)\right) < m$.
6: Include the eigenvalue $\lambda = 0$ if $1 \in \sigma\left(-\widetilde{A_2}\widetilde{A_1}\right)$.
7: Determine α given by (3.6) of Ref. [16].
8: Determine vectors C_{λ_n} defined by (3.9) of Ref. [16].
9: Determine solution $u(x,t,l')$ given by (43). Return to start the procedure from step 2 taking a different matrix $H\left(0, a_2(k)\right)$. This cycle will be repeated until maximum of r times.
10: Determine solution $u(x,t)$ defined by (45).

Example 3.1. *We will consider the homogeneous parabolic problem with homogeneous conditions (1)–(4), where matrix $A \in \mathbb{C}^{4\times 4}$ is chosen as*

$$A = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 1 & 0 & -2 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{46}$$

and the 4×4 *matrices* $A_i, B_i, i \in \{1, 2\}$ *are*

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{47}$$

We will follow the Algorithm 1 step by step.

1. *Matrix A satisfies condition (5) because* $\sigma(A) = \{1, 2\}$.

2. *The block matrix*

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

is regular.

3. *Note that although* A_1 *is singular, taking* $\rho_0 = 1 \in \mathbb{R}$ *the matrix pencil* $A_1 + \rho_0 B_1 = I_{4\times 4}$ *is regular. Therefore, we take* $\rho_0 = 1$.

4. *Matrices* $\widetilde{A_1}$ *and* $\widetilde{B_1}$ *are given by*

$$\widetilde{A_1} = (A_1 + \rho_0 B_1)^{-1} A_1 = A_1, \; \widetilde{B_1} = (A_1 + \rho_0 B_1)^{-1} B_1 = B_1, \tag{48}$$

Matrices $\widetilde{A_2}$ *and* $\widetilde{B_2}$ *are given by*

$$\left.\begin{aligned} \widetilde{A_2} &= \left(B_2 - (A_2 + \rho_0 B_2)\widetilde{B_1}\right)^{-1} A_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \widetilde{B_2} &= \left(B_2 - (A_2 + \rho_0 B_2)\widetilde{B_1}\right)^{-1} B_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}\right\} \tag{49}$$

5. *We have*

$$\sigma\left(\widetilde{B_1}\right) = \{0,1\}\ ,\ \sigma\left(\widetilde{B_2}\right) = \{0,-1,1\}\ ,\ \sigma\left(\widetilde{A_2}\right) = \{0,-1\}$$

By (22) we define

$$\left.\begin{array}{lcl} G_1 &=& \{b_1(1)=0, b_1(2)=1\}\ ,\ s_1=2 \\ G_2 &=& \{b_2(1)=0, b_2(2)=-1, b_2(3)=1\}\ ,\ s_2=3 \\ G_3 &=& \{a_2(1)=0, a_2(2)=-1\}\ ,\ s_3=2. \end{array}\right\}$$

6. *We have to consider matrices* $H(1,0), H(1,-1)$, $H(1,1)$ *defined by (23) and, as* $0 \in G_1$, *we have also to consider matrices* $H(0,0), H(0,-1)$ *defined by (25):*

$$H(1,0) = \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}\ ,\ H(1,-1) = \begin{pmatrix} 1/2&1/2&0&0\\1/2&1/2&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$$

$$H(1,1) = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}\ ,\ H(0,0) = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}$$

$$H(0,-1) = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}.$$

We discard null matrices $H(1,1)$ *and* $H(0,-1)$.

7. *We check if subspaces* $ImH(1,0), ImH(1,-1)$ *are invariant by matrix A, i.e., matrices* $H(1,0), H(1,-1)$ *satisfy condition (40) and if subspace* $ImH(0,0)$ *is invariant by matrix A, i.e., matrix* $H(0,0)$ *satisfies condition (41).*

$$\left[I - H(1,0)\left(H(1,0)\right)^{\dagger}\right] AH(1,0) = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$$

thus $ImH(1,0)$ *is invariant by matrix A.*

$$\left[I - H(1,-1)\left(H(1,0)\right)^{\dagger}\right] AH(1,-1) = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\-1/2&-1/2&0&0\\0&0&0&0 \end{pmatrix}$$

thus $ImH(1,-1)$ *is not invariant by matrix A. Then* $q = 1$.

$$\left[I - H(0,0)\left(H(0,0)\right)^{\dagger}\right] AH(0,0) = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$$

thus $ImH(0,0)$ *is invariant by matrix* A*. Then* $r = 1$.

