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Additional Information

An EWMA Control Chart for the Multivariate Coefficient of Variation

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Abstract

Monitoring the multivariate coefficient of variation over time is a natural choice when the focus is on stabilising the relative variability of a multivariate process, as is the case in a significant number of real situations in engineering, health sciences and finance, to name but a few areas. However, not many tools are available to practitioners with this aim. This paper introduces a new control chart to monitor the multivariate coefficient of variation through an EWMA scheme. Concrete methodologies to calculate the limits and evaluate the performance of the chart proposed, and determine the optimal values of the chart's parameters are derived, based on a theoretical study of the statistic being monitored. Computational experiments reveal that our proposal clearly outperforms existing alternatives, in terms of the average run length to detect an out-of-control state. A numerical example is included to show the efficiency of our chart when operating in practice.

Keywords: multivariate coefficient of variation, EWMA, average run length, doubly noncentral F distribution, Nelder-Mead method, trimmed mean

1. Introduction

Statistical process monitoring (SPM) is an essential, powerful instrument for ensuring quality in companies and industry. Research on this topic has provided

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practitioners with a wide range of different SPM tools adapted to the features of the process and the specific quality characteristic being monitored, as well as the requirements set for them.

In the past decade, more than twenty control charts aimed at monitoring the coefficient of variation (CV; the ratio between the standard deviation and the mean) of a normal distribution have been proposed, starting with the seminal Shewhart-like chart by Kang et al.[1]. Some recent works worth mentioning include a short-run variable sample size chart[2], a short-run variable sampling interval chart[3] and a variable-sampling-interval exponentially weighted moving average chart[4]. Monitoring the CV is of interest in processes where the mean and the standard deviation may vary but the ratio between them is expected to remain constant under in-control conditions. In other words, since the CV is a relative measure of dispersion, a control chart for the CV permits detection of unexpected shifts in the variability of the process with regards to its mean. Teoh et al.[5], among others, report examples from different domains in which monitoring the CV and keeping it constant is a goal, including material, mechanical, manufacturing and landfill engineering, as well as health sciences. The CV is also used in finance as a way to measure the investment risk with regards to the expected return, as shown by Sharpe[6].

Several authors have attempted to extend the CV to the multivariate case. The first definition of a multivariate coefficient of variation was given by Reyment[7]. Other proposals were later formulated by Van Valen[8], Voinov and Nikulin[9] and Albert and Zhang[10]; the latter also includes a review of all the former. A recent work by Aerts et al.[11] highlights the importance of using a multivariate coefficient of variation to measure the relative dispersion of multidimensional data and examines the sample properties (such as variability and robustness) of all the aforementioned proposals. They also adduce the difficulties of working with a coefficient of variation matrix rather than a single scalar, which is another possible approach[12–14].

SPM is not only applied to univariate quality characteristics but also to situations in which several (correlated) random variables are involved in the quality requirements being set and, therefore, have to be jointly monitored in order to ensure stable process performance. As pointed out by MacGregor and Kourti[15], this is actually a more realistic and appropriate approach to quality surveillance and improvement, since all the key quality characteristics for a given process or product are not usually independent of each other. A review on multivariate control charts can be found in Bersimis et al.[16].

In this context, far from being a marginal or narrow-scope subject, the development of specific tools for monitoring the multivariate coefficient of variation (MCV) is relevant insofar as it deals with process variability; and in fact, has been gaining attention recently. However, in contrast to the significant

amount of research generated by the univariate case, few papers addressing surveillance of the multivariate setting are available. Yeong et al.[17] introduced the first chart aimed at monitoring the MCV. Due to the skewness of the MCV, they proposed two one-sided, Shewhart-like control charts to separately control undesired increases and decreases in the MCV. In both charts, the statistic being plotted for each sample is the sample MCV (see the definition later in Section 2). They derived the cumulative distribution function of the sample MCV and its inverse, which were then used to define control limits based on a given assumed probability of false alarm. They also provided theoretical formulae to calculate performance measures, such as the average run length and the standard deviation of the run length for a given unacceptable shifted MCV, and they showed how to compute the so-called *expected average run length* for random variations of the MCV. Lim et al.[18] developed a control chart for the MCV based on run sums[19]. They also suggested simultaneously implementing two one-sided charts. Lim et al.[18] investigated the optimisation of their chart, in terms of the minimum average run length for different shift sizes. Their proposal was shown to outperform the chart introduced by Yeong et al.[17]. Besides this, Lim et al.[18] presented an interesting, real example about how their chart can be applied to monitoring the relative dispersion of the inner diameters of steel sleeves.

As far as we know, no other alternatives to monitor the MCV have been published; there is clearly room for improvement. Specifically, SPM for the MCV could benefit from relatively simple, well-known strategies, such as the exponentially weighted moving average chart (EWMA), which is known to improve the sensitivity of Shewhart-like charts for small shifts[20]. More precisely, EWMA charts[21] include information of past observations $Y_0, Y_1, ..., Y_{t-1}$ into each new sampled value Y_t by calculating a weighted average of them, recursively expressed as $Z_t = \lambda Y_t + (1 - \lambda)Z_{t-1}$, which is the actual value to be plotted and monitored. See more on this in Section 3.

The control chart presented in the current paper aims to provide a *new alternative* for monitoring the MCV using an EWMA scheme. It is based on the definition of the MCV made by Voinov and Nikulin[9], mainly because of comparability with the two previously proposed charts. Moreover, this formalisation of the MCV is concluded to be an advisable choice by Aerts et al.[11] for several reasons, namely: (1) it has an intuitive definition (see Section 2 below); (2) it is invariant under any linear scale transformation (as also highlighted by[10]); (3) its asymptotic variance is independent of the dimension of data being studied; and (4) it outperforms other alternatives in terms of robustness.

The rest of the paper is structured as follows: In Section 2, basic assumptions and definitions are given. Then, in Section 3, the chart to monitor the MCV based on an EWMA scheme is presented and described. Some necessary theoretical

results regarding the statistic being monitored are derived in Section 4. Sections 5 and 6 are aimed at further developing the implementation our chart, focusing on how to measure its performance and how to optimise the selection of the chart's parameters. Computational experiences including a comparison with previous proposals are reported in Section 7, and a numerical example aimed at showing the usefulness and efficiency of the chart being constructed can be found in Section 8. Conclusions about the performance of our proposal are gathered in Section 9.

2. Basic definitions and assumptions

Consider a p-dimensional quality characteristic $\mathbf{X}$ following a multivariate normal distribution with mean vector $\boldsymbol{\mu} \neq \mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$; that is, $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. We will assume a surveillance plan consisting of drawing independent n-unit random samples of $\mathbf{X}$ at equal time intervals. The i-th observation of the j-th component of $\mathbf{X}$ will be denoted by X_{ij}, $1 \leq i \leq n, 1 \leq j \leq p$; and the (column) vector $\mathbf{X}_i = \left(X_{i1}, \ldots, X_{ip}\right)^{\mathsf{T}}$ will stand for the i-th observation of $\mathbf{X}$, $1 \leq i \leq n$. The (population) *multivariate coefficient of variation* (MCV) of $\mathbf{X}$ is a scalar defined as[9]:

$$\gamma = \left(\boldsymbol{\mu}^{\mathsf{T}} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}\right)^{-\frac{1}{2}}. \tag{1}$$

Note that, by definition, $\gamma > 0$. In particular, when $p = 1$, this definition of γ reduces to the well-known one-dimensional coefficient of variation expressed in absolute value, $\sigma / |\mu|$, for $X \sim N(\mu, \sigma)$.

A (biased) natural estimator of the MCV is the sample MCV $\hat{\gamma}$,

$$\hat{\gamma} = \left(\bar{\mathbf{X}}^{\mathsf{T}} \mathbf{S}^{-1} \bar{\mathbf{X}}\right)^{-\frac{1}{2}}, \tag{2}$$

where

$$\bar{\mathbf{X}} = \left(\frac{1}{n} \sum_{i=1}^{n} X_{i1}, \ldots, \frac{1}{n} \sum_{i=1}^{n} X_{ip}\right)^{\mathsf{T}} \quad \text{and} \tag{3}$$

$$\mathbf{S} = \frac{1}{n-1} \sum_{i=1}^{n} \left(\mathbf{X}_i - \bar{\mathbf{X}}\right) \left(\mathbf{X}_i - \bar{\mathbf{X}}\right)^{\mathsf{T}}. \tag{4}$$

In both existing proposals by Yeong et al.[17] and Lim et al.[18] referred to in the introduction, the underlying process and the sampling strategy meet the hypothesis just stated, and the definition of MCV they use is also the one given by Voinov and Nikulin[9].

4

3. An EWMA chart for monitoring $\hat{\gamma}^2$

In this paper, we propose using an EWMA scheme to monitor $\hat{\gamma}^2$, the square of the MCV. EWMA control charts were first introduced by Roberts[21]. Generally speaking, they are known to improve the performance of Shewhart-like control charts against small shifts by adding information from past observations to the current ones. More precisely, according to an EWMA design, the following statistic will be computed and plotted in our chart for the MCV-squared for each sample $t \geq 1$:

$$Z_t = \lambda \hat{\gamma}_t^2 + (1 - \lambda)Z_{t-1}, \tag{5}$$

where $\hat{\gamma}_t^2$, calculated as in (2), is the value of the computed MCV-squared in sample t, and λ is a *smoothing parameter* to be fixed, $0 < \lambda \leq 1$. Z_0 will be initialised to $\mu_0(\hat{\gamma}^2)$, the expected value of $\hat{\gamma}^2$ when the process is *in control* or *on target*, i.e. when the value of the parameter γ is constant and equals a given *target* or assumed value γ_0; that is to say,

$$Z_0 = \mu_0(\hat{\gamma}^2) \equiv E[\hat{\gamma}^2 \mid n, p, \gamma = \gamma_0]. \tag{6}$$

This value $\mu_0(\hat{\gamma}^2)$ would also be considered as the *central line* (CL) of the control chart. The asymptotic limits of an EWMA chart are generally defined in terms of the asymptotic mean $\mu(Z_{+\infty})$ and the asymptotic standard deviation $\sigma(Z_{+\infty})$ of the statistic Z_t when the process is in control (i.e., when $\gamma = \gamma_0$), which are known to be[22]

$$\mu(Z_{+\infty}) = \mu_0(\hat{\gamma}^2),$$

$$\sigma(Z_{+\infty}) = \sqrt{\frac{\lambda}{2 - \lambda}} \sigma_0(\hat{\gamma}^2),$$

where $\sigma_0(\hat{\gamma}^2)$ is the standard deviation of $\hat{\gamma}^2$ when the process is *in control* or *on target*:

$$\sigma_0(\hat{\gamma}^2) \equiv \sqrt{\text{Var}[\hat{\gamma}^2 \mid n, p, \gamma = \gamma_0]}. \tag{7}$$

Exact analytic expressions to calculate both $\mu_0(\hat{\gamma}^2)$ and $\sigma_0(\hat{\gamma}^2)$ will be provided in Section 4 (see Equations (16) and (17)). By definition (see (2)), the value of $\hat{\gamma}^2$ cannot take values below zero; and, therefore, neither can Z_t, while neither of them are upperly bounded. Moreover, practitioners will be mainly interested in avoiding unforeseen rises in the relative variability of the multivariate process being monitored. This is why, in this case, we propose using a one-sided control chart to detect *unexpected increases* in γ. More precisely, we define the following *upper control limit* (UCL) for our chart:

$$\text{UCL} = \mu_0(\hat{\gamma}^2) + K\sqrt{\frac{\lambda}{2 - \lambda}} \sigma_0(\hat{\gamma}^2), \tag{8}$$

where $K > 0$ is a chart parameter that represents the distance between the CL and the UCL expressed in number of standard deviations of Z_t.

Both λ and K are constant parameters to be determined by the final user. The value of these parameters directly affects the performance of the chart —more specifically, (i) its ability to detect *abnormal or undesired shifts* of the parameter γ and (ii) its *expected false alarm rate*. In Section 6, we propose a means to automatically determine the values of λ and K in an optimal way, according to any given performance requirements stated by the user. Beforehand, Section 5 will detail how the performance of this chart can be measured.

4. Sampling distribution of $\hat{\gamma}^2$

In this section, necessary theoretical results regarding the sampling distribution of the MCV are derived. More precisely, an expression for the cumulative distribution function (cdf) of the square of the sample MCV will be obtained, as well as for its mean and its standard deviation.

4.1. cdf of $\hat{\gamma}^2$

Based on Wijsman[23], Yeong et al.[17] deduced that

$$\frac{n(n-p)}{(n-1)p\hat{\gamma}^2} \sim F\left(p, n-p, \frac{n}{\gamma^2}\right),\tag{9}$$

where $F(\nu_1, \nu_2, \delta)$ stands for the *(singly) noncentral F distribution* with ν_1 and ν_2 degrees of freedom and noncentrality parameter δ[24,25]. This distribution can be seen as a particular case of the *doubly noncentral F distribution* $F(\nu_1, \nu_2, \delta_1, \delta_2)$[25] with the second noncentrality parameter $\delta_2 = 0$. Actually, we will always use the notation with four parameters from now on, even if the last parameter $\delta_2 = 0$. As in the *ordinary* version of the F distribution, the following equivalence holds (by definition; see Equation (A.1) in Appendix A):

$$\left(F(\nu_1, \nu_2, \delta_1, \delta_2)\right)^{-1} \sim F(\nu_2, \nu_1, \delta_2, \delta_1).\tag{10}$$

More details on the singly and doubly noncentral F distributions can be found in Johnson et al.[26], Paolella[27], Walck[28] and Forbes et al.[29].

From (9) and (10), it follows that

$$\frac{(n-1)p}{n(n-p)}\hat{\gamma}^2 \sim F\left(n-p, p, 0, \frac{n}{\gamma^2}\right).\tag{11}$$

The cdf $F_{\hat{\gamma}^2}(u \mid n, p, \gamma)$ of $\hat{\gamma}^2$ can be easily derived from (9) in a similar way to that used for $\hat{\gamma}$ by Yeong et al.[17]:

$$F_{\hat{\gamma}^2}(u \mid n, p, \gamma) = 1 - F_{F'}\left(\frac{n(n-p)}{(n-1)pu} \,\middle|\, p, n-p, \frac{n}{\gamma^2}, 0\right),\tag{12}$$

where $F_{F'}$ stands for the cdf of a singly noncentral F distribution with degrees of freedom p and $n - p$ and noncentrality parameter n/γ^2.

4.2. Raw moments. Mean and standard deviation of $\hat{\gamma}^2$

4.2.1. General case

In order to calculate the mean and the standard deviation of the sample MCV-squared, $\hat{\gamma}^2$ (which will be mainly applied to obtain $\mu_0(\hat{\gamma}^2)$ and $\sigma_0(\hat{\gamma}^2)$), we introduce the following workable expressions for the *first* and *second raw moments* of $F'' \sim F(n - p, p, 0, n/\gamma^2)$:

$$\mu_1'(F'') = \frac{p}{2}\mathcal{C}\left(\frac{p}{2} - 1, -\frac{n}{2\gamma^2}\right), \tag{13}$$

$$\mu_2'(F'') = \frac{p^2}{4(p-4)}\left(\frac{2}{n-p} + 1\right)\left(2 - \left(\frac{n}{\gamma^2} + p - 4\right)\mathcal{C}\left(\frac{p}{2} - 1, -\frac{n}{2\gamma^2}\right)\right). \tag{14}$$

The term $\mathcal{C}(a, z)$ can be calculated as the following continued fraction:

$$\mathcal{C}(a, z) = \cfrac{1}{a + \cfrac{-az}{a + 1 + \cfrac{z}{a + 2 + \cfrac{-(a+1)z}{a + 3 + \cfrac{2z}{a + 4 + \cfrac{-(a+2)z}{a + 5 + \cfrac{3z}{a + 6 + \cdots}}}}}}}. \tag{15}$$

Appendix A shows how these equations were derived. Other equivalent expressions for either the raw moments of a doubly noncentral F variable or its mean and standard deviation are available in the literature (see[26] and[27], for instance). Expressions (13) to (15) have been successfully tested for computational accuracy by us, which is why we use and report them here.

Therefore, and taking Equation (11) into account, the average and the standard deviation of $\hat{\gamma}^2$ can be easily computed using

$$\mu(\hat{\gamma}^2) = \frac{n(n-p)}{(n-1)p}\mu_1'(F'') \quad \text{and} \tag{16}$$

$$\sigma(\hat{\gamma}^2) = \frac{n(n-p)}{(n-1)p}\sqrt{\mu_2'(F'') - (\mu_1'(F''))^2}. \tag{17}$$

4.2.2. Case $p \leq 4$

Note that, as mentioned in Appendix A, the first raw moment of F'', $\mu_1'(F'')$, is not defined when $p \leq 2$, while the second raw moment, $\mu_2'(F'')$, is not defined when $p \leq 4$. Therefore, expressions (16) and (17) are supposed to be valid only for $p > 2$ and $p > 4$, respectively; formally speaking, $\mu(\hat{\gamma}^2)$ and $\sigma(\hat{\gamma}^2)$ *do not exist* when $p \leq 2$ and $p \leq 4$, respectively.

In this paper, we only use $\mu(\hat{\gamma}^2)$ and $\sigma(\hat{\gamma}^2)$ as a means to calculate the CL and UCL of the chart being constructed (see Section 3), following the traditional scheme of EWMA charts. Continuing with this approach, we propose addressing those specific cases in which the dimension p of the quality characteristic being studied is less than or equal to 4 in the following way:

Case $p = 1$ (unidimensional) The expressions (13) and (14) we have derived for the raw moments of F'' *can* be calculated to a finite value when $p = 1$, and so do the mean and the standard deviation of the sample MCV-squared according to Equations (16) and (17), as long as $\mu_2'(F'') \geq (\mu_1'(F''))^2$. The values obtained in this way were validated using Monte-Carlo simulation, and therefore, we consider that they could still be used in their roles, despite the fact that according to probability theory, the distribution of $\hat{\gamma}^2$ does not have any mean or standard deviation in this case. The behaviour of this *extension* of $\mu(\hat{\gamma}^2)$ and $\sigma(\hat{\gamma}^2)$ to cases in which they do not theoretically exist can be seen in an example in Appendix B.

Case $p = 2$ In this case, both the first and the second raw moments of F'' compute to infinity, mostly because of the presence of a very skewed right-hand tail in the distribution of F''. We propose skipping this issue by calculating a *truncated* or *trimmed* version of them, according to the following procedure:

Step 1 Fix a small value ε (for instance, 10^{-4}, 10^{-5}, etc.) that will be used for right-trimming the distribution of F''.

Step 2 Calculate $u_0 = F_{F''}^{-1}(1 - \varepsilon | n - p, p, 0, n/\gamma^2)$ or, equivalently (recall (10)), $u_0 = 1/F_{F'}^{-1}(\varepsilon | p, n - p, n/\gamma^2, 0)$, where $F'' \sim F(n - p, p, 0, n/\gamma^2)$ and $F' \sim F(p, n - p, n/\gamma^2, 0)$. To minimise computational errors, the second way to calculate u_0 is recommended, since most of the available implementations of $F_{F''}^{-1}$ show accuracy issues when evaluated at values close to 1.

