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The periodic points of ε-contractive maps in fuzzy metric spaces

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Abstract

In this paper, we introduce the notion of ε-contractive maps in fuzzy metric space $(X, M, *)$ and study the periodicities of ε-contractive maps. We show that if $(X, M, *)$ is compact and $f : X \longrightarrow X$ is ε-contractive, then $P(f) = \cap_{n=1}^{\infty} f^n(X)$ and each connected component of X contains at most one periodic point of f, where $P(f)$ is the set of periodic points of f. Furthermore, we present two examples to illustrate the applicability of the obtained results.

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1. Introduction

The notion of fuzzy metric spaces was introduced by Kramosil and Michalek [10] and later was modified by George and Veeramani [3] in order to obtain a Hausdorff topology in a fuzzy metric space. Recently there has been a great interest in discussing some properties on discrete dynamical systems in fuzzy

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metric spaces. Many authors introduced and investigated the different types of fuzzy contractive maps and obtained a lot of fixed point theorems (see [1, 2, 5, 6, 8, 11, 12, 14, 15, 16, 17, 18, 19]). Until now, there are very little of works that investigates the periodic points of discrete dynamical systems in fuzzy metric spaces because it is much more difficult to find various conditions to obtain the periodic points of discrete dynamical systems.

In the present paper, we introduce the notion of ε-contractive map in fuzzy metric space $(X, M, *)$ and obtain the following results: (1) If $(X, M, *)$ is compact and $f : X \longrightarrow X$ is ε-contractive, then $P(f) = \cap_{n=1}^{\infty} f^n(X)$, where $P(f)$ is the set of periodic points of f. (2) If $(X, M, *)$ is compact and $f : X \longrightarrow X$ is ε-contractive, then each connected component of X contains at most one periodic point of f. Furthermore, if $(X, M, *)$ is also connected, then f has at most one fixed point.

2. Preliminaries

Throughout the paper, let $\mathbf{N}$ be the set of all positive integers. Firstly, we recall the basic definitions and the properties about fuzzy metric spaces.

Definition 2.1 (Schweizer and Sklar [13])**.** A binary operation $T : [0,1]^2 \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions (i)-(v):

(i) $T(a,b) = T(b,a)$;
(ii) $T(a,b) \leq T(c,d)$ for $a \leq c$ and $b \leq d$;
(iii) $T(T(a,b),c) = T(a,T(b,c))$;
(iv) $T(a,0) = 0$ and $T(a,1) = a$;
(v) T is continuous on $[0,1]^2$,

where $a, b, c, d \in [0,1]$.

For any $a, b \in [0,1]$, we will use the notation $a*b$ instead of $T(a,b)$. $T(a,b) = \min\{a,b\}$, $T(a,b) = ab$ and $T(a,b) = \max\{a+b-1, 0\}$ are the most commonly used t-norms.

In the present paper, we also use the following definition of a fuzzy metric space.

Definition 2.2 (George and Veeramani [3])**.** A triple $(X, M, *)$ is called a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm and M is a map defined on $X^2 \times (0, +\infty)$ into $[0,1]$ satisfying the following conditions (i)-(v) for any $x, y, z \in X$ and $s, t \in (0, +\infty)$:

(i) $M(x,y,t) > 0$;
(ii) $M(x,y,t) = 1$ (for any $t > 0$) $\Longleftrightarrow x = y$;
(iii) $M(x,y,t) = M(y,x,t)$;
(iv) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$;
(v) $M_{xy} : (0,+\infty) \longrightarrow [0,1]$ is a continuous (where $M_{xy}(t) = M(x,y,t)$).

Remark 2.3 (Grabiec [4]). M_{xy} is non-decreasing for all $x, y \in X$.

If $(X, M, *)$ is a fuzzy metric space, then $(M, *)$, or simply M, is called a fuzzy metric on X. In [3], George and Veeramani showed that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0,1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for any $x \in X, \varepsilon \in (0,1)$ and $t > 0$.

