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Equicontinuous local dendrite maps

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Abstract

Let X be a local dendrite, and $f : X \to X$ be a map. Denote by $E(X)$ the set of endpoints of X. We show that if $E(X)$ is countable, then the following are equivalent:

(1) *f is equicontinuous;*
(2) $\bigcap_{n=1}^{\infty} f^n(X) = R(f)$;
(3) *$f|\bigcap_{n=1}^{\infty} f^n(X)$ is equicontinuous;*
(4) *$f|\bigcap_{n=1}^{\infty} f^n(X)$ is a pointwise periodic homeomorphism or is topologically conjugate to an irrational rotation of $\mathbb{S}^1$;*
(5) *$\omega(x, f) = \Omega(x, f)$ for all $x \in X$.*

This result generalizes [17, Theorem 5.2], [24, Theorem 2] *and* [11, Theorem 2.8].

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1. Introduction

A *map* is a continuous function between two topological spaces. A *topological dynamical system* is a pair (X, f), where X is a compact metric space (d is a metric on X) and f is a map from X to itself. Let $\mathbb{N}$ be the set of positive integers. Let f^0 be the identity map of X. Define, inductively, $f^n = f \circ f^{n-1}$ for any non-zero positive integer n. For $x \in X$, $\{f^n(x) : n \in \mathbb{N}\}$ is called *the orbit* of x and is denoted by $O(x, f)$. For any $x \in X$, write

$$\omega(x, f) = \{y : \exists\, n_k \in \mathbb{N}, n_k \to \infty, \lim_{k\to\infty} f^{n_k}(x) = y\}$$

called the ω*-limit set* of x under f, and write

$$\Omega(x, f) = \{y : \exists\, x_k \in X \texttt{ and } n_k \in \mathbb{N}, n_k \to \infty, \lim_{k\to\infty} x_k = x \texttt{ and } \lim_{k\to\infty} f^{n_k}(x_k) = y\}.$$

x is a *periodic point* if $f^n(x) = x$ for some non-zero positive integer n. Note that, if $n = 1$, then x is a *fixed pint*. Also, x is called a *recurrent point* of f if for any neighborhood U of x and any $m \in \mathbb{N}$ there exists $n > m$ such that $f^n(x) \in U$. Note that, x is a recurrent point of f if and only if $x \in \omega(x, f)$. Let $Fix(f)$, $P(f)$ and $R(f)$ denote the set of fixed points, periodic points and recurrent points, respectively. We say that the map f is *pointwise periodic* if $P(f) = X$. Also, f is said to be *pointwise recurrent* if $R(f) = X$. A subset A of X is called f*-invariant* if $f(A)$ is a subset of A. It is called a *minimal set* of f if it is non-empty, closed, f-invariant and minimal (in the sense of inclusion) for these properties. If X is a minimal set, then f is called a *minimal map*. f is said to be *equicontinuous* (with respect to d) if for each $\varepsilon > 0$, there exists $0 < \alpha < \varepsilon$ such that for any non zero-integer n and any $x, y \in X$ with $d(x, y) < \alpha$, one has $d(f^n(x), f^n(y)) < \varepsilon$.

It is interesting to give some characterizations of equicontinuous maps [12, 13, 16, 17, 24]. In [3, Theorem 2.1], by means of the orbit map $O_f : X \to C(\mathbb{N}, X)$ and the metric d_f, Akin, Auslander and Berg gave some necessary and sufficient conditions for a map f of a compact metric space (X, d) to be equicontinuous. In [6, Proposition 2.2], Blanchard, Host and Maass discussed the topological complexity, and showed that a surjective map f of a compact metric space X is equicontinuous if and only if any finite open cover of X under f has bounded complexity. On one-dimensional spaces, one has some still finer results. In [12], Cano proved that if f is an equicontinuous map from the interval $I = [0, 1]$ to itself, then $Fix(f)$ is connected, and furthermore, if $Fix(f)$ is non-degenerate, then f has no periodic points except fixed points. Bruckner and Hu (only if) and Boyce (if) proved that a map $f : I \to I$ is equicontinuous if and only if $\bigcap_{n=1}^{\infty} f^n(I) = Fix(f^2)$; see [9, 10]. This result was also proved by Blokh in [8]. Valaristos [25] described the characters of equicontinuous circle maps: a map f of the unit circle $\mathbb{S}^1$ to itself is equicontinuous if and only if one of the following four statements holds:

(1) f is topologically conjugate to a rotation;
(2) $Fix(f)$ contains exactly two points and $Fix(f^2) = \mathbb{S}^1$;

(3) $Fix(f)$ contains exactly one point and $Fix(f^2) = \bigcap_{n=1}^{\infty} f^n(\mathbb{S}^1)$;

(4) $Fix(f) = \bigcap_{n=1}^{\infty} f^n(\mathbb{S}^1)$.

In [21], Sun obtained some necessary and sufficient conditions of equicontinuous σ-maps. When X is a graph, Mai proved in [17, Theorem 5.2] that the following properties are equivalent:

(1) f is equicontinuous;

(2) $R(f) = \bigcap_{n=1}^{\infty} f^n(X)$;

(3) the restriction of f on $\bigcap_{n=1}^{\infty} f^n(X)$ is equicontinuous;

(4) the restriction of f on $\bigcap_{n=1}^{\infty} f^n(X)$ is a periodic homeomorphism, or is topologically conjugate to the irrational rotation of the unit circle $\mathbb{S}^1$.

In [11, 22, 23], the authors studied equicontinuous dendrite maps. For a dendrite X with countable set of endpoints, in [22, Theorem 2.8] it is shown that f is equicontinuous if and only if $\Omega(x, f^n) = \omega(x, f^n)$ for any $x \in X$ and each $n \in \mathbb{N}$. For a dendrite X with finite branch points, in [23, Theorem 28] it is proved that the following statements are equivalent:

(1) f is equicontinuous;

(2) $\Omega(x, f) = \omega(x, f)$ for any $x \in X$;

(3) $P(f) = \bigcap_{n=1}^{\infty} f^n(X)$, and $\omega(x, f)$ is a periodic orbit for every $x \in X$ and the function $\omega_f : x \to \omega(x, f)$ $(x \in X)$ is continuous;

(4) $\Omega(x, f)$ is a periodic orbit for any $x \in T$.

Recently, for general dendrites, in [11, Theorem 4.12] it is shown that the following statements are equivalent:

(1) f is equicontinuous;

(2) $\Omega(x, f) = \omega(x, f)$ for any $x \in X$;

(3) ω_f is continuous and $\overline{P(f)} = \bigcap_{n=1}^{\infty} f^n(X)$.

In this paper we will give some equivalent conditions of equicontinuity for local dendrite maps, whose dynamical behavior is both important and interesting in the study of Discrete Dynamical Systems and Continuum Theory. Our main results are the following:

Theorem 1.1. *Let X be a compact metric space and $f : X \to X$ be a map. Consider the following statement:*

(1) *f is equicontinuous;*

(2) *$\Omega(x, f^n) = \omega(x, f^n)$ for all $x \in X$ and $n \in \mathbb{N}$.*

Then (1) *implies* (2) *and, if* $\bigcap_{n=1}^{\infty} f^n(X) = P(f)$, *then* (2) *implies* (1).

Theorem 1.1 generalizes [22, Theorem 2.8].

Proposition 1.2. *Every pointwise recurrent local dendrite map is equicontinuous.*

The following result generalizes [17, Theorem 5.2], [24, Theorem 2] and [11, Theorem 2.8].

Theorem 1.3. *Let* $f : X \to X$ *be a local dendrite map. If* $E(X)$ *is countable, then the following are equivalent:*

(1) f *is equicontinuous;*
(2) $\bigcap_{n=1}^{\infty} f^n(X) = R(f)$*;*
(3) $f|\bigcap_{n=1}^{\infty} f^n(X)$ *is equicontinuous and surjective;*
(4) $f|\bigcap_{n=1}^{\infty} f^n(X)$ *is either a pointwise periodic homeomorphism or is topologically conjugate to an irrational rotation of* $\mathbb{S}^1$*;*
(5) $\omega(x, f) = \Omega(x, f)$ *for all* $x \in X$.