8. *Taking into account that* $0 \in G_1$*, we build the block matrix* $\mathcal{H}$ *defined by (30):*

$$\mathcal{H} = [H(1,0)\, H(0,0)] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

9. *Let* $f(x) = (f_1(x), f_2(x), f_3(x), f_4(x))^T$ *be a vector valued function which verifies* $f \in C^2([0,1])$ *and* $f(x) \in Im(\mathcal{H})$*, i.e.,* $f(x)$ *satisfies (31):*

$$\left(I - \mathcal{H}\mathcal{H}^{\dagger}\right) f(x) = \begin{pmatrix} f_1(x) \\ 0 \\ 0 \\ f_4(x) \end{pmatrix}, 0 \le x \le 1$$

so then $f(x) \in Im(\mathcal{H})$ *only if* $f(x) = (0, f_2(x), f_3(x), 0)^t$.

10. *As we have only a non null matrix of the form* $H(b_1(i), b_2(j))$ *invariant by matrix* A*,* $H(1,0)$*, we have to check condition (32) for values* $b_1 = 1, b_2 = 0$*. Condition (32) takes the form*

$$[H(1,0)\ \Theta]\, \mathcal{H}^{\dagger} \left((1-\rho_0 b_1) f(0) + b_1 f'(0)\right) = \begin{pmatrix} 0 \\ f_2'(0) \\ 0 \\ 0 \end{pmatrix}$$

then, we have to take $f_2'(0) = 0$.

Furthermore, the second identity of (32) take the form

$$[H(1,0)\ \Theta]\, \mathcal{H}^{\dagger} \left(-\left(\frac{1-b_2+\rho_0 b_1 b_2}{b_1}\right) f(1) + b_2 f'(1)\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which does not impose any new restriction on function $f(x)$*. As we have only a non null matrix of the form* $H(0, a_2(k))$ *invariant by matrix* A*,* $H(0,0)$*, we have to check condition (33) for the value* $a_2 = 0$*, i.e.:*

$$[\Theta\, H(0,0)]\, \mathcal{H}^{\dagger} f(0) = \begin{pmatrix} 0 \\ 0 \\ f_3(0) \\ 0 \end{pmatrix},$$

then, we have to take $f_3(0) = 0$*. Also, condition (33) takes the form*

$$[\Theta\,, H(0,0)]\, \mathcal{H}^{\dagger} \left(a_2 f(1) + f'(1)\right) = \begin{pmatrix} 0 \\ 0 \\ f_3'(1) \\ 0 \end{pmatrix},$$

thus we have to take $f_3'(1) = 0$. *Then, vector valued function* $f(x)$ *have to satisfy the following conditions:*

$$f(x) = (0, f_2(x), f_3(x), 0)^T \ , \ f_2'(0) = f_3(0) = f_3'(1) = 0. \tag{50}$$

11 − 12. *Projection* $f_l(x)$ *of* $f(x)$ *on subspace* $ImH(1,0)$, *by (36), is given by*

$$f_l(x) = [H(1,0)\ 0_{4\times 4}]\,\mathcal{H}^{\dagger} f(x) = \begin{pmatrix} 0 \\ f_2(x) \\ 0 \\ 0 \end{pmatrix},$$

and projection $f_{l'}(x)$ *of* $f(x)$ *on subspace* $ImH(0,0)$, *by (37), is given by*

$$f_{l'}(x) = [0_{4\times 4}\, H(0,0)]\,\mathcal{H}^{\dagger} f(x) = \begin{pmatrix} 0 \\ 0 \\ f_3(x) \\ 0 \end{pmatrix},$$

trivially fulfilled $f(x) = f_l(x) + f_{l'}(x)$. *Each projection have to satisfy conditions (38) and (39) respectively. For* $f_l(x)$, *condition (38) indicates that, taking* $\rho_0 = 1, b_1 = 1, b_2 = 0$, *one gets*

