SOME METRIC AND TOPOLOGICAL PROPERTIES OF NEARLY STRONGLY AND NEARLY VERY CONVEX SPACES

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Abstract. We obtain characterizations of nearly strong convexity and nearly very convexity by using the dual concept of S and WS points, related to the so-called Rolewicz's property (α). We give a characterization of those points in terms of continuity properties of the identity mapping. The connection between these two geometric properties is established, and finally an application to approximative compactness is given.

1. Introduction and preliminaries

A Banach space $(X, \|\cdot\|)$ (whose closed unit ball and sphere are denoted by B_X and S_X, respectively) is said to have *property* (α) whenever

$$\alpha(S(x^*, 1/n)) \to 0, \text{ as } n \to \infty,$$

for all $x^* \in S_{X^*}$, where α stands for the Kuratowski index of non-compactness (i.e., $\alpha(S)$ is the infimum of numbers $r > 0$ such that S can be covered by a finite number of subsets having diameter less that r) and $S(x^*, \varepsilon) := \{x \in B_X : x^*(x) > 1-\varepsilon\}$ is the ε-*slice* of B_X defined by x^*. Property (α) was introduced by S. Rolewicz. The second named author proved that it is equivalent to X having the so-called *drop property*, which in turn is equivalent to X being reflexive and having the Radon–Riesz property (H) [M87] (a Banach space X has the Radon–Riesz property (H) whenever the w-convergence of a sequence in S_X to a point x in S_X is equivalent to the $\|\cdot\|$-convergence to x).

In order to avoid the restriction of reflexivity —itself a consequence of James' compactness theorem— in the previous concept, J. H. Wang and the first named author introduced in [WZ97] the so-called *nearly strong convexity*, where the (α) property is checked only on the set NA(X) of norm-attaining functionals. Precisely, if D denotes the duality mapping, i.e., $D(x) := \{x^* \in S_{X^*} : x^*(x) = 1\}$ for $x \in S_X$, the definition reads:

Definition 1.1 (Wang, Zhang)**.** Let $(X, \|\cdot\|)$ be a Banach space, and let $x_0 \in S_X$. The norm $\|\cdot\|$ is said to be *strongly convex at* x_0 (*nearly strongly convex at* x_0) if given $x_0^* \in D(x_0)$ and a sequence $\{x_n\}$ in B_X such that $x_0^*(x_n) \to 1$, then $x_n \to x_0$ (respectively, the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact). If the set $\{x_n : n \in \mathbb{N}\}$ is just weakly relatively compact, the norm is said to be *nearly very convex at* x_0. The norm is said to be *strongly convex* if it is strongly convex at every point of

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S_X. The same applies to the nearly strong convexity and nearly very convexity concepts.

That the concept of nearly strong convexity can be written in terms of the property (α) is stated in Proposition 2.8 below. A trivial observation is that we may replace B_X by S_X in the previous definition.

We follow the notation as in standard texts (see, e.g., [FHHMZ11]). For example, if X is a Banach space and $x \in X$, then $B(x, \varepsilon)$ denotes the closed ball centered at x and having radius $\varepsilon > 0$. The action of an element $x^* \in X^*$ on an element $x \in X$ will be denoted by $x^*(x)$ or, alternatively, $\langle x, x^* \rangle$. We shall consider X canonically embedded in its bidual X^{**}. The weak topology on X will be denoted by $w(X, X^*)$ or just w if there is no risk of misunderstanding. The same applies to the weak* topology on X^* (denoted $w(X^*, X)$ or just, simply, w^*).

Remark 1.2. To place the concepts introduced in Definition 1.1 in a proper context, and to see a few connections with some standard geometrical properties of Banach spaces, notice the following almost trivial implications. (Below, LUR is the usual acronym for locally uniformly rotund: The norm $\|\cdot\|$ of a Banach space X is said to be *locally uniformly rotund* if given a sequence $\{x_n\}$ in X such that $\|x_n\| \to 1$ and $\|x_n + x_0\| \to 2$, then $x_n \to x_0$; if the convergence of $\{x_n\}$ to x_0 is in the weak topology, the norm is said to be wLUR.) Then we have LUR $\Rightarrow$ strongly convex $\Rightarrow$ nearly strongly convex $\Rightarrow$ nearly very convex, and these four concepts are different (see [ZL11, Examples 2.5, 2.6, and 2.7]) (for examples outside the context of reflexive spaces —all of them nearly very smooth— consider [Dr14, Theorem 1], where it is proved that every infinite-dimensional Banach space with separable dual admits an equivalent wLUR norm which is not LUR: It is obvious that every wLUR space is nearly very convex; this wLUR equivalent norm cannot be nearly strongly convex, since this last property implies property (H) that, together with wLUR, implies LUR, see below). The concepts nearly strongly convex and nearly very convex are discussed, e.g., in [BLLN08], [FW01], [GM11], [ZL11], [ZL12], [ZMLG15], and [ZS09], and they are related to questions of approximation in Banach spaces. We may mention, for example, a characterization of nearly strict convexity in terms of the preduality mapping:

