LIMIT ORDERS AND MULTILINEAR FORMS ON ℓ_p SPACES

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Abstract. Since the concept of limit order is a useful tool to study operator ideals, we propose an analogous definition for ideals of multilinear forms. From the limit orders of some special ideals (of nuclear, integral, r-dominated and extendible multilinear forms) we derive some properties of them and show differences between the bilinear and n-linear cases ($n \geq 3$).

Introduction

The theory of operator ideals between Banach spaces has had a remarkable impact in functional analysis since its development, in 1968, by Pietsch and his school. The concept of ideal of multilinear functionals was also introduced by Pietsch [16] in 1983 and has been developed by several authors. The ideals of nuclear, integral or r-summing operators, for example, have found their analogues in the multilinear setting. However, it is important to note that the multilinear theory is far from being a translation of the linear one: it presents very different situations and involves new techniques. In [10, 11], general results about ideals of multilinear mappings are presented.

In the linear theory, a tool that proved useful to study different properties of particular ideals is the concept of limit order (see [15]). Motivated by this, we propose an analogous definition for ideals of multilinear forms. As an application of this new concept, we present some properties of the ideals of nuclear, integral, r-dominated and extendible multilinear forms. We show that there are important differences between bilinear and n-linear situations for $n \geq 3$.

In the first section, we give the definitions of limit orders and show their values for the ideals of continuous, nuclear and integral multilinear forms. The second section deals with r-dominated multilinear forms. We compute their limit orders and study their attainment. We show a structural difference between bilinear and n-linear mappings with $n \geq 3$: on the one hand, every r-dominated bilinear form is 2-dominated for $r > 2$; on the other, if $n \geq 3$ there is no r_0 such that for $r \geq r_0$, every r-dominated n-linear form is r_0-dominated. In the third section we focus on the ideal of extendible multilinear forms. We study the existence of extendible multilinear forms which are not nuclear (these last being trivially extendible). While every extendible bilinear form on a space with cotype 2 is integral [4, 6], we show that this is not the case for n-linear forms with $n \geq 3$. We also improve some results in [4] for homogeneous polynomials.

2000 *Mathematics Subject Classification.* 46G25, 46A45.

Key words and phrases. Ideals of multilinear mappings, limit orders.

The third author was supported by the MCYT and FEDER Project BFM2002-01423.

Given X, Y Banach spaces, we denote by $\mathcal{L}(X,Y)$ the space of continuous linear mappings $T: X \to Y$. If $X_1,\dots,X_n$ and Y are Banach spaces, $\mathcal{L}(X_1,\dots,X_n;Y)$ denotes the space of n-linear mappings $T: X_1\times\cdots\times X_n \to Y$. Whenever $X_1=\cdots=X_n=X$ and $Y=\mathbb{C}$, the space of n-linear mappings is simply denoted by $\mathcal{L}(^nX)$. We are going to deal with mappings $T\in\mathcal{L}(^n\ell_p)$. We denote by $x_1,\dots,x_n$ the elements in ℓ_p. If x is a sequence we write $x=(x(k))_{k=1}^\infty$, with $x(k)\in\mathbb{C}$.

Let us recall that $T\in\mathcal{L}(^nX)$ is **nuclear** if there are sequences $(x^*_{1,k})_k,\dots,(x^*_{n,k})_k$ in X^* with $\|x^*_{i,k}\|\le 1$ for all k and $i=1,\dots,n$ and there is $(\lambda(k))_k\in\ell_1$ so that for every $x_1,\dots,x_n\in X$

$$T(x_1,\dots,x_n)=\sum_k \lambda(k)\cdot x^*_{1,k}(x_1)\cdots x^*_{n,k}(x_n).$$

We denote by $\mathcal{N}(^nX)$ the space of nuclear n-linear forms on X.

A mapping $T\in\mathcal{L}(^nX)$ is called **integral** if there exists a positive Borel-Radon measure μ on $B_{X^*}\times\cdots\times B_{X^*}$ (with the weak*-topologies) such that

$$T(x_1,\dots,x_n)=\int_{B_{X^*}\times\cdots\times B_{X^*}} x_1^*(x_1)\cdots x_n^*(x_n)\ d\mu(x_1^*,\dots,x_n^*)$$

for all $x_1,\dots,x_n\in X$ (see [7, 4.5]). The space of integral n-linear forms on X is denoted by $\mathcal{I}(^nX)$.

A sequence $(x_n)_n$ in a Banach space X is **strongly p-summable** if $(\|x_n\|)_n\in\ell_p$. The space of strongly p-summable sequences is a Banach space with the norm

$$\|(x_n)_n\|_p=\left(\sum_n \|x_n\|^p\right)^{1/p}.$$

A sequence in a Banach space is **weakly p-summable** if $(x^*(x_n))_n\in\ell_p$ for all $x^*\in X^*$. The space of weakly p-summable sequences endowed with the norm

$$w_p((x_n)_n)=\sup_{x^*\in B_{X^*}}\left(\sum_n |x^*(x_n)|^p\right)^{1/p}.$$

is a Banach space. These concepts can also be considered for finite sequences $(x_1,\dots,x_n)$ by means of the natural identification with $(x_1,\dots,x_n,0,0,\dots)$.

