On products of generalised supersoluble finite groups

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Abstract

In this paper, mutually sn-permutable subgroups of groups belonging to a class of generalised supersoluble groups are studied. Some analogs of known theorems on mutually sn-permutable products are established.

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1 Introduction and statement of results.

All groups considered here will be finite.

This paper is a natural continuation of a series of articles of Vasilev, Vasileva and Tyutyanov ([5, 6, 7]) where an interesting generalisation of the class of all supersoluble groups associated with a particular embedding of the Sylow subgroups is introduced and widely studied.

Following [5], we say that a subgroup H of a group G is $\mathbb{P}$-*subnormal* in G whenever either $H = G$ or there exists a chain of subgroups $H = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for every $i = 1, \dots, n$.

In [5, Lemma 1.4] some useful properties of the $\mathbb{P}$-subnormal subgroups are exhibited. They allow us to prove that a subgroup H of a soluble group G is $\mathbb{P}$-subnormal in G if and only if H is $\mathcal{U}$-subnormal in G in the sense of [3, Definition 6.1.2], where $\mathcal{U}$ is the class of all supersoluble groups.

By a well-known result of Huppert [10, Satz VI.9.2(b)], a group G is supersoluble if and only if every maximal subgroup of G is $\mathbb{P}$-subnormal. As a consequence, the supersoluble groups are exactly those groups in which every subgroup is $\mathbb{P}$-subnormal. Bearing in mind this result and the strong influence of the embedding of the Sylow subgroups on the structure of a group, the following extension of the class of all supersoluble groups introduced in [5] turns out to be natural.

Definition 1. A group G is called *widely supersoluble*, w-supersoluble for short, if every Sylow subgroup of G is $\mathbb{P}$-subnormal in G.

The results of [5, Section 2] showed that the class of all w-supersoluble groups, denoted by $w\mathcal{U}$, is a subgroup-closed saturated formation of soluble groups containing $\mathcal{U}$, which is locally defined by a formation function f such that for every prime p, $f(p)$ is composed of all soluble groups G whose Sylow subgroups are abelian of exponent dividing $p-1$.

Not every group in $w\mathcal{U}$ is supersoluble (see [5, Example 1]). However, every group in $w\mathcal{U}$ has an ordered Sylow tower of supersoluble type (see [5, Proposition 2.8]).

In [6, Section 4], the authors considered products of w-supersoluble groups, and proved the following remarkable result.

Theorem 1. *[6, Theorem 4.7] Let $G = AB$ be a group which is the product of two w-supersoluble subgroups A and B. If A and B are $\mathbb{P}$-subnormal in G and $G^{\mathcal{A}}$ is nilpotent, then G is w-supersoluble.*

Here $G^{\mathcal{A}}$ denotes the residual of G with respect to the formation $\mathcal{A}$ of all groups with abelian Sylow subgroups.

The example in [2, Example 4.1.32] shows that the nilpotency of the $\mathcal{A}$-residual is necessary in Theorem 1.

The main aim of this paper is to analyse mutually sn-permutable products of w-supersoluble groups. The idea to consider these products arises naturally from [6, Lemma 4.5]: if $G = AB$ is a product of two subgroups A and B, and B permutes with every subnormal subgroup of A and A is soluble, then B is $\mathbb{P}$-subnormal in G.

We recall that two subgroups A and B of a group G are said to be *mutually sn-permutable* if A permutes with all subnormal subgroups of B and B permutes with all subnormal subgroups of A. If every subnormal subgroup of A permutes with every subnormal subgroup of B, then we say that A and B are *totally sn-permutable.* If A and B are mutually (respectively totally) sn-permutable subgroups of a group $G = AB$, then we say that G is a mutually (respectively totally) sn-permutable product of A and B (see [2, Section 4.1] for more general definition).

Mutually and totally sn-permutable products were first considered by Carocca [4]; they were also studied in [1].

Unfortunately, the class of all w-supersoluble groups is not closed under taking mutually sn-permutable product as the following example shows.

Example 1. Let $X = \langle a, b : a^4 = 1 = b^2, a^b = a^{-1} \rangle$ be a dihedral group of order 8, and let $V = \langle v_1, v_2 \rangle$ be a vector space of dimension 2 over the field of 5 elements. Then V can be considered as X-module with the following action:

$$v_1^a = 3v_1,\ v_1^b = v_2,\ v_2^a = 2v_2,\ v_2^b = v_1$$

Let $G = V \rtimes X$ be the corresponding semidirect product, and consider the following subgroups of G:

$$A = V\langle a \rangle;\ B = \langle v_1 v_2 \rangle \times \langle b \rangle$$

Note that $G = AB$. It is clear that A is supersoluble, B is nilpotent and it is easy to see that G is the mutually sn-permutable product of A and B. But G is not w-supersoluble.