Corollary 1.4. *Under the assumptions of Theorem 1.3, if* $\omega(x, f) = \Omega(x, f)$ *for all* $x \in X$*, then* $\omega(x, f^n) = \Omega(x, f^n)$ *for all* $x \in X$ *and every* $n \in \mathbb{N}$.

Recently, several authors have been interested in studying local dendrite maps (for example one can see [1, 2, 4]).

2. Preliminaries

A compact connected metric space is called *a continuum*. A *Peano continuum* is a locally connected continuum. An *arc* is any space homeomorphic to the interval $I = [0, 1]$. We mean by a *simple closed cure* every continuum homeomorphic to the circle $\mathbb{S}^1$.

Recall that a *graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their endpoints (i.e., it is a one-dimensional compact connected polyhedron). A *tree* is a graph which contains no simple closed curves.

A *dendrite* is a Peano continuum which contains no simple closed curves. Not that every dendrite is uniquely arcwise connected continuum, that is any two distinct points in a dendrite D can be joined by a unique arc $[x, y]$. We define by $(x, y) = [x, y] \setminus \{x, y\}$. For more properties of dendrites (see [19, Chapter X]). Let D be a dendrite and $x \in D$, x is called an*endpoint* of D if $D \setminus \{x\}$ is connected. The set of all endpoints of D is denoted by $E(D)$. We say that x is a *cut point* of D if $x \in D \setminus E(D)$ [19, Theorem 10.7, p. 168].

By a *local dendrite*, we mean a continuum every point of which has a dendrite neighborhood. Each local dendrite is a Peano continuum which has a finite number of circles [15]. Since every subcontinuum of a dendrite is a dendrite [15, º51, VI, Theorem 4], a subcontinuum of a local dendrite is a local dendrite. The simple examples of local dendrites are graphs and dendrites. Let X be a local dendrite. A point $e \in X$ is called an endpoint of X if it admits a neighborhood U in X such that U is an arc and $U \setminus \{e\}$ is connected. The set of endpoints of X is denoted by $E(X)$. A point $x \in X$ is called a *branch point* of X if there exists a closed neighborhood D of x which is a dendrite such that x is a branch point of D (i.e. $D \setminus \{x\}$ has more than two connected components). We denoted by $B(X)$ the set of branch points of X. By [15, Theorem 6,304 and Theorem 7, 302], $B(X)$ is at most countable. A *local dendrite map* is a map from a local dendrite into itself.

Lemma 2.1 ([1])**.** *Let X be a local dendrite and Y be a sub-local dendrite of X distinct of X such that any arc J of X joining two distinct points of Y is included in Y. Then for any connected component C of $X \setminus Y$, $\overline{C} \cap Y$ is degenerate (i.e. reduced to a point).*

If S_1, S_2, ..., S_r are the circles in a local dendrite X, then $\Gamma(X)$ is the intersection of all subgraphs in X containing the union of S_i's. Therefore, $\Gamma(X)$ is the smallest graph containing all circles of the local dendrite X. Define $X \setminus \Gamma(X) = \bigcup_{i \in \mathcal{A}} C_i$ where C_i are the connected components of $X \setminus \Gamma(X)$. Since $B(X)$ is at most countable, the set $\mathcal{A}$ is at most countable. By Lemma 2.1, for any $i \in \mathcal{A}$, $\overline{C_i} \cap \Gamma(X)$ is reduced to a point z_i. Let $\mathcal{A}_k$ be a subset of $\mathcal{A}$ such that, for each $i \in \mathcal{A}_k$, $\overline{C_i} \cap \Gamma(X) = \{z_k\}$. Put $C^k = \bigsqcup_{i \in \mathcal{A}_k} \overline{C_i}$.

Since $\Gamma(X)$ contains all circles of X, by [1] and [4, Lemma 2.4], we obtain the following lemma.