An operator $T\in\mathcal{L}(X,Y)$ is **absolutely r-summing** if there exists $c>0$ such that for any finite choice of elements $x_1,\dots,x_n\in X$ we have

$$\|(T(x_i))_{i=1}^n\|_r\le c\ w_r((x_i)_{i=1}^n).$$

We denote by $\Pi_r(X,Y)$ the space of absolutely r-summing operators between X and Y.

A map $T\in\mathcal{L}(X_1,\dots,X_n;Y)$ is said to be **absolutely $(s;r_1,\dots,r_n)$-summing** (where $\frac{1}{s}\le\frac{1}{r_1}+\cdots+\frac{1}{r_n}$) if there exists $c>0$ such that for any finite choice of elements $x^i_j\in X_j$,

$j = 1, \dots, n$, $i = 1, \dots, m$ we have

$$\|(T(x_1^i, \dots, x_n^i)_{i=1}^m\|_s \le c \prod_{j=1}^n w_{r_j}\left((x_j^i)_{i=1}^m\right).$$

A map $T \in \mathcal{L}((X_1, \dots, X_n; Y)$ is said to be r**-dominated** if it is absolutely $(r/n; r, \dots, r)$-summing; that is, there exists $c > 0$ such that for every $x_l^i \in X_l$, $l = 1, \dots, n$, $i = 1, \dots, m$,

$$\left(\sum_{i=1}^m \|T(x_1^i, \dots, x_n^i)\|^{r/n}\right)^{n/r} \le c w_r(x_1^i) \cdots w_r(x_n^i).$$

We denote by $\mathcal{D}_r(^nX)$ the space of r-dominated n-linear forms on X.

1. Limit orders for multilinear forms

If $T \in \mathcal{L}(^n\ell_p)$, we call it **diagonal** if there exists a sequence $\alpha = (\alpha(k))_k$ such that for all $x_1, \dots, x_n \in \ell_p$ we can write

$$T(x_1, \dots, x_n) = \sum_k \alpha(k) x_1(k) \cdots x_n(k).$$

We denote by T_α the diagonal multilinear mapping given by the sequence α. On the other hand, the diagonal operator from ℓ_p to ℓ_q associated to a sequence σ is defined by $D_\sigma(x) = (\sigma(k)x(k))_k$.

Given a diagonal multilinear form $T_\alpha \in \mathcal{L}(^n\ell_p)$, we consider a sequence σ such that $\sigma(k)^n = \alpha(k)$ for all k. We take the diagonal operator $D_\sigma : \ell_p \to \ell_n$ associated to σ and define a mapping $\Phi : \ell_n \times \cdots \times \ell_n \to \mathbb{C}$ by $\Phi(x_1, \dots, x_n) = \sum_k x_1(k) \cdots x_n(k)$. The fact that T is well defined on ℓ_p guarantees that $D_\sigma(\ell_p) \subset \ell_n$. Now, the diagonal n-linear mapping T can be rewritten as

$$T_\alpha(x_1, \dots, x_n) = \Phi(D_\sigma(x_1), \dots, D_\sigma(x_n)). \tag{1}$$

We use this decomposition several times.

Given $N \in \mathbb{N}$, we define the n-linear form Φ_N on $\mathbb{C}^N$ by:

$$\Phi_N(x_1, \dots, x_n) = \sum_{k=1}^N x_1(k) \cdots x_n(k).$$

We recall the notion of limit order for operators ideals (see [15, Section 14.4]). Given an operator ideal $\mathfrak{A}$, the limit order $\lambda(\mathfrak{A}; p, q)$ is the infimum over all $\lambda \ge 0$ such that every diagonal operator $D_\sigma : \ell_p \to \ell_q$ with $\sigma \in \ell_{1/\lambda}$ belongs to $\mathfrak{A}(\ell_p, \ell_q)$.

Ideals of multilinear forms were introduced in [16]. Now, we define the concept of limit order for ideals of multilinear forms:

Definition 1.1. *Let $\mathfrak{A}$ be an ideal of multilinear forms. For $1 \le p \le \infty$, the limit order $\lambda_n(\mathfrak{A}; p)$ is given by:*

$$\lambda_n(\mathfrak{A}; p) = \inf\{\lambda : \text{ for each } \alpha \in \ell_{1/\lambda},\ T_\alpha \text{ belongs to } \mathfrak{A}(^n\ell_p)\}$$

With almost the same proof as in [15, Section 14.4], we obtain alternative expressions for $\lambda_n(\mathfrak{A}; p)$. First, we have:

$$\lambda_n(\mathfrak{A}; p) = \inf\{\lambda : \text{ if } \alpha = (k^{-\lambda})_k, \text{ then } T_\alpha \text{ belongs to } \mathfrak{A}({}^n\ell_p)\}.$$

Also, if $\mathfrak{A}$ is quasi-normed, then $\lambda_n(\mathfrak{A}; p)$ is the infimum of all $\lambda \geq 0$ such that

$$\|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} \leq C N^\lambda \tag{2}$$

for all $N \geq 1$, where $C > 0$ is a constant.