(i) [ZMLG15] X is nearly strongly convex (respectively, nearly very convex) $\Leftrightarrow$ the predual mapping D^{-1} is ($\|\cdot\|$-$\|\cdot\|$-) (respectively, ($\|\cdot\|$-w-)) upper semicontinuous on $S(X^*) \cap NA(X)$ with norm-compact images (respectively, weak-compact images),

and how in nearly strongly convex Banach spaces proximinality and approximative compactness agree:

(ii) [ZL12] X is nearly strongly convex $\Rightarrow$ every proximinal closed convex subset in X is approximatively compact (and conversely). ®

Remark 1.3. Observe that it is equivalent to say that $\|\cdot\|$ is nearly strongly convex (nearly very convex) at x_0 that for every $x_0^* \in D(x_0)$, every sequence $\{x_n\}$

in B_X such that $x_0^*(x_n) \to 1$ has a $\|\cdot\|$-convergent (respectively, w-convergent) subsequence. The statement for the norm topology is a simple consequence of the $\|\cdot\|$-compactness (and for the weak topology, of the Eberlein–Šmulyan theorem). ®

Remark 1.4. Observe that the concepts introduced in Definition 1.1 are metric in nature. Indeed, and as an example, if $\|\cdot\|$ is nearly very convex at some $x_0 \in S_X$, then every $x_0^* \in D(x_0)$ defines a w-compact face $\{x \in S_X : x_0^*(x) = 1\}$ by the Eberlein–Šmulyan theorem. Let us consider the space c_0 endowed with the supremum norm $\|\cdot\|_\infty$. For $n \in \mathbb{N}$, let e_n (e_n^*) be the n-th vector of the canonical basis of c_0 (respectively, of ℓ_1). Then e_n^* ($\in D(e_n)$) defines a face of B_{c_0} that is homeomorphic to B_{c_0}, hence not w-compact, so $\|\cdot\|_\infty$ is not nearly very convex at e_n. However, c_0 —as every separable Banach space— has an equivalent LUR norm $\|\cdot\|$ by a result of M. I. Kadec, and, as it was mentioned in Remark 1.2, $\|\cdot\|$ is then nearly very convex. ®

The symbol $\widehat{\ }$ stands for the canonical embedding of a Banach space into its bidual.

The following lemma has a simple proof. We provide it for the sake of completeness.

Lemma 1.5. *Let $\{S_n\}$ be a decreasing sequence of subsets of a complete metric space (M, d). Then, the following statements are equivalent:*
(i) $\alpha(S_n) \to 0$.
(ii) *If $x_n \in S_n$, $n \in \mathbb{N}$, the sequence $\{x_n\}$ has a convergent subsequence.*
(iii) *If $x_n \in S_n$, $n \in \mathbb{N}$, the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact.*

Proof. (i)⇒(ii) If $S := \{x_n : n \in \mathbb{N}\}$ is finite we are done. Otherwise, find a finite covering of S_1 by sets of diameter less than $\alpha(S_1) + 1$. One of them must contain an infinite subsequence $\{x_n^{(1)}\}$ of $\{x_n\}$. This subsequence but its first element is in S_2. The same argument gives a subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ in a set of diameter less than $\alpha(S_2) + 1/2$. Continue in this way. A diagonal procedure gives a Cauchy subsequence of $\{x_n\}$, and we are done.

(ii)⇒(iii) The statement (ii) implies easily that the set $S := \{x_n : n \in \mathbb{N}\}$ is relatively sequentially compact, and so relative compact.

(iii)⇒(i) If $\alpha(S_n) \geq \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$, take $x_1 \in S_1$. The ball $B(x_1, \varepsilon/4)$ cannot cover S_2, so there exists $x_2 \in S_2 \setminus B(x_1, \varepsilon/4)$. By the same token, there exists $x_3 \in S_3 \setminus \bigcup_{k=1}^2 B(x_k, \varepsilon/4)$. By induction we get an $\varepsilon/4$-separated sequence $\{x_n\}$ such that $x_n \in S_n$ for all $n \in \mathbb{N}$, and this contradicts the relative compactness of $\{x_n : n \in \mathbb{N}\}$. □

Lemma 1.6 below is a simple but useful result. It appeared already in a restricted form in [HMZ12] and more precisely in [ZMLG15], where the proof is provided. Given $n \in \mathbb{N}$, put $I_n := \{1, 2, \ldots, n\}$.