Step 3 Calculate the trimmed versions of $\mu_1'(F'')$ and $\mu_2'(F'')$ as:

$$\tilde{\mu}_1'(F'') = \frac{1}{1 - \varepsilon} \int_0^{u_0} u f_{F''}(u) du, \tag{18}$$

$$\tilde{\mu}_2'(F'') = \frac{1}{1 - \varepsilon} \int_0^{u_0} u^2 f_{F''}(u) du, \tag{19}$$

where $f_{F''}$ stands for the probability density function of F''. These integrals can be numerically calculated using the usual methods such as the adaptive Gauss-Kronrod rules (see[30], for instance).

Step 4 Similar to Equations (16) and (17) in the general case, calculate the trimmed versions of $\mu(\hat{\gamma}^2)$ and $\sigma(\hat{\gamma}^2)$ as follows:

$$\tilde{\mu}(\hat{\gamma}^2) = \frac{n(n-p)}{(n-1)p}\tilde{\mu}'_1(F''), \tag{20}$$

$$\tilde{\sigma}(\hat{\gamma}^2) = \frac{n(n-p)}{(n-1)p}\sqrt{\tilde{\mu}'_2(F'') - (\tilde{\mu}'_1(F''))^2}. \tag{21}$$

The reason why this is an appropriate way to substitute the nonexistent values of $\mu'_1(F'')$ and $\mu'_2(F'')$ is further developed in Appendix B.

Case $p = 3$ In this case, $\mu'_1(F'')$ and, therefore, $\mu(\hat{\gamma}^2)$ exist and can be normally computed to finite values using Equations (13) and (16), respectively. Besides, as in **Case $p = 1$**, $\mu'_2(F'')$ and $\sigma(\hat{\gamma}^2)$ do not theoretically exist, but the value returned by expressions (14) and (17) are acceptable according to Monte-Carlo simulation.

Case $p = 4$ In this case, $\mu'_1(F'')$ and $\mu(\hat{\gamma}^2)$ can be calculated once more as in the general case (Equations (13) and (16)), while $\mu'_2(F'')$ computes to infinity and, therefore, as in **Case $p = 2$**, the trimmed version of $\sigma(\hat{\gamma}^2)$ should be used, in a similar way to Equation (21):

$$\tilde{\sigma}(\hat{\gamma}^2) = \frac{n(n-p)}{(n-1)p}\sqrt{\tilde{\mu}'_2(F'') - (\mu'_1(F''))^2}, \tag{22}$$

with $\tilde{\mu}'_2(F'')$ calculated as in Equation (19).

To sum up, in the case $p \in \{1, 2, 3, 4\}$, appropriate *substitutes* for $\mu_0(\hat{\gamma}^2)$ and $\sigma_0(\hat{\gamma}^2)$ can be found, either by directly applying Equations (16) and (17) or by calculating their trimmed versions in Equations (20) and (21), or (22), as explained previously.

5. Performance measures

The *performance* of a given control chart with fixed sample size and fixed sampling interval is usually measured in terms of its *average run length* (ARL), that is to say, how many samples have to be drawn, on average, in order to obtain a point Z_t outside the control limits, depending on the actual value of the parameter under study (γ, in our case). The standard deviation of the run length (SDRL) can also be given as additional information.

In this paper, the approach based on *discrete-state Markov chains* is used to estimate the ARL and SDRL of a control chart such as the one described in Section 3 for a process operating under a given value of the MCV γ. Simulation will also be used as an auxiliary method in order to double check the accuracy of the estimations obtained with the Markov approach.

The ARL approximation by discrete-state Markov chains was first introduced by Brook and Evans[31] for the case of cumulative-sum control charts. Later, Lucas and Saccucci[22] explicitly extended it to EWMA charts. In the following, we briefly describe how to adapt it to the one-sided EWMA chart for $\hat{\gamma}^2$ that we are proposing here.

From a Markov-chain point of view, the ARL is the expected number of steps or transitions needed to reach the *absorbing state* of being outside the control limits starting from the *initial state*. More precisely, a control chart is actually seen as a *discrete-time continuous-state Markov process*, in which each sample t would be a step, each possible value of Z_t within the control limits would be a different *transient state*, and the range of values outside the control limits would represent a unique absorbing state. For the sake of computability, this model is approximated by a *discrete-state Markov chain*, in which the range within control limits is divided into s equally wide subintervals, each of them representing a transient state (see Figure 1). The larger the number of subintervals s, the better the approximation to the continuous original setting. Consequently, existing formulae can be used in order to calculate the expected number of steps to absorption. These formulae require the computation of the $s \times s$ matrix $\mathbf{Q}$ of *transition probabilities* between transient states and the column vector $\mathbf{q}$ of s initial probabilities. The following algorithm summarises the procedure of computing the ARL and SDRL of the EWMA chart proposed in Section 3, for a shifted MCV $\gamma = \gamma_1$.

Algorithm 1: Calculation of the ARL and SDRL of the EWMA chart for the MCV-squared

Input information: n, p, γ_0, γ_1, λ, UCL, s.

Step 1 Calculate the width of each subinterval $w = \text{UCL}/s$.

Step 2 Calculate the lower and upper limits and midpoint of each subinterval $[e_j^{\text{L}}, e_j^{\text{U}}]$:

$$
\begin{aligned}
e_j^{\text{L}} &= (j-1)w, \\
e_j^{\text{U}} &= jw, \\
h_j &= (e^{\text{L}} + e^{\text{U}})/2,
\end{aligned}
$$

$j = 1, \ldots, s$ (i.e., $e_{j-1}^U = e_j^L$, for $j = 2, \ldots, s$). The midpoints h_j will be considered as the *representatives* of each subinterval when calculating the transition probabilities (see **Step 3**).

Step 3 Calculate the elements of the matrix of transition probabilities $\mathbf{Q} = [q_{i,j}]_{i,j=1,\ldots,s}$:

$$q_{i,j} = \Pr(e_j^L \leq Z_t \leq e_j^U \mid Z_{t-1} = h_i)$$

using Equation (5):
$$= F_{\hat{\gamma}^2}\left(\frac{e_j^U - (1-\lambda)h_i}{\lambda} \,\middle|\, n, p, \gamma_1\right) - F_{\hat{\gamma}^2}\left(\frac{e_j^L - (1-\lambda)h_i}{\lambda} \,\middle|\, n, p, \gamma_1\right),$$

for $i, j = 1, \ldots, s$, using Equation (12).

Step 4 Calculate the vector of initial probabilities $\mathbf{q} = (q_1, \ldots, q_s)^\mathsf{T}$. All the components of this vector are equal to zero except the one corresponding to the subinterval containing the value $Z_0 = \mu_0(\hat{\gamma}^2)$ (which has already been calculated using Equation (16) or (20) with $\gamma = \gamma_0$ when computing
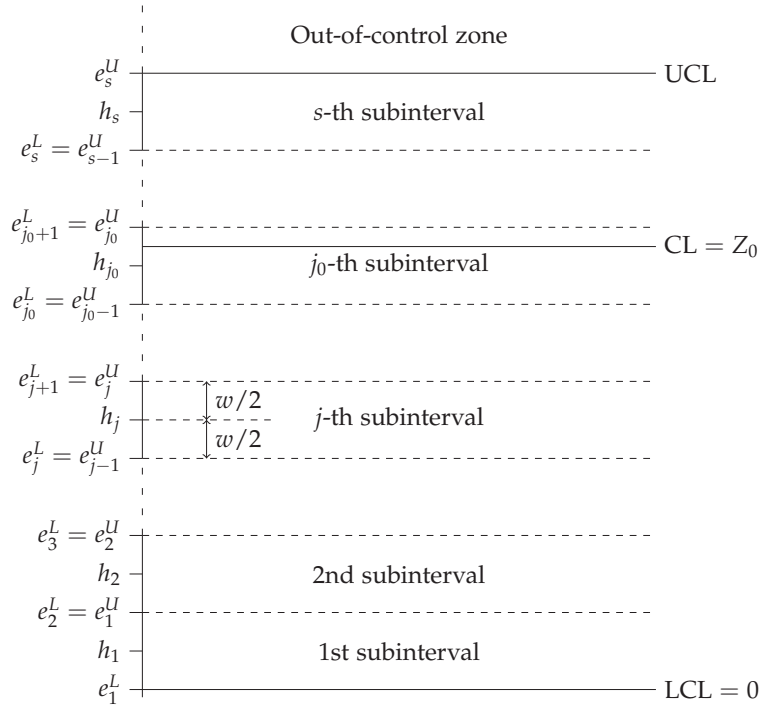

Figure 1: Subdivision of the in-control range into s equally spaced subintervals, according to the Markov approach explained in Algorithm 1.

11

the UCL). Therefore:

$$q_j = \begin{cases} 1, & \text{if } j = j_0 \\ 0, & \text{otherwise} \end{cases},$$

where $j_0 = \lceil \mu_0(\hat{\gamma}^2)/w \rceil$; $\lceil \cdot \rceil$ stands for the *ceiling* function, which returns the lowest integer being greater than or equal to a given real number.

Step 5 Return the ARL and SDRL, which can be calculated as follows:

$$\text{ARL}(\gamma_1 \mid n, p, \lambda, K) = \mathbf{q}^{\mathsf{T}}(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{1}, \tag{23}$$

$$\text{SDRL}(\gamma_1 \mid n, p, \lambda, K) = \sqrt{2\mathbf{q}^{\mathsf{T}}(\mathbf{I} - \mathbf{Q})^{-2}\mathbf{Q}\mathbf{1} - \text{ARL}^2 + \text{ARL}}, \tag{24}$$

where $\mathbf{I}$ stands for the $s \times s$ identity matrix and $\mathbf{1} = (1, \ldots, 1)^{\mathsf{T}}$.

The equations in Step 5 were introduced by Brook and Evans[31] and are either totally or partially reproduced by Montgomery[20], Lucas and Saccucci[22] and Castagliola et al.[32], among others.

Also, note that the initial probabilities calculated in Step 4 correspond to the so-called initial starting or zero state. Alternatively, a vector of steady-state probabilities could also be given; Lucas and Saccucci[22] explained how this vector can be calculated if necessary.

6. Optimal determination of the chart's parameters

6.1. Optimisation model and solving strategies

As explained in Section 3, it is advisable to provide practitioners with a method to determine the optimal values of the parameters λ and K that make the proposed EWMA chart meet some given performance requirements. Generally speaking, any control chart with a fixed sampling interval should present a small false alarm rate, which means a large enough ARL when the process is in control; at the same time, the ARL when the process is operating under undesired conditions should be as small as possible (see[33], for instance). In the case of our EWMA chart for the MCV-squared, both conditions can be simultaneously achieved by means of the following optimisation model:

$$\begin{cases} \text{Min} & f(\lambda, K) := \text{ARL}(\lambda, K \mid n, p, \gamma_1) \\ \text{s.t.:} & \text{ARL}(\lambda, K \mid n, p, \gamma_0) = \text{ARL}_0 \\ & 0 < \lambda \leq 1, \quad K > 0 \end{cases}. \tag{25}$$

In model (25), γ_0 represents the in-control value of the MCV, γ, while $\gamma_1 > \gamma_0$ stands for an unacceptably shifted value of γ; ARL_0 is the desired in-control

ARL (presumably high). All of them are parameters to be provided by the final user (representing the requirements to be fulfilled by the control chart being designed), together with n and p. It is, therefore, a *constrained nonlinear optimisation problem* with two continuous variables to be determined, λ and K (the chart's parameters).

Actually, since the only constraint in model (25) is an equality, it is somehow equivalent to an *unconstrained univariate problem*, as long as the value of any of the decision variables, λ or K, can be obtained from and is determined by the value of the other one through the equation in the model. That is why we suggest using the following approach.

Algorithm 2: Determination of the optimal values of the parameters of the EWMA chart for the MCV-squared

Stage 1 Implement a method to determine the value $K_0(\lambda)$ of K such that $\mathrm{ARL}(\lambda, K_0(\lambda) \mid n, p, \gamma_0) = \mathrm{ARL}_0$, for any given λ. This can be done by means of any appropriate root-finding technique. This method will be intensively used in **Stage 2**.

Stage 2 Look for the value λ^* such that $\lambda^* = \arg\min f(\lambda, K_0(\lambda))$, the only constraints being the bounds on λ: $0 < \lambda \le 1$ (in practice, a lower bound for λ such as 0.01, for instance, can be considered). This can be done by means of any suitable unconstrained nonlinear global optimisation method.

Stage 3 The optimal solution to model (25) is given by $(\lambda^*, K^* = K_0(\lambda^*))$.

Since an analytic, explicit expression for the ARL of our chart is not available (see Algorithm 1 in Section 5), the use of derivative-free methods in Stages 1 and 2 becomes almost necessary. As a result of its own definition, the behaviour of the ARL as a function of K —the rest of the parameters being fixed— is expected to be monotonic and smooth, which makes any well-performed (derivative-free) root-finding method suitable for Stage 1. Besides, the optimal value of λ in Stage 2 is also expected to be findable without much trouble using any classical method, since the search is limited to a bounded set, $]0, 1]$. Computationally speaking, the calculations of the ARL and $K_0(\lambda)$ at each iteration of the numerical method being applied in Stage 2 are, presumably, the main bottlenecks to pay attention to.

6.2. Implementation of the chart

As a guide for practitioners, we now summarise the steps to be followed to carry out a monitoring plan based on our proposal.

PHASE I

Step 1 Collect exploratory data from the process being studied, organised as m n-unit independent random samples, equally spaced in time. For each sample t, statistics such as $\hat{\gamma}_t$, $\bar{\mathbf{X}}_t$ and $\mathbf{S}_t$ can be calculated using Equations (2), (3) and (4). It is traditionally suggested that $m = 20$. See also general recommendations by Vining [34] regarding Phase I.

Step 2 Check whether it can be assumed that the MCV is constant in our process. As suggested by Yeong et al. [17], this can be done by performing a regression analysis based on the pairs of data $\{(\bar{\mathbf{X}}_t^\mathsf{T}\bar{\mathbf{X}}_t, \hat{\gamma}_t^2)\}_{t=1,2,\ldots,m}$ collected in **Step 1**. If no significant relationship between $\hat{\gamma}^2$ and $\bar{\mathbf{X}}^\mathsf{T}\bar{\mathbf{X}}$ can be inferred, then the MCV can be assumed to be constant.

Step 3 Determine or estimate the in-control MCV, γ_0. This can be carried out using the in-control samples collected in **Step 1**; additional random samples (up to a total of m in-control samples) can be drawn. More precisely, the following estimation for γ_0 was suggested by both Yeong et al. [17] and Lim et al. [18]: $\hat{\gamma}_0 = (\sum_{t=1}^m \hat{\gamma}_t^2 / m)^{1/2}$.

CHART DESIGN

Step 4 Decide the values for ARL_0 (ARL to be reached by the chart when in control) and γ_1 (an unacceptable, shifted value of γ to be detected, $\gamma_1 > \gamma_0$), according to the stated quality requirements to be met by the process.

Step 5 Calculate $\mu_0(\hat{\gamma}^2)$ and $\sigma_0(\hat{\gamma}^2)$ as explained Subsection 4.2, taking the value of γ_0 estimated in **Step 3**. These calculated values are used in **Steps 6** and **7**.

Step 6 Find λ^* and K^*, the *optimal values* for λ and K, by solving model (25) using Algorithm 2 as described in Subsection 6.1 above.

Step 7 Calculate the CL as $\mu_0(\hat{\gamma}^2)$ and the UCL of the chart according to Equation (8), using the optimal values for λ and K found in **Step 6**.

PHASE II

Step 8 Initialise the chart by setting $Z_0 = \text{CL} = \mu_0(\hat{\gamma}^2)$.

Step 9 Start the regular monitoring process, taking n-unit random samples at equal time intervals. For each sample t, calculate the EWMA statistic Z_t according to Equation (5). An out-of-control signal must be raised whenever $Z_t > \text{UCL}$.

This is the complete scheme to be followed to apply the chart from the beginning as operating under a *zero-state* situation. The steps and calculations (specially the computations of the ARL in Step 6) should be adapted for the *steady-state* case.

7. Computational experiments

7.1. Implementation details

The methodology described in the preceding sections was implemented in Scilab 6.0.0[35] on a computer running a Windows operating system. The following three functions were programmed:

- **Mean_and_StDev**(n, p, γ). This function returns the mean and the standard deviation of the sample MCV-squared of a n-sized random sample of a p-dimensional normal variable with a population MCV equal to γ, following the formulae deduced in Subsection 4.2.

- **ARL_and_SDRL**$(n, p, \gamma_0, \gamma_1, \lambda, K)$. This function returns the ARL and SDRL of the chart proposed in this paper, following Algorithm 1 in Section 5. Note that, in accordance with Algorithm 1, this function calls **Mean_and_StDev** to calculate the CL and the UCL of the chart by using Equations (6) and (8).

- **Optimal_λ_and_K**$(n, p, \gamma_0, \gamma_1, \text{ARL}_0)$. This function returns the optimal value of parameters of the chart proposed in this paper, following Algorithm 2 in Subsection 6.1. In accordance with Algorithm 2, this function calls **ARL_and_SDRL** when needed.

A total of 300 nested fractions to calculate $\mathcal{C}$ according to Equation (15) in **Mean_and_StDev** were found to converge with sufficient accuracy in preliminary tests, as well as a number of $s = 400$ Markov-chain states in ARL computations (see Section 5) in **ARL_and_SDRL**. Built-in Scilab functions were used for Stages 1 and 2 of the optimisation strategy (see Section 6) in **Optimal_λ_and_K**. More precisely, a modification of the Powell hybrid method[36] was applied as a root-finding technique, while the Nelder-Mead algorithm[37] was used as a derivative-free minimisation technique. All these details are given for the sake of reproducibility.

7.2. Benchmark instances

Below, we show the outcomes from calculating the optimal values of λ and K for the set of instances resulting from combining these parameter values: $p \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, $n \in \{5, 10, 15, 20\}$, $\gamma_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$,

$\tau \in \{1.1, 1.2, 1.25, 1.5, 2.0\}$, where $\tau = \gamma_1/\gamma_0$; cases not satisfying $n > p$ have been omitted. For the sake of comparison with the charts by Yeong et al. [17] and Lim et al. [18], ARL_0 was set to 370.4, which is a usual reference value since it happens to be the ARL for a Shewhart $\bar{X}$ chart with typical control limits $\mu_0 \pm 3\sigma_0/\sqrt{n}$ for a normally-distributed in-control process.

The purpose of this extensive set of performance calculations under different input settings is three-fold: it allows us to show the efficiency of our method and to compare it against existing proposals; it also facilitates comparison with future research on this topic; and finally, it provides practitioners with useful and easy-to-use look-up tables.