Lemma 2.2. *Under the notation above, the $(C^k)_k$ are pairwise disjoint sub-dendrites of X.*

Lemma 2.3. *If X is a local dendrite with $E(X)$ is countable then every sub-local dendrite of X has a countable set of endpoints.*

Proof. Let Y be a sub-local dendrite of X. If $E(Y)$ is uncountable, then there exists a connected component C of $Y \setminus \Gamma(Y)$ such that $\overline{C}$ is a dendrite with uncountable set of endpoints. Since $Y \cap \Gamma(X) = \Gamma(Y)$, $Y \setminus \Gamma(Y) = Y \setminus \Gamma(X) \subset X \setminus \Gamma(X)$. Therefore, there exists a connected component L of $X \setminus \Gamma(X)$ such that $C \subset L$. Now we define a one-to-one function $f : E(\overline{C}) \to E(\overline{L})$ by: let $e \in E(\overline{C})$. If $e \in E(\overline{L})$, then $f(e) = e$. If $e \notin E(\overline{L})$, let C_e be a connected component of $\overline{L} \setminus \overline{C}$ such that $\overline{C_e} \cap \overline{C} = \{e\}$. We take $f(e) \in E(\overline{C_e}) \setminus \{e\}$. It is easy to see that f is a one-to-one function. Thus the cardinality of $E(\overline{C})$ is less than or equal to the cardinality of $E(\overline{L})$. Consequently, $E(\overline{L})$ is uncountable which implies that $E(X)$ is uncountable, a contradiction. □

Lemma 2.4 ([2, Lemma 4.3])**.** *Let (X,d) be a local dendrite. Thus, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for any x and y in X with $d(x, y) < \delta$, the diameter $diam([x, y]) < \varepsilon$.*

3. Proof of Theorem 1.1

To prove Theorem 1.1 we will use the following two results. The first one is proved in [11, Lemma 3.5].

Lemma 3.1. *Let X be a compact metric space and $f : X \to X$ be an equicontinuous map. Then $\omega(x, f^n) = \Omega(x, f^n)$ for all $x \in X$ and for all $n \in \mathbb{N}$.*

Part (2) of the following result is shown in [17, Proposition 2.4], so we prove only part (1).

Lemma 3.2. *Let X be a compact metric space and $f : X \to X$ be a map. Then the following assertions hold:*

(1) *If $\omega(x, f) = \Omega(x, f)$ for all $x \in X$, then $\bigcap_{n=1}^{\infty} f^n(X) = R(f)$;*

(2) *If f is equicontinuous, then $\bigcap_{n=1}^{\infty} f^n(X) = R(f)$.*

Proof. Since X is compact and $f(R(f)) = R(f)$, $\bigcap_{n\in\mathbb{N}} f^n(X) \supseteq R(f)$. Conversely, given $x \in \bigcap_{n\in\mathbb{N}} f^n(X)$, there is a sequence $x_k \in X$ and $x_0 \in X$ with $x_k \to x_0$ and $n_k \to +\infty$ such that $f^{n_k}(x_k) = x$ for all $k \in \mathbb{N}$. Hence $x \in \Omega(x_0, f) = \omega(x_0, f)$. Since $x \in \omega(x_0, f)$, there exists a sequence $(p_k)_k \in \mathbb{N}$, $p_k \to \infty$ such that $f^{p_k}(x_0) \to x$. By choosing a subsequence, we can suppose that $p_{k+1} - p_k > k$ for all $k \in \mathbb{N}$. Then $f^{p_{k+1}-p_k}(f^{p_k}(x_0)) = f^{p_{k+1}}(x_0) \to x$. Therefore, $x \in \Omega(x, f) = \omega(x, f)$ which implies that $x \in R(f)$. Consequently, $\bigcap_{n=0}^{\infty} f^n(X) = R(f)$. □

Remark that, in the above Lemma, if we suppose that f is an onto map, we infer that f is pointwise recurrent that is, $R(f) = X$.

Proof of Theorem 1.1. By Lemma 3.1 we have (1) ⇒ (2).

To show that (2) ⇒ (1), assume that $\bigcap_{n=1}^{\infty} f^n(X) = P(f)$ and that (2) holds. Note that, in [22, Lemma 2.4] it is proved that (2) ⇒ (1) if X is a dendrite. We will extend this result for every compact metric space X. Since X is compact then

$$\Omega(x, f^n) = \omega(x, f^n) \subset \bigcap_{n=1}^{\infty} f^n(X) = P(f)$$

for all $x \in X$ and every $n \in \mathbb{N}$. If f is not equicontinuous, then there exist $x \in X$, a sequence $(x_n)_n$ in X with $x_n \to x$ and a sequence $(p_n)_n$ in $\mathbb{N}$,