In [1, Theorem C], the authors prove that if $G = AB$ is the mutually sn-permutable product of the supersoluble subgroups A and B and the derived subgroup G' of G is nilpotent, then G is supersoluble.

The w-supersoluble version of this result follows directly from [6, Lemma 4.5] and Theorem 1, bearing in mind that every w-supersoluble group is soluble.

Theorem 2. *Let G be the mutually sn-permutable product of the subgroups A and B. If A and B are w-supersoluble and $G^{\mathcal{A}}$ is nilpotent, then G is w-supersoluble.*

On the other hand, the behaviour of minimal normal subgroups of factorized groups has been an important source of information about their structure. Stonehewer [11] proves that if N is a minimal normal subgroup of a group that can be written as the product $G = AB$ of two nilpotent subgroups A and B, then either AN or BN is nilpotent. This result is not true if we replace nilpotent by supersoluble or w-supersoluble. For instance, PSL(2,7) can be written as the product of two supersoluble subgroups. In [1, Theorem A], the authors obtain a supersoluble version of Stonehewer's result by asuming that the product is mutually sn-permutable. Our next theorem confirms that an analogous result holds for mutually sn-permutable products of w-supersoluble groups.

Theorem 3. *let $G = AB$ be the mutually sn-permutable product of the w-supersoluble subgroups A and B. If N is a minimal normal subgroup of G, then both AN and BN are w-supersoluble.*

As Example 1 illustrates, the mutually sn-permutable product of a nilpotent group and a w-supersoluble group is not necessarily w-supersoluble. However, if the nilpotent factor permutes with the Sylow subgroups of the w-supersoluble one, w-supersolubility is guaranteed.

Theorem 4. *Let $G = AB$ be the mutually sn-permutable product of the subgroups A and B, where A is w-supersoluble and B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G is w-supersoluble.*

In [1, Theorem D], the authors study metanilpotent mutually sn-permutable products of supersoluble groups and proved that they are supersoluble provided that the largest nilpotent quotients of the factors have coprime orders.

We obtain a result in this spirit, but we required that the factors are w-supersoluble.

Theorem 5. *Let $G = AB$ be the mutually sn-permutable product of the w-supersoluble subgroups A and B. If $(|A/A^{\mathcal{A}}|, |B/B^{\mathcal{A}}|) = 1$, then G is w-supersoluble.*

2 Proofs.

Proof of Theorem 3 By [5, Theorem 2.3], A and B are soluble. Therefore G is soluble by [4, Theorem 3]. Applying [6, Lemma 4.5], we have that A and B are $\mathbb{P}$-subnormal subgroups of G. Let N be a minimal normal subgroup of G. Then N is an elementary abelian p-group for some prime p. Next we see that BN is w-supersoluble. First of all, BN/N is w-supersoluble since $w\mathcal{U}$ is closed under taking epimorphic images. Let Q denote a Sylow q-subgroup of BN. Then QN/N is a Sylow q-subgroup of BN/N and so QN/N is $\mathbb{P}$-subnormal in BN/N. By [6, Lemma 3.1], QN is $\mathbb{P}$-subnormal in BN. If $q = p$, it follows that $N \leq Q$ and Q is $\mathbb{P}$-subnormal in BN. Therefore we may assume that $q \neq p$. Then every Sylow q-subgroup of BN is conjugate to a Sylow q-subgroup of B. Hence we may assume without loss of generality that Q is contained in B. Then Q is $\mathbb{P}$-subnormal in B since B is w-supersoluble. But B is $\mathbb{P}$-subnormal in G. Consequently, Q is $\mathbb{P}$-subnormal in G by [6, Lemma 3.1]. We can apply again [6, Lemma 3.1] to conclude that Q is $\mathbb{P}$-subnormal in BN. In both cases, Q is $\mathbb{P}$-subnormal in BN. Analogously, AN is w-supersoluble. This completes the proof of the theorem.

Proof of Theorem 4 Assume the result is not true and G is a minimal counterexample. Let L be a minimal normal subgroup of G. Then, by [2, Lemmas 4.1.8 and 4.1.10], $G/L = (AL/L)(BL/L)$ is a mutually sn-permutable product of the subgroups AL/L and BL/L and BL/L permutes with the Sylow subgroups of AL/L. Moreover, AL/L is w-supersoluble and BL/L is nilpotent. The minimal choice of G implies that G/L is a $w\mathcal{U}$-group. Since $w\mathcal{U}$ is a saturated formation of soluble groups, it follows that G is a primitive soluble group, and hence G has a unique minimal normal subgroup, N say; N is an elementary abelian p-group and $N = C_G(N) = F(G) = O_p(G)$. Applying Theorem 3, we know that AN and BN are w-supersoluble. Let q be the largest prime dividing the order of G and assume that $q \neq p$. We can suppose without loss of generality that q divides $|AN|$. Since AN has a Sylow tower of supersoluble type, we have that AN has a unique Sylow q-subgroup, $(AN)_q$ say. Then $(AN)_q$ centralizes N and thus $(AN)_q = 1$ since $C_G(N) = N$, which is a contradiction. Consequently p is the largest prime dividing $|G|$. Since G is a primitive soluble group, we can write $G = NM$, where M is a maximal subgroup of G and $N \cap M = 1$. Then M is w-supersoluble. Since $O_p(M) = 1$ by [8, Theorem A.15.6], and M is a Sylow tower group of supersoluble type, it follows that p does not divide the order of M and so N is a Sylow p-subgroup of G.