If $\mathcal{L}$ is the ideal of continuous multilinear forms, it is easy to check that

$$\lambda_n(\mathcal{L}; p) = \begin{cases} 0 & \text{if } p \leq n \\ 1 - \frac{n}{p} & \text{if } p > n \end{cases}$$

Note that in this case the limit order is attained (i.e., the infimum in definition 1.1 is actually a minimum).

We compute now the limit orders for the ideals of nuclear and integral multilinear forms. Since nuclear and integral norms coincide in finite-dimensional spaces, the equivalence in inequality (2) implies that both limit orders are the same.

Next Lemma generalizes [4, Lemma 2.1] to n-linear forms. Since it is proved in the same way, apart from some slight technical modifications, we state it here without a proof.

Lemma 1.2. *Let $T \in \mathcal{L}({}^n\ell_p)$ be nuclear.*
(i) If $1 < p < n'$, then $(T(e_k, \dots, e_k))_k \in \ell_{p'/n}$.
(ii) If $n' \leq p < \infty$, then $(T(e_k, \dots, e_k))_k \in \ell_1$.

Next Proposition is again a generalization of [4, Proposition 2.2] to any degree. We present here a different proof.

Proposition 1.3. *Let $T_\alpha \in \mathcal{L}({}^n\ell_p)$ be diagonal.*
(i) For $1 < p < n'$, T_α is nuclear if and only if $\alpha \in \ell_{p'/n}$.
(ii) For $n' \leq p \leq \infty$, T_α is nuclear if and only if $\alpha \in \ell_1$.

Proof. Since $T(e_k, \dots, e_k) = \alpha(k)$ for every k, necessity is already proved by Lemma 1.2 for both cases. We only need to prove sufficiency in case (i). Let us consider a decomposition of T_α as that in (1), but with $\Phi : \ell_1 \times \cdots \times \ell_1 \to \mathbb{C}$ and $D_\sigma : \ell_p \to \ell_1$.

By [9, Example 2.25] Φ is integral and $\|\Phi\|_I = 1$. The diagonal operator D_σ is well defined; indeed, if $1 < p < n'$, we have $(\alpha(k))_k \in \ell_{p'/n}$. Hence $(\sigma(k))_k \in \ell_{p'}$ and $(\sigma(k)x(k))_k \in \ell_1$.

Using this decomposition we have $T_\alpha \in \mathcal{I}({}^n\ell_p)$. By [9, Proposition 2.27], $\mathcal{I}({}^n\ell_p) = \mathcal{N}({}^n\ell_p)$ and so T_α is nuclear. □

Proceeding as in the previous proof, we obtain $T_\alpha = \Phi \circ (D_\sigma, \dots, D_\sigma)$ is integral on ℓ_1 whenever σ (or equivalently α) is bounded. Moreover, with the same proof as [4, Proposition 2.3] we can see that T is nuclear on ℓ_1 if and only if $\alpha \in c_0$. Therefore, we have:

Proposition 1.4. *Let $T_\alpha \in \mathcal{L}(^n\ell_1)$ be diagonal. Then:*
(i) T_α *is integral;*
(ii) T_α *is nuclear if and only if* $\alpha \in c_0$.

As a consequence, we obtain the limit orders:

$$\lambda_n(\mathcal{N}; p) = \lambda_n(\mathcal{I}; p) = \begin{cases} \frac{n}{p'} & \text{if } 1 \leq p < n' \\ \\ 1 & \text{if } n' \leq p \end{cases}$$

Again, in this case the limit order is attained (if we consider, for $p = 1$, $\ell_{p'/n} = c_0$ for nuclear mappings and $\ell_{p'/n} = \ell_\infty$ for integral mappings).

2. Diagonal r-dominated mappings

In this section we compute limit orders for the ideal of r-dominated multilinear forms. This allows us to compare r-domination for different values of r and to relate this with other ideals of multilinear forms.