$$\left.\begin{array}{rcl} (1-\rho_0 b_1) f_l(0) + b_1 f_l'(0) & = & \begin{pmatrix} 0 \\ f_2'(0) \\ 0 \\ 0 \end{pmatrix} \\ -\left(\dfrac{1-b_2+\rho_0 b_1 b_2}{b_1}\right) f_l(1) + b_2 f_l'(1) & = & \begin{pmatrix} 0 \\ -f_2(1) \\ 0 \\ 0 \end{pmatrix} \end{array}\right\}$$

which means that at the conditions given in (50) we must to add that $f_2(1) = 0$, *i.e.,* $f(x)$ *have to verify the following conditions:*

$$f(x) = \begin{pmatrix} 0 \\ f_2(x) \\ f_3(x) \\ 0 \end{pmatrix}, \ f_2(1) = f_2'(0) = f_3(0) = f_3'(1) = 0. \tag{51}$$

For $f_{l'}(x)$, *condition (39) indicates that, taking* $\rho_0 = 1, a_2 = 0$, *one gets*

$$\left.\begin{array}{rcl} f_{l'}(0) & = & \begin{pmatrix} 0 \\ 0 \\ f_3(0) \\ 0 \end{pmatrix} \\ a_2(k_0) f_{l'}(1) + f_{l'}'(1) & = & \begin{pmatrix} 0 \\ 0 \\ f_3'(1) \\ 0 \end{pmatrix} \end{array}\right\}$$

which does not add any new restrictions on function $f(x)$.

We consider, for example, the vector valued function

$$f(x) = \begin{pmatrix} 0 \\ x^2(x-1)^3 \\ x^2-2x \\ 0 \end{pmatrix}, \tag{52}$$

which satisfy all the conditions given in (51).

As $0 \in G_1$, we are in the option B of algorithm 1, and we have to go to algorithm 3.

We continue with the algorithm 3.

1. *We must to determine, using algorithm 2, each solution $u(x,t,l)$ associated with non null matrices $H(b1(i), b2(j))$ invariant by matrix A. In this example we have only one matrix, given by $H(1,0)$, then we continue with algorithm 2.*

We continue with the algorithm 2.

1. *We take $b_1 = 1$, $b_2 = 0$ and $f_l(x)$ given by* $f_l(x) = \begin{pmatrix} 0 \\ x^2(x-1)^3 \\ 0 \\ 0 \end{pmatrix}$.

2. *Equation (16) of Ref. [15] take the form*

$$\lambda \ \cot(\lambda) = 0\,. \tag{53}$$

It is easy to show that for each $k \geq 1$, there exists an exact solution of (53) given by $\lambda_k = \frac{\pi}{2} + k\pi \in \,]k\pi, (k+1)\pi[$, with an additional solution $\lambda_0 = \frac{\pi}{2} \in \,]0, \pi[$ because

$$\frac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1} = 0 < 1.$$

Thus, we have a numerable family of solutions of (53), $\mathcal{F}$, given by formula (27) of Ref. [15]:

$$\mathcal{F} = \left\{ \lambda_k = \frac{\pi}{2} + k\pi; \lambda_k \in (k\pi, (k+1)\pi)\,,\ k \geq 1 \right\} \cup \mathcal{F}_0\,,\ \mathcal{F}_0 = \left\{ \lambda_0 = \frac{\pi}{2} \right\}. \tag{54}$$

3. *The minimal polynomial of matrix A is given by*

$$p(x) = (x-2)^2(x-1)^2,$$

then $p = 4$.

4. *If λ is a positive solution of (53), the block matrix $G_\lambda(\rho_0)$ defined by (31) of Ref. [15] takes the form:*

$$G_\lambda(1) = \left(\begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -5 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & -10 \\ 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -\lambda^2 & 0 & 0 & 0 \\ -\lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline -2\lambda^2 & 0 & 0 & \lambda^2 \\ -2\lambda^2 & 0 & 0 & \lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline -4\lambda^2 & 0 & 0 & 3\lambda^2 \\ -4\lambda^2 & 0 & 0 & 3\lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline -8\lambda^2 & 0 & 0 & 7\lambda^2 \\ -8\lambda^2 & 0 & 0 & 7\lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

5. *Since the second and third columns of $G_\lambda(1)$ are zero,we have that Rank $(G_\lambda(1)) < 4$. Thus, each one of the positive solutions given by (54) is an eigenvalue, see Ref. [15].*

6. *It is trivial to check that $1 \notin \sigma\left(-\widetilde{A}_2\widetilde{A}_1\right)$, because*

$$-\widetilde{A}_2\widetilde{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \sigma\left(-\widetilde{A}_2\widetilde{A}_1\right) = \{0\},$$

Then we do not include 0 as an eigenvalue.