Lemma 1.6. *Given a non-empty subset A of a Banach space X, $n \in \mathbb{N}$, a subset $\{f_i :\ i \in I_n\}$ of X^*, and a set of real numbers $\{\alpha_i :\ i \in I_n\}$, we have*

$$\{x^{**} \in \overline{A}^{w^*} :\ f_i(x^{**}) > \alpha_i,\ i \in I_n\}$$
$$\subset \overline{\{x \in A :\ f_i(x) > \alpha_i,\ i \in I_n\}}^{w^*} \tag{1.1}$$
$$\subset \{x^{**} \in \overline{A}^{w^*} :\ f_i(x^{**}) \geq \alpha_i,\ i \in I_n\}, \tag{1.2}$$

and the second inclusion is an equality if A is convex and the intermediate set in (1.1) *is non-empty.*

Remark 1.7. (i) A straightforward consequence of Lemma 1.6 is that, if $f \in S_{X^*}$ and $0 < \varepsilon < 1$,

$$S(f,\varepsilon) := \{x^{**} \in B_{X^{**}} :\ f(x^{**}) > 1-\varepsilon\}$$
$$\subset \overline{\{x \in B_X :\ f(x) > 1-\varepsilon\}}^{w^*} = \{x^{**} \in B_{X^{**}} :\ f(x^{**}) \geq 1-\varepsilon\} = \overline{S(f,\varepsilon)}^{\|\cdot\|}.$$

(ii) Note, too, that if $A \subset X$, then $\alpha(A) = \alpha(\overline{A}^{w^*})$, where $\overline{A}^{w^*}$ denotes the closure of A in (X^{**}, w^*). This is a consequence of the trivial fact that $B_{X^{**}}$ is the w^*-closure of B_X in X^{**}. ®

2. Main results

The following concepts are introduced for dualizing the nearly strong (respectively, very) convexity.

Definition 2.1. Let X be a Banach space. Let $x_0 \in S_X$. We say that x_0 *is an S-point* (a *WS-point*) if given a sequence $\{x_n^*\}$ in B_{X^*} such that $x_n^*(x_0) \to 1$ then the set $\{x_n^* :\ n \in \mathbb{N}\}$ is $\|\cdot\|$- (respectively, w-) relatively compact.

Remark 2.2. Observe that, according to Lemma 1.5 above, it is the same to say that x_0 is an S-point that x_0 *gives property* (α) *on* B_{X^*}, i.e., that $\alpha(S(x_0, 1/n)) \to 0$ as $n \to \infty$. An easy consequence of Lemmata 1.5 and 1.6 is that $x_0^* \in S_{X^*}$ gives property (α) on $B_{X^{**}}$ if, and only if, it gives property (α) on B_X, i.e., if $\alpha(S(x_0^*, 1/n) \cap X) \to 0$ as $n \to \infty$. An observation similar to Remark 1.3 is that x_0 is an S- (WS-) point if, and only if, any sequence $\{x_n^*\}$ in B_{X^*} such that $x_n^*(x_0) \to 1$ has a $\|\cdot\|$- (respectively, w-) convergent subsequence. ®

The connection between S- (WS-) points and points of continuity is given by the following result. In a sense it is somehow surprising, since a sequential continuity condition turns out to be equivalent to a continuity condition —even in absence of metrizability—. This is true for the norm and for the weak topology as well. Recall that if A is a subset of the dual X^* of a Banach space X, a point $x^* \in A$ is said to be a w^*-$\|\cdot\|$- (respectively, a w^*-w-) *point of continuity* of A if x^* is a point of continuity of the identity map from (A, w^*) to $(A, \|\cdot\|)$ (respectively, from (A, w^*) to (A, w)).

Proposition 2.3. *Let $(X, \|\cdot\|)$ be a Banach space, and let $x \in S_X$. Then, the following are equivalent:*

(i) *x is an S- (respectively, WS-) point.*

(ii) *Every* $x^* \in D(x)$ *is a* w^*-$\|\cdot\|$- *(respectively* w^*-w-*) point of continuity of* B_{X^*}.

Proof. First, we shall prove the case of the S property and the w^*-$\|\cdot\|$ continuity.