Proposition 2.1. *Let $T_\alpha \in \mathcal{L}(^n\ell_p)$ be diagonal and D_σ its associated diagonal operator. Then T_α is r-dominated if and only if D_σ is absolutely r-summing.*

Proof. Let us begin by assuming that T_α is r-dominated and choose $x_1^i = \cdots = x_{n-1}^i = x^i$ and $x_n^i(k) = \mathrm{sg}(\sigma(k)x^i(k))x^i(k)$. Since T_α is r-dominated

$$\begin{aligned} w_r((x^i)_i)^n C &\geq \left(\sum_{i=1}^N |T_\alpha(x^i, \ldots, x^i, x_n^i)|^{r/n} \right)^{n/r} \\ &= \left(\sum_{i=1}^N \left| \sum_k \sigma(k)^n x^i(k)^n \mathrm{sg}(\sigma(k)x^i(k)) \right|^{r/n} \right)^{n/r} \\ &= \left(\sum_{i=1}^N \left(\sum_k |\sigma(k)x^i(k)|^n \right)^{r/n} \right)^{n/r} \\ &= \left(\sum_{i=1}^N \|D_\sigma(x^i)\|_{\ell_n}^r \right)^{n/r}. \end{aligned}$$

This gives

$$\left(\sum_{i=1}^N \|D_\sigma(x^i)\|_{\ell_n}^r \right)^{1/r} \leq C w_r((x^i)_i)$$

and D_σ is absolutely r-summing.

The converse is an immediate consequence of [17, Proposition 3.6]. □

This proposition allows us to relate limit orders of r-dominated multilinear forms with those of absolutely r-summing operators:

Corollary 2.2. *For $1 \leq p \leq \infty$ and $n \geq 2$, we have:*

$$\lambda_n(\mathcal{D}_r, p) = n\ \lambda(\Pi_r, p, n)$$

A full classification of limit orders for r-summing operators can be found in [15, Section 22.4]. Using this classification and the previous corollary we obtain:

$$\lambda_n(\mathcal{D}_r; p) = \begin{cases} \frac{n}{p'} & \text{if } 1 \leq r \leq p' & (A) \\ \frac{n}{r} & \text{if } 1 \leq p' \leq r \leq n & (B) \\ 1 & \text{if } p' \leq 2 \text{ and } n \leq r & (C) \\ n\varepsilon & \text{if } 2 < p' \leq r \text{ and } n \leq r & (D) \end{cases} \tag{3}$$

where

$$\varepsilon = \frac{1}{r} + \frac{\left(\frac{1}{r'} - \frac{1}{p}\right)\left(\frac{1}{n} - \frac{1}{r}\right)}{\frac{1}{2} - \frac{1}{r}}$$

Now we see that this limit order is attained. In other words, every diagonal n-linear mapping T_α, with $\alpha \in \ell_{1/\lambda_n(\mathcal{D}_r;p)}$, is r-dominated on ℓ_p. By Proposition 2.1, we only need to deal with limit orders of r-summing operators. This is done in the following two propositions.

Proposition 2.3. *If $1 \leq r \leq p'$ and $q \geq 2$, then for any $\sigma \in \ell_{1/\lambda(\Pi_r;p;q)}$, the diagonal operator $D_\sigma : \ell_p \to \ell_q$ is r-summing (i.e., the limit order is attained).*

Proof. In this case $\lambda(\Pi_r; p; q) = 1/p'$. The fact that, for $\sigma \in \ell_{p'}$, the operator D_σ actually takes its values in ℓ_1 allows us to factor D_σ as:

$$\ell_p \to \ell_1 \hookrightarrow \ell_2 \hookrightarrow \ell_q$$

Since $i : \ell_1 \hookrightarrow \ell_2$ is 1-summing it follows that D_σ is 1-summing and therefore r-summing. $\square$

In the next proposition we follow some ideas of [8].

Proposition 2.4. *If either $r \leq 2 \leq p'$ or $p' \leq r$, then for any $\sigma \in \ell_{1/\lambda(\Pi_r;p;q)}$, the diagonal operator $D_\sigma : \ell_p \to \ell_q$ is r-summing (i.e., the limit order is attained).*

Proof. We set $\lambda_0 = \lambda(\Pi_r; p; q)$. Let $\mathcal{D}iag$ be the set of all diagonal operators $D_\sigma^N : \mathbb{C}^N \to \mathbb{C}^N$, for any $N \geq 1$. We define the following functions on $\mathcal{D}iag$:

$$A(D_\sigma^N) := \|D_\sigma^N\|_{\Pi_r(\ell_p;\ell_q)} \quad , \quad B(D_\sigma^N) := \|\sigma\|_{\ell_{1/\lambda_0}}.$$

Let us check that the functions A and B verify the conditions in [7, Lemma 34.12.1].

By the definition of limit order, for every $\sigma \in \ell_{1/(\lambda_0+\epsilon)}$, we have $D_\sigma \in \Pi_r(\ell_p; \ell_q)$. Since the application $\sigma \mapsto D_\sigma$ has closed graph, it is continuous. In particular, there exists c_ϵ such that

$$\|D_\sigma^N\|_{\Pi_r(\ell_p;\ell_q)} \leq \|\sigma\|_{\ell_{1/(\lambda_0+\epsilon)}} \leq c_\epsilon N^\epsilon \|\sigma\|_{\ell_{1/\lambda_0}}.$$

Therefore, $A(D_\sigma^N) \leq c_\epsilon N^\epsilon B(D_\sigma^N)$, which is the first condition in [7, Lemma 34.12.1].

