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Disconnection in the Alexandroff duplicate

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Abstract

It was demonstrated in [2] that the Alexandroff duplicate of the Čech-Stone compactification of the naturals is not extremally disconnected. The question was raised as to whether the Alexandroff duplicate of a non-discrete extremally disconnected space can ever be extremally disconnected. We answer this question in the affirmative; an example of van Douwen is significant. In a slightly different direction we also characterize when the Alexandroff duplicate of a space is a P-space as well as when it is an almost P-space.

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1. Introduction

In 1929, Alexandroff [1] constructed a non-metrizable first countable compact Hausdorff space. Alexandroff's construction was generalized by Engelking [6] in 1966, and since then the construction has been a constant force in the supply of counter-examples to topological questions. Our aim here is to consider some well-known peculiar topological properties and classify when the Alexandroff duplicate of a space X has such properties. Throughout, we assume that X **is a Tychonoff space**, that is, completely regular and Hausdorff. For such

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a space X, $C(X)$ denotes the collection of real-valued continuous functions on X. Since X is Tychonoff, it is also the case that $A(X)$ is Tychonoff (see [5]).

Definition 1.1. Let X be a topological space. The *Alexandroff duplicate of* X is the space created by taking two (disjoint) copies of X, say $A(X) = X \cup X'$. For any subset $T \subseteq X$, we let T' be its copy in X'. Then we define basic open sets of $A(X)$ to be the singletons of X' as well as any subset of the form $O \cup O' \smallsetminus \{x'\}$ for some open subset $O \subseteq X$ and $x \in X$. (The space $A(X)$ is also known as the Alexandroff double.)

Observe that X' is open in $A(X)$, so X is a closed subspace of $A(X)$. The space $A(X)$ always has a dense set of isolated points. Also, X has no isolated points if and only if X' is dense in $A(X)$. Moreover, every $f \in C(X)$ has a continuous extension to all of $A(X)$, that is, X is C-embedded in $A(X)$. Simply extend f to X' by assigning to each $x' \in X'$ the value of its original $x \in X$.

Since there will be a lot of movement between the spaces X and $A(X)$, we use $\mathrm{cl}\,T$ to denote the closure of T in X and $\mathrm{Cl}\,Z$ to denote the closure of a subset $Z \subseteq A(X)$. Given a subset T of X, we use $\mathrm{acc}(T)$ to denote the derived set of T, that is, the set of accumulation points of T (i.e. points $x \in X$ such that $x \in \mathrm{cl}\,(T \smallsetminus \{x\})$). If needed, we shall add a subscript, as in $\mathrm{acc}_X(T)$. Recall that T is closed if and only if $\mathrm{acc}(T) \subseteq T$, and furthermore, for any set T, $\mathrm{cl}\,T = T \cup \mathrm{acc}(T)$. The condition $\mathrm{acc}(T) = \varnothing$ is equivalent to saying that T is closed and discrete.

Lemma 1.2. *Let X be a space. For any subset $T \subseteq X$. $\mathrm{Cl}\,T' = T' \cup \mathrm{acc}_X(T)$.*

Subsets of X which are closed and discrete are of utmost importance in the study of clopen subsets of $A(X)$. For the sake of convenience, we call a set that is both closed and discrete *disclosed*.

In this article we shall use the term *σ-disclosed* to describe a countable union of disclosed subsets. It is straightforward to show that a σ-disclosed subset is a countable union of disclosed subsets which are pairwise disjoint. Clearly, every countable set is σ-disclosed, and hence it is obvious that σ-disclosed sets need not be closed. In a Lindelöf space, the σ-disclosed subsets are precisely the countable subsets.

In the second section, we answer the question raised in [2] by characterizing those X for which $A(X)$ is extremally disconnected. An example of van Douwen [8] is used to show that there are crowded such X, see Example 2.5. In the last section we consider other peculiar properties such as basically disconnected spaces, (weak) P-spaces, and almost P-spaces. We end this section with a characterization of open (and clopen) subsets of $A(X)$, which will be helpful in later sections.

Proposition 1.3. *A subset of $A(X)$ is clopen if and only if it is of the form*

$$K \cup (K' \smallsetminus C') \cup D'$$

where K is a clopen subset of X, $C \subseteq K$, $D \cap K = \varnothing$, and both C and D are disclosed.

Proof. First, let $Y = K \cup (K' \setminus C') \cup D'$ where K is a clopen subset of X, $C \subseteq K$, and $D \cap K = \varnothing$, and both satisfy $\operatorname{acc}_X(C) = \varnothing = \operatorname{acc}_X(D)$. Then D' and C' are clopen in $A(X)$ by Lemma 1.2 since $\operatorname{acc}_X(C) = \varnothing = \operatorname{acc}_X(D)$, and $K \cup K'$ is clopen in $A(X)$ by Lemma 1.2. Hence $K \cup (K' \setminus C') = (K \cup K') \setminus C'$ is clopen in $A(X)$ and so is Y.