(i)⇒(ii). Assume that $x \in S_X$ is an S-point. Note that, trivially, $D(x) := \{x^* \in B_{X^*} :\ x^*(x) = 1\}$ is a $\|\cdot\|$-compact set. Fix $x^* \in D(x)$. Let $\{x_i^*, i \in I, \leq\}$ be a net in B_{X^*} that w^*-converges to x^*. Let us choose an arbitrary subnet of $\{x_i^*, i \in I, \leq\}$, that in order to simplify the notation will be written again $\{x_i^*, i \in I, \leq\}$. In particular, $x_i^*(x) \to 1$. Given $n \in \mathbb{N}$, we can find $i_n \in I$ such that $x_i^*(x) \geq 1 - 1/n$ for all $i \geq i_n$. Without loss of generality, we may assume that the sequence $\{i_n\}_{n=1}^\infty$ is increasing. We have now two possibilities:

a) There exists $i_0 \in I$ such that $i_n \leq i_0$ for all $n \in \mathbb{N}$. This shows, in particular, that $x_i^* \in D(X)$ for all $i \geq i_0$. Since $D(x)$ is $\|\cdot\|$-compact, we conclude that $\{x_i^*, i \in I, \leq\}$ has a $\|\cdot\|$-convergent subnet.

b) Assume now the opposite: For all $i \in I$ there exists $n \in \mathbb{N}$ such that $i < i_n$. Then $\{x_{i_n}^*, n \in \mathbb{N}, \leq\}$ is a subnet of $\{x_i^*, i \in I, \leq\}$. Since $x_i^*(x) \to 1$, we obtain $x_{i_n}^*(x) \to 1$. Since x is an S-point we get that $\{x_{i_n}^* :\ n \in \mathbb{N}\}$ is a $\|\cdot\|$-relatively compact subset of X^*. This shows that $\{x_{i_n}^* : n \in \mathbb{N}\}$ has a $\|\cdot\|$-convergent subnet, and this in turn is a $\|\cdot\|$-convergent subnet of $\{x_i^*, i \in I, \leq\}$.

In both cases we got a $\|\cdot\|$-convergent (obviously to x^*) subnet. Since this applies to an arbitrary subnet of $\{x_i^*, i \in I, \leq\}$, we get that $x_i^* \to x^*$ in the norm topology.

(ii)⇒(i) Assume now that every $x^* \in D(x)$ is a w^*-$\|\cdot\|$-continuity point of B_{X^*}. Let $\{x_n^*\}$ be a sequence in B_{X^*} such that $x_n^*(x) \to 1$. We shall show that $\{x_n^* :\ n \in \mathbb{N}\}$ is $\|\cdot\|$-relatively compact. It is enough to check that every infinite subset A of $\{x_n^* :\ n \in \mathbb{N}\}$ has a $\|\cdot\|$-cluster point. The set A is w^*-relatively compact, hence it has a w^*-cluster point x_0^*. Obviously, $x_0^*(x) = 1$, hence x_0^* is a w^*-$\|\cdot\|$-point of continuity. This implies that x_0^* is a $\|\cdot\|$-cluster point of A.

The case of an WS-point and the w^*-w continuity is similar. □

We quote here a result in [GGS78]: *For* $x \in S_X$, *the* w^*- *and* w- *topologies agree on* S_{X^*} *at points of* $D(x)$ *if, and only if,* $(X^{***} \supset)\ D(\hat{x}) = \widehat{D(x)}\ (\subset X^*)$. In view of Proposition 2.3, we readily obtain the following consequence (note that the w^*- and the w- topologies agree on S_{X^*} at points of $D(x)$, for a given $x \in S_X$, if, and only if, they agree on B_{X^*} at those points).

Corollary 2.4. *Let* $(X, \|\cdot\|)$ *be a Banach space. Let* $x \in S_X$. *Then the two following statements are equivalent:*

(i) x *is a WS-point.*

(ii) $(X^{***} \supset)\ D(\hat{x}) = \widehat{D(x)}\ (\subset X^*)$.

Compare the characterization in Corollary 2.4 with the one in the following Proposition from [ZMLG15]. In a sense, they are dual to each other.

Proposition 2.5 ([ZMLG15], Corollary 3.14)**.** *A Banach space* X *is nearly very convex if, and only if,* $(X^{**} \supset)\ D(x^*) = \widehat{D^{-1}(x^*)}\ (\subset X)$ *for all* $x^* \in D(S_X)$.