The tensor product of two diagonal operators is also diagonal and the second condition is fulfilled. For the third condition, we actually have that $B(D_\sigma^N \otimes D_\sigma^N) = B(D_\sigma^N)^2$, so it is also verified.

As a consequence of [5, Corollary 1.4.5], since $r \leq 2 \leq p'$ or $p' \leq r$, there exists a constant $a > 0$ such that $A(D_\sigma^N)^2 \leq aA(D_\sigma^N \otimes D_\sigma^N)$; hence the fourth condition is verified.

Therefore, by [7, Lemma 34.12.1], we have $A(D_\sigma^N) \leq aB(D_\sigma^N)$ for all N and σ. By continuity, we have:

$$\|D_\sigma\|_{\Pi_r(\ell_p;\ell_q)} \leq a\|\sigma\|_{\ell_{1/\lambda_0}}$$

which completes the proof. □

Note that to study r-dominated n-linear forms we consider $q = n \geq 2$. So we have:

Corollary 2.5. *The limit order $\lambda_n(\mathcal{D}_r; p)$ is attained.*

Let us focus now on a reciprocal property of limit orders. Our aim is to determine if an r-dominated operator T_α is necessarily given by $\alpha \in \ell_{1/\lambda_n(\mathcal{D}_r;p)}$. Again, we first study the situation for linear operators:

Proposition 2.6. *Suppose one of the following conditions holds:*
(i) $1 \leq r \leq p'$,
(ii) $1 \leq p' \leq r \leq n$,
(iii) $p' \leq 2$ *and* $n \leq r$.
If $D_\sigma : \ell_p \to \ell_n$ is absolutely r-summing, then $\sigma \in \ell_{1/\lambda(\Pi_r;p;n)}$

Proof. First, we show that if D_σ is absolutely r-summing, then σ belongs to $\ell_{\max(r,p')}$. The canonical basis $(e_k)_k$ on ℓ_p is weakly p'-summing. If $p' \leq r$, $(e_k)_k$ is also weakly r-summing. Since D_σ is absolutely r-summing, $(D_\sigma(e_k))_k$ is r-summing and $\sigma \in \ell_r$. On the other hand, if $r < p'$, D_σ is p'-summing and therefore we obtain $\sigma \in \ell_{p'}$.

Now, if either condition (i) or (ii) holds, the limit order $\lambda(\Pi_r; p; n)$ coincide with $1/\max(r, p')$, and the conclusion follows for both cases.

The result for condition (iii) follows from [13, Theorem 4] and Proposition 2.1. □

Proposition 2.1 together with Proposition 2.6 give:

Proposition 2.7. *For each of the cases (A), (B) and (C) of equation (3), if T_α is r-dominated, then $\alpha \in \ell_{1/\lambda_n(\mathcal{D}_r;p)}$.*

Corollary 2.8. *If either (A) or (B) or (C) of equation (3) holds:*
(i) $\sigma \in \ell_{1/\lambda(\Pi_r,p,n)}$ *if and only if* $D_\sigma : \ell_p \to \ell_n$ *is absolutely r-summing.*
(ii) $\alpha \in \ell_{1/\lambda_n(\mathcal{D}_r,p)}$ *if and only if* $T_\alpha \in \mathcal{L}(^n\ell_p)$ *is r-dominated.*

As an application of the limit orders computed above, we show a structural difference between r-dominated bilinear and n-linear forms for $n \geq 3$. First, we have:

Remark 2.9. *If X is a Banach space and $r \geq 2$, then r-dominated and 2-dominated bilinear forms on X coincide.*

Proof. A bilinear form is r-dominated ($r \geq 2$) if and only if it is $\alpha_{r',r'}$-continuous [7, Theorem 19.2]. Since $r' \leq 2$, by [7, Proposition 12.8], the $\alpha_{r',r'}$ tensor norm is equivalent to the w_2 tensor norm. Again by [7, Theorem 19.2], a bilinear form is w_2-continuous if and only if it is 2-dominated. $\square$

A natural question now is if there is an analogous result for n-linear mappings: is there any r_0 such that for $r \geq r_0$, every r-dominated n-linear form is r_0-dominated? Or at least, does there exist an interval of r such that all r-dominated n-linear mappings coincide? Both questions can be answered in the negative. Moreover, the answer is negative even if we restrict ourselves to diagonal n-linear mappings:

Proposition 2.10. *Let $n \geq 3$. Given $r \geq 1$, there exists p such that, for any $s > r$, there are diagonal s-dominated n-linear forms on ℓ_p which are not r-dominated.*

Proof. First, we consider $r < n$ and take p such that $p' < r$. It is enough to prove the statement for $r < s < n$. In this case, $\lambda_n(\mathcal{D}_r; p) = \frac{n}{r} > \frac{n}{s} = \lambda_n(\mathcal{D}_s; p)$, which means that there are s-dominated n-linear forms on ℓ_p which are not r-dominated.