$p_n \to \infty$ such that $f^{p_n}(x_n) \to a \in X$ and $f^{p_n}(x) \to b \in X$ with $a \neq b$. Then $a \in \Omega(x, f) \subset P(f)$ and $b \in \omega(x, f) \subset P(f)$. Since a and b are in $P(f)$, there exists $k \in \mathbb{N}$ such that $a, b \in Fix(f^k)$. We can assume, without loss of generality, that modulus k, p_n is congruent with r ($p_n = kq_n + r$), for every $n \in \mathbb{N}$. Note that $f^r(x_n) \to f^r(x)$, $kq_n \to \infty$ and $f^{kq_n}(f^r(x_n)) \to a$. Thus $a \in \Omega(f^r(x), f^k) = \omega(f^r(x), f^k)$. Consequently, there is a sequence $s_n \to \infty$ (by choosing a subsequence, we can suppose $s_n - q_n > n$) such that $(f^k)^{s_n}(f^r(x)) \to a$. Since $r = p_n - kq_n$, $f^{ks_n}(f^{p_n - kq_n}(x)) = f^{ks_n - kq_n}(f^{p_n}(x)) \to a$. Since $f^{p_n}(x) \to b$, $a \in \Omega(b, f^k) = \omega(b, f^k) = \{b\}$. Therefore, $a = b$ which leads to a contradiction. Then f is equicontinuous. □

The following examples show that the condition $\bigcap_{n=1}^{\infty} f^n(X) = P(f)$ of Theorem 1.1 is essential. Example 3.3 is inspired from [14, page 92].

Example 3.3. There exist a compact metric space X and a homeomorphism $fX \to X$ such that $\bigcap_{n=1}^{\infty} f^n(X) \neq P(f)$, $\Omega((x,y), f^n) = \omega(x, f^n)$ for all $x \in X$ and $n \in \mathbb{N}$, and f is not equicontinuous.

Let $y_n \in \mathbb{R} \setminus \mathbb{Q}$ be a sequence which converges to $y_\infty \in \mathbb{R} \setminus \mathbb{Q}$, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be a unit circle, and $X = \mathbb{S}^1 \times \{y_n : n \in \mathbb{N} \cup \{\infty\}\}$. Then X is a compact metric space with the metric d induced from one of the Euclidean space $\mathbb{R}^2$. Let $f : X \to X$ by $f(x, y) = (x + y, y)$. Note that, f is a homeomorphism. Thus $\bigcap_{n=1}^{\infty} f^n(X) = X$. Since $\{y_n : n \in \mathbb{N} \cup \{\infty\}\} \subset \mathbb{R} \setminus \mathbb{Q}$, $P(f) = \varnothing$. Therefore, $\bigcap_{n=1}^{\infty} f^n(X) \neq P(f)$.

Let $(x, y) \in X$ and $n \in \mathbb{N}$. Since $f^n(x, y) = (x + ny, y)$, on $\mathbb{S}^1$, $x \mapsto x + ny$ is an irrational rotation. Then $\omega((x, y), f^n) = \mathbb{S}^1 \times \{y\}$ and $\Omega((x, y), f^n) = \mathbb{S}^1 \times \{y\}$. Consequently, $\Omega((x, y), f^n) = \omega(x, f^n)$ for all $x \in X$ and $n \in \mathbb{N}$.

f is not equicontinuous; indeed, fix a small $\varepsilon > 0$ (e.g. $\varepsilon = \frac{1}{4}$) and arbitrary $\delta > 0$. Choose a pair $p = (x, y)$; $q = (x, y_\infty)$ with $0 < |y - y_\infty| < \varepsilon$ and $d(q, p) < \delta$. Fix an integer $N > 1$ such that $k|y - y_\infty| < \varepsilon < N|y - y_\infty| < 1$ for any integer $k < N$. Then $d(f^N(p), f^N(q)) > N|y - y_\infty| > \varepsilon$. Consequently, f is not equicontinuous.

Remark that the action of f is distal; indeed, let $(x, y) \neq (x', y')$ be two points of X. If $\lim f^{k_i}(x, y) = \lim f^{k_i}(x', y')$, then $\lim(x + k_i y, y) = \lim(x' + k_i y', y')$ which implies that $y = y'$ and $x = x'$ which is impossible.