Proposition 2.7 below is a local version of the previous proposition. In order to prove it we need the following statement, that collects several results in [ZMLG15]:

Theorem 2.6. *Let $(X, \|\cdot\|)$ be a Banach space. Let $x_0 \in S_X$ and $x_0^* \in D(x_0)$. Then, each of the following statements is equivalent to the fact that D^{-1} is $\|\cdot\|$-w-usco at x_0^*:*

(i) [ZMLG15, Theorem 3.13] $\widehat{D^{-1}(x_0^*)} = D(x_0^*)$.

(ii) [ZMLG15, Theorem 3.5] $D^{-1}(x_0^*)$ *is w-compact and for every w-neighborhood N of 0, the set $D^{-1}(x_0^*) + N$ contains a nonempty slice of B_X defined by x_0^*.*

(iii) [ZMLG15, Theorem 3.6] *For every net $\{x_i : i \in I, \leq\}$ such that $\langle x_i, x_0^*\rangle \to 1$, there exists a subnet $\{x_{i_j} : j \in J, \preceq\}$ of $\{x_i : i \in I, \leq\}$ that w-converges.*

Proposition 2.7. *Let $(X, \|\cdot\|)$ be a Banach space. Let $x_0 \in S_X$. Then $\|\cdot\|$ is nearly very convex at x_0 if, and only if, $D(x_0^*) = \widehat{D^{-1}(x_0^*)}$ for all $x_0^* \in D(x_0)$.*

Proof. Assume that $\|\cdot\|$ is nearly very convex at x_0, and let $x_0^* \in D(x_0)$. If $\{x_n\}$ is a sequence in $D^{-1}(x_0^*)$, by definition the set $\{x_n : n \in \mathbb{N}\}$ is w-relatively compact. It follows then, by the Eberlein–Šmulyan theorem, that the set $D^{-1}(x_0^*)$ is w-compact. Assume now that there exists a w-neighborhood N of 0 (that can be taken to be w-open) such that $D^{-1}(x_0^*) + N$ does not contain any nonempty slice of B_X defined by x_0^*. Thus, there exists a sequence $\{x_n\}$ in B_X such that $\langle x_n, x_0^*\rangle \to 1$ and $x_n \notin D^{-1}(x_0^*) + N$ for all $n \in \mathbb{N}$. Again by definition, the set $\{x_n : n \in \mathbb{N}\}$ is w-relatively compact, hence there exists a w-cluster point $x \in B_X$. Note that $\langle x, x_0^*\rangle = 1$, and that $x \notin D^{-1}(x_0^*) + N$, a contradiction. This shows, thanks to the equivalence (i)⇔(ii) in Theorem 2.6, that $\widehat{D^{-1}(x_0^*)} = D(x_0^*)$.

Assume now that $\widehat{D^{-1}(x_0^*)} = D(x_0^*)$. Take a sequence $\{x_n\}$ in B_X such that $\langle x_n, x_0^*\rangle \to 1$, and let $x^{**} \in \overline{\{x_n : n \in \mathbb{N}\}}^{w^*}$. Note that if $x^{**} \notin X$ then $\langle x^{**}, x_0^*\rangle = 1$. There exists a net $\{y_i : i \in I, \leq\}$ in $\{x_n : n \in \mathbb{N}\}$ that w^*-converges to x^{**}. Thus, $\langle y_i, x_0^*\rangle \to 1$. By the equivalence (i)⇔(iii) in Theorem 2.6, there exists, then, a subnet $\{y_{i_j} : j \in J, \preceq\}$ of $\{y_i : i \in I, \leq\}$ that w-converges (to some element $y_0 \in X$). Obviously, $x^{**} = y_0$, and we reach a contradiction. This shows that $x^{**} \in X$, hence $\{x_n : n \in \mathbb{N}\}$ is a w-relatively compact subset of X. Thus, $\|\cdot\|$ is nearly very convex at x_0. □

We include down a simple proof of the necessary condition in Proposition 2.7 in a particular case, based just on the very definition of near very convexity, in order to have a taste of the argument behind the general proof above.

Proof. (of the necessary condition in Proposition 2.7 for separable spaces that do not contain an isomorphic copy of ℓ_1.) Assume that $\|\cdot\|$ is nearly very convex at x_0, and let $x_0^* \in D(x_0)$. Given $x_0^{**} \in D(x_0^*)$, there exists, by the Odell–Rosenthal theorem (see, e.g., [Di84, page 215]), a sequence $\{x_n\}$ in B_X that w^*-converges to x_0^{**}. Since $\langle x_n, x_0^*\rangle \to 1$, we know that $\{x_n : n \in \mathbb{N}\}$ is w-relatively compact in X. It follows that $x_0^{**} \in X$ (i.e., $x_0^{**} \in \widehat{D^{-1}(x_0^*)}$). On the other hand (and note that this implication holds without any restriction on the space), if $D(x_0^*) \subset X$, let $\{x_n\}$ be a sequence in B_X such that $\langle x_n, x_0^*\rangle \to 1$. Let x_0^{**} be a w^*-cluster point of $\{x_n : n \in \mathbb{N}\}$ in X^{**}. We have $x_0^{**} \in D(x_0^*)$, so $x_0^{**} \in X$ and $\{x_n : n \in \mathbb{N}\}$ is w-relatively compact. □