7. *Taking into account that $\dfrac{(1-b_2+\rho_0 b_1 b_2)(1-\rho_0 b_1)}{b_1} = 0 < 1$, one gets $\alpha = 0$.*

8. *Vectors C_{λ_n} defined by (47) of Ref. [15] take the values*

$$C_{\lambda_n} = \frac{384\left(80 - 16(-1)^n(2n+1)\pi - 3(\pi+2n\pi)^2\right)}{\pi^7(2k+1)^7}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \quad (55)$$

9. *Eigenfunctions asociates to eigenvalues $\lambda_n > 0$, are given by:*

$$X_{\lambda_n} = \left(\sin(\lambda_n x)\left(1-\rho_0 b_1\right) - \lambda_n \cos(\lambda_n x) b_1\right) C_{\lambda_n}. \tag{56}$$

Using the minimal theorem [4, p.571], one gets that

$$e^{Au} = \begin{pmatrix} e^{2u} & 0 & 0 & -e^{u}\left(-1+e^{u}\right) \\ e^{u}\left(-1+e^{u}\right) & e^{u} & 0 & -e^{u}\left(-1+e^{u}+u\right) \\ -e^{2u}u & 0 & e^{2u} & e^{2u}u \\ 0 & 0 & 0 & e^{u} \end{pmatrix}, \tag{57}$$

Next, by considering (57) with $u = -\left(\frac{\pi}{2}+n\pi\right)^2 t$ and simplifying, we obtain the value of $e^{-\left(\frac{\pi}{2}+n\pi\right)^2 At}$.

10. *Replacing C_{λ_n} given by (55) in (56), multiplying by the matrix $e^{-\left(\frac{\pi}{2}+n\pi\right)^2 At}$, we finally obtain the solution $u(x,t,l)$ given by*

$$u(x,t,l) = \sum_{n\geq 0} \frac{192 e^{-\frac{1}{4}(\pi+2n\pi)^2 t}\left(-80+(2n+1)\pi\left(16(-1)^n+(3+6n)\pi\right)\right)\cos\left(\frac{1}{2}(2n+1)\pi x\right)}{\pi^6(2n+1)^6}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

We avoid the step 11 of Algorithm 2.

We continue with the step 2 of the algorithm 3.

2. *We must to determine, using algorithm 3, each solution $u(x,t,l')$ associated with non null matrices $H(0, a_2(k))$ invariant by matrix A. In this example we have only one, given by $H(0,0)$, then we define*

$$a_2 = 0.$$

3. *Equation (2.5) given in Ref. [16] is the same equation (53) and have the same positive solutions $\lambda_k = \frac{\pi}{2} + k\pi \in (k\pi,(k+1)\pi)$, with an additional solution $\lambda_0 = \frac{\pi}{2} \in \,]0,\pi[$ because $-a_2 = 0 < 1$. Thus, set $\mathcal{F}$ defined by (2.8) of Ref. [16], and denoted $\mathcal{F}_\star$, is given by:*

$$\mathcal{F}_\star = \left\{\lambda_k = \frac{\pi}{2} + k\pi; \lambda_k \in (k\pi,(k+1)\pi),\ k \geq 1\right\} \cup \mathcal{F}_0 \tag{58}$$

where

$$\mathcal{F}_0 = \left\{\lambda_0 = \frac{\pi}{2}\right\}.$$

4. *If λ is a positive solution of (53) and $p = 4$, matrix $G_\lambda(\rho_0)$ defined by (2.12) of Ref. [16] take the form:*

$$G_\lambda(1) = \left(\begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -5 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & -10 \\ 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -\lambda^2 & 0 & 0 & 0 \\ -\lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline -2\lambda^2 & 0 & 0 & -\lambda^2 \\ -2\lambda^2 & 0 & 0 & -\lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline -4\lambda^2 & 0 & 0 & 3\lambda^2 \\ -4\lambda^2 & 0 & 0 & 3\lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline -8\lambda^2 & 0 & 0 & 7\lambda^2 \\ -8\lambda^2 & 0 & 0 & 7\lambda^2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