Definition 2.4 (Gregori and Sapena [7])**.** Let $(X, M, *)$ be a fuzzy metric space. A sequence of points $x_n \in X$ is called to converge to x (denoted by $x_n \longrightarrow x$) $\Longleftrightarrow \lim_{n\longrightarrow+\infty} M(x_n, x, t) = 1$ (for any $t > 0$), i.e. for each $\delta \in (0,1)$ and $t > 0$, there exists $N \in \mathbf{N}$ such that $M(x_n, x, t) > 1 - \delta$ for all $n \geq N$.

Definition 2.5. (1) A fuzzy metric space $(X, M, *)$ is said to be compact if each sequence of points in X has a convergent subsequence. A subset A of X is said to be compact if A as a fuzzy metric subspace is compact.

(2) A fuzzy metric space $(X, M, *)$ is said to be connected if there exist two nonempty closed sets U, V of X with $U \cap V = \varnothing$ such that $X = U \cup V$. A subset A of X is said to be connected if A as a fuzzy metric subspace is connected.

Definition 2.6. Let $(X, M, *)$ be a fuzzy metric space and $\varepsilon \in (0,1)$. A map $f : X \longrightarrow X$ is said to be ε-contractive if $M(x, y, t) > 1 - \varepsilon$ for any $x, y \in X$ with $x \neq y$ and $t > 0$, then $M(f(x), f(y), t) > M(x, y, t)$.

Denote by $C(X, \varepsilon)$ the set of all ε-contractive maps in X.

Remark 2.7.

(1) Let $(X, M, *)$ be compact and A be a subset of X. Then A is compact $\Longleftrightarrow A$ is closed (see [9]).
(2) If $f \in C(X, \varepsilon)$, then $f^n \in C(X, \varepsilon)$ for any $n \in \mathbf{N}$.
(3) If A is a connected component of X, then A is closed.
(4) If $f \in C(X, \varepsilon)$ and A is a closed subset of X, then $f^{-1}(A)$ is also closed. Indeed, let a sequence of points $x_n \in f^{-1}(A)$ with $x_n \longrightarrow x$. Then there exists $N \in \mathbf{N}$ such that $1 \geq M(f(x_n), f(x), t) \geq M(x_n, x, t) > 1 - \varepsilon$ for any $n \geq N$, which implies $1 \geq \lim_{n\longrightarrow\infty} M(f(x_n), f(x), t) \geq \lim_{n\longrightarrow\infty} M(x_n, x, t) = 1$ and $f(x_n) \longrightarrow f(x)$. Since $f(x_n) \in A$ and A is closed, we see $f(x) \in A$. Thus $x \in f^{-1}(A)$, which implies that $f^{-1}(A)$ is closed.
(5) By (4) we see that if $f \in C(X, \varepsilon)$ and A is a connected subset of X, then $f(A)$ is also connected.

Let $(X, M, *)$ be a fuzzy metric space and $f : X \longrightarrow X$. Write $f^0(x) = x$ and $f^n = f \circ f^{n-1}$ for any $x \in X$ and $n \in \mathbf{N}$. We write

$$P(f) \quad = \quad \{x : \text{there exists some } n \in \mathbf{N} \text{ such that } f^n(x) = x\}.$$

For any $x \in X$, we write

$$\omega(x, f) \quad = \quad \{y : \text{ there exists a sequence of positive integers } n_1 < n_2 < \cdots \text{ such that } \lim_{k\longrightarrow\infty} M(f^{n_k}(x), y, t) = 1 \text{ for any } t > 0\}.$$

$P(f)$ is called the set of periodic points of f. $\omega(x, f)$ is called the set of ω-limit points of x under f. If $f(x) = x$, then x is called the fixed point of f.

3. Main results

In this section, we study the set of periodic points of ε-contractive maps in fuzzy metric spaces.