Proposition 2.8 below is somehow the "predual" version of Theorem 3.3 in [ZMLG15], and collects some information that was given above.

Proposition 2.8. *Let $(X, \|\cdot\|)$ be a Banach space. Let $x_0 \in S_X$. Then the following are equivalent:*

(i) *$\|\cdot\|$ is nearly strongly convex at x_0.*
(ii) *For every $x_0^* \in D(x_0)$, $\alpha(S(x_0^*, 1/n)) \to 0$ as $n \to \infty$.*
(iii) *Every $x_0^* \in D(x_0)$ gives property (α) on $B_{X^{**}}$.*
(iv) *Every $x_0^* \in D(x_0)$ is an S-point.*
(v) *For $x_0^* \in D(x_0)$, every $x_0^{**} \in D(x_0^*)$ $(\subset X^{**})$ is a point of w^*-$\|\cdot\|$-continuity of $B_{X^{**}}$.*

Proof. (i)⇔(ii) is just the equivalence (i)⇔(iii) in Lemma 1.5 applied to the sequence $\{S(x_0^*; \varepsilon_n)\}_{n=1}^{\infty}$, where $\{\varepsilon_n\}_{n=1}^{\infty}$ is any decreasing sequence of positive numbers.

(ii)⇔(iii)⇔(iv) was already mentioned in Remark 2.2.

(iv)⇔(v) is Proposition 2.3. □

Note that (i)⇔(v) in Proposition 2.8 above shows that if the norm is nearly strongly convex then every $x \in S_X$ is a point of w^*-$\|\cdot\|$-continuity of $B_{X^{**}}$ (i.e., a point of w-$\|\cdot\|$ continuity of B_X), in particular that every nearly strongly convex norm has the *Kadec property*, i.e., the property that the weak and norm topologies coincide on its unit sphere. (It is then very easy to give examples of Banach spaces that are nearly very convex and not nearly strongly convex: Every reflexive Banach space is obviously nearly very convex. However, every infinite-dimensional Banach space has an equivalent norm that fails the Kadec property.)

It is precisely the Kadec property —or its sequential version, the Radon–Riesz property (H), defined above— what makes the difference between the nearly strong and very convexity, as the next result shows. As we mentioned above, neither the Kadec property nor the (H) property can be removed from the statement. Note in passing that the two properties, the Kadec property and property (H), are in general different, and that they coincide if the Banach space has no isomorphic copy of ℓ_1 (for separable spaces this is a result of Troyanski; in fact, it holds for arbitrary Banach spaces).

Proposition 2.9. *Let X be a Banach space. Then, the three following conditions are equivalent.*

(i) *X is nearly strongly convex.*
(ii) *X is nearly very convex and any $x \in S_X$ is a point of w^*-$\|\cdot\|$-continuity of $B_{X^{**}}$.*
(iii) *X is nearly very convex and has the Kadec property.*
(iv) *X is nearly very convex and has property (H).*

Proof. (i)⇒(ii)⇒(iii) follows from the previous observation, and (iii)⇒(iv) is obvious. (iv)⇒(i) follows easily from the definition and Remark 1.3. □

Remark 2.10. In [ZL12] (respectively, [ZMLG15]) it was proved that a Banach space X is nearly strongly convex (respectively, nearly very convex) if, and only if, the predual mapping D^{-1} is $(\|\cdot\|$-$\|\cdot\|$-$)$ (respectively, $(\|\cdot\|$-w-$)$) upper semicontinuous on $S(X^*)\cap NA(X)$ with $\|\cdot\|$-compact images (respectively, w-compact images). ®

Remark 2.11. Note that Proposition 2.9 shows that (v) in Proposition 2.8 cannot be substituted by checking w-$\|\cdot\|$-continuity with respect to B_X at points in $D^{-1}(x_0^*)$ $(\subset X)$ instead of w^*-$\|\cdot\|$-continuity with respect to $B_{X^{**}}$ at points in $D(x_0^*)$ $(\subset X^{**})$. In fact, the former is just the Kadec property of the norm, something that clearly does not imply nearly strong convexity. ®