The results of a performance comparison with Yeong et al. [17] and Lim et al. [18] in terms of the out-of-control ARL and SDRL (i.e., $ARL(\gamma_1)$ and $SDRL(\gamma_1)$) are also presented. More precisely, our chart (which will be identified as *EWMA chart* for short) was separately compared against the *upward* chart (the one designed to detect increases in γ) from each of these papers, only using those instances with results reported by the authors in each case; namely, those combining $p \in \{2, 3, 4\}$, $n \in \{5, 10, 15\}$, $\gamma_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$, $\tau \in \{1.2, 1.5, 2.0\}$ for the chart by Yeong et al. [17] (abbreviated *MCV chart* when needed); and $p \in \{2, 3\}$, $n \in \{5, 10\}$, $\gamma_0 \in \{0.1, 0.3, 0.5\}$, $\tau \in \{1.1, 1.25, 1.5\}$ in the case of Lim et al. [18] (also referred to here as *RS chart*).

7.3. Numerical results

Table 1 contains the values of $\mu_0(\hat{\gamma}^2)$ and $\sigma_0(\hat{\gamma}^2)$, the mean and the standard deviation of the sample MCV-squared when the process operates under $\gamma = \gamma_0$, calculated using Equations (16) and (17), respectively, for the values of n, p and γ_0 being considered (note that $\mu_0(\hat{\gamma}^2)$ will also become the CL of the chart in each case). According to what was explained in Subsection 4.2.2, Equation (20) was used instead of Equation (16) to calculate $\tilde{\mu}_0(\hat{\gamma}^2)$ when $p = 2$, and Equations (21) and (22) were used instead of Equation (17) to calculate $\tilde{\sigma}_0(\hat{\gamma}^2)$ for $p = 2$ and $p = 4$, respectively. A value of $\varepsilon = 10^{-5}$ was used in those cases.

Tables 2 to 9 show, for each of the values of p being considered, the outcomes of solving model (25) using Algorithm 2 for the aforementioned set of instances. More precisely, the optimal values λ^* and K^* of λ and K are shown, together with the resulting UCL, $ARL(\gamma_1)$ and $SDRL(\gamma_1)$.

As expected, the optimal out-of-control ARL decreases when the shift to detect τ increases (from an average number of 37.2651 samples when $\tau = 1.1$ to 2.6883 for $\tau = 2.0$; these average values are calculated considering the results of all the tested instances). A similar behaviour is observed with regards to n: the larger the sample size, the smaller the number of samples needed to detect an out-of-control situation on average (from 24.8985 samples when $n = 5$ to 7.19384 when $n = 20$, which are average results based on the tested instances

with $p \leq 4$). On the other hand, an increase in the dimension p results in an increase in the optimal $\text{ARL}(\gamma_1)$, provided that the rest of parameters remain constant (8.60874 for $p = 1$ versus 15.8993 for $p = 8$ on average, based on the instances with $n \geq 10$). The effect of a higher in-control MCV γ_0 is a slight increase in the average number of samples needed to detect the same relative shift (from 13.5346 for $\gamma_0 = 0.1$ to 17.6951 for $\gamma_0 = 0.5$, on average). Similar conclusions can be drawn for the out-of-control SDRL.

7.4. Comparison with existing alternatives

The results shown in Subsection 7.3 compare very favourably with the performance entries reported by Yeong et al.[17] and Lim et al.[18] for their respective charts. Table 10 shows the relative variation of the ARL and the SDRL (both in percentages) in the EWMA chart with regards to the MCV chart for each of the instances considered in the paper by Yeong et al.[17]. Table 11 contains the same information regarding the RS chart for the instances considered by Lim et al.[18].

Decreases in the out-of-control ARL of 38.5–76.8% are recorded in the EWMA chart with respect to the MCV chart, for all the cases being compared. The outperformance of our chart is observed to be higher for smaller shifts τ, larger sample sizes, higher values of γ_0 and larger values of p, in decreasing order of significance. As for the SDRL, an increase of up to 148.2% was observed in 24% of the instances being compared, mostly associated with the case $\tau = 2.0$.

Similar results can be seen from the comparison with the RS chart: our chart reduces the ARL by 41.5 to 67.4% in all the compared instances. The decrease in the ARL is significantly higher for smaller shifts τ; the rest of the parameters (n, p and γ_0) do not exert any relevant effect. Some 25% of the cases register an increase in the SDRL, mostly when τ takes the highest value considered in the comparison.

8. Illustrative example

A funding company makes investments in $p = 3$ industrial sectors S_1 (automotive), S_2 (aeronautic) and S_3 (electronic), in $n = 5$ world regions R_1 (Africa), R_2 (North America), R_3 (South America), R_4 (Asia) and R_5 (Europe). The rates of return (in %) of this company are provided in Table 12 from 2000 to 2016 for the $p = 3$ industrial sectors $\{S_1, S_2, S_3\}$ and the $n = 5$ world regions $\{R_1, R_2, R_3, R_4, R_5\}$. For instance, in 2000, the rate of return for sector S_1 and region R_1 was 17.8% which means that, if this company invested V, then the gain was $(1 + 0.178) \times V = 1.178 \times V$ (no matter the real value of V). A negative rate of return implies a loss for the company (this does not happen here). For a single characteristic (a particular industrial sector), the volatility of an investment

can be quantified by the standard deviation of the rate of return. The greater the volatility the greater the investment risk (you can gain a lot *but* you can also gain little or even lose a lot). If volatility can be used as a measure of risk for an investment, the coefficient of variation is a better measure of risk. It quantifies the dispersion of an investment's returns *in relation to* the expected return, and, thus, the relative risk of the investment. Hence, the coefficient of variation permits comparison between different investments. This is also the same for the multivariate coefficient of variation when several characteristics are combined (here, $p = 3$ industrial sectors).

In Table 12, for years 2000–2016, we also present the estimated mean vectors $\bar{\mathbf{X}}$, the estimated variance-covariance matrices $\mathbf{S}$ and the estimated multivariate coefficients of variation squared $\hat{\gamma}^2$. The values for $\hat{\gamma}^2$ are also plotted in Figure 2(top). Years 2000–2009 are considered by the company as satisfactory in terms of investment returns and these years are used here as a Phase I sample for estimating the in-control multivariate coefficients of variation squared γ_0^2. Using the average of the values obtained during the period 2000–2009, we have $\hat{\gamma}_0^2 = 0.00163769$ (i.e. $\hat{\gamma} = 0.0404684$). In order to detect a change of $\tau = 2$ from γ_0 to $\gamma_1 = 2 \times \gamma_0$, an EWMA γ^2 chart was designed for $n = 5$, $p = 3$ and $ARL_0 = 370.4$. Based on Equations (16) and (17), we have $\mu_0(\hat{\gamma}^2) = 0.000819114$ and $\sigma_0(\hat{\gamma}^2) = 0.000820298$. Moreover, the optimal values (obtained using the optimization algorithm presented in Section 6) for this EWMA γ^2 chart are $\lambda = 0.2314$ and $K = 3.622$. This gives the upper control limit of the EWMA γ^2 chart as

$$\text{UCL} = 0.000819114 + 3.622 \times \sqrt{\frac{0.2314}{2 - 0.2314}} \times 0.000820298 = 0.001894,$$

according to Equation (8).

The rightmost column of Table 12 provides the EWMA values Z computed from the $\hat{\gamma}^2$ values. These values are also plotted in Figure 2(bottom) along with the upper control limit UCL = 0.001894. As it can immediately be noticed, from 2012 to 2016, several out-of-control situations (in bold in Table 12) are detected by the EWMA γ^2 chart clearly illustrating that a change occurred in the rates of return of the funding company.

9. Conclusions and future research

A *new* one-sided EWMA chart for the MCV-squared of a multidimensional normal process was developed, including optimisation of the chart parameters λ and K with respect to performance requirements in terms of the ARL using derivative-free algorithms. The derivation of explicit expressions for the CL and UCL of the chart was addressed by means of a theoretical study of the

p	n	γ_0	$\mu_0(\hat{\gamma}^2)$	$\sigma_0(\hat{\gamma}^2)$	p	n	γ_0	$\mu_0(\hat{\gamma}^2)$	$\sigma_0(\hat{\gamma}^2)$	p	n	γ_0	$\mu_0(\hat{\gamma}^2)$	$\sigma_0(\hat{\gamma}^2)$	p	n	γ_0	$\mu_0(\hat{\gamma}^2)$	$\sigma_0(\hat{\gamma}^2)$
1	5	0.1	0.010061	0.007201	3	5	0.1	0.005010	0.005051										
1	5	0.2	0.041001	0.030476	3	5	0.2	0.020164	0.020840										
1	5	0.3	0.095364	0.076005	3	5	0.3	0.045858	0.049559										
1	5	0.4	0.178635	0.161619	3	5	0.4	0.082858	0.096345										
1	5	0.5	0.303758	0.335545	3	5	0.5	0.132594	0.175019										
1	10	0.1	0.010030	0.004781	3	10	0.1	0.007786	0.004199	5	10	0.1	0.005550	0.003535	7	10	0.1	0.003323	0.002727
1	10	0.2	0.040490	0.019945	3	10	0.2	0.031237	0.017306	5	10	0.2	0.022133	0.014392	7	10	0.2	0.013174	0.010971
1	10	0.3	0.092547	0.048173	3	10	0.3	0.070648	0.040917	5	10	0.3	0.049546	0.033331	7	10	0.3	0.029197	0.024903
1	10	0.4	0.168375	0.094844	3	10	0.4	0.126540	0.077986	5	10	0.4	0.087443	0.061629	7	10	0.4	0.050815	0.044755
1	10	0.5	0.271636	0.170682	3	10	0.5	0.199726	0.133565	5	10	0.5	0.135322	0.101141	7	10	0.5	0.077244	0.070742
1	15	0.1	0.010020	0.003828	3	15	0.1	0.008577	0.003534	5	15	0.1	0.007138	0.003218	7	15	0.1	0.005703	0.002870
1	15	0.2	0.040324	0.015896	3	15	0.2	0.034378	0.014560	5	15	0.2	0.028495	0.013150	7	15	0.2	0.022675	0.011637
1	15	0.3	0.091671	0.038037	3	15	0.3	0.077614	0.034362	5	15	0.3	0.063898	0.030618	7	15	0.3	0.050508	0.026739
1	15	0.4	0.165416	0.073605	3	15	0.4	0.138655	0.065169	5	15	0.4	0.113053	0.056957	7	15	0.4	0.088534	0.048826
1	15	0.5	0.263686	0.128197	3	15	0.5	0.218053	0.110367	5	15	0.5	0.175543	0.093988	7	15	0.5	0.135835	0.078650
1	20	0.1	0.010015	0.003283	3	20	0.1	0.008952	0.003100	5	20	0.1	0.007891	0.002906	7	20	0.1	0.006832	0.002700
1	20	0.2	0.040242	0.013605	3	20	0.2	0.035861	0.012766	5	20	0.2	0.031516	0.011896	7	20	0.2	0.027205	0.010987
1	20	0.3	0.091243	0.032408	3	20	0.3	0.080894	0.030102	5	20	0.3	0.070731	0.027771	7	20	0.3	0.060751	0.025396
1	20	0.4	0.164003	0.062226	3	20	0.4	0.144332	0.056960	5	20	0.4	0.125297	0.051808	7	20	0.4	0.106868	0.046729
1	20	0.5	0.260019	0.106976	3	20	0.5	0.226592	0.096010	5	20	0.5	0.194869	0.085708	7	20	0.5	0.164719	0.075942
2	5	0.1	**0.007530**	**0.006207**	4	5	0.1	0.002500	**0.003553**										
2	5	0.2	**0.030493**	**0.025933**	4	5	0.2	0.010000	**0.014473**										
2	5	0.3	**0.070117**	**0.063086**	4	5	0.3	0.022500	**0.033624**										
2	5	0.4	**0.128885**	**0.127323**	4	5	0.4	0.040000	**0.062773**										
2	5	0.5	**0.211854**	**0.252166**	4	5	0.5	0.062497	**0.105388**										
2	10	0.1	**0.008906**	**0.004497**	4	10	0.1	0.006667	**0.003878**	6	10	0.1	0.004436	0.003155	8	10	0.1	0.002213	0.002222
2	10	0.2	**0.035843**	**0.018644**	4	10	0.2	0.026667	**0.015883**	6	10	0.2	0.017636	0.012770	8	10	0.2	0.008748	0.008887
2	10	0.3	**0.081490**	**0.044544**	4	10	0.3	0.060000	**0.037159**	6	10	0.3	0.039280	0.029276	8	10	0.3	0.019293	0.019975
2	10	0.4	**0.147085**	**0.086228**	4	10	0.4	0.106667	**0.069713**	6	10	0.4	0.068836	0.053356	8	10	0.4	0.033353	0.035414
2	10	0.5	**0.234625**	**0.150994**	4	10	0.5	0.166667	**0.116672**	6	10	0.5	0.105556	0.085887	8	10	0.5	0.050278	0.055025
2	15	0.1	**0.009298**	**0.003683**	4	15	0.1	0.007857	**0.003378**	6	15	0.1	0.006420	0.003049	8	15	0.1	0.004987	0.002681
2	15	0.2	**0.037342**	**0.015231**	4	15	0.2	0.031429	**0.013858**	6	15	0.2	0.025577	0.012409	8	15	0.2	0.019788	0.010828
2	15	0.3	**0.084596**	**0.036191**	4	15	0.3	0.070714	**0.032483**	6	15	0.3	0.057163	0.028701	8	15	0.3	0.043933	0.024718
2	15	0.4	**0.151878**	**0.069313**	4	15	0.4	0.125714	**0.060996**	6	15	0.4	0.100663	0.052892	8	15	0.4	0.076659	0.044730
2	15	0.5	**0.240440**	**0.118960**	4	15	0.5	0.196429	**0.101905**	6	15	0.5	0.155357	0.086215	8	15	0.5	0.116944	0.071232
2	20	0.1	**0.009483**	**0.003192**	4	20	0.1	0.008421	**0.003003**	6	20	0.1	0.007361	0.002804	8	20	0.1	0.006303	0.002591
2	20	0.2	**0.038047**	**0.013186**	4	20	0.2	0.033684	**0.012330**	6	20	0.2	0.029356	0.011447	8	20	0.2	0.025062	0.010514
2	20	0.3	**0.086042**	**0.031248**	4	20	0.3	0.075789	**0.028926**	6	20	0.3	0.065719	0.026591	8	20	0.3	0.055828	0.024185
2	20	0.4	**0.154080**	**0.059551**	4	20	0.4	0.134737	**0.054337**	6	20	0.4	0.116008	0.049263	8	20	0.4	0.097871	0.044200
2	20	0.5	**0.243073**	**0.101338**	4	20	0.5	0.210526	**0.090705**	6	20	0.5	0.179605	0.080766	8	20	0.5	0.150197	0.071220

Table 1: Expected value ($\mu_0(\hat{\gamma}^2)$) and standard deviation ($\sigma_0(\hat{\gamma}^2)$) of the sample MCV-squared, $\hat{\gamma}^2$, for different values of n, p and γ_0, calculated as in Equations (16) and (17). Entries in **bold** represent nonexistent values that have been replaced by their trimmed versions, computed according to Equations (20), (21) (for the trimmed mean and standard deviation when $p = 2$) and (22) (for the trimmed standard deviation when $p = 4$); see also Subsection 4.2.2.