Lemma 3.1. *Let $(X, M, *)$ be a compact fuzzy metric space and $f \in C(X, \varepsilon)$. Then $f(\omega(x, f)) = \omega(x, f)$ for any $x \in X$ and $\omega(x, f) = \omega(y, f)$ for any $y \in \omega(x, f)$.*

Proof. Let $u \in \omega(x, f)$. Then there exists a sequence of positive integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that $\lim_{k \longrightarrow \infty} M(f^{n_k}(x), u, t) = 1$ for any $t > 0$. Thus for any $0 < \delta < \varepsilon$ and $t > 0$, there exists $N \in \mathbf{N}$ such that $M(f^{n_k}(x), u, t) > 1 - \delta$ for all $n \geq N$. From which it follows that

$$M(f^{n_k+1}(x), f(u), t) \geq M(f^{n_k}(x), u, t) > 1 - \delta.$$

Therefore $f(u) \in \omega(x, f)$ and $f(\omega(x, f)) \subset \omega(x, f)$.

On the other hand, by taking subsequence we may assume that $f^{n_k-1}(x) \longrightarrow v$ for some $v \in \omega(x, f)$ since $(X, M, *)$ is a compact. Then for any $0 < \delta < \varepsilon$ and $t > 0$, there exist $0 < \delta_1 < \delta$ and $N \in \mathbf{N}$ such that $(1-\delta_1)*(1-\delta_1) > 1-\delta$ and $M(f^{n_k}(x), u, t/2) > 1 - \delta_1$ and $M(f^{n_k-1}(x), v, t/2) > 1 - \delta_1$ for all $n \geq N$. Note that $f \in C(X, \varepsilon)$, we see that when $n \geq N$, one has

$$\begin{aligned} M(u, f(v), t) &\geq M(u, f^{n_k}(x), \frac{t}{2}) * M(f^{n_k}(x), f(v), \frac{t}{2}) \\ &\geq M(u, f^{n_k}(x), \frac{t}{2}) * M(f^{n_k-1}(x), v, \frac{t}{2}) \\ &\geq (1 - \delta_1) * (1 - \delta_1) > 1 - \delta. \end{aligned}$$

Thus we obtain $M(u, f(v), t) = 1$ for any $t > 0$ and $u = f(v) \in f(\omega(x, f))$, which implies $\omega(x, f) \subset f(\omega(x, f))$.

Now we show that $\omega(y, f) \subset \omega(x, f)$ for any $y \in \omega(x, f)$. Let $z \in \omega(y, f)$. Then there exist two sequences of positive integers $k_1 < k_2 < \cdots$ and $r_1 < r_2 < \cdots$ such that $f^{k_n}(x) \longrightarrow y$ and $f^{r_n}(y) \longrightarrow z$. Thus for any $0 < \delta < \varepsilon$ and $t > 0$, there exist $0 < \delta_1 < \delta$ and $N \in \mathbf{N}$ such that $(1 - \delta_1) * (1 - \delta_1) > 1 - \delta$ and $M(y, f^{k_n}(x), t/2) > 1 - \delta_1$ and $M(z, f^{r_n}(y), t/2) > 1 - \delta_1$ for any $n > N$. Note that $f \in C(X, \varepsilon)$, we see that when $n > N$, one has

$$\begin{aligned} M(z, f^{k_n+r_n}(x), t) &\geq M(z, f^{r_n}(y), \frac{t}{2}) * M(f^{r_n}(y), f^{k_n+r_n}(x), \frac{t}{2}) \\ &\geq M(z, f^{r_n}(y), \frac{t}{2}) * M(y, f^{k_n}(x), \frac{t}{2}) \\ &\geq (1 - \delta_1) * (1 - \delta_1) > 1 - \delta. \end{aligned}$$

Therefore we obtain $z \in \omega(x, f)$, which implies $\omega(y, f) \subset \omega(x, f)$.