If $r \geq n$, let us choose p such that $2 < p' \leq r$. For $s \geq r$, we have $\lambda_n(\mathcal{D}_s; p) = n\left(\frac{1}{s} + \frac{\left(\frac{1}{s'} - \frac{1}{p}\right)\left(\frac{1}{n} - \frac{1}{s}\right)}{\frac{1}{2} - \frac{1}{s}}\right)$. Differentiating and taking into account that $1 \leq p < 2$ and $n \geq 3$, we obtain $\frac{\partial \lambda_n(\mathcal{D}_s;p)}{\partial s} = \frac{(p-2)(n-2)}{p(s-2)^2} < 0$. Therefore, $\lambda_n(\mathcal{D}_s; p)$ is strictly decreasing on s for $s \geq r$ and this completes the proof . $\square$

Although the classes of r and s-dominated diagonal multilinear forms are different for $r \neq s$, in some particular cases many of them coincide. We present some examples in the following corollary. Stronger results can be found on [13, Theorems 16 and 17].

Corollary 2.11. *Let $T_\alpha \in \mathcal{L}(^n\ell_p)$ be diagonal. Then,*
(i) If $p \geq 2$ and $r \geq n$, T_α is r-dominated if and only if it is n-dominated.
(ii) If $1 \leq r \leq p'$, T_α is r-dominated if and only if it is 1-dominated.

Proof. It follows from Corollary 2.8 and the fact that in both cases the limit order does not depend on r. $\square$

Let us now relate the concepts of domination, nuclearity and integrality for multilinear mappings. Meléndez and Tonge [13, Theorem 2] showed that every diagonal n-linear form on ℓ_1 is 1-dominated. Proposition 1.4 states that they are also integral. On the other hand, since integral multilinear forms are ε-continuous, it is easy to see that they are necessarily n-dominated. Therefore, we can combine Proposition 2.7 and Proposition 1.3 to obtain:

Corollary 2.12. *Let $T_\alpha \in \mathcal{L}(^n\ell_p)$ be diagonal. Then,*
(i) For $p = 1$, T_α is 1-dominated and integral.
(ii) For $p > 1$, T_α is n-dominated if and only if T_α is nuclear.

3. Extendible n-linear mappings

A mapping $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is called **extendible** (see e.g. [3, 4, 12]) if for all Banach spaces $Z_1, \dots, Z_n$ such that each X_j is contained in Z_j, there exists $\tilde{T} \in \mathcal{L}(Z_1, \dots, Z_n; Y)$ that extends T. The extendible norm of an extendible multilinear form is defined as

$$\|T\|_e = \inf\{c > 0 : \quad \begin{array}{l} \text{for all } Z_i \supseteq X_i \text{ there is an extension of } T \\ \text{to } Z_1 \times \cdots \times Z_n \text{ with norm} \leq c\}. \end{array}$$

First examples of extendible multilinear mappings are nuclear mappings.

If X is a Banach space and $T \in \mathcal{L}(^n X)$ is extendible, then it can be clearly extended to some $C(K)$ space. An application of Grothendieck's multilinear inequality gives that if T is extendible then T is absolutely $(1; 2, \dots, 2)$-summing (see [2] and also [14, Corollary 2.6] for a formulation more akin to our approach). Using this fact we can give a following generalization of [4, Proposition 2.4] to any degree $n \geq 2$.

Proposition 3.1. *Let $T_\alpha \in \mathcal{L}(^n \ell_p)$ diagonal with $2 \leq p \leq \infty$. Then T_α is extendible if and only if T_α is nuclear.*

Proof. If T_α is extendible, then it is absolutely $(1; 2, \dots, 2)$-summing and, for any $x_1^i, \dots, x_n^i \in \ell_p$ with $i = 1, \dots, N$,

$$\sum_{i=1}^N |T_\alpha(x_1^i, \dots, x_n^i)| \leq C \cdot w_2((x_1^i)_i) \cdots w_2((x_n^i)_i).$$

We choose now $x_1^i = \cdots = x_n^i = e_i$. Since $2 \leq p$, the sequence $(e_i)_i$ is weakly 2-summable in ℓ_p; therefore

$$\sum_{i=1}^N |\alpha(i)| \leq C \cdot w_2((e_i)_i)^n \leq K$$

for every N. Hence $(\alpha(k))_k \in \ell_1$ and, by Proposition 1.3, T_α is nuclear. $\square$

One may still ask if there are extendible multilinear forms on ℓ_p (with $2 \leq p \leq \infty$) which are not nuclear. By Proposition 3.1, one must look for them outside the class of diagonal multilinear forms. We devote some lines to answer this question. Since we also answer some questions posed in [4] for homogeneous polynomials, we state our results both in multilinear and polynomial settings.