γ_0	$\tau = 1.1$	$\tau = 1.2$	$\tau = 1.25$	$\tau = 1.5$	$\tau = 2.0$
			$n = 5$		
0.1	(0.0106 , 1.3316); 0.0108 (38.0189 , 26.8192)	(0.0406 , 2.2023); 0.0123 (17.9581 , 11.6940)	(0.0587 , 2.4348); 0.0131 (13.5069 , 8.7772)	(0.1875 , 3.2105); 0.0175 (5.3865 , 3.6508)	(0.3875 , 3.7770); 0.0234 (2.3168 , 1.4813)
0.2	(0.0106 , 1.3357); 0.0440 (38.9156 , 27.7666)	(0.0376 , 2.1693); 0.0502 (18.4706 , 12.1147)	(0.0532 , 2.3961); 0.0531 (13.9435 , 9.1023)	(0.1625 , 3.1764); 0.0698 (5.6265 , 3.8044)	(0.3000 , 3.6840); 0.0882 (2.4527 , 1.5543)
0.3	(0.0102 , 1.3166); 0.1025 (40.4097 , 29.4577)	(0.0102 , 1.3166); 0.1025 (18.7833 , 11.4164)	(0.0422 , 2.2720); 0.1207 (14.7524 , 9.6377)	(0.1297 , 3.1102); 0.1576 (6.0743 , 4.1136)	(0.2586 , 3.7461); 0.2051 (2.6939 , 1.7538)
0.4	(0.0103 , 1.2918); 0.1936 (43.2610 , 32.7561)	(0.0103 , 1.2919); 0.1936 (19.9824 , 12.8041)	(0.0103 , 1.2919); 0.1936 (15.5318 , 9.5622)	(0.0797 , 2.7949); 0.2707 (6.8155 , 4.5428)	(0.1735 , 3.5502); 0.3555 (3.0784 , 2.0311)
0.5	(0.0102 , 1.2106); 0.3329 (48.3997 , 39.1636)	(0.0121 , 1.3241); 0.3384 (22.4349 , 15.5963)	(0.0141 , 1.4320); 0.3442 (17.5563 , 11.8017)	(0.0324 , 2.0767); 0.3932 (7.9684 , 5.2304)	(0.1001 , 3.1622); 0.5473 (3.6787 , 2.4830)
			$n = 10$		
0.1	(0.0150 , 1.5416); 0.0107 (24.9511 , 15.0345)	(0.1002 , 2.6525); 0.0129 (10.5149 , 6.6706)	(0.1255 , 2.7759); 0.0135 (7.7627 , 4.7910)	(0.3508 , 3.3623); 0.0174 (3.0448 , 1.9039)	(0.6439 , 3.7207); 0.0223 (1.4151 , 0.7006)
0.2	(0.0172 , 1.6399); 0.0435 (25.9095 , 15.9654)	(0.1000 , 2.6804); 0.0528 (10.9668 , 7.0974)	(0.1379 , 2.8654); 0.0560 (8.1195 , 5.2406)	(0.2750 , 3.2846); 0.0666 (3.2160 , 1.9491)	(0.6129 , 3.8083); 0.0910 (1.4979 , 0.7803)
0.3	(0.0141 , 1.5195); 0.0987 (26.9127 , 16.7618)	(0.0795 , 2.5867); 0.1179 (11.7107 , 7.4590)	(0.1125 , 2.8005); 0.1255 (8.7165 , 5.5702)	(0.2422 , 3.3090); 0.1517 (3.5093 , 2.1608)	(0.5131 , 3.8773); 0.2023 (1.6419 , 0.8972)
0.4	(0.0141 , 1.5233); 0.1806 (28.7275 , 18.5241)	(0.0492 , 2.3297); 0.2035 (12.8344 , 7.8193)	(0.0767 , 2.6186); 0.2180 (9.6064 , 5.9548)	(0.1938 , 3.2866); 0.2705 (3.9422 , 2.4513)	(0.4004 , 3.9218); 0.3545 (1.8543 , 1.0579)
0.5	(0.0106 , 1.3296); 0.2882 (30.4751 , 20.2903)	(0.0422 , 2.2599); 0.3283 (14.3244 , 8.9872)	(0.0659 , 2.5753); 0.3528 (10.8470 , 6.9214)	(0.1469 , 3.2136); 0.4261 (4.5424 , 2.8601)	(0.2721 , 3.8150); 0.5300 (2.1451 , 1.2576)
			$n = 15$		
0.1	(0.0329 , 2.0031); 0.0110 (19.4996 , 11.5686)	(0.1516 , 2.8079); 0.0131 (7.7417 , 4.8249)	(0.1827 , 2.9008); 0.0135 (5.6801 , 3.3998)	(0.4652 , 3.3725); 0.0171 (2.2384 , 1.3128)	(0.8258 , 3.6271); 0.0217 (1.1585 , 0.4155)
0.2	(0.0376 , 2.0942); 0.0449 (20.2008 , 12.4072)	(0.1456 , 2.8194); 0.0529 (8.1040 , 5.1063)	(0.1750 , 2.9155); 0.0547 (5.9564 , 3.6098)	(0.4189 , 3.3885); 0.0681 (2.3696 , 1.3909)	(0.7756 , 3.7041); 0.0872 (1.2072 , 0.4781)
0.3	(0.0438 , 2.1970); 0.1042 (21.4423 , 13.8735)	(0.1253 , 2.7848); 0.1191 (8.7011 , 5.4667)	(0.1377 , 2.8377); 0.1210 (6.4331 , 3.8158)	(0.3625 , 3.4146); 0.1528 (2.5914 , 1.5337)	(0.6762 , 3.8025); 0.1950 (1.2968 , 0.5753)
0.4	(0.0137 , 1.5000); 0.1746 (22.4988 , 13.0925)	(0.0986 , 2.7024); 0.2107 (9.5489 , 5.9329)	(0.1193 , 2.8183); 0.2177 (7.1009 , 4.2767)	(0.2720 , 3.3620); 0.2636 (2.9076 , 1.7014)	(0.5730 , 3.9190); 0.3482 (1.4350 , 0.7054)
0.5	(0.0166 , 1.6270); 0.2828 (24.5927 , 15.1044)	(0.0771 , 2.6082); 0.3306 (10.6794 , 6.6304)	(0.0970 , 2.7600); 0.3436 (7.9933 , 4.8654)	(0.2002 , 3.2844); 0.4041 (3.3273 , 1.9468)	(0.4379 , 3.9631); 0.5327 (1.6275 , 0.8571)
			$n = 20$		
0.1	(0.0470 , 2.1856); 0.0111 (16.0667 , 9.4078)	(0.1600 , 2.7910); 0.0127 (6.2407 , 3.6156)	(0.2443 , 2.9863); 0.0137 (4.5669 , 2.7069)	(0.5393 , 3.3465); 0.0167 (1.8206 , 0.9958)	(0.9145 , 3.5287); 0.0206 (1.0622 , 0.2547)
0.2	(0.0532 , 2.2688); 0.0453 (16.6641 , 10.1286)	(0.1607 , 2.8225); 0.0516 (6.5319 , 3.8727)	(0.2220 , 2.9793); 0.0546 (4.7972 , 2.8337)	(0.5068 , 3.3845); 0.0671 (1.9262 , 1.0706)	(0.9010 , 3.6107); 0.0847 (1.0890 , 0.3073)
0.3	(0.0477 , 2.2267); 0.1025 (17.6957 , 10.8133)	(0.1469 , 2.8206); 0.1170 (7.0316 , 4.2210)	(0.2130 , 3.0162); 0.1250 (5.1907 , 3.1518)	(0.4648 , 3.4462); 0.1527 (2.1050 , 1.2009)	(0.8125 , 3.7241); 0.1911 (1.1429 , 0.3914)
0.4	(0.0313 , 2.0033); 0.1797 (19.1156 , 11.2651)	(0.1255 , 2.7902); 0.2089 (7.7366 , 4.6724)	(0.1726 , 2.9738); 0.2209 (5.7356 , 3.4575)	(0.3625 , 3.4326); 0.2645 (2.3567 , 1.3366)	(0.7313 , 3.8754); 0.3471 (1.2336 , 0.5063)
0.5	(0.0314 , 2.0198); 0.2873 (20.8678 , 12.8141)	(0.0986 , 2.7096); 0.3260 (8.6511 , 5.2017)	(0.1440 , 2.9473); 0.3479 (6.4540 , 3.9303)	(0.2976 , 3.4512); 0.4144 (2.6839 , 1.5492)	(0.5844 , 3.9760); 0.5333 (1.3686 , 0.6375)

Table 2: Optimal setting of the EWMA chart for the MCV-squared for $p = 1$, $n \in \{5, 10, 15, 20\}$, $\gamma_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ and $\tau \in \{1.1, 1.2, 1.25, 1.5, 2.0\}$, obtained according to Algorithm 2. The first row of each block contains the pair (λ^*, K^*) followed by the resulting UCL; the second row shows the pair $(\text{ARL}(\gamma_1), \text{SDRL}(\gamma_1))$, with $\gamma_1 = \tau\gamma_0$.

γ_0	$\tau = 1.1$	$\tau = 1.2$	$\tau = 1.25$	$\tau = 1.5$	$\tau = 2.0$
			$n = 5$		
0.1	(0.0103 , 1.3181); 0.0081 (44.3806 , 33.0765)	(0.0344 , 2.1162); 0.0093 (21.4777 , 14.5503)	(0.0449 , 2.2913); 0.0097 (16.3060 , 10.8373)	(0.1361 , 3.0559); 0.0127 (6.6035 , 4.5163)	(0.3127 , 3.7348); 0.0175 (2.8162 , 1.8941)
0.2	(0.0103 , 1.3238); 0.0330 (45.4549 , 34.2081)	(0.0103 , 1.3238); 0.0330 (21.2595 , 13.2806)	(0.0500 , 2.3826); 0.0404 (16.8291 , 11.5726)	(0.1314 , 3.0817); 0.0517 (6.9007 , 4.7919)	(0.2610 , 3.6713); 0.0674 (2.9889 , 2.0084)
0.3	(0.0105 , 1.3315); 0.0762 (47.3744 , 36.3898)	(0.0112 , 1.3732); 0.0766 (22.1696 , 14.3144)	(0.0112 , 1.3732); 0.0766 (17.2814 , 10.6502)	(0.1079 , 3.0067); 0.1154 (7.4520 , 5.1832)	(0.2129 , 3.6305); 0.1492 (3.3053 , 2.2475)
0.4	(0.0103 , 1.3098); 0.1409 (50.5746 , 39.9269)	(0.0110 , 1.3471); 0.1416 (23.6727 , 15.9188)	(0.0110 , 1.3471); 0.1416 (18.4432 , 11.8793)	(0.0752 , 2.7879); 0.1991 (8.3547 , 5.7646)	(0.1501 , 3.4487); 0.2540 (3.8200 , 2.6146)
0.5	(0.0104 , 1.2211); 0.2341 (56.2020 , 46.4424)	(0.0131 , 1.3766); 0.2401 (26.8470 , 19.1351)	(0.0141 , 1.4247); 0.2421 (20.9426 , 14.3669)	(0.0141 , 1.4248); 0.2421 (9.6378 , 6.1310)	(0.0938 , 2.9752); 0.3783 (4.6406 , 3.1926)
			$n = 10$		
0.1	(0.0134 , 1.4730); 0.0094 (26.5016 , 16.2784)	(0.0799 , 2.5416); 0.0112 (11.4078 , 7.1010)	(0.1191 , 2.7653); 0.0120 (8.4288 , 5.3033)	(0.3007 , 3.3061); 0.0152 (3.3071 , 2.0595)	(0.6000 , 3.7349); 0.0199 (1.5101 , 0.7897)
0.2	(0.0141 , 1.5186); 0.0382 (27.4531 , 17.1119)	(0.0806 , 2.5695); 0.0457 (11.8752 , 7.5478)	(0.1127 , 2.7646); 0.0484 (8.8084 , 5.5907)	(0.2783 , 3.3229); 0.0607 (3.4963 , 2.1955)	(0.5482 , 3.7842); 0.0792 (1.6047 , 0.8702)
0.3	(0.0110 , 1.3630); 0.0860 (28.2635 , 17.8397)	(0.0785 , 2.5898); 0.1048 (12.7026 , 8.2880)	(0.0894 , 2.6701); 0.1072 (9.4668 , 5.9211)	(0.2316 , 3.2997); 0.1347 (3.8219 , 2.4008)	(0.4912 , 3.8782); 0.1801 (1.7700 , 1.0105)
0.4	(0.0141 , 1.5245); 0.1582 (30.7563 , 20.2624)	(0.0565 , 2.4244); 0.1827 (13.8751 , 8.8564)	(0.0766 , 2.6254); 0.1923 (10.4279 , 6.6149)	(0.1910 , 3.2862); 0.2392 (4.3044 , 2.7364)	(0.3631 , 3.8446); 0.3032 (2.0143 , 1.1796)
0.5	(0.0118 , 1.4046); 0.2509 (32.9725 , 22.4195)	(0.0501 , 2.3866); 0.2924 (15.5487 , 10.2086)	(0.0501 , 2.3866); 0.2924 (11.7599 , 7.2991)	(0.1407 , 3.1778); 0.3666 (4.9720 , 3.1632)	(0.2649 , 3.7822); 0.4578 (2.3539 , 1.4181)
			$n = 15$		
0.1	(0.0376 , 2.0835); 0.0104 (20.4035 , 12.5808)	(0.1406 , 2.7811); 0.0121 (8.1546 , 5.0872)	(0.1790 , 2.9033); 0.0127 (5.9879 , 3.6415)	(0.4523 , 3.3816); 0.0160 (2.3570 , 1.4060)	(0.8133 , 3.6530); 0.0204 (1.1916 , 0.4609)
0.2	(0.0424 , 2.1659); 0.0422 (21.1889 , 13.5120)	(0.1255 , 2.7509); 0.0482 (8.5264 , 5.2774)	(0.1721 , 2.9180); 0.0510 (6.2818 , 3.8685)	(0.4195 , 3.4097); 0.0641 (2.4976 , 1.5021)	(0.7250 , 3.7030); 0.0799 (1.2469 , 0.5228)
0.3	(0.0313 , 2.0041); 0.0937 (22.3474 , 13.9039)	(0.1064 , 2.7007); 0.1078 (9.1580 , 5.6367)	(0.1441 , 2.8718); 0.1136 (6.7786 , 4.1417)	(0.3379 , 3.3847); 0.1398 (2.7325 , 1.6236)	(0.6568 , 3.8066); 0.1809 (1.3475 , 0.6286)
0.4	(0.0133 , 1.4837); 0.1603 (23.4608 , 13.8336)	(0.0860 , 2.6245); 0.1904 (10.0602 , 6.1664)	(0.1193 , 2.8234); 0.2012 (7.4915 , 4.5874)	(0.2848 , 3.4021); 0.2480 (3.0722 , 1.8511)	(0.5127 , 3.8452); 0.3084 (1.5026 , 0.7573)
0.5	(0.0149 , 1.5577); 0.2565 (25.5753 , 15.7867)	(0.0753 , 2.5946); 0.3015 (11.2576 , 7.0523)	(0.1000 , 2.7820); 0.3164 (8.4421 , 5.2424)	(0.2156 , 3.3419); 0.3786 (3.5227 , 2.1222)	(0.4250 , 3.9297); 0.4833 (1.7175 , 0.9307)
			$n = 20$		
0.1	(0.0469 , 2.1887); 0.0106 (16.6261 , 9.8643)	(0.1602 , 2.7989); 0.0121 (6.4812 , 3.8152)	(0.2344 , 2.9772); 0.0129 (4.7478 , 2.8263)	(0.5189 , 3.3456); 0.0158 (1.8878 , 1.0431)	(0.9127 , 3.5482); 0.0199 (1.0750 , 0.2811)
0.2	(0.0548 , 2.2884); 0.0431 (17.2626 , 10.6952)	(0.1578 , 2.8209); 0.0489 (6.7893 , 4.0674)	(0.2253 , 2.9952); 0.0521 (4.9901 , 3.0071)	(0.4953 , 3.3876); 0.0637 (1.9985 , 1.1258)	(0.8641 , 3.6196); 0.0797 (1.1057 , 0.3352)
0.3	(0.0478 , 2.2309); 0.0969 (18.3289 , 11.3457)	(0.1508 , 2.8400); 0.1114 (7.3147 , 4.4845)	(0.1788 , 2.9297); 0.1147 (5.3941 , 3.1695)	(0.4506 , 3.4396); 0.1440 (2.1867 , 1.2596)	(0.7816 , 3.7238); 0.1792 (1.1666 , 0.4233)
0.4	(0.0408 , 2.1596); 0.1726 (19.7947 , 12.3171)	(0.1125 , 2.7323); 0.1938 (8.0461 , 4.7954)	(0.1565 , 2.9206); 0.2048 (5.9663 , 3.5550)	(0.3630 , 3.4390); 0.2505 (2.4521 , 1.4135)	(0.6695 , 3.8342); 0.3161 (1.2676 , 0.5394)
0.5	(0.0391 , 2.1541); 0.2739 (21.7176 , 13.9539)	(0.0989 , 2.7122); 0.3058 (9.0014 , 5.4756)	(0.1318 , 2.8911); 0.3209 (6.7161 , 4.0504)	(0.3007 , 3.4578); 0.3905 (2.8002 , 1.6421)	(0.5701 , 3.9537); 0.4961 (1.4166 , 0.6826)

Table 3: Optimal setting of the EWMA chart for the MCV-squared for $p = 2$, $n \in \{5, 10, 15, 20\}$, $\gamma_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ and $\tau \in \{1.1, 1.2, 1.25, 1.5, 2.0\}$, obtained according to Algorithm 2. The first row of each block contains the pair (λ^*, K^*) followed by the resulting UCL; the second row shows the pair $(\text{ARL}(\gamma_1), \text{SDRL}(\gamma_1))$, with $\gamma_1 = \tau\gamma_0$.

γ_0	$\tau = 1.1$	$\tau = 1.2$	$\tau = 1.25$	$\tau = 1.5$	$\tau = 2.0$
			$n = 5$		
0.1	(0.0114 , 1.3775); 0.0055 (55.4228 , 44.5164)	(0.0114 , 1.3775); 0.0055 (26.0340 , 17.5602)	(0.0145 , 1.5473); 0.0057 (20.6702 , 13.3567)	(0.1079 , 2.9778); 0.0086 (8.7773 , 6.3244)	(0.2250 , 3.6188); 0.0115 (3.7371 , 2.6591)
0.2	(0.0114 , 1.3816); 0.0223 (56.7578 , 45.9490)	(0.0141 , 1.5303); 0.0228 (27.2341 , 18.7375)	(0.0141 , 1.5303); 0.0228 (21.1757 , 13.8764)	(0.0953 , 2.9137); 0.0337 (9.1662 , 6.5868)	(0.2006 , 3.5774); 0.0451 (3.9764 , 2.8420)
0.3	(0.0110 , 1.3473); 0.0508 (58.8348 , 48.5082)	(0.0141 , 1.5321); 0.0523 (28.4920 , 20.0550)	(0.0141 , 1.5321); 0.0523 (22.1663 , 14.8958)	(0.0680 , 2.6858); 0.0708 (9.8812 , 6.9481)	(0.1627 , 3.4708); 0.0970 (4.4210 , 3.1712)
0.4	(0.0106 , 1.3121); 0.0921 (62.6297 , 52.7296)	(0.0106 , 1.3121); 0.0921 (29.7295 , 21.4705)	(0.0106 , 1.3121); 0.0921 (23.2528 , 16.0660)	(0.0501 , 2.4677); 0.1210 (11.0343 , 7.7305)	(0.1095 , 3.1684); 0.1563 (5.1521 , 3.6603)
0.5	(0.0103 , 1.2360); 0.1481 (68.5426 , 59.4569)	(0.0106 , 1.2499); 0.1486 (32.8520 , 24.7297)	(0.0106 , 1.2499); 0.1486 (25.7392 , 18.5598)	(0.0106 , 1.2499); 0.1486 (12.2691 , 8.0960)	(0.0719 , 2.7538); 0.2257 (6.3035 , 4.4508)
			$n = 10$		
0.1	(0.0156 , 1.5723); 0.0084 (28.8210 , 18.4048)	(0.0803 , 2.5571); 0.0100 (12.4802 , 8.0402)	(0.0970 , 2.6655); 0.0103 (9.2626 , 5.7680)	(0.2719 , 3.2818); 0.0133 (3.6356 , 2.2967)	(0.5521 , 3.7438); 0.0175 (1.6322 , 0.8987)
0.2	(0.0126 , 1.4453); 0.0332 (29.3366 , 18.7203)	(0.0766 , 2.5512); 0.0400 (13.0004 , 8.4440)	(0.1063 , 2.7463); 0.0425 (9.6684 , 6.2638)	(0.2401 , 3.2578); 0.0521 (3.8455 , 2.4215)	(0.5135 , 3.7913); 0.0698 (1.7409 , 0.9907)
0.3	(0.0126 , 1.4465); 0.0754 (30.7626 , 20.0631)	(0.0766 , 2.5835); 0.0918 (13.9240 , 9.3148)	(0.0766 , 2.5835); 0.0918 (10.3930 , 6.5012)	(0.2190 , 3.2823); 0.1177 (4.2101 , 2.7015)	(0.4133 , 3.7764); 0.1495 (1.9310 , 1.1264)
0.4	(0.0118 , 1.4049); 0.1350 (32.6530 , 21.9021)	(0.0438 , 2.2608); 0.1529 (15.1591 , 9.5212)	(0.0705 , 2.5724); 0.1649 (11.4350 , 7.3367)	(0.1627 , 3.1684); 0.2001 (4.7476 , 3.0103)	(0.3252 , 3.7565); 0.2556 (2.2149 , 1.3300)
0.5	(0.0113 , 1.3767); 0.2136 (35.3934 , 24.5958)	(0.0399 , 2.2236); 0.2421 (16.8976 , 10.9446)	(0.0547 , 2.4426); 0.2544 (12.8775 , 8.2868)	(0.1158 , 3.0048); 0.2992 (5.4979 , 3.4588)	(0.2314 , 3.6279); 0.3750 (2.6135 , 1.5955)
			$n = 15$		
0.1	(0.0104 , 1.3067); 0.0089 (20.7776 , 11.5969)	(0.1377 , 2.7800); 0.0112 (8.6313 , 5.4737)	(0.2001 , 2.9730); 0.0121 (6.3602 , 4.0763)	(0.4322 , 3.3812); 0.0149 (2.4925 , 1.5069)	(0.7633 , 3.6623); 0.0187 (1.2318 , 0.5086)
0.2	(0.0378 , 2.1030); 0.0386 (22.2438 , 14.1678)	(0.1283 , 2.7706); 0.0449 (9.0287 , 5.7383)	(0.1627 , 2.8980); 0.0469 (6.6508 , 4.1260)	(0.3625 , 3.3455); 0.0573 (2.6403 , 1.5653)	(0.7131 , 3.7231); 0.0747 (1.2944 , 0.5776)
0.3	(0.0378 , 2.1160); 0.0877 (23.5725 , 15.3752)	(0.0877 , 2.5964); 0.0967 (9.7003 , 5.8309)	(0.1252 , 2.7981); 0.1025 (7.1827 , 4.3322)	(0.3134 , 3.3500); 0.1272 (2.8941 , 1.7271)	(0.6500 , 3.8209); 0.1687 (1.4078 , 0.6916)
0.4	(0.0150 , 1.5599); 0.1475 (24.7092 , 14.9220)	(0.0844 , 2.6141); 0.1744 (10.6400 , 6.6123)	(0.1125 , 2.7894); 0.1830 (7.9355 , 4.8844)	(0.2596 , 3.3406); 0.2227 (3.2590 , 1.9620)	(0.5002 , 3.8339); 0.2829 (1.5807 , 0.8269)
0.5	(0.0126 , 1.4464); 0.2307 (26.4993 , 16.4589)	(0.0703 , 2.5469); 0.2717 (11.9027 , 7.4702)	(0.0938 , 2.7353); 0.2850 (8.9440 , 5.5664)	(0.1908 , 3.2399); 0.3342 (3.7484 , 2.2397)	(0.4010 , 3.8664); 0.4317 (1.8216 , 1.0099)
			$n = 20$		
0.1	(0.0485 , 2.2075); 0.0100 (17.2319 , 10.4305)	(0.1566 , 2.7947); 0.0115 (6.7484 , 4.0147)	(0.2128 , 2.9403); 0.0121 (4.9470 , 2.9142)	(0.5111 , 3.3541); 0.0150 (1.9617 , 1.1028)	(0.8902 , 3.5635); 0.0188 (1.0905 , 0.3098)
0.2	(0.0563 , 2.3041); 0.0409 (17.9244 , 11.3201)	(0.1547 , 2.8164); 0.0463 (7.0711 , 4.2818)	(0.2126 , 2.9740); 0.0490 (5.1984 , 3.1332)	(0.4830 , 3.3884); 0.0603 (2.0784 , 1.1865)	(0.8627 , 3.6366); 0.0763 (1.1256 , 0.3677)
0.3	(0.0376 , 2.0973); 0.0896 (19.0097 , 11.4259)	(0.1344 , 2.7831); 0.1034 (7.6141 , 4.6006)	(0.1757 , 2.9251); 0.1082 (5.6209 , 3.3392)	(0.4189 , 3.4075); 0.1337 (2.2763 , 1.3094)	(0.7752 , 3.7333); 0.1703 (1.1940 , 0.4600)
0.4	(0.0243 , 1.8523); 0.1560 (20.4858 , 11.9471)	(0.1049 , 2.6931); 0.1804 (8.3860 , 4.9841)	(0.1662 , 2.9572); 0.1950 (6.2276 , 3.8212)	(0.3260 , 3.3721); 0.2291 (2.5581 , 1.4596)	(0.6754 , 3.8442); 0.3007 (1.3062 , 0.5838)
0.5	(0.0121 , 1.4241); 0.2373 (21.8196 , 12.5015)	(0.0969 , 2.6971); 0.2850 (9.3831 , 5.7513)	(0.1315 , 2.8867); 0.3001 (7.0104 , 4.2780)	(0.2724 , 3.3799); 0.3554 (2.9260 , 1.7008)	(0.5063 , 3.8555); 0.4421 (1.4715 , 0.7209)