Remark 2.12.
(1) Note that (iv)⇔(v) in Proposition 2.8 holds when "S-point" in (iv) is replaced by "WS-point", and w^*-$\|\cdot\|$-continuity in (v) by w^*-w-continuity (this is just Proposition 2.3).
(2) In view of Proposition 2.8 and item (1) in this remark, it is natural to conjecture that (i)⇔(iv) holds whenever "S-point" and "nearly strong convexity" are replaced by "WS-point" and "nearly very convexity", respectively. This fails in general. For the right equivalence see Proposition 2.13, and for an example of a nearly very convex space with a point $x_0\in S_X$ that is not a point of w^*-w-continuity of $B_{X^{**}}$ see Remark 2.14. ®

The following result *is* the right counterpart of Proposition 2.8 for nearly very convexity (see also item (2) in Remark 2.12 above). In the proof we shall need the local version of Proposition 2.5 above, given as Proposition 2.7.

Proposition 2.13. *Let $(X,\|\cdot\|)$ be a Banach space. Let $x_0\in S_X$. Then the following are equivalent:*

(i) *The norm $\|\cdot\|$ is nearly very convex at x_0 and every $x\in D(x_0^*)\cap X$ for $x_0^*\in D(x_0)$, is a point of w^*-w-continuity of $B_{X^{**}}$.*

(ii) *Every point $x_0^*\in D(x_0)$ is a WS-point.*

Proof. (ii)⇒(i). Assume that (ii) holds. By definition, given a sequence $\{x_n^{**}\}$ in $B_{X^{**}}$ such that $\langle x_n^{**},x_0^*\rangle\to 1$, the set $\{x_n^{**}:\ n\in\mathbb{N}\}$ is w-relatively compact in X^{**}. In particular, this happens for any sequence $\{x_n\}$ in B_X such that $\langle x_n,x_0^*\rangle\to 1$, so $\{x_n:\ n\in\mathbb{N}\}$ is w-relatively compact in X^{**}. Thus, $\overline{\{x_n:\ n\in\mathbb{N}\}}^w$ is w-compact in X^{**}. Since $\overline{\operatorname{conv}}^w\{x_n:\ n\in\mathbb{N}\}=\overline{\operatorname{conv}}^{\|\cdot\|}\{x_n:\ n\in\mathbb{N}\}$ $(\subset X)$ by Mazur's theorem, we get that $\{x_n:\ n\in\mathbb{N}\}$ is w-relatively compact in X. This shows that $\|\cdot\|$ is nearly very convex at x_0. Proposition 2.3 above shows that given $x_0^*\in D(x_0)$, every point $x^{**}\in D(x_0^*)$ is of w^*-w-continuity in $B_{X^{**}}$. In particular, this happens for every $x\in D(x_0^*)\cap X$.

(i)⇒(ii). Assume that (i) holds. Proposition 2.7 shows that for every $x_0^*\in D(x_0)$ we have $D(x_0^*)=D^{-1}(x_0^*)$. Let $\{x_n^{**}\}$ be a sequence in $B_{X^{**}}$ such that $\langle x_n^{**},x_0^*\rangle\to 1$. Let x_0^{**} be any w^*-cluster point of $\{x_n^{**}:\ n\in\mathbb{N}\}$. Obviously, $x_0^{**}\in D(x_0^*)$. Thus, $x_0^{**}\in D^{-1}(x_0^*)$ $(\subset X)$. Since x_0^{**} is a point of w^*-w-continuity in $B_{X^{**}}$, we get that the set $\{x_n^{**}:\ n\in\mathbb{N}\}$ is, in fact, w-relatively compact in $B_{X^{**}}$. This proves that x_0^* is a WS-point. □

Remark 2.14. We provide an example to show that the condition on w^*-w-continuity in item (i) of Proposition 2.13 cannot be dropped in order to get (ii) there. In [MOTV09] an example of a (R), not LUR norm $|||\cdot|||$ on c_0 was given. Later on, it was proved in [Dr14] that $|||\cdot|||$ is, in fact, wLUR (hence $(c_0, |||\cdot|||)$ is nearly very convex). The norm $|||\cdot|||$ is defined as

$$|||x||| = \|x\|_\infty + \left(\sum_{n=1}^{\infty} 2^{-n}|x(n)|^2\right)^{1/2}, \quad \text{for } x \in c_0, \tag{2.1}$$