In [4, Example 1.3] examples of extendible non nuclear 2-homogeneous polynomials on ℓ_p are presented for $p > 4$. A refinement of the proof shows that the same construction works for $p > 2$ (answering a question posed in that article). Indeed, we define

$$t_N = \frac{1}{\sqrt{N}} \sum_{j,k=1}^N e^{-2\pi i \frac{jk}{N}} e_j \otimes e_k \in \ell_p^N \otimes \ell_p^N$$

and $A_N \in \mathcal{L}(^2 \ell_p^N)$ by

$$A_N(x, y) = \frac{1}{\sqrt{N}} \sum_{j,k=1}^N e^{2\pi i \frac{jk}{N}} x(j) y(k). \tag{4}$$

From [7, Exercise 4.3] we get $\|t_N\|_\epsilon \leq N^{1/p-1/2}$ and then

$$N = |A_N(t_N)| \leq \|A_N\|_N \, \|t_N\|_\epsilon \leq \|A_N\|_N \;\; N^{1/p-1/2}.$$

Therefore, $\|A_N\|_N \geq N^{3/2-1/p}$ and the result follows just as in [4, Example 1.3].

Note that the symmetric bilinear form associated to this example is also extendible and not nuclear. In order to conclude that there are extendible n-linear forms (and n-homogeneous polynomials) which are not nuclear for any degree $n \geq 2$ we need the following:

Lemma 3.2. *(i) Let $T \in \mathcal{L}(^nX)$ be an n-linear form and $x^* \in X^*$. Then T is nuclear if and only if $x^*T \in \mathcal{L}(^{n+1}X)$ is nuclear.*
(ii) Let $P : X \to \mathbb{C}$ be an n-homogeneous polynomial and $x^* \in X^*$. Then P is nuclear if and only if x^*P is nuclear.

Proof. We only show (ii) since (i) is much simpler. If P is nuclear, the polynomial x^*P is clearly nuclear. Now we assume that x^*P is nuclear and fix $x_0 \in X$ with $x^*(x_0) = 1$. We consider a mapping $\xi : \mathcal{P}(^{n+1}X) \to \mathcal{P}(^{n+1}X)$ defined in [1] by

$$\xi(Q)(x) = Q(x) - Q(x - x^*(x)x_0)$$

for $x \in X$. Then

$$\begin{aligned}\xi(x^*P)(x) &= (x^*P)(x) - (x^*P)(x - x^*(x)x_0)\\ &= x^*(x)P(x) - (x^*(x) - x^*(x)x^*(x_0))P(x - x^*(x)x_0) = (x^*P)(x)\end{aligned}$$

and $\xi(x^*P) = x^*P$. Now, since x^*P is a nuclear $(n+1)$-homogeneous polynomial, a representation $x^*(x)P(x) = \sum_k x_k^*(x)^{n+1}$ can be found with $\sum_k \|x_k^*\|^{n+1} < \infty$. Applying ξ to this representation we get

$$\begin{aligned}x^*(x)P(x) &= \sum_{k=1}^{\infty} \xi((x_k^*)^{n+1})(x) = \sum_{k=1}^{\infty} \left(x_k^*(x)^{n+1} - (x_k^*(x) - x^*(x)x_k^*(x_0))^{n+1}\right)\\
&= \sum_{k=1}^{\infty} x_k^*(x)^{n+1} - \sum_{j=0}^{n+1}\sum_{k=1}^{\infty} \binom{n+1}{j} x_k^*(x)^j(-1)^{n+1-j}x^*(x)^{n+1-j}x_k^*(x_0)^{n+1-j}\\
&= -\sum_{k=1}^{\infty}\sum_{j=0}^{n} \binom{n+1}{j} x_k^*(x)^j(-1)^{n+1-j}x^*(x)^{n+1-j}x_k^*(x_0)^{n+1-j}\\
&= x^*(x)\Big(-\sum_{k=1}^{\infty}\sum_{j=1}^{n} \binom{n+1}{j} x_k^*(x)^j(-1)^{n+1-j}x^*(x)^{n-j}x_k^*(x_0)^{n+1-j}\Big).\end{aligned}$$

The last expression gives a representation of P that satisfies

$$\begin{aligned}&\sum_{k=1}^{\infty}\sum_{j=1}^{n} \binom{n+1}{j} \|x_k^*\|^j\|x^*\|^{n-j}|x_k^*(x_0)|^{n+1-j}\\
&\leq \left(\sum_{k=1}^{\infty} \|x_k^*\|^{n+1}\right)\left(\sum_{j=1}^{n} \binom{n+1}{j} \|x_k^*\|^j\|x^*\|^{n-j}\|x_0\|^{n+1-j}\right) < \infty.\end{aligned}$$

And P is nuclear. $\square$

Lemma 3.2, [4, Proposition 2.7] and the example above allow us to state the following:

Proposition 3.3. *Let $p > 2$.*
(i) For all $n \geq 2$, there are extendible non nuclear $n-$linear mappings on ℓ_p.
(ii) For all $n \geq 2$, there are extendible non nuclear $n-$homogeneous polynomials on ℓ_p.