Table 4: Optimal setting of the EWMA chart for the MCV-squared for $p = 3$, $n \in \{5, 10, 15, 20\}$, $\gamma_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ and $\tau \in \{1.1, 1.2, 1.25, 1.5, 2.0\}$, obtained according to Algorithm 2. The first row of each block contains the pair (λ^*, K^*) followed by the resulting UCL; the second row shows the pair $(\text{ARL}(\gamma_1), \text{SDRL}(\gamma_1))$, with $\gamma_1 = \tau\gamma_0$.

γ_0	$\tau = 1.1$	$\tau = 1.2$	$\tau = 1.25$	$\tau = 1.5$	$\tau = 2.0$
			$n = 5$		
0.1	(0.0108 , 1.3348); 0.0028 (78.8677 , 70.4443)	(0.0108 , 1.3348); 0.0028 (38.1383 , 29.7481)	(0.0108 , 1.3348); 0.0028 (29.8234 , 22.2537)	(0.0500 , 2.4987); 0.0039 (14.0641 , 10.4950)	(0.1298 , 3.3654); 0.0056 (6.1277 , 4.7226)
0.2	(0.0105 , 1.3248); 0.0114 (80.6827 , 72.3257)	(0.0105 , 1.3248); 0.0114 (39.2290 , 30.8341)	(0.0105 , 1.3248); 0.0114 (30.7171 , 23.1254)	(0.0105 , 1.3248); 0.0114 (14.3112 , 9.7865)	(0.1126 , 3.2544); 0.0215 (6.5253 , 5.0148)
0.3	(0.0110 , 1.3474); 0.0259 (83.8713 , 75.9344)	(0.0110 , 1.3475); 0.0259 (41.0687 , 32.7700)	(0.0110 , 1.3475); 0.0259 (32.2280 , 24.6620)	(0.0110 , 1.3475); 0.0259 (15.1750 , 10.5689)	(0.0969 , 3.1481); 0.0464 (7.2686 , 5.6026)
0.4	(0.0102 , 1.3020); 0.0458 (88.3605 , 80.7605)	(0.0102 , 1.3020); 0.0458 (44.0353 , 35.6913)	(0.0102 , 1.3020); 0.0458 (34.7301 , 27.0474)	(0.0107 , 1.3244); 0.0461 (16.5947 , 11.8345)	(0.0657 , 2.8003); 0.0724 (8.4806 , 6.4326)
0.5	(0.0104 , 1.2790); 0.0723 (95.7768 , 89.3745)	(0.0104 , 1.2790); 0.0723 (48.4857 , 40.4801)	(0.0104 , 1.2791); 0.0723 (38.4139 , 30.8581)	(0.0104 , 1.2791); 0.0723 (18.8168 , 13.7506)	(0.0104 , 1.2791); 0.0723 (10.0491 , 7.0645)
			$n = 10$		
0.1	(0.0110 , 1.3482); 0.0071 (30.5817 , 19.8644)	(0.0626 , 2.4290); 0.0084 (13.8447 , 8.8161)	(0.0914 , 2.6531); 0.0089 (10.2933 , 6.5848)	(0.2387 , 3.2450); 0.0113 (4.0566 , 2.5967)	(0.5180 , 3.7764); 0.0153 (1.7932 , 1.0439)
0.2	(0.0106 , 1.3374); 0.0282 (31.4925 , 20.6407)	(0.0626 , 2.4462); 0.0336 (14.4063 , 9.3265)	(0.0891 , 2.6608); 0.0358 (10.7511 , 6.9677)	(0.2269 , 3.2614); 0.0452 (4.2932 , 2.7816)	(0.4566 , 3.7720); 0.0593 (1.9194 , 1.1370)
0.3	(0.0117 , 1.4018); 0.0640 (33.3128 , 22.3644)	(0.0564 , 2.4059); 0.0752 (15.3727 , 10.0360)	(0.0862 , 2.6739); 0.0811 (11.5550 , 7.6603)	(0.1880 , 3.2045); 0.0984 (4.7036 , 3.0361)	(0.3756 , 3.7484); 0.1270 (2.1409 , 1.2958)
0.4	(0.0106 , 1.3389); 0.1135 (35.2467 , 24.2538)	(0.0470 , 2.3148); 0.1317 (16.7756 , 10.9938)	(0.0627 , 2.5056); 0.1381 (12.7027 , 8.2346)	(0.1378 , 3.0616); 0.1647 (5.3171 , 3.3783)	(0.3002 , 3.7142); 0.2155 (2.4741 , 1.5342)
0.5	(0.0103 , 1.3214); 0.1778 (38.2003 , 27.1806)	(0.0345 , 2.1314); 0.1996 (18.6313 , 12.1399)	(0.0384 , 2.2045); 0.2026 (14.2765 , 8.9401)	(0.1250 , 3.0784); 0.2594 (6.1711 , 4.0416)	(0.2158 , 3.5703); 0.3115 (2.9448 , 1.8437)
			$n = 15$		
0.1	(0.0121 , 1.4025); 0.0082 (22.0245 , 12.5806)	(0.1125 , 2.6873); 0.0101 (9.1622 , 5.6618)	(0.1722 , 2.9110); 0.0109 (6.7485 , 4.2581)	(0.4002 , 3.3677); 0.0135 (2.6481 , 1.6112)	(0.7447 , 3.6908); 0.0175 (1.2812 , 0.5668)
0.2	(0.0281 , 1.9348); 0.0346 (23.3752 , 14.5211)	(0.1253 , 2.7698); 0.0414 (9.6050 , 6.2079)	(0.1503 , 2.8687); 0.0428 (7.0745 , 4.4046)	(0.3503 , 3.3496); 0.0528 (2.8075 , 1.6940)	(0.6824 , 3.7362); 0.0687 (1.3520 , 0.6380)
0.3	(0.0125 , 1.4425); 0.0744 (24.0701 , 14.2056)	(0.0941 , 2.6444); 0.0898 (10.2917 , 6.4177)	(0.1252 , 2.8083); 0.0943 (7.6374 , 4.7081)	(0.3008 , 3.3424); 0.1164 (3.0810 , 1.8655)	(0.6256 , 3.8258); 0.1546 (1.4796 , 0.7607)
0.4	(0.0167 , 1.6270); 0.1348 (26.1162 , 16.1745)	(0.0758 , 2.5554); 0.1567 (11.3047 , 7.0048)	(0.1095 , 2.7795); 0.1665 (8.4460 , 5.2704)	(0.2350 , 3.2816); 0.1988 (3.4762 , 2.0919)	(0.5084 , 3.8628); 0.2633 (1.6731 , 0.9148)
0.5	(0.0141 , 1.5240); 0.2095 (28.1184 , 17.8549)	(0.0626 , 2.4753); 0.2418 (12.6384 , 7.8879)	(0.0781 , 2.6184); 0.2502 (9.5231 , 5.8110)	(0.2034 , 3.2928); 0.3093 (4.0079 , 2.4708)	(0.3629 , 3.7790); 0.3777 (1.9431 , 1.0936)
			$n = 20$		
0.1	(0.0548 , 2.2776); 0.0096 (17.9237 , 11.2752)	(0.1719 , 2.8486); 0.0110 (7.0525 , 4.3772)	(0.2221 , 2.9719); 0.0116 (5.1677 , 3.1373)	(0.4959 , 3.3590); 0.0142 (2.0440 , 1.1648)	(0.8754 , 3.5840); 0.0179 (1.1092 , 0.3419)
0.2	(0.0476 , 2.2178); 0.0380 (18.6163 , 11.5436)	(0.1542 , 2.8232); 0.0437 (7.3856 , 4.5440)	(0.2001 , 2.9539); 0.0458 (5.4291 , 3.2741)	(0.4695 , 3.3912); 0.0568 (2.1675 , 1.2536)	(0.8502 , 3.6548); 0.0724 (1.1492 , 0.4033)
0.3	(0.0313 , 1.9938); 0.0831 (19.7603 , 11.6971)	(0.1346 , 2.7909); 0.0975 (7.9524 , 4.8842)	(0.1722 , 2.9222); 0.1017 (5.8723 , 3.5273)	(0.4256 , 3.4302); 0.1274 (2.3774 , 1.4009)	(0.7502 , 3.7374); 0.1595 (1.2259 , 0.4987)
0.4	(0.0188 , 1.6967); 0.1437 (21.2014 , 12.1845)	(0.1127 , 2.7384); 0.1711 (8.7549 , 5.3692)	(0.1610 , 2.9442); 0.1821 (6.5109 , 4.0228)	(0.3382 , 3.4043); 0.2182 (2.6739 , 1.5664)	(0.6500 , 3.8326); 0.2792 (1.3505 , 0.6280)
0.5	(0.0119 , 1.4135); 0.2204 (22.5823 , 13.0843)	(0.0986 , 2.7099); 0.2665 (9.8110 , 6.1161)	(0.1130 , 2.7938); 0.2725 (7.3309 , 4.3772)	(0.2813 , 3.4055); 0.3355 (3.0675 , 1.8236)	(0.5379 , 3.9034); 0.4253 (1.5320 , 0.7851)

Table 5: Optimal setting of the EWMA chart for the MCV-squared for $p = 4$, $n \in \{5, 10, 15, 20\}$, $\gamma_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ and $\tau \in \{1.1, 1.2, 1.25, 1.5, 2.0\}$, obtained according to Algorithm 2. The first row of each block contains the pair (λ^*, K^*) followed by the resulting UCL; the second row shows the pair $(\text{ARL}(\gamma_1), \text{SDRL}(\gamma_1))$, with $\gamma_1 = \tau\gamma_0$.

γ_0	$\tau = 1.1$	$\tau = 1.2$	$\tau = 1.25$	$\tau = 1.5$	$\tau = 2.0$
			$n = 10$		
0.1	(0.0141 , 1.5109); 0.0060 (34.5843 , 23.5649)	(0.0656 , 2.4747); 0.0072 (15.6248 , 10.4836)	(0.0844 , 2.6281); 0.0075 (11.6567 , 7.6742)	(0.2222 , 3.2495); 0.0096 (4.6193 , 3.0571)	(0.4699 , 3.7941); 0.0130 (2.0138 , 1.2353)
0.2	(0.0102 , 1.3115); 0.0235 (34.6688 , 23.5699)	(0.0532 , 2.3615); 0.0278 (16.2369 , 10.6467)	(0.0813 , 2.6249); 0.0299 (12.1771 , 8.0913)	(0.1911 , 3.1875); 0.0370 (4.8894 , 3.2017)	(0.3875 , 3.7195); 0.0484 (2.1626 , 1.3276)
0.3	(0.0117 , 1.4010); 0.0531 (36.8462 , 25.6531)	(0.0438 , 2.2554); 0.0608 (17.2973 , 11.2562)	(0.0704 , 2.5599); 0.0658 (13.0644 , 8.6760)	(0.1690 , 3.1611); 0.0816 (5.3627 , 3.5392)	(0.3379 , 3.7167); 0.1054 (2.4252 , 1.5265)
0.4	(0.0102 , 1.3179); 0.0933 (38.8727 , 27.6520)	(0.0353 , 2.1312); 0.1050 (18.8188 , 12.2457)	(0.0557 , 2.4346); 0.1128 (14.3571 , 9.4668)	(0.1424 , 3.1073); 0.1405 (6.0704 , 4.0348)	(0.2614 , 3.6219); 0.1740 (2.8224 , 1.7981)
0.5	(0.0110 , 1.3562); 0.1455 (42.5545 , 31.3094)	(0.0141 , 1.5232); 0.1483 (20.4015 , 12.5872)	(0.0392 , 2.2196); 0.1671 (16.0887 , 10.4461)	(0.1001 , 2.9032); 0.2027 (7.0471 , 4.5944)	(0.1941 , 3.4763); 0.2506 (3.3865 , 2.1733)
			$n = 15$		
0.1	(0.0137 , 1.4802); 0.0075 (23.4017 , 13.7107)	(0.1048 , 2.6601); 0.0092 (9.7890 , 6.1028)	(0.1657 , 2.9057); 0.0099 (7.2251 , 4.6342)	(0.3632 , 3.3408); 0.0122 (2.8305 , 1.7283)	(0.6750 , 3.6854); 0.0156 (1.3421 , 0.6266)
0.2	(0.0125 , 1.4362); 0.0300 (24.1239 , 14.2223)	(0.1065 , 2.6906); 0.0369 (10.2415 , 6.5255)	(0.1380 , 2.8336); 0.0386 (7.5681 , 4.7333)	(0.3188 , 3.3167); 0.0475 (3.0035 , 1.8204)	(0.6129 , 3.7129); 0.0609 (1.4225 , 0.7006)
0.3	(0.0134 , 1.4815); 0.0676 (25.4861 , 15.3977)	(0.0940 , 2.6505); 0.0819 (10.9992 , 7.0185)	(0.1157 , 2.7706); 0.0849 (8.1716 , 5.0661)	(0.2719 , 3.2938); 0.1039 (3.2996 , 2.0025)	(0.5586 , 3.7803); 0.1360 (1.5655 , 0.8292)
0.4	(0.0133 , 1.4809); 0.1200 (27.2505 , 16.9582)	(0.0751 , 2.5522); 0.1418 (12.0744 , 7.6276)	(0.1002 , 2.7290); 0.1488 (9.0373 , 5.6537)	(0.2294 , 3.2733); 0.1802 (3.7290 , 2.2855)	(0.4625 , 3.8038); 0.2319 (1.7827 , 0.9979)
0.5	(0.0149 , 1.5587); 0.1882 (29.8455 , 19.3646)	(0.0564 , 2.4074); 0.2141 (13.4925 , 8.4179)	(0.0750 , 2.5909); 0.2236 (10.1899 , 6.2882)	(0.1750 , 3.1765); 0.2680 (4.3086 , 2.6284)	(0.3754 , 3.8076); 0.3476 (2.0865 , 1.2204)
			$n = 20$		
0.1	(0.0476 , 2.2041); 0.0089 (18.6650 , 11.6065)	(0.1544 , 2.8042); 0.0102 (7.3752 , 4.5295)	(0.1971 , 2.9234); 0.0107 (5.4104 , 3.2378)	(0.4795 , 3.3604); 0.0134 (2.1364 , 1.2337)	(0.8770 , 3.6074); 0.0172 (1.1319 , 0.3788)
0.2	(0.0472 , 2.2147); 0.0356 (19.4026 , 12.1957)	(0.1502 , 2.8163); 0.0411 (7.7339 , 4.8098)	(0.1844 , 2.9205); 0.0426 (5.6860 , 3.4163)	(0.4379 , 3.3700); 0.0527 (2.2666 , 1.3136)	(0.8004 , 3.6568); 0.0670 (1.1772 , 0.4397)
0.3	(0.0376 , 2.1009); 0.0788 (20.5629 , 12.7200)	(0.1192 , 2.7301); 0.0898 (8.3229 , 5.0437)	(0.1564 , 2.8760); 0.0940 (6.1540 , 3.6698)	(0.3883 , 3.3877); 0.1169 (2.4876 , 1.4585)	(0.7002 , 3.7212); 0.1466 (1.2634 , 0.5384)
0.4	(0.0281 , 1.9417); 0.1373 (22.1314 , 13.5158)	(0.1002 , 2.6734); 0.1571 (9.1662 , 5.5556)	(0.1501 , 2.9057); 0.1682 (6.8203 , 4.2070)	(0.3100 , 3.3523); 0.1997 (2.8037 , 1.6366)	(0.6027 , 3.7940); 0.2544 (1.4016 , 0.6724)
0.5	(0.0377 , 2.1312); 0.2202 (24.4381 , 16.1878)	(0.0813 , 2.5914); 0.2406 (10.2648 , 6.2199)	(0.1251 , 2.8559); 0.2581 (7.6936 , 4.7696)	(0.2391 , 3.2860); 0.2987 (3.2234 , 1.8754)	(0.5199 , 3.8739); 0.3917 (1.6015 , 0.8438)

Table 6: Optimal setting of the EWMA chart for the MCV-squared for $p = 5$, $n \in \{10, 15, 20\}$, $\gamma_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ and $\tau \in \{1.1, 1.2, 1.25, 1.5, 2.0\}$, obtained according to Algorithm 2. The first row of each block contains the pair (λ^*, K^*) followed by the resulting UCL; the second row shows the pair $(\mathrm{ARL}(\gamma_1), \mathrm{SDRL}(\gamma_1))$, with $\gamma_1 = \tau \gamma_0$.