Finally we show that $\omega(y, f) = \omega(x, f)$ for any $y \in \omega(x, f)$. Let $z \in \omega(x, f)$. Then there exist two sequences of positive integers $k_1 < k_2 < \cdots$ and $r_1 < r_2 < \cdots$ such that $f^{k_n}(x) \longrightarrow y$ and $f^{r_n}(x) \longrightarrow z$ with $r_n - k_n > n$. Thus for any $0 < \delta < \varepsilon$ and $t > 0$, there exist $0 < \delta_1 < \delta$ and $N \in \mathbf{N}$ such that $(1-\delta_1)*(1-\delta_1) > 1-\delta$ and $M(y, f^{k_n}(x), t/2) > 1-\delta_1$ and $M(z, f^{r_n}(x), t/2) >$

$1-\delta_1$ for any $n > N$. Note that $f \in C(X, \varepsilon)$, we see that when $n > N$, one has

$$\begin{aligned} M(z, f^{r_n - k_n}(y), t) &\geq M(z, f^{r_n}(x), \frac{t}{2}) * M(f^{r_n}(x), f^{r_n-k_n}(y), \frac{t}{2}) \\ &\geq M(z, f^{r_n}(x), \frac{t}{2}) * M(y, f^{k_n}(x), \frac{t}{2}) \\ &\geq (1-\delta_1) * (1-\delta_1) > 1-\delta. \end{aligned}$$

Therefore we obtain $z \in \omega(y, f)$, which implies $\omega(y, f) = \omega(x, f)$. The proof is completed. □

Theorem 3.2. *Let $(X, M, *)$ be a compact fuzzy metric space and $f \in C(X, \varepsilon)$. Then $P(f) = \cap_{n=1}^{\infty} f^n(X)$.*

Proof. Let $Y = \cap_{n=1}^{\infty} f^n(X)$. Then $f(Y) = Y$. It is easy to see that $P(f) \subset Y$. In the following we show that $Y \subset P(f)$.

Let $x_0 \in Y$. Then there exists a sequence of points $x_n \in Y$ such that $f(x_n) = x_{n-1}$ for every $n \in \mathbf{N}$. Since X is compact, there exist a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and a point $u \in Y$ satisfying $x_{n_k} \longrightarrow u$. Thus, for any $0 < \delta < \varepsilon$ and $t > 0$, there exits $N \in \mathbf{N}$ such that $M(x_{n_k}, u, t) \geq 1-\delta$ for all $k \geq N$. Note that $f \in C(X, \varepsilon)$, we have

$$M(x_0, f^{n_k}(u), t) = M(f^{n_k}(x_{n_k}), f^{n_k}(u), t) > \cdots > M(x_{n_k}, u, t) \geq 1-\delta$$

for all $k \geq N$. Therefore $x_0 \in \omega(u, f)$, which with Lemma 3.1 implies $x_0 \in \omega(u, f) = \omega(x_0, f)$.

By $x_0 \in \omega(x_0, f)$ we see that there exist $1 < n_1 < n_2 < \cdots < n_k < \cdots$ such that $\lim_{k \longrightarrow \infty} M(f^{n_k}(x_0), x_0, t) = 1$. Choose n_r with $M(f^{n_r}(x_0), x_0, t) > 1-\varepsilon$.

In the following we show $f^{n_r}(x_0) = x_0$, which implies $x_0 \in P(f)$. Indeed, if $f^{n_r}(x_0) \neq x_0$, then $M(f^{n_r}(x_0), x_0, t_0) < 1$ for some $t_0 > 0$. Since $M(f^{n_r}(x_0), x_0, t)$ is continuous and non-decreasing in $(0, +\infty)$ and $f \in C(X, \varepsilon)$, we see that there exists $\gamma > 0$ such that $M(f^{n_r}(x_0), x_0, t_0) \leq M(f^{n_r}(x_0), x_0, t_0 + \gamma) < M(f^{n_r+1}(x_0), f(x_0), t_0)$. Thus