Example 3.4. There exist a compact metric space X and a non surjective map $f : X \to X$ such that $\bigcap_{n=1}^{\infty} f^n(X) \neq P(f)$, $\Omega((x,y), f^n) = \omega(x, f^n)$ for all $x \in X$ and $n \in \mathbb{N}$, and f is not equicontinuous.

As in the Example 3.3, we put $X = \mathbb{S}^1 \times \{y_n : n \in \mathbb{N} \cup \{\infty\}\}$. Let $f : X \to X$ by $f(x, y) = (x + y, y_\infty)$. Note that, f is a non surjective map and $\bigcap_{n=1}^{\infty} f^n(X) = \mathbb{S}^1 \times \{y_\infty\}$. Since $\{y_n : n \in \mathbb{N} \cup \{\infty\}\} \subset \mathbb{R} \setminus \mathbb{Q}$, $P(f) = \varnothing$. Therefore, $\bigcap_{n=1}^{\infty} f^n(X) \neq P(f)$.

Let $(x, y) \in X$ and $n \in \mathbb{N}$. Since $f^n(x, y) = (x + ny, y_\infty)$, on $\mathbb{S}^1$, $x \mapsto x + ny$ is an irrational rotation. Then $\omega((x, y), f^n) = \mathbb{S}^1 \times \{y_\infty\}$ and $\Omega((x, y), f^n) = \mathbb{S}^1 \times \{y_\infty\}$. Consequently, $\Omega((x, y), f^n) = \omega(x, f^n)$ for all $x \in X$ and $n \in \mathbb{N}$.

As in the Example 3.3, f is not equicontinuous.

4. Proofs of Proposition 1.2 and Theorem 1.3

Lemma 4.1. *Let X be a compact metric space and $f : X \to X$ be a map. Consider the following statements:*

(1) *$R(f) = X$;*
(2) *f is an equicontinuous homeomorphism.*

Then (2) implies (1) and if X is a dendrite, then (1) implies (2).

Proof. By [18, Lemma 2.4], a homeomorphism of a compact metric space is equicontinuous if and only if it is distal and locally almost periodic. The last condition implies that $R(f) = X$. Hence (2) implies (1).

(1) $\Rightarrow$ (2) From [7, Theorem 1.18] we have f is a homeomorphism and $X \setminus E(X) \subset P(f)$ hence, by the equivalence between clauses (1) and (8) of [20, Theorem 3.8], f is equicontinuous. $\square$

According to [4, Lemma 2.8], we get the following lemma.

Lemma 4.2. *Let $f : X \to X$ be a local dendrite map. If $R(f) = X$, then $f(\Gamma(X)) = \Gamma(X)$.*

Note that, by [1, Proposition 3.6], if f is a monotone onto local dendrite map, then $f(\Gamma(X)) = \Gamma(X)$.

Now we introduce some notations used in the proof of Lemma 4.3. A space X is said to be *almost totally disconnected* if the set of its degenerate components, considered as a subset of X, is dense in X. A compact metric space X is called a *cantoroid* if it is almost totally disconnected and has no isolated point. A *generalized brain* is a cantoroid whose nondegenerate components form a null family, they are local dendrites and only finitely many of them contain circles.

Lemma 4.3. *Let $f : X \to X$ be a local dendrite map. If $X \neq \mathbb{S}^1$ then f is not minimal.*

Proof. If $X \neq \mathbb{S}^1$, by [16, Theorem 3.2] and [19, Theorems 10.31], the result holds whenever X is either a graph or a dendrite. If X is neither a dendrite nor a graph, then X contains at least an attached dendrite. According to [5, Theorem C], a minimal set on local dendrites is either a finite set or a finite union of disjoint circles or a generalized brain. Since a generalized brain is not connected, a local dendrite can not be a minimal set. This ends the proof of Lemma 4.3. □

According to [17, Proposition 2.5], we get the following lemma.

Lemma 4.4. *Let $f : X \to X$ be a local dendrite map and let $p \in \mathbb{N}$. Then f is equicontinuous if and only if f^p is equicontinuous.*

According to [4, Theorem 2.1], we have the following result.