γ_0	$\tau = 1.1$	$\tau = 1.2$	$\tau = 1.25$	$\tau = 1.5$	$\tau = 2.0$
			$n = 10$		
0.1	(0.0102 , 1.3077); 0.0047	(0.0501 , 2.3302); 0.0056	(0.0626 , 2.4699); 0.0058	(0.1875 , 3.1974); 0.0077	(0.3885 , 3.7564); 0.0103
	(37.9565 , 26.7642)	(18.0310 , 12.1353)	(13.5485 , 8.8941)	(5.4135 , 3.6626)	(2.3320 , 1.4917)
0.2	(0.0113 , 1.3763); 0.0190	(0.0391 , 2.1837); 0.0216	(0.0596 , 2.4537); 0.0231	(0.1691 , 3.1579); 0.0299	(0.3381 , 3.7035); 0.0390
	(39.5100 , 28.2135)	(18.7208 , 12.2766)	(14.1362 , 9.3400)	(5.7326 , 3.8715)	(2.5128 , 1.6170)
0.3	(0.0110 , 1.3584); 0.0422	(0.0406 , 2.2209); 0.0486	(0.0493 , 2.3469); 0.0502	(0.1377 , 3.0558); 0.0636	(0.2876 , 3.6546); 0.0831
	(41.3637 , 30.0385)	(19.9420 , 13.4569)	(15.1454 , 9.9315)	(6.2891 , 4.2049)	(2.8323 , 1.8499)
0.4	(0.0110 , 1.3576); 0.0742	(0.0110 , 1.3576); 0.0742	(0.0313 , 2.0600); 0.0827	(0.1148 , 2.9733); 0.1080	(0.2126 , 3.4872); 0.1330
	(44.2043 , 32.8699)	(20.8426 , 12.8372)	(16.6188 , 10.5353)	(7.1214 , 4.7737)	(3.3186 , 2.1650)
0.5	(0.0104 , 1.3265); 0.1138	(0.0133 , 1.4863); 0.1160	(0.0188 , 1.7210); 0.1199	(0.0751 , 2.6983); 0.1513	(0.1690 , 3.3766); 0.1937
	(47.8654 , 36.5159)	(23.2007 , 14.8772)	(18.4161 , 11.4790)	(8.2854 , 5.3816)	(4.0089 , 2.6445)
			$n = 15$		
0.1	(0.0158 , 1.5719); 0.0068	(0.0877 , 2.5751); 0.0081	(0.1381 , 2.8238); 0.0088	(0.3225 , 3.3035); 0.0108	(0.6273 , 3.6935); 0.0140
	(24.9964 , 15.1322)	(10.5372 , 6.4885)	(7.7707 , 4.9024)	(3.0485 , 1.8640)	(1.4178 , 0.6999)
0.2	(0.0306 , 1.9905); 0.0287	(0.0861 , 2.5836); 0.0324	(0.1192 , 2.7660); 0.0342	(0.2879 , 3.2819); 0.0423	(0.6193 , 3.7609); 0.0568
	(26.4510 , 17.3830)	(11.0100 , 6.8701)	(8.1533 , 5.0655)	(3.2372 , 1.9732)	(1.5088 , 0.7904)
0.3	(0.0134 , 1.4820); 0.0607	(0.0845 , 2.5976); 0.0728	(0.0985 , 2.6877); 0.0747	(0.2469 , 3.2523); 0.0922	(0.5299 , 3.7785); 0.1223
	(27.0436 , 16.7130)	(11.8212 , 7.5582)	(8.8096 , 5.4116)	(3.5603 , 2.1746)	(1.6706 , 0.9212)
0.4	(0.0133 , 1.4833); 0.1071	(0.0752 , 2.5593); 0.1274	(0.0861 , 2.6424); 0.1303	(0.2221 , 3.2648); 0.1617	(0.4322 , 3.7722); 0.2054
	(28.9453 , 18.4072)	(12.9896 , 8.4048)	(9.7373 , 6.0442)	(4.0303 , 2.5175)	(1.9151 , 1.1029)
0.5	(0.0126 , 1.4466); 0.1653	(0.0470 , 2.2946); 0.1861	(0.0815 , 2.6481); 0.2024	(0.1814 , 3.2091); 0.2427	(0.3344 , 3.7121); 0.2988
	(31.3021 , 20.5332)	(14.4971 , 8.9503)	(10.9836 , 7.0412)	(4.6675 , 2.9349)	(2.2567 , 1.3363)
			$n = 20$		
0.1	(0.0469 , 2.1998); 0.0083	(0.1409 , 2.7681); 0.0095	(0.1938 , 2.9263); 0.0100	(0.4500 , 3.3494); 0.0124	(0.8125 , 3.6138); 0.0157
	(19.5047 , 12.2963)	(7.7418 , 4.7335)	(5.6836 , 3.4547)	(2.2404 , 1.3011)	(1.1592 , 0.4156)
0.2	(0.0423 , 2.1570); 0.0330	(0.1375 , 2.7795); 0.0380	(0.1755 , 2.9050); 0.0397	(0.4067 , 3.3485); 0.0487	(0.7756 , 3.6700); 0.0628
	(20.2551 , 12.6831)	(8.1157 , 5.0268)	(5.9740 , 3.6123)	(2.3784 , 1.3815)	(1.2106 , 0.4821)
0.3	(0.0313 , 1.9969); 0.0724	(0.1188 , 2.7342); 0.0840	(0.1696 , 2.9277); 0.0894	(0.3817 , 3.3920); 0.1095	(0.7197 , 3.7542); 0.1406
	(21.4451 , 13.0905)	(8.7391 , 5.3863)	(6.4702 , 4.0121)	(2.6137 , 1.5583)	(1.3071 , 0.5916)
0.4	(0.0114 , 1.3822); 0.1212	(0.0985 , 2.6674); 0.1459	(0.1345 , 2.8463); 0.1537	(0.3068 , 3.3542); 0.1863	(0.5826 , 3.7844); 0.2355
	(22.4755 , 12.8881)	(9.6248 , 5.9078)	(7.1655 , 4.3763)	(2.9495 , 1.7506)	(1.4603 , 0.7267)
0.5	(0.0135 , 1.4932); 0.1896	(0.0876 , 2.6380); 0.2252	(0.1129 , 2.7931); 0.2348	(0.2322 , 3.2674); 0.2753	(0.5008 , 3.8472); 0.3592
	(24.6029 , 14.7188)	(10.7875 , 6.7444)	(8.0859 , 4.9709)	(3.3976 , 1.9967)	(1.6805 , 0.9088)

Table 7: Optimal setting of the EWMA chart for the MCV-squared for $p = 6$, $n \in \{10, 15, 20\}$, $\gamma_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ and $\tau \in \{1.1, 1.2, 1.25, 1.5, 2.0\}$, obtained according to Algorithm 2. The first row of each block contains the pair (λ^*, K^*) followed by the resulting UCL; the second row shows the pair $(\text{ARL}(\gamma_1), \text{SDRL}(\gamma_1))$, with $\gamma_1 = \tau\gamma_0$.

γ_0	$\tau = 1.1$	$\tau = 1.2$	$\tau = 1.25$	$\tau = 1.5$	$\tau = 2.0$
			$n = 10$		
0.1	(0.0104 , 1.3173); 0.0036 (44.4142 , 33.1917)	(0.0137 , 1.4971); 0.0037 (21.1225 , 13.1176)	(0.0461 , 2.3028); 0.0043 (16.3594 , 10.9110)	(0.1375 , 3.0523); 0.0056 (6.6360 , 4.5407)	(0.3127 , 3.7144); 0.0077 (2.8351 , 1.9063)
0.2	(0.0102 , 1.3131); 0.0142 (45.8149 , 34.4236)	(0.0102 , 1.3131); 0.0142 (21.5439 , 13.3797)	(0.0500 , 2.3676); 0.0173 (17.0582 , 11.6724)	(0.1408 , 3.0960); 0.0225 (7.0307 , 4.9089)	(0.2877 , 3.6900); 0.0298 (3.0645 , 2.0827)
0.3	(0.0103 , 1.3210); 0.0316 (48.1157 , 36.7599)	(0.0103 , 1.3210); 0.0316 (22.7313 , 14.4230)	(0.0375 , 2.1865); 0.0367 (18.2156 , 12.2023)	(0.1001 , 2.8736); 0.0456 (7.7153 , 5.2049)	(0.2445 , 3.6138); 0.0628 (3.4705 , 2.3798)
0.4	(0.0103 , 1.3228); 0.0551 (51.4161 , 40.1260)	(0.0103 , 1.3228); 0.0551 (24.4775 , 15.9561)	(0.0108 , 1.3426); 0.0552 (19.2344 , 11.9543)	(0.0907 , 2.8361); 0.0785 (8.7282 , 5.9729)	(0.2000 , 3.5091); 0.1032 (4.0902 , 2.8277)
0.5	(0.0103 , 1.3190); 0.0840 (55.8428 , 44.6551)	(0.0156 , 1.5989); 0.0873 (27.7330 , 18.9556)	(0.0156 , 1.5989); 0.0873 (21.7497 , 14.1392)	(0.0719 , 2.6917); 0.1140 (10.1367 , 6.9146)	(0.1318 , 3.1975); 0.1373 (4.9672 , 3.3563)
			$n = 15$		
0.1	(0.0141 , 1.5029); 0.0061 (26.5664 , 16.3929)	(0.0805 , 2.5403); 0.0072 (11.4194 , 7.1170)	(0.1377 , 2.8413); 0.0079 (8.4498 , 5.4960)	(0.2719 , 3.2360); 0.0094 (3.3171 , 2.0151)	(0.6187 , 3.7391); 0.0129 (1.5130 , 0.7965)
0.2	(0.0165 , 1.6102); 0.0244 (27.7521 , 17.4610)	(0.0876 , 2.6055); 0.0292 (11.9336 , 7.6966)	(0.1000 , 2.6815); 0.0298 (8.8660 , 5.4623)	(0.2753 , 3.2850); 0.0379 (3.5196 , 2.1949)	(0.5525 , 3.7396); 0.0496 (1.6167 , 0.8800)
0.3	(0.0126 , 1.4461); 0.0536 (28.7741 , 18.1895)	(0.0768 , 2.5515); 0.0641 (12.8071 , 8.2530)	(0.1111 , 2.7719); 0.0685 (9.5659 , 6.2092)	(0.2346 , 3.2433); 0.0821 (3.8746 , 2.4141)	(0.4750 , 3.7407); 0.1063 (1.8004 , 1.0227)
0.4	(0.0127 , 1.4503); 0.0942 (30.8202 , 20.0293)	(0.0501 , 2.3171); 0.1067 (14.0698 , 8.6522)	(0.0866 , 2.6547); 0.1161 (10.5704 , 6.7478)	(0.1908 , 3.1746); 0.1389 (4.3932 , 2.7333)	(0.3812 , 3.6987); 0.1762 (2.0774 , 1.2187)
0.5	(0.0126 , 1.4474); 0.1449 (33.4895 , 22.4700)	(0.0368 , 2.1407); 0.1589 (15.7032 , 9.5499)	(0.0688 , 2.5426); 0.1736 (11.9133 , 7.5728)	(0.1487 , 3.0696); 0.2043 (5.0969 , 3.1587)	(0.3127 , 3.6664); 0.2600 (2.4646 , 1.4928)
			$n = 20$		
0.1	(0.0391 , 2.1019); 0.0076 (20.4094 , 12.6801)	(0.1254 , 2.7190); 0.0087 (8.1583 , 4.9456)	(0.1800 , 2.9014); 0.0093 (5.9921 , 3.6479)	(0.4375 , 3.3571); 0.0116 (2.3585 , 1.3931)	(0.8264 , 3.6488); 0.0151 (1.1925 , 0.4632)
0.2	(0.0377 , 2.0956); 0.0304 (21.2017 , 13.2441)	(0.1253 , 2.7403); 0.0350 (8.5480 , 5.2774)	(0.1724 , 2.9064); 0.0370 (6.2995 , 3.8708)	(0.4070 , 3.3685); 0.0459 (2.5052 , 1.4906)	(0.7568 , 3.6876); 0.0588 (1.2505 , 0.5304)
0.3	(0.0346 , 2.0575); 0.0677 (22.4705 , 14.1647)	(0.1000 , 2.6476); 0.0762 (9.2125 , 5.5573)	(0.1532 , 2.8805); 0.0818 (6.8185 , 4.2016)	(0.3500 , 3.3557); 0.1000 (2.7547 , 1.6411)	(0.6820 , 3.7499); 0.1293 (1.3586 , 0.6428)
0.4	(0.0180 , 1.6690); 0.1143 (23.9303 , 14.3223)	(0.0985 , 2.6714); 0.1353 (10.1437 , 6.3370)	(0.1313 , 2.8379); 0.1420 (7.5600 , 4.6753)	(0.2945 , 3.3373); 0.1717 (3.1146 , 1.8676)	(0.5609 , 3.7741); 0.2170 (1.5283 , 0.7872)
0.5	(0.0154 , 1.5737); 0.1752 (25.8658 , 15.8301)	(0.0830 , 2.6068); 0.2059 (11.3668 , 7.1384)	(0.1111 , 2.7856); 0.2160 (8.5368 , 5.3171)	(0.2375 , 3.2871); 0.2564 (3.5943 , 2.1618)	(0.4691 , 3.8013); 0.3245 (1.7709 , 0.9767)

Table 8: Optimal setting of the EWMA chart for the MCV-squared for $p = 7$, $n \in \{10, 15, 20\}$, $\gamma_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ and $\tau \in \{1.1, 1.2, 1.25, 1.5, 2.0\}$, obtained according to Algorithm 2. The first row of each block contains the pair (λ^*, K^*) followed by the resulting UCL; the second row shows the pair $(\text{ARL}(\gamma_1), \text{SDRL}(\gamma_1))$, with $\gamma_1 = \tau\gamma_0$.

γ_0	$\tau = 1.1$	$\tau = 1.2$	$\tau = 1.25$	$\tau = 1.5$	$\tau = 2.0$
			$n = 10$		
0.1	(0.0111 , 1.3565); 0.0024 (55.3228 , 44.5806)	(0.0111 , 1.3565); 0.0024 (25.9977 , 17.5742)	(0.0142 , 1.5305); 0.0025 (20.7007 , 13.3671)	(0.1032 , 2.9362); 0.0037 (8.8174 , 6.3026)	(0.2414 , 3.6735); 0.0052 (3.7639 , 2.7028)
0.2	(0.0110 , 1.3502); 0.0096 (56.9625 , 46.2461)	(0.0110 , 1.3502); 0.0096 (26.8989 , 18.3728)	(0.0110 , 1.3502); 0.0096 (21.0311 , 13.6582)	(0.0971 , 2.9056); 0.0146 (9.3322 , 6.6915)	(0.2006 , 3.5314); 0.0192 (4.0764 , 2.9046)
0.3	(0.0114 , 1.3802); 0.0214 (60.0330 , 49.2267)	(0.0114 , 1.3802); 0.0214 (28.5770 , 19.8141)	(0.0114 , 1.3802); 0.0214 (22.4036 , 14.7828)	(0.0845 , 2.8191); 0.0311 (10.2264 , 7.3226)	(0.1626 , 3.3797); 0.0394 (4.6335 , 3.2913)
0.4	(0.0114 , 1.3820); 0.0371 (64.1148 , 53.4807)	(0.0114 , 1.3820); 0.0371 (30.8560 , 21.8914)	(0.0114 , 1.3820); 0.0371 (24.2821 , 16.4061)	(0.0609 , 2.5870); 0.0496 (11.5487 , 8.0846)	(0.1321 , 3.2360); 0.0638 (5.4803 , 3.8970)
0.5	(0.0114 , 1.3787); 0.0560 (69.5283 , 59.1928)	(0.0114 , 1.3787); 0.0560 (33.9474 , 24.7490)	(0.0114 , 1.3787); 0.0560 (26.8411 , 18.6455)	(0.0535 , 2.5046); 0.0731 (13.3692 , 9.4530)	(0.1016 , 3.0402); 0.0890 (6.6768 , 4.7230)
			$n = 15$		
0.1	(0.0126 , 1.4275); 0.0053 (28.3665 , 17.9327)	(0.0808 , 2.5569); 0.0064 (12.4945 , 8.0578)	(0.1048 , 2.7064); 0.0067 (9.2645 , 5.8641)	(0.2849 , 3.3041); 0.0086 (3.6430 , 2.3252)	(0.5000 , 3.6689); 0.0107 (1.6372 , 0.8873)
0.2	(0.0110 , 1.3587); 0.0209 (29.1223 , 18.4818)	(0.0782 , 2.5545); 0.0254 (13.0616 , 8.4983)	(0.1126 , 2.7676); 0.0271 (9.7223 , 6.3578)	(0.2532 , 3.2673); 0.0333 (3.8709 , 2.4569)	(0.5158 , 3.7506); 0.0437 (1.7544 , 1.0006)
0.3	(0.0142 , 1.5221); 0.0471 (31.2859 , 20.4578)	(0.0502 , 2.3087); 0.0531 (14.0466 , 8.6161)	(0.0883 , 2.6479); 0.0580 (10.4848 , 6.6899)	(0.2000 , 3.1649); 0.0700 (4.2678 , 2.6593)	(0.4252 , 3.7051); 0.0915 (1.9645 , 1.1506)
0.4	(0.0142 , 1.5254); 0.0824 (33.5562 , 22.5271)	(0.0375 , 2.1443); 0.0899 (15.4178 , 9.3186)	(0.0829 , 2.6383); 0.1012 (11.5993 , 7.5648)	(0.1784 , 3.1479); 0.1207 (4.8424 , 3.0682)	(0.3566 , 3.6782); 0.1533 (2.2806 , 1.3803)
0.5	(0.0141 , 1.5228); 0.1261 (36.5324 , 25.2762)	(0.0250 , 1.8960); 0.1321 (17.1612 , 10.1957)	(0.0595 , 2.4544); 0.1476 (13.0574 , 8.3102)	(0.1406 , 3.0409); 0.1765 (5.6268 , 3.5516)	(0.2783 , 3.5845); 0.2196 (2.7230 , 1.6760)
			$n = 20$		
0.1	(0.0346 , 2.0359); 0.0070 (21.4299 , 13.3009)	(0.1255 , 2.7292); 0.0081 (8.6273 , 5.3460)	(0.1754 , 2.9014); 0.0086 (6.3443 , 3.9248)	(0.4125 , 3.3504); 0.0107 (2.4935 , 1.4873)	(0.7756 , 3.6606); 0.0139 (1.2328 , 0.5111)
0.2	(0.0409 , 2.1444); 0.0283 (22.3203 , 14.3955)	(0.1256 , 2.7506); 0.0325 (9.0468 , 5.7081)	(0.1627 , 2.8876); 0.0341 (6.6691 , 4.1274)	(0.4000 , 3.3808); 0.0428 (2.6518 , 1.6106)	(0.7193 , 3.6962); 0.0542 (1.2985 , 0.5826)
0.3	(0.0377 , 2.1097); 0.0629 (23.6674 , 15.3979)	(0.1031 , 2.6713); 0.0709 (9.7367 , 6.0369)	(0.1376 , 2.8305); 0.0744 (7.2189 , 4.4244)	(0.3504 , 3.3741); 0.0934 (2.9190 , 1.7815)	(0.6504 , 3.7497); 0.1188 (1.4195 , 0.7016)
0.4	(0.0135 , 1.4889); 0.1033 (24.7690 , 14.7770)	(0.0940 , 2.6500); 0.1239 (10.7300 , 6.7615)	(0.1253 , 2.8171); 0.1301 (8.0077 , 4.9948)	(0.2813 , 3.3189); 0.1572 (3.3031 , 2.0018)	(0.5375 , 3.7627); 0.1987 (1.6077 , 0.8556)
0.5	(0.0142 , 1.5228); 0.1594 (26.9909 , 16.7025)	(0.0672 , 2.4824); 0.1832 (12.0168 , 7.3534)	(0.1049 , 2.7534); 0.1963 (9.0444 , 5.6627)	(0.2224 , 3.2459); 0.2320 (3.8188 , 2.3071)	(0.4322 , 3.7448); 0.2902 (1.8750 , 1.0506)