where $\|\cdot\|_\infty$ stands for the supremum norm on c_0. Observe that formula (2.1) above gives also the norm on ℓ_∞ that is bidual to $|||\cdot|||$. In order to show that there are points in $S_{(c_0,|||\cdot|||)}$ that are not of w^*-w-continuity of $B_{(\ell_\infty,|||\cdot|||)}$, put $(\ell_\infty \ni)\ x_n^{**} := e_1 + (e_n + e_{n+1} + \ldots)$ for $n \in \mathbb{N}$, where e_n is the n-th canonical basis vector of c_0. Observe that $|||x_n^{**}||| \to 1 + 1/\sqrt{2}$. Clearly, $x_n^{**} \to e_1$ in the w^*-topology of ℓ_∞, and $|||e_1||| = 1 + 1/\sqrt{2}$.

Let us define now a particular vector in $\ell_\infty^* \setminus \ell_1$. The sequence $\{e_n^*\}$, where e_n^* denotes the n-th canonical vector basis of ℓ_1, is clearly bounded, w^*-null, and not w-null (consider the vector $(1, 1, \ldots)$ in ℓ_∞), so it has a $w(\ell_\infty^*, \ell_\infty)$-cluster point $x_0^{***} \in \ell_\infty^* \setminus \ell_1$. Let us compute now the action of x_n^{**} on x_0^{***} for each $n \in \mathbb{N}$. Fixing $n \in \mathbb{N}$, the sequence $\{\langle x_n^{**}, e_m^*\rangle\}_{m=1}^\infty$ is eventually 1, so $\langle x_n^{**}, x_0^{***}\rangle = 1$ for all $n \in \mathbb{N}$. However, the sequence $\{\langle e_1, e_m^*\rangle\}_{m=1}^\infty$ is eventually 0, so $\langle e_1, x_0^{***}\rangle = 0$. It follows that $\{x_n^{**}/|||x_n^{**}|||\}$ in $B_{(\ell_\infty,|||\cdot|||)}$ is w^*-convergent to $e_1/|||e_1||| \in c_0$, but not $w(\ell_\infty, \ell_\infty^*)$-convergent. Ⓡ

3. Some applications to optimization

The rest of the paper deals with some optimization results. Let us collect in the next two definitions the relevant concepts related to approximation. If X is a Banach space, C is a nonempty subset of X, and $x \in X$, then $d(x, C)$ denotes the distance from x to C.

Definition 3.1 ([ES61])**.** A subset C of a Banach space X is said to be *proximinal* if $P_C(x) = \{z \in C : \|x - z\| = d(x, C)\} \neq \emptyset$ for every $x \in X$.

Definition 3.2. A nonempty subset C of a Banach space X is said to be *approximatively compact* if for any $\{y_n\}_{n=1}^\infty \subset C$ and any $x \in X$ satisfying $\|x - y_n\| \to d(x, C)$, there exists a subsequence of $\{y_n\}_{n=1}^\infty$ converging to an element in C. X is called *approximatively compact* if every nonempty closed convex subset of X is approximatively compact.

The following results relates the concepts of approximate compactness and nearly strong convexity.

Theorem 3.3 ([ZS09, FW01, BLLN08])**.** *Let X be a Banach space. Then, the following are equivalent:*

(i) *X is nearly strongly convex.*
(ii) *Every proximinal closed convex subset in X is approximatively compact.*
(iii) *Every proximinal closed subspace in X is approximatively compact.*
(iv) *Every proximinal hyperplane in X is approximatively compact.*

In [HKL06] it was proved the equivalence between (i) and (ii) in Theorem 3.4 below, that is the reflexive version of Theorem 3.3. Note that the class of spaces verifying Theorem 3.4 coincides with the class of spaces having the drop property, as it has been mentioned above.

Theorem 3.4. *Let X be a Banach space. Then, the following are equivalent.*
(i) *X is approximatively compact.*
(ii) *X is reflexive and has the property (H).*
(iii) *X is reflexive and has the Kadec property.*
(iv) *X is reflexive and nearly strongly convex.*

Proof. As we mentioned above, the equivalence (i)$\Leftrightarrow$(ii) is in [HKL06]. That (ii) implies (iii) follows from the trivial fact that every reflexive space is nearly very convex and by Proposition 2.9, or alternatively from the fact that no reflexive space contains an isomorphic copy of ℓ_1 and then by using the result of properties (H) and Kadec mentioned immediately before Proposition 2.9. (iii)$\Rightarrow$(iv) is again in Proposition 2.9. In a reflexive space, obviously every nonempty closed convex subset of X is proximinal; then (iv)$\Rightarrow$(i) follows from Theorem 3.3. ☐

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