$$\begin{aligned} & M(f^{n_r}(x_0), x_0, t_0+\gamma) \\ \geq\; & M(f^{n_r}(x_0), f^{n_k+n_r}(x_0), \frac{\gamma}{2}) * M(f^{n_k+n_r}(x_0), f^{n_k}(x_0), t_0) * M(f^{n_k}(x_0), x_0, \frac{\gamma}{2}) \\ \geq\; & M(x_0, f^{n_k}(x_0), \frac{\gamma}{2}) * M(f^{1+n_r}(x_0), f(x_0), t_0) * M(f^{n_k}(x_0), x_0, \frac{\gamma}{2}). \end{aligned}$$

Taking the limit on both sides in the above as $k \longrightarrow \infty$ we obtain

$$M(f^{n_r}(x_0), x_0, t_0+\gamma) \geq 1 * M(f^{1+n_r}(x_0), f(x_0), t) * 1 = M(f^{1+n_r}(x_0), f(x_0), t_0).$$

This leads a contradiction. The proof is completed. □

Remark 3.3. It is easy to see that if $(X, M, *)$ is a compact fuzzy metric space and $f \in C(X, \varepsilon)$, then $\cup_{x \in X} \omega(x, f) = P(f)$ since $f(\omega(x, f)) = \omega(x, f)$ and $P(f) \subset \cup_{x \in X} \omega(x, f) \subset \cap_{n=1}^{\infty} f^n(X) = P(f)$.

Lemma 3.4. *Let $(X, M, *)$ be a fuzzy metric space, $u \in X$ and $f \in C(X, \varepsilon)$. Write $J(u, f) = \{x \in X : u \in \omega(x, f)\}$. Then $J(u, f)$ is a closed subset of X. Furthermore, if $(X, M, *)$ is compact, then $J(u, f)$ is a open subset of X.*

Proof. Let $x_1, x_2, \cdots, x_n, \cdots \in J(u, f)$ and $x \in X$ such that $\lim_{n\longrightarrow\infty} M(x_n, x, t) = 1$ for any $t > 0$. For any $n \in \mathbf{N}$, there exist $1 < k_{1n} < k_{2n} < \cdots < k_{rn} < \cdots$ such that $\lim_{r\longrightarrow\infty} M(f^{k_{rn}}(x_n), u, t) = 1$ for any $t > 0$. Then for any $0 < \delta < \varepsilon$ and $t > 0$, we can choose $0 < \delta_1 < \delta$ with $(1 - \delta_1) * (1 - \delta_1) > 1 - \delta$ and $N \in \mathbf{N}$ such that (by taking subsequence) $M(f^{k_{nn}}(x_n), u, t/2) \geq 1 - \delta_1$ and $M(x_n, x, t/2) > 1 - \delta_1$ for any $n \geq N$. Then

$$\begin{aligned} M(f^{k_{nn}}(x), u, t) &\geq M(f^{k_{nn}}(x), f^{k_{nn}}(x_n), \frac{t}{2}) * M(f^{k_{nn}}(x_n), u, \frac{t}{2}) \\ &\geq M(x, x_n, \frac{t}{2}) * M(f^{k_{nn}}(x_n), u, \frac{t}{2}) \\ &\geq (1 - \delta_1) * (1 - \delta_1) > 1 - \delta. \end{aligned}$$

Thus $u \in \omega(x, f)$ and $J(u, f)$ is closed.

Now we prove the second part of the lemma. Assume that X is compact. Then for any $0 < \delta < \varepsilon$, there exists $0 < \delta_1 < \delta < \varepsilon$ such that $(1-\delta_1)*(1-\delta_1) > 1 - \delta$.