Theorem 4.5. *Let $f : X \to X$ be a local dendrite map. Then f is pointwise recurrent if and only if f is a homeomorphism such that one of the following statements holds:*

1. *If f is not minimal, then every non endpoint has a finite orbit;*
2. *If f is minimal, then $X = \mathbb{S}^1$ and f is topologically conjugate to an irrational rotation.*

Proof of Proposition 1.2. Assume that $R(f) = X$. By [17, Corollary 5.1] and Lemma 4.1, the result holds whenever X is either a graph or a dendrite. Assume that X is neither a dendrite nor a graph. Assume that X is neither a dendrite nor a graph.

If f is minimal, then, by Theorem 4.5 f is topologically conjugate to an irrational rotation. It is well known that every rotation is an isometric, consequently, it is equicontinuous. Thus, by [17, Lemma 3.1] f is equicontinuous.

If f is not minimal, then, by Theorem 4.5, f is a homeomorphism and every non endpoint has finite orbit. By Lemma 4.2, $\Gamma(X)$ is invariant. Then, by [17, Corollary 5.1], $f|_{\Gamma(X)}$ is equicontinuous. We further have $X \setminus \Gamma(X)$ is the union of pairwise disjoint dendrites (C^k) such that $C^k \cap \Gamma(X) = \{z_k\}$ and as $f^{n_0}(z_k) = z_k$ hence by [1, Lemma 3.8] one has $f^{n_0}(C^k) = C^k$ and since $R(f) = R(f^{n_0}) = X$, $f^{n_0} : C^k \to C^k$ is pointwise recurrent. Thus, by Lemma 4.1 and Lemma 4.4, $f|_{C^k}$ is equicontinuous for each k which implies $f|_{X\setminus\Gamma(X)}$ is equicontinuous ((C^k) are pairwise disjoint dendrites). Consequently, f is equicontinuous on X. □

According to [4, Corollary 2.2] we obtain the following result.

Proposition 4.6. *Let X be a local dendrite and $f : X \to X$ be a local dendrite map. Assume that $E(X)$ is countable. Then f is pointwise recurrent if and only if one the following statements holds:*

1. *f is a pointwise periodic homeomorphism;*
2. *$X = \mathbb{S}^1$ and f is topologically conjugate to an irrational rotation.*

Proof of Theorem 1.3. Since X is compact and connected, for every $n \in \mathbb{N}$, $f^n(X)$ is also compact and connected. Let $M(f) = \bigcap_{n=1}^{\infty} f^n(X)$. It follows from $X \supset f(X) \supset f^2(X) \supset \cdots$ that $M(f)$ is a nonempty sub-local dendrite of X. By [11, Lemma 3.1], $f(M(f)) = M(f)$.

Since $M(f)$ is an f-invariant sub-local dendrite, by Proposition 4.6, (2) is equivalent to (4).

If $R(f) = M(f)$, then $f|_{M(f)} : M(f) \to M(f)$ is pointwise recurrent. Consequently, by Proposition 1.2, $f|_{M(f)}$ is equicontinuous. Therefore, (2) $\Rightarrow$ (3).

By Lemma 3.2, (1) implies (2).

(3) $\Rightarrow$ (1). If $U = M(f) \cap \overline{X \setminus M(f)}$, then $U \subset E(M(f))$. By Lemma 2.3, $E(M(f))$ is countable. Since U is compact, it is a finite set. Obviously, for every $k \in \mathbb{N}$, there exists $p_k \in \mathbb{N}$ such that $f^{p_k}(X) \subset M(f) \cup B(U, 2^{-k})$. Since $f|_{M(f)}$ is equicontinuous, by [17, Theorem 5.1], f is equicontinuous.

By Lemma 3.1, (1) $\Rightarrow$ (5).

By Lemma 3.2, (5) $\Rightarrow$ (2). □

The following example shows that $E(X)$ is countable cannot be removed from the hypothesis of Theorem 1.3.

Example 4.7. By [11, Example 5.4], there exist a dendrite X with uncountable set of endpoints and a homeomorphism $f : X \to X$ such that f satisfies (1) and does not satisfies (4).

By applying Theorem 1.3 and Lemma 3.1, we obtain Corollary 1.4.

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