Assume that B is a p-group. Then $G = AN$ is w-supersoluble, contrary to assumption. Since B is nilpotent and N is self-centralising in G, it follows N is not contained in B and B has a non-trivial Hall p'-subgroup, $B_{p'}$ say. Then $AB_{p'}$ is a subgroup of G because the product is mutually sn-permutable. Then $1 \neq B_{p'}^G \leq AB_{p'}$ and hence $N \leq AB_{p'}$. Since N is a p-group, we have that N is contained in A.

Let $A_{p'}$ be a Hall p'-subgroup of A. Note that $1 \neq A_{p'}$ because BN is a proper subgroup of G. Since B permutes with every Sylow subgroup of A amd N is not contained in B, it follows that $A_{p'}B$ is a proper subgroup of G. However, $G = NA_{p'}B$ since $A_{p'}B$ contains a Hall p'-subgroup of G. Since $N \cap A_{p'}B = N \cap B$ is a normal subgroup of G and N is a minimal normal subgroup of G that is not contained in B, it follows that $N \cap B = 1$ and so B is a p'-group. Let N_1 be a minimal normal subgroup of A such that $N_1 \leq N$. Then N_1B is a subgroup of G and $N = N_1^G \leq N_1B$. Consequently

$N = N_1$. Since N is self-centralising in A, we have that N is the unique minimal normal subgroup of A and $O_{p'}(A) = 1$.

On the other hand, note A permutes with every Sylow subgroup of B. Thus G is the mutually Syl-permutable product of A and B. Applying [2, Proposition 4.1.16], we have that $A \cap B$ is a subnormal subgroup of G. Moreover $A \cap B$ is a p'-group and $O_{p'}(G) = 1$. This implies that $A \cap B = 1$. By [2, Proposition 4.1.16], $G = AB$ is the totally sn-permutable product of A and B.

Since A has a Sylow tower of supersoluble type and A is not a p-group, there exists a prime $q \neq p$ and a Sylow q-subgroup A_q of A such that NA_q is a normal subgroup of A. The hypotheses of the theorem implies that $X = (NA_q)B$ is a subgroup of G which is the totally sn-permutable product of NA_q and B. By [5, Theorem 2.13], $NA_q \leq A$ is supersoluble since it is metanilpotent. Applying [1, Theorem 1], we have that X is supersoluble. Also $O_{p'}(X) = 1$ and $O_p(X) = N$. Thus A_qB is an abelian group with exponent dividing $p - 1$. Hence B centralises A_q. Arguing analogously with every A-conjugate of A_q we obtain that B centralises the normal closure A_q^A. Since $N \leq A_q^A$, we have that $B \leq C_G(N) = N$. Consequently $B = 1$ and $G = A$. This final contradiction proves the theorem.

Proof of Theorem 5 Assume that the theorem is false and take a minimal counterexample $G = AB$. Arguing as in Theorem 4, we have that G is a primitive soluble group. Then $G = NM$, where N is the unique minimal normal subgroup of G, M is a maximal subgroup of G, $N \cap M = 1$ and $C_G(N) = N$. We also know that N is a p-group for some prime p. Similar arguments to those used in the proof of Theorem 4 allow us to conclude that p is the largest prime dividing the order of G and N is a Sylow p-subgroup of G. Applying Theorem 3, we know that AN and BN belong to $w\mathcal{U}$. Moreover $O_{p'}(AN) = O_{p'}(BN) = 1$. The local definition of the saturated formation $w\mathcal{U}$ implies that $A/A \cap N \simeq AN/N$ and $B/B \cap N \simeq BN/N$ have abelian Sylow subgroups. Therefore A and B have abelian Sylow subgroups. Furthermore $A^{\mathcal{A}} \leq N$ and $B^{\mathcal{A}} \leq N$ so that $(|AN/N|, |BN/N|) = (|A/A \cap N|, |B/B \cap N|) = 1$. Hence G has abelian Sylow subgroups. Consequently we can apply Theorem 2 to conclude that G is w-supersoluble. This final contradiction proves the theorem.

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