Table 9: Optimal setting of the EWMA chart for the MCV-squared for $p = 8$, $n \in \{10, 15, 20\}$, $\gamma_0 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ and $\tau \in \{1.1, 1.2, 1.25, 1.5, 2.0\}$, obtained according to Algorithm 2. The first row of each block contains the pair (λ^*, K^*) followed by the resulting UCL; the second row shows the pair $(\text{ARL}(\gamma_1), \text{SDRL}(\gamma_1))$, with $\gamma_1 = \tau\gamma_0$.

p = 2

γ_0	$\tau = 1.2$	$\tau = 1.5$	$\tau = 2.0$
		$n = 5$	
0.1	(−71.2 , −57.0)	(−56.5 , −33.2)	(−41.7 , 4.3)
0.2	(−74.6 , −59.0)	(−56.8 , −34.7)	(−42.9 , 0.3)
0.3	(−74.3 , −59.9)	(−58.1 , −37.2)	(−44.2 , −5.6)
0.4	(−74.0 , −61.1)	(−60.4 , −40.5)	(−47.0 , −13.2)
0.5	(−72.5 , −61.2)	(−66.2 , −45.3)	(−50.8 , −22.3)
		$n = 10$	
0.1	(−74.0 , −57.4)	(−53.3 , −14.8)	(−49.1 , 62.4)
0.2	(−73.9 , −58.2)	(−54.3 , −18.1)	(−47.9 , 51.4)
0.3	(−73.9 , −59.4)	(−56.4 , −23.3)	(−46.5 , 37.2)
0.4	(−75.4 , −60.9)	(−58.7 , −29.4)	(−47.3 , 21.3)
0.5	(−75.6 , −62.4)	(−61.8 , −35.8)	(−49.0 , 6.0)
		$n = 15$	
0.1	(−72.4 , −54.4)	(−49.6 , 5.7)	(−61.6 , 148.2)
0.2	(−73.2 , −55.6)	(−50.6 , 0.3)	(−58.5 , 118.7)
0.3	(−74.3 , −57.3)	(−53.3 , −7.1)	(−54.1 , 89.8)
0.4	(−75.6 , −59.4)	(−55.7 , −15.6)	(−51.5 , 61.6)
0.5	(−76.3 , −61.5)	(−59.1 , −24.4)	(−49.7 , 37.4)

p = 3

γ_0	$\tau = 1.2$	$\tau = 1.5$	$\tau = 2.0$
		$n = 5$	
0.1	(−71.4 , −57.3)	(−55.2 , −35.6)	(−41.0 , −5.9)
0.2	(−70.5 , −56.8)	(−56.1 , −36.8)	(−41.9 , −8.8)
0.3	(−70.2 , −57.4)	(−58.4 , −39.0)	(−43.9 , −13.7)
0.4	(−70.8 , −59.3)	(−60.5 , −42.1)	(−47.6 , −20.4)
0.5	(−70.1 , −60.1)	(−66.7 , −48.4)	(−52.4 , −28.8)
		$n = 10$	
0.1	(−73.3 , −57.8)	(−54.1 , −18.8)	(−47.1 , 49.7)
0.2	(−73.5 , −58.5)	(−55.6 , −21.8)	(−46.2 , 40.4)
0.3	(−73.3 , −59.5)	(−56.8 , −26.7)	(−46.4 , 27.0)
0.4	(−75.9 , −61.1)	(−60.1 , −32.4)	(−47.2 , 13.0)
0.5	(−76.1 , −62.6)	(−63.4 , −38.5)	(−50.0 , −1.0)
		$n = 15$	
0.1	(−72.3 , −55.1)	(−49.9 , 1.3)	(−59.0 , 128.1)
0.2	(−72.8 , −56.2)	(−52.3 , −3.3)	(−55.9 , 102.3)
0.3	(−75.2 , −57.8)	(−54.2 , −10.4)	(−52.0 , 76.0)
0.4	(−75.5 , −59.8)	(−56.8 , −18.7)	(−49.9 , 52.0)
0.5	(−76.5 , −61.9)	(−60.4 , −27.1)	(−49.3 , 30.1)

p = 4

γ_0	$\tau = 1.2$	$\tau = 1.5$	$\tau = 2.0$
		$n = 5$	
0.1	(−63.4 , −52.8)	(−53.1 , −35.7)	(−38.5 , −14.5)
0.2	(−63.1 , −52.8)	(−58.6 , −38.1)	(−39.8 , −16.6)
0.3	(−62.7 , −53.0)	(−59.4 , −40.5)	(−41.6 , −20.0)
0.4	(−62.3 , −53.2)	(−60.6 , −43.7)	(−45.7 , −25.2)
0.5	(−61.3 , −53.5)	(−62.2 , −47.6)	(−55.0 , −33.9)
		$n = 10$	
0.1	(−73.7 , −58.0)	(−55.0 , −22.7)	(−45.1 , 36.9)
0.2	(−73.6 , −58.7)	(−55.7 , −25.5)	(−45.1 , 28.8)
0.3	(−74.1 , −59.7)	(−57.9 , −29.8)	(−45.6 , 17.6)
0.4	(−74.8 , −61.1)	(−61.2 , −35.2)	(−47.3 , 5.3)
0.5	(−75.9 , −62.6)	(−63.1 , −40.9)	(−50.6 , −7.7)
		$n = 15$	
0.1	(−73.3 , −55.8)	(−50.7 , −2.6)	(−56.1 , 106.6)
0.2	(−72.7 , −56.7)	(−52.5 , −7.3)	(−53.4 , 87.8)
0.3	(−74.6 , −58.4)	(−54.6 , −13.9)	(−50.0 , 66.3)
0.4	(−75.8 , −60.3)	(−57.8 , −21.7)	(−48.3 , 43.0)
0.5	(−76.8 , −62.3)	(−60.3 , −29.7)	(−49.4 , 22.2)

Table 10: Performance comparison between the EWMA and the MCV charts. Each cell contains the relative difference in ARL (in percent) of the EWMA chart with respect to the MCV chart, followed by the relative difference in SDRL (in percent). Only the cases considered by Yeong et al. [17] are included in the comparison.

p = 2

γ_0	$\tau = 1.10$	$\tau = 1.25$	$\tau = 1.50$
		$n = 5$	
0.1	(−62.5 , −48.1)	(−54.4 , −23.8)	(−41.9 , 5.8)
0.3	(−61.4 , −48.3)	(−59.9 , −28.5)	(−41.9 , 3.8)
0.5	(−57.1 , −46.8)	(−57.1 , −32.5)	(−48.6 , −2.5)
		$n = 10$	
0.1	(−66.7 , −42.5)	(−49.3 , 2.3)	(−42.5 , 41.3)
0.3	(−67.4 , −45.6)	(−51.2 , −3.7)	(−43.0 , 34.1)
0.5	(−66.4 , −48.4)	(−53.9 , −12.3)	(−42.8 , 30.2)

p = 3

γ_0	$\tau = 1.10$	$\tau = 1.25$	$\tau = 1.50$
		$n = 5$	
0.1	(−58.4 , −47.0)	(−58.6 , −31.3)	(−41.5 , −3.5)
0.3	(−57.2 , −46.8)	(−58.5 , −34.3)	(−44.2 , −7.5)
0.5	(−53.4 , −45.2)	(−58.6 , −39.4)	(−52.2 , −18.2)
		$n = 10$	
0.1	(−65.7 , −43.4)	(−50.9 , −2.4)	(−42.1 , 35.2)
0.3	(−66.6 , −46.3)	(−52.4 , −8.3)	(−42.3 , 34.1)
0.5	(−65.8 , −48.9)	(−53.5 , −16.1)	(−44.4 , 23.5)

Table 11: Performance comparison between the EWMA and the RS charts. Each cell contains the relative difference in ARL (in percent) of the EWMA chart with respect to the RS chart, followed by the relative difference in SDRL (in percent). Only the cases considered by Lim et al. [18] are included in the comparison.

		R_1	R_2	R_3	R_4	R_5	$\bar{\mathbf{X}}$	$\mathbf{S}$			$\hat{\gamma}^2$	Z
	S_1	17.8	25.2	18.1	19.0	19.0	19.82	9.3320	1.3540	0.1205		
2000	S_2	42.0	40.7	35.5	42.0	40.5	40.14	1.3540	7.2230	1.1710	0.004082	0.001574
	S_3	8.3	9.4	8.6	10.5	12.1	9.78	0.1205	1.1710	2.4070		
	S_1	21.5	22.5	22.0	18.1	19.1	20.64	3.7180	3.6455	0.0435		
2001	S_2	40.5	36.9	42.0	36.2	35.1	38.14	3.6455	8.7530	4.1410	0.001739	0.001612
	S_3	11.9	8.5	12.8	11.4	9.3	10.78	0.0435	4.1410	3.2770		
	S_1	17.5	18.9	19.1	21.8	22.7	20.00	4.7000	−0.5675	0.3900		
2002	S_2	39.9	38.2	38.0	39.8	37.8	38.74	−0.5675	1.0480	−0.0355	0.000539	0.001364
	S_3	8.9	11.0	7.8	9.6	9.5	9.36	0.3900	−0.0355	1.3530		
	S_1	19.1	18.7	21.4	20.8	18.9	19.78	1.5170	0.5765	0.4550		
2003	S_2	38.8	42.4	42.5	39.6	39.5	40.56	0.5765	3.0730	0.6325	0.001422	0.001377
	S_3	9.7	10.3	11.0	10.4	10.1	10.30	0.4550	0.6325	0.2250		
	S_1	19.0	21.6	19.5	19.0	19.2	19.66	1.2180	1.2920	−0.1385		
2004	S_2	39.4	40.8	35.9	35.0	41.6	38.54	1.2920	8.6780	−2.0265	0.002000	0.001522
	S_3	8.2	9.7	12.6	10.5	10.9	10.38	−0.1385	−2.0265	2.6070		
	S_1	18.9	19.1	21.3	17.0	20.2	19.30	2.5750	3.3900	−0.6550		
2005	S_2	41.1	38.1	42.8	36.2	39.2	39.48	3.3900	6.5970	−0.1985	0.001470	0.001510
	S_3	10.8	9.5	8.7	9.6	9.2	9.56	−0.6550	−0.1985	0.6030		
	S_1	17.9	20.0	20.5	18.5	19.4	19.26	1.1330	−1.6770	−0.5400		
2006	S_2	43.0	41.2	36.5	39.1	41.0	40.16	−1.6770	6.0930	2.0675	0.000603	0.001300
	S_3	8.2	9.6	6.3	9.6	9.8	8.70	−0.5400	2.0675	2.2100		
	S_1	20.6	18.7	18.5	23.6	19.7	20.22	4.2770	3.6525	3.8680		
2007	S_2	40.3	36.9	35.4	40.8	41.6	39.00	3.6525	7.2650	3.9925	0.001834	0.001423
	S_3	9.0	8.7	6.8	12.4	9.5	9.28	3.8680	3.9925	4.0870		
	S_1	19.0	20.4	21.6	20.3	18.4	19.94	1.5880	1.0640	1.3085		
2008	S_2	37.3	44.8	40.5	38.9	40.6	40.42	1.0640	7.8170	4.2605	0.001383	0.001414
	S_3	8.3	11.7	10.7	7.0	8.2	9.18	1.3085	4.2605	3.7870		
	S_1	21.2	16.5	18.2	21.2	21.2	19.66	4.8080	1.9075	0.2025		
2009	S_2	38.9	39.6	36.8	40.6	41.6	39.50	1.9075	3.3200	−0.9650	0.001305	0.001389
	S_3	10.9	8.6	9.1	7.0	8.9	8.90	0.2025	−0.9650	1.9350		
	S_1	9.6	8.8	8.4	6.9	7.4	8.22	1.1720	−2.0755	−2.4445		
2010	S_2	19.5	17.9	18.9	23.7	21.6	20.32	−2.0755	5.4020	4.0830	0.000499	0.001183
	S_3	2.2	5.0	5.3	8.9	6.0	5.48	−2.4445	4.0830	5.7470		
	S_1	11.0	11.8	15.6	11.2	10.3	11.98	4.3820	−1.7770	−0.5615		
2011	S_2	18.9	21.6	19.0	20.1	22.5	20.42	−1.7770	2.5370	0.7490	0.002599	0.001510
	S_3	4.6	6.2	4.8	6.1	5.5	5.44	−0.5615	0.7490	0.5330		
	S_1	9.5	10.0	8.3	8.8	12.3	9.78	2.4070	2.2845	−0.6710		
2012	S_2	22.1	17.1	19.6	23.4	25.2	21.48	2.2845	10.1570	1.0565	0.007852	**0.002978**
	S_3	4.5	3.8	6.2	5.8	5.0	5.06	−0.6710	1.0565	0.9380		
	S_1	10.0	8.0	8.2	11.5	7.1	8.96	3.1230	−1.5155	−1.8780		
2013	S_2	21.1	21.0	21.2	17.9	20.0	20.24	−1.5155	1.9430	2.0330	0.001588	**0.002656**
	S_3	4.4	7.0	7.3	3.0	4.0	5.14	−1.8780	2.0330	3.6380		
	S_1	12.2	9.6	7.7	11.3	13.2	10.80	4.7550	−1.3900	−0.4325		
2014	S_2	20.0	18.2	18.4	19.6	14.8	18.20	−1.3900	4.2000	0.2850	0.004144	**0.003000**
	S_3	4.6	4.5	4.4	3.5	3.8	4.16	−0.4325	0.2850	0.2330		
	S_1	11.6	9.8	12.4	11.0	9.4	10.84	1.5480	0.6550	0.7060		
2015	S_2	17.2	20.1	21.5	18.6	18.1	19.10	0.6550	2.9050	1.8425	0.003456	**0.003106**
	S_3	4.6	6.4	6.4	4.7	3.3	5.08	0.7060	1.8425	1.7570		
	S_1	11.2	5.4	9.4	8.4	6.9	8.26	4.9980	−0.5210	1.9820		
2016	S_2	20.0	22.1	24.5	17.2	20.6	20.88	−0.5210	7.2470	1.8835	0.006183	**0.003818**
	S_3	5.8	4.0	7.3	4.7	4.9	5.34	1.9820	1.8835	1.6130		

Table 12: Illustrative example. Rates of return (in %) from 2000 to 2016 for $p = 3$ industrial sectors $\{S_1, S_2, S_3\}$ and $n = 5$ world regions $\{R_1, R_2, R_3, R_4, R_5\}$.
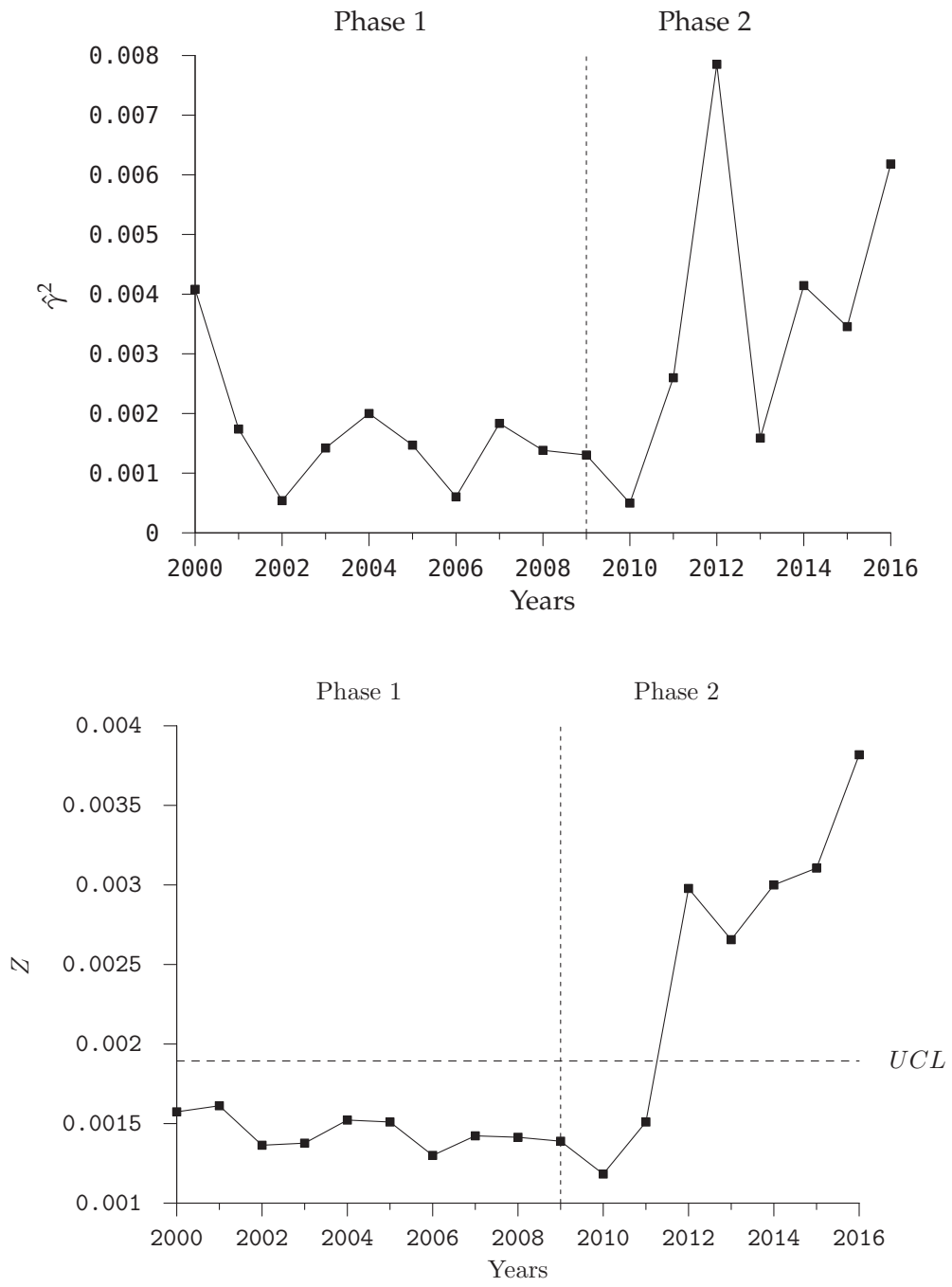
Figure 2: Values of $\hat{\gamma}$ (top) and values Z of the EWMA γ^2 chart (bottom) corresponding to the data in Table 12.

properties of the sample MCV. A practical example from the financial domain was provided to illustrate how our chart can be used to monitor the MCV and to successfully detect unwanted shifts in the joint relative variability of the system being considered.