5. *Since the second and third columns of $G_\lambda(1)$ are zero, we have that Rank $(G_\lambda(1)) < 4$. Thus, each one of the positive solutions given by (58) is an eigenvalue, see Ref. [16].*

6. *As shown as $1 \notin \sigma\left(-\widetilde{A}_2\widetilde{A}_1\right)$, eigenvalue 0 is not added.*

7. *Taking into account that $0 \notin \mathcal{F}_\star$, one gets that α given by (3.6) of Ref. [16] takes the value $\alpha = 0$.*

8. *Vectors C_{λ_n} defined by (3.9) of Ref. [16] take the value:*

$$C_{\lambda_n} = -\frac{32}{\pi^3(2k+1)^3}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{59}$$

9. *Eigenfunctions X_{λ_n} asociates to eigenvalues $\lambda_n > 0$, are given by:*

$$X_{\lambda_n} = \sin(\lambda_n x). \tag{60}$$

Matrix e^{Au} has already been calculated in (57), the taking in (57) the value $u = -\left(\frac{\pi}{2} + n\pi\right)^2 t$ and simplifying, we obtain the value of $e^{-\left(\frac{\pi}{2}+n\pi\right)^2 At}$. Replacing values of C_{λ_n} given by (59) in (60), multiplying by the matrix $e^{-\left(\frac{\pi}{2}+n\pi\right)^2 At}$ and simplifying, we finally obtain the solution $u(x,t,l')$ given by

$$u(x,t,l') = \sum_{n\geq 0} -\frac{32e^{-\frac{1}{2}(\pi+2n\pi)^2 t}\sin\left(\frac{1}{2}(1+2n)\pi x\right)}{\pi^3(2n+1)^3}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{61}$$

10. *By (45), after simplification, we finally obtain the solution of problem (1)–(4) which is given by*

$$u(x,t) = u(x,t,l) + u(x,t,l')$$

$$= \sum_{n\geq 0} \begin{pmatrix} 0 \\ \dfrac{192e^{-\frac{1}{4}(\pi+2n\pi)^2 t}\left(-80+(2n+1)\pi(16(-1)^n+(3+6n)\pi)\right)\cos\left(\frac{1}{2}(2n+1)\pi x\right)}{\pi^6(2n+1)^6} \\ -\dfrac{32e^{-\frac{1}{2}(\pi+2n\pi)^2 t}\sin\left(\frac{1}{2}(2n+1)\pi x\right)}{\pi^3(2n+1)^3} \\ 0 \end{pmatrix}.$$

It is not difficult to show that the problem (1)–(4) with matrix A and matrices $A_i, B_i, i \in \{1,2\}$ defined by (46) and (47) respectively, for vector valued function $f(x)$ defined by (52) can not be solved *using the algorithms given in references [15] and [16]. In effect, to apply the proposed method in [15], condition (14), with $b_1 = 1, b_2 = 0$, have to be satisfied. In this case one gets*

$$Ker\left(\widetilde{B}_1 - I\right) \cap Ker\left(\widetilde{B}_2\right) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

then $f(x) \notin Ker\left(\widetilde{B}_1 - I\right) \cap Ker\left(\widetilde{B}_2\right)$. In the same form, to apply the proposed method in [16] condition (17) with $a_2 = 0$, have to be satisfied. But in this case one gets

$$Ker\left(\widetilde{B}_1\right) \cap Ker\left(\widetilde{A}_2\right) = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

then $f(x) \notin Ker\left(\widetilde{B}_1\right) \cap Ker\left(\widetilde{A}_2\right)$. ***Thus, we have achieved our objective to build the exact series solution of the problem (1)–(4) for more general vector valued functions $f(x)$ that the method proposed in references [15] and [16].***

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