Let $x \in J(u, f)$. We prove that $B(x, \delta_1, t/2) \subset J(u, f)$. Let $y \in B(x, \delta_1, t/2)$. Since $x \in J(u, f)$, there exist $n_1 < n_2 < \cdots < n_k < \cdots$ such that $\lim_{k\longrightarrow\infty} M(f^{n_k}(x), u, t/2) = 1$ for any $t > 0$. Thus for any $t > 0$, there exists $N \in \mathbf{N}$ such that $M(f^{n_k}(x), u, t/2) \geq 1 - \delta_1$ for any $k > N$. Therefore we have that for $k > N$ and $t > 0$,

$$\begin{aligned} M(f^{n_k}(y), u, t) &\geq M(f^{n_k}(y), f^{n_k}(x), \frac{t}{2}) * M(f^{n_k}(x), u, \frac{t}{2}) \\ &\geq M(y, x, \frac{t}{2}) * M(f^{n_k}(x), u, \frac{t}{2}) \\ &\geq (1 - \delta_1) * (1 - \delta_1) > 1 - \delta. \end{aligned}$$

By taking subsequence we may assume that $f^{n_k}(y) \longrightarrow v$. Taking the limit on both sides in the above as $k \longrightarrow \infty$ we obtain

$$M(v, u, t) \geq 1 - \delta.$$

If $J(u, f) = \varnothing$, then $J(u, f)$ is open.

If $J(u, f) \neq \varnothing$, then by Remark 3.3 we see that $v, u \in P(f)$. Let m and n be the periods of u and v, respectively. Note that $f \in C(X, \varepsilon)$, we have

$$M(u, v, t) = M(f^{mn}(u), f^{mn}(v), t) > M(u, v, t),$$

which is impossible unless $u = v$. Hence $u \in \omega(y, f)$ and $B(x, \delta_1, t/2) \subset J(u, f)$. Since x is an arbitrarily chosen point of $J(u, f)$, we see that $J(u, f)$ is an open subset of X. The proof is completed. □

Theorem 3.5. *If $(X, M, *)$ is a compact fuzzy metric space and $f \in C(X, \varepsilon)$, then each connected component of X contains at most one periodic point of f. Furthermore, if $(X, M, *)$ is also connected, then f has at most one fixed point.*

Proof. Let $p \in P(f)$ and $Y(p)$ be the connected component of X containing p and r be the period of p. Write $h = f^r$. By Remark 2.7 we see that $h \in C(X, \varepsilon)$ and $h(Y(p))$ is connected and $Y(p)$ is a compact. Since $p \in Y(p) \cap h(Y(p))$, we have $h(Y(p)) \subset Y(p)$. Replace X and f of Lemma 3.4 by $Y(p)$ and h, respectively, and write $J(p,h) \subset Y(p)$ to be as in Lemma 3.4. By Lemma 3.4 we see that $J(p,h)$ is both closed and open in $Y(p)$. Also, since $p \in \omega(p,h)$ and $Y(p)$ is connected, we have that $J(p,h) = Y(p)$

In the following, we show $P(f) \cap Y(p) = \{p\}$. Indeed, if $P(f) \cap Y(p) \neq \{p\}$ and $q \in P(f) \cap Y(p) - \{p\}$, then $q \in P(h) \cap Y(p) = P(h) \cap J(p,h)$ and $p \in \omega(q,h)$. Let $n_1 < n_2 < \cdots < n_k < \cdots$ such that $h^{n_k}(q) \longrightarrow p$. Then for any $0 < \delta < \varepsilon$ and $t > 0$, there exists $N \in \mathbf{N}$ such that $M(h^{n_k}(q), p, t) > 1 - \delta$ for any $k \geq N$. Let s be the period of q. we have

$$M(p, h^{n_k}(q), t) = M(h^{sn_k}(p), h^{(s+1)n_k}(q), t) > M(p, h^{n_k}(q), t).$$

This will lead a contradiction. The proof is completed. □

In the following we present two examples to illustrate the applicability of the obtained results.