According to the computational experiments carried out, the chart is shown to clearly outperform the existing alternatives for monitoring the MCV (including a run-sum approach), especially when aimed at detecting relatively small shifts, as usually happens with SPM tools based on an EWMA scheme. Consequently, our proposal provides practitioners with an enhanced, yet simple tool to monitor the MCV in an efficient way.

The optimisation of the chart proposed with regards to an unknown shift τ to be detected was not dealt with in this paper due to writing space constraints. This would be an interesting alternative for practitioners. It can be addressed by assuming a certain probability distribution for (the undesired) τ and then looking for the values of the chart parameters λ and K that minimise the so-called *expected ARL* (EARL) of the chart while ensuring a required in-control ARL. More details about this approach can be found in Castagliola et al.[32] for instance.

Future research on control charts for the MCV could also benefit from adapting another well-known monitoring scheme such as the cumulative sum chart (CUSUM;[38]). Avoiding the normality constraints by using a nonparametric approach is another interesting path to be explored for both the univariate and the multivariate case.

Acknowledgements

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Appendix A. First and second raw moments of the doubly noncentral *F* distribution

The doubly noncentral *F* distribution $F(\nu_1, \nu_2, \delta_1, \delta_2)$ arises as the ratio of two independent noncentral χ^2 variables; more precisely:

$$F(\nu_1, \nu_2, \delta_1, \delta_2) \equiv \frac{\chi^2(\nu_1, \delta_1)/\nu_1}{\chi^2(\nu_2, \delta_2)/\nu_2}. \tag{A.1}$$

Here, $\chi^2(\nu, \delta)$ stands for the noncentral χ^2 distribution with $\nu > 0$ degrees of freedom and noncentrality parameter $\delta \geq 0$[24]; ν_1 and ν_2 are not restricted to integers, although they are traditionally considered so[27]. The distribution resulting from $\delta_2 = 0$ is referred to as the (singly) noncentral *F*. The case $\delta_1 = 0$ (as it happens for $\hat{\gamma}^2$; see (11)) can be regarded as the reciprocal of a singly noncentral *F*. Finally, the case in which both δ_1 and δ_2 equal zero reduces to the *ordinary F* distribution.

Walck[28] gives the following expressions for $\mu'_1(F'')$ and $\mu'_2(F'')$, the first and second raw moments of a $F(\nu_1, \nu_2, \delta_1, \delta_2)$ variable:

$$\mu'_1(F'') = \frac{\nu_2}{\nu_1} \cdot \frac{\nu_1 + \delta_1}{\nu_2 - 2} e^{-\frac{\delta_2}{2}} {}_1F_1\left(\frac{\nu_2}{2} - 1; \frac{\nu_2}{2}; \frac{\delta_2}{2}\right), \quad \text{if } \nu_2 > 2, \tag{A.2}$$

$$\mu'_2(F'') = \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{\delta_1^2 + (2\delta_1 + \nu_1)(\nu_1 + 2)}{(\nu_2 - 2)(\nu_2 - 4)} e^{-\frac{\delta_2}{2}} {}_1F_1\left(\frac{\nu_2}{2} - 2; \frac{\nu_2}{2}; \frac{\delta_2}{2}\right), \quad \text{if } \nu_2 > 4. \tag{A.3}$$

The values of $\mu_1'(F'')$ and $\mu_2'(F'')$ are undefined for $\nu_2 \leq 2$ and $\nu_2 \leq 4$, respectively. The so-called *confluent hypergeometric function of the first kind* ${}_1F_1$ [39] is defined as

$$
{}_1F_1(a;b;z) = \sum_{k=0}^{+\infty} \left(\frac{z^k}{k!} \cdot \frac{(a)_k}{(b)_k} \right), \tag{A.4}
$$

where $(c)_k = \Gamma(c+k)/\Gamma(c)$. Recall that the *gamma* function is defined as

$$
\Gamma(a) = \int_0^{+\infty} t^{a-1} e^{-t} dt, \tag{A.5}
$$

for any $a \in \mathbb{C}$ except for $a \in \{0, -1, -2, \dots\}$. A deeper insight into the gamma and the hypergeometric functions can be obtained from Lebedev [40], for instance.

Using some properties of ${}_1F_1$, we can express Equation (A.3) as a function of ${}_1F_1(\nu_2/2 - 1\,;\, \nu_2/2\,;\, \delta_2/2)$, like Equation (A.2). Specifically, the following equivalence holds (as seen in [41] Equation 13.3.1):

$$
(b-a){}_1F_1(a-1;b;z) + (2a-b+z){}_1F_1(a;b;z) - a{}_1F_1(a+1;b;z) = 0, \tag{A.6}
$$

for any values a, b and z. Using Equation (A.6) with $b = a + 1$ and taking into account that ${}_1F_1(b;b;z) = \sum_{k=0}^{+\infty}(z^k/k!) = e^z$ for any b (see also [41] Equation 13.6.1), it can be deduced that

$$
{}_1F_1(a-1;a+1;z) = ae^z - (a-1+z){}_1F_1(a;a+1;z). \tag{A.7}
$$

Applying Equation (A.7) to Equation (A.3) with $a = \nu_2/2 - 1$ and $z = \delta_2/2$, we obtain:

$$
\begin{aligned}
\mu_2'(F'') &= \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{\delta_1^2 + (2\delta_1 + \nu_1)(\nu_1 + 2)}{(\nu_2 - 2)(\nu_2 - 4)} e^{-\frac{\delta_2}{2}} \left(\frac{\nu_2 - 2}{2} e^{\frac{\delta_2}{2}} - \left(\frac{\nu_2 - 4}{2} + \frac{\delta_2}{2}\right) {}_1F_1\left(\frac{\nu_2}{2} - 1\,;\, \frac{\nu_2}{2}\,;\, \frac{\delta_2}{2}\right) \right) \\
&= \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{\delta_1^2 + (2\delta_1 + \nu_1)(\nu_1 + 2)}{2(\nu_2 - 4)} \left(1 - \frac{\nu_2 + \delta_2 - 4}{\nu_2 - 2} e^{-\frac{\delta_2}{2}} {}_1F_1\left(\frac{\nu_2}{2} - 1\,;\, \frac{\nu_2}{2}\,;\, \frac{\delta_2}{2}\right) \right),
\end{aligned} \tag{A.8}
$$

which finally lets us express the second raw moment $\mu_2'(F'')$ as a function of ${}_1F_1(\nu_2/2 - 1\,;\, \nu_2/2\,;\, \delta_2/2)$ instead of ${}_1F_1(\nu_2/2 - 2\,;\, \nu_2/2\,;\, \delta_2/2)$.

Numerically evaluating expressions (A.2) and (A.8) with enough accuracy can be really challenging under some concrete settings. We suggest applying the following approach to numerically calculate ${}_1F_1$ in an effective way:

$$
a^{-1} e^z {}_1F_1(a\,;\, a+1\,;\, -z) = e^z z^{-a} \Gamma_{\mathrm{L}}(a, z) = \mathcal{C}(a, z), \tag{A.9}
$$

where Γ_{L} stands for the lower incomplete gamma function, defined as $\Gamma_{\mathrm{L}}(a, z) = \int_0^z t^{a-1} e^{-t} dt$, and $\mathcal{C}(a, z)$ can be computed as the following continued fraction:

$$
\mathcal{C}(a, z) = \cfrac{f_0(a, z)}{g_0(a, z) + \cfrac{f_1(a,z)}{g_1(a,z) + \cfrac{f_2(a,z)}{g_2(a,z) + \cdots}}}, \tag{A.10}
$$

with

$$f_k(a,z) = \begin{cases} 1, & \text{if } k = 0 \\ \dfrac{k}{2}z, & \text{if } k \text{ is even, } k \neq 0 \\ -\left(a + \dfrac{k-1}{2}\right)z, & \text{if } k \text{ is odd} \end{cases} \tag{A.11}$$

and

$$g_k(a,z) = a + k, \quad \text{for } k \geq 0 \tag{A.12}$$

(see also (15)). The leftmost equality in Equation (A.9) comes from Olver et al. [41], Equation 8.5.1, while the expression of Γ_{L} as a continued fraction can be found in Olver et al. [41], Equation 8.9.1, and is valid except for $a \in \{-1, -2, \dots\}$. By applying Equation (A.9) in Equations (A.2) and (A.8), we reach the following workable expressions for $\mu'_1(F'')$ and $\mu'_2(F'')$:

$$\mu'_1(F'') = \frac{\nu_2(\nu_1 + \delta_1)}{2\nu_1} \mathcal{C}\left(\frac{\nu_2}{2} - 1, -\frac{\delta_2}{2}\right), \tag{A.13}$$

$$\mu'_2(F'') = \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{\delta_1^2 + (2\delta_1 + \nu_1)(\nu_1 + 2)}{4(\nu_2 - 4)}\left(2 - (\nu_2 + \delta_2 - 4)\mathcal{C}\left(\frac{\nu_2}{2} - 1, -\frac{\delta_2}{2}\right)\right), \tag{A.14}$$

with $\mathcal{C}(a,z)$ calculated as in Equation (A.10). The particular case in which $\nu_1 = n - p$, $\nu_2 = p$, $\delta_1 = 0$ and $\delta_2 = n/\gamma^2$ leads to Equations (13) and (14), which were to be proven.

Appendix B. On the suitability of using trimmed means for cases $p = 2$ and $p = 4$

When working either with sampled data or with a theoretical probability distribution, the effect of too extreme or abnormal values on computed statistics can be minimised by directly truncating or trimming them off, among other possibilities.

In the case of $F'' \sim F(n - p, p, 0, n/\gamma^2)$ with $p = 2$, the values of the first and second raw moments $\mu'_1(F'') = E[F'']$ and $\mu'_2(F'') = E[(F'')^2]$ compute to infinity, mainly due to the highly positive skewness of the distribution; and the same happens to $\mu'_2(F'')$ when $p = 4$. Therefore, calculating an upperly trimmed version of $\mu'_1(F'')$ and $\mu'_2(F'')$ —that will be denoted by $\tilde{\mu}'_1(F'')$ and $\tilde{\mu}'_2(F'')$, respectively— when these quantities cannot be computed to a finite number seems a reasonable strategy. How this can be done has been explained in Subsection 4.2.2.

This way to replace the nonexistent values of $\mu'_1(F'')$ and $\mu'_2(F'')$ can be assumed to be valid for our purposes if the resulting values of $\tilde{\mu}'_1(F'')$ and

$\tilde{\mu}'_2(F'')$ properly fit the gap left by them (see an example of this in Figures B.1(a)–(b)).

In order to graphically validate this, the trimmed versions of the raw moments, $\tilde{\mu}'_1(F'')$ and $\tilde{\mu}'_2(F'')$, were plotted against their respective original versions, $\mu'_1(F'')$ and $\mu'_2(F'')$, as functions of p, the rest of parameters being fixed (recall that the degrees of freedom of a doubly noncentral F distribution are not meant to take only integer values). Figures B.1(c)–(d) depict a concrete example. As shown in the figures, the behaviour of $\tilde{\mu}'_1(F'')$ and $\tilde{\mu}'_2(F'')$ only significantly differs from that of $\mu'_1(F'')$ and $\mu'_2(F'')$, respectively, in those points where discontinuities in the nontrimmed moments occur. This allows us to accept the trimmed versions of the raw moments as valid substitutes in cases $p = 2$ and $p = 4$.

As it should be noted from Figures B.1(c)–(d), $\mu'_1(F'')$ in Equation (13) can be effectively calculated for values $p < 2$, and $\mu'_2(F'')$ in Equation (14) can be computed for values of p lower than 4 other than 2. This is despite the fact that, from a mathematical point of view, the first and second raw moments of F'' do not formally exist when the second degrees of freedom —p, in our case— are less than 2 and 4, respectively. This enables the strategy we have proposed to be followed for $p = 1$ and $p = 3$ in Subsection 4.2.2.
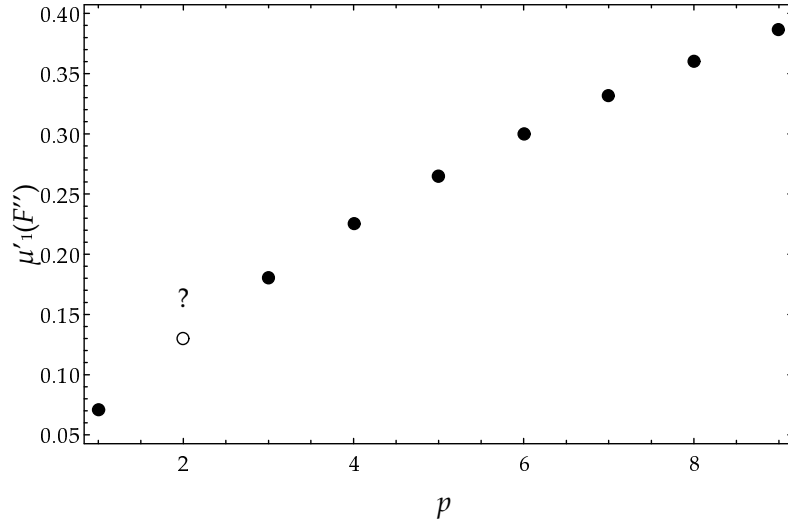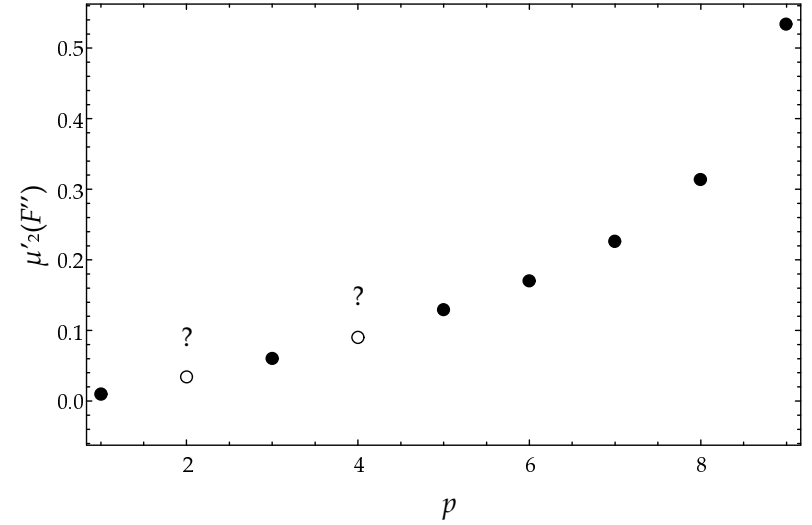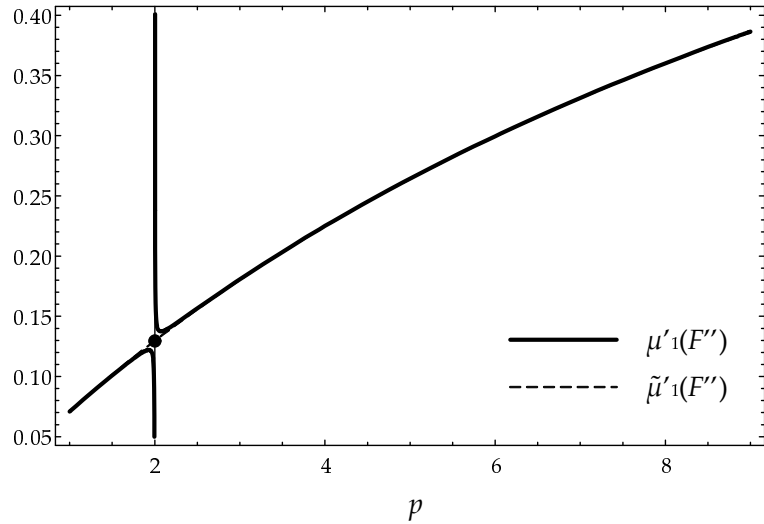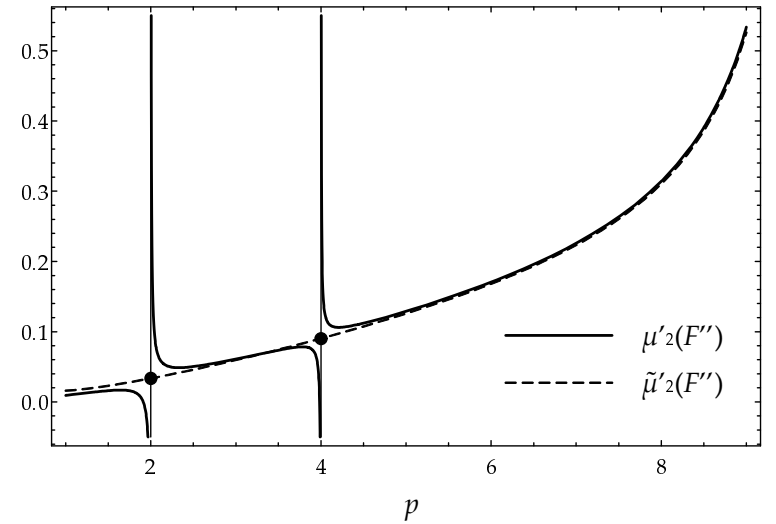
Figure B.1: Example of the use of trimmed raw moments of $F'' \sim F(n - p, p, 0, n/\gamma^2)$ to replace the nonexistent nontrimmed counterparts, with $n = 10$ and $\gamma = 0.75$. Figures (a) and (b) show a discrete plot of the first and second raw moments of F'', respectively. White-colour dots with a question mark represent missing points that evaluate to infinity (in $p = 2$ for the fist raw moment, and in $p = 2$ and $p = 4$ for the second raw moment). Figures (c) and (d) represent the nontrimmed and trimmed first and second raw moments of F'' as a continuous function of p; $\varepsilon = 3 \cdot 10^{-5}$ was used for trimming. Black dots are used to represent the values that $\tilde{\mu}'_1(F'')$ and $\tilde{\mu}'_2(F'')$ will take to respectively substitute the nonexistent values of $\mu'_1(F'')$ and $\mu'_2(F'')$.