Now we turn back our attention to diagonal multilinear forms and limit orders. Let $\mathcal{E}$ denote the ideal of extendible multilinear forms. From [4, Corollary 1.4, Proposition 2.4], we have

$$\lambda_2(\mathcal{E}, p) = \lambda_2(\mathcal{N}, p) \quad \text{for } 1 \leq p \leq \infty.$$

Moreover, Proposition 3.1 implies

$$\lambda_n(\mathcal{E}, p) = \lambda_n(\mathcal{N}, p) \quad \text{for } 2 \leq p \leq \infty.$$

Now we show that this equality does not hold for every p if $n \geq 3$. More precisely, if $(2(n-1))' < p < 2$, we have that $\lambda_n(\mathcal{E}, p) < \lambda_n(\mathcal{N}, p)$. This shows that, unlike the bilinear case, for $n \geq 3$ there are diagonal extendible n-linear forms which are not nuclear in some ℓ_p.

Lemma 3.4. *$\lambda_n(\mathcal{E}, p) \leq \frac{1}{2} + \frac{1}{p'}$ for all p.*

Proof. We begin by considering, for each $N \in \mathbb{N}$, $\xi_N : \ell_p^N \to \ell_\infty^N$ defined by

$$\xi_N(x) = \left(\sum_{s=1}^N e^{-2\pi i \frac{sk}{N}} x(s) \right)_{k=1}^N .$$

Using Hölder's inequality we get

$$\begin{aligned} \|\xi_N(x)\|_{\ell_\infty^N} &= \sup_{1 \leq k \leq N} \left| \sum_{s=1}^N e^{-2\pi i \frac{sk}{N}} x(s) \right| \\ &\leq \sup_{1 \leq k \leq N} \left(\sum_{s=1}^N \left| e^{-2\pi i \frac{sk}{N}} \right| \right)^{1/p'} \|x\|_{\ell_p^N} = N^{1/p'} \|x\|_{\ell_p^N}. \end{aligned}$$

Hence $\|\xi_N\| \leq N^{1/p'}$.

We consider the bilinear mapping A_N given by equation (4), but acting on $\ell_\infty^N \times \ell_\infty^N$. This mapping satisfies $\|A_N\| \leq N$ [7, Exercise 4.3]. Inspired by this we define now $S_N \in \mathcal{L}(^n \ell_\infty^N)$ by

$$S_N(x_1, \ldots, x_n) = \sum_{j,k=1}^N e^{2\pi i \frac{jk}{N}} x_1(j) x_2(k) \cdots x_n(k)$$

which satisfies $\|S_N\| = \sqrt{N} \|A_N\| \leq N\sqrt{N}$.

Now, the n-linear form $\Phi_N : \ell_p^N \times \cdots \times \ell_p^N \to \mathbb{C}$ given by $\Phi_N(x_1, \ldots, x_n) = \sum_{k=1}^N x_1(k) \cdots x_n(k)$ can be written as

$$\Phi_N(x_1, \ldots, x_n) = \frac{1}{N} S_N(\xi_N(x_1), x_2, \ldots, x_n).$$

Therefore, by the metric extension property of ℓ_∞^N, the extendible norm of Φ_N satisfies

$$\|\Phi_N\|_{\mathcal{E}(^n\ell_p^N)} \leq \frac{1}{N}\|S_N\|_{\mathcal{E}(^n\ell_\infty^N)}\|\xi_N\| = \frac{1}{N}\|S_N\| \ \|\xi_N\| \leq N^{1/2+1/p'}.$$

By the equivalence given in equation (2), we obtain the desired inequality. □

Corollary 3.5. *If $(2(n-1))' < p < 2$, then $\lambda_n(\mathcal{E}, p) < \lambda_n(\mathcal{N}, p)$. Thus, for $(2(n-1))' < p < 2$ there are extendible multilinear forms on ℓ_p which are not nuclear.*

Proof. For $n' \leq p < 2$, $1/2+1/p' < 1 = \lambda_n(\mathcal{N}, p)$ and for $(2(n-1))' < p < n'$, $1/2+1/p' < \frac{n}{p'} = \lambda_n(\mathcal{N}, p)$. □

Remark 3.6. If X is a Banach space with cotype 2, every extendible bilinear form (and 2-homogeneous polynomial) on X is integral [4, 6]. For $(2(n-1))' < p < 2$, nuclear and integral multilinear forms coincide on ℓ_p (and also nuclear and integral polynomials). Therefore, Corollary 3.5 shows that the result for cotype 2 spaces cannot be extended to degrees greater than 2.

Acknowledgements

We would like to thank Andreas Defant for all the helpful conversations and suggestions regarding this work.

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