Example 3.6. Let $X = [0, 1/3] \cup [2/3, 1] \subset (-\infty, +\infty)$. Define $s * t = st$ for any $s, t \in [0,1]$, and let $M : X \times X \times (0, \infty) \longrightarrow [0,1]$ by, for any $x, y \in X$ and $t > 0$,

$$M(x,y,t) = \begin{cases} \frac{1}{1+|x-y|}, & \text{if } t \geq 1, \\ \frac{t}{t+|x-y|}, & \text{if } 0 < t < 1. \end{cases}$$

Then $(X, M, *)$ is a compact fuzzy metric space. Take $k \in (0,1)$ and define $f : X \longrightarrow X$ by, for any $x \in X$,

$$f(x) = \begin{cases} kx + \frac{2}{3}, & \text{if } x \in [0, \frac{1}{3}], \\ k(1-x), & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

We claim that $f \in C(X, 1/4)$. Indeed, for any $x, y \in X$ and $t > 0$ with $M(x,y,t) > 1 - 1/4$, we have $|x - y| < 1/3$. Then $x, y \in [0, 1/3]$ or $x, y \in [2/3, 1]$, from which it follows that

$$|f(x) - f(x)| = k|x - y| < |x - y|.$$

Thus $M(f(x), f(y), t) > M(x,y,t)$ and $f \in C(X, 1/4)$. By Theorem 3.2 and Theorem 3.5 we see that $P(f) = \cap_{n=1}^{\infty} f^n(X)$ contains at most 2 points, and $[0, 1/3] \cap P(f)$ contains at most one point, and $[2/3, 1] \cap P(f)$ contains at most one point. In fact, we have $P(f) = \{k/3(k^2+1), (3k^2+2)/3(k^2+1)\}$ with $f(k/3(k^2+1)) = (3k^2+2)/3(k^2+1)$ and $f((3k^2+2)/3(k^2+1)) = k/3(k^2+1)$.

Example 3.7. Let $X = [0, 1/3] \cup [2/3, 1] \subset (-\infty, +\infty)$. Define $s * t = st$ for any $s, t \in [0,1]$, and let $M : X \times X \times (0, \infty) \longrightarrow [0,1]$ by, for any $x, y \in X$ and $t > 0$,

$$M(x,y,t) = \frac{t}{t + |x - y|}.$$

Then $(X, M, *)$ is a compact fuzzy metric space. Take $k \in (0,1)$ and define $f : X \longrightarrow X$ by, for any $x \in X$,

$$f(x) = \begin{cases} kx + \frac{2}{3}, & \text{if } x \in [0, \frac{1}{3}], \\ k(1-x) + \frac{2}{3}, & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

We claim that $f \in C(X, 1/4)$. Indeed, if $x, y \in [0, 1/3]$ or $x, y \in [2/3, 1]$, then $|f(x) - f(y)| = k|x - y| < |x - y|$, which follows that

$$M(f(x), f(y), t) > M(x, y, t).$$

If $x \in [0, 1/3]$ and $y \in [2/3, 1]$, then $|f(x) - f(y)| = 1/3 < |x - y|$, which also follow that

$$M(f(x), f(y), t) > M(x, y, t).$$

By Theorem 3.2 and Theorem 3.5 we see that $P(f) = \cap_{n=1}^{\infty} f^n(X)$ contains at most 2 points, and $[0, 1/3] \cap P(f)$ contains at most one point, and $[2/3, 1] \cap P(f)$ contains at most one point. In fact, we have $P(f) = \{3k+2/(3k+3)\} \subset [2/3, 1]$.

4. Conclusion

In this paper, we introduce the notion of ε-contractive maps in a fuzzy metric space, and study the periodicities of ε-contractive maps, and obtain the following result: If f is a ε-contractive map in compact fuzzy metric space $(X, M, *)$, then $P(f) = \cap_{n=1}^{\infty} f^n(X)$ and each connected component of X contains at most one periodic point of f. Furthermore, we present two examples to illustrate the applicability of the obtained results.

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