Now, let Y be a clopen subset of $A(X)$. If $Y \subseteq X'$, then simply let $K = C = \varnothing$ and $D' = Y$. Otherwise, let $K = Y \cap X$, so K is clopen in X. It follows that $K \cup K'$ is clopen in $A(X)$. Let $D' = (Y \cap X') \setminus K'$, then $D \cap K = \varnothing$ since $D' \cap K' = \varnothing$. We need to show that $\operatorname{acc}_X(D) = \varnothing$, so consider $\operatorname{Cl} D' = D' \cup \operatorname{acc}_X(D)$. If there is some $x \in \operatorname{acc}_X(D)$, then $x \in \operatorname{acc}_{A(X)} Y \cap X$ implies that $x \in K$. But $K \cup K'$ is an open neighborhood of x disjoint from D', a contradiction. Thus $\operatorname{acc}_X(D) = \varnothing$.

Next, let $C' = K' \setminus Y$, and observe that C' is open in $A(X)$ with $C \subseteq K$ since $C' \subseteq K'$. We also want to show that $\operatorname{acc}_X(C) = \varnothing$. Note that $\operatorname{Cl} C' = C' \cup \operatorname{acc}_X(C)$, and suppose there exists $x \in \operatorname{acc}_X(C)$. But $x \notin \operatorname{Cl} C'$ because Y is an open neighborhood of x disjoint from C' in $A(X)$. Therefore $\operatorname{acc}_X(C) = \varnothing$.

Finally, it is clear that

$$\begin{aligned} K \cup (K' \setminus C') \cup D' &= K \cup (K' \setminus (K' \setminus Y)) \cup D' \\ &= K \cup (K' \cap Y) \cup [(Y \cap X') \setminus K'] \\ &= Y. \end{aligned}$$

□

In generalizing from clopen subsets to open subsets the proof of the following should be evident.

Lemma 1.4. *Let $U \subseteq A(X)$ be an open subset of $A(X)$ and set $V = (U \cap X) \cup (U \cap X)'$. Then $V \setminus U = C'$ for some $C \subseteq U \cap X$ which is discrete in X and closed in $U \cap X$. Furthermore, $U \setminus V = D'$ for some $D \subseteq X$ with the property that for each subset $E \subseteq U$ which is closed in $A(X)$, the set $\{x \in D : x' \in E\}$ is disclosed in X.*

2. Extremally Disconnected Spaces

Recall that X is *extremally disconnected* (e.d. for short) if the closure of each open set is clopen. This is equivalent to saying that no point is in the simultaneous closure of two disjoint open subsets. The compact e.d. spaces are precisely the Stone duals of complete boolean algebras. A space X is e.d. if and only if βX, the Čech-Stone compactification of X, is e.d. Locally, a point $x \in X$ is said to be an *extremally disconnected point* if it is not in the closure of two disjoint open sets.

In [2] the authors demonstrate that $A(\beta\mathbb{N})$ is not an extremally disconnected space. The authors raise the question of whether there is a non-discrete space whose Alexandroff duplicate is extremally disconnected. In our attempt to answer this question, we created a condition on a topological space that characterized the situation: Definition 2.1. We then set off to find a non-discrete example. In looking for such a space we happened upon a particular piece of work of van Douwen ([8]). Not only was this article invaluable to our efforts, moreover, we realized that our "new" condition had already been discovered and studied by van Douwen. (The reader should be aware that in [8] it is not assumed that spaces are Tychonoff.)

Definition 2.1. We call a point $x \in X$ *perfectly disconnected in* X if it is not simultaneously an accumulation point of two disjoint subsets. If every point of X is perfectly disconnected, then in line with Definition 1.3 of [8], we say that X is *perfectly disconnected*. Notice that every perfectly disconnected point is extremally disconnected.

Recall that a space is hereditarily extremally disconnected if every subspace is extremally disconnected. We prove that a perfectly disconnected space is hereditarily extremally disconnected.

Proposition 2.2. *Suppose X is perfectly disconnected. Let $T \subseteq X$. If $x \in X$ is an accumulation point of T, then it belongs to the interior of* $\operatorname{cl} T$*. It follows that for all $T \subseteq X$,*

$$\operatorname{cl} T \smallsetminus T \subseteq \operatorname{int}(\operatorname{cl} T).$$

Moreover, a perfectly disconnected space is hereditarily extremally disconnected.

Proof. Suppose X is perfectly disconnected and $x \in \operatorname{acc}(T)$. Notice that $x \notin \operatorname{acc}(X \smallsetminus T)$, so there exists an open neighborhood O of x such that $(O \smallsetminus \{x\}) \cap (X \smallsetminus T) = \varnothing$. It follows that $O \subseteq \operatorname{cl} T$.

As for the claim that a perfectly disconnected space is hereditarily extremally disconnected, consider that in [4, Proposition 3.1], the authors prove that a topological space is hereditarily extremally disconnected if and only if for all $A, B \subseteq X$

$$\operatorname{cl}(A \smallsetminus \operatorname{cl} B) \cap \operatorname{cl}(B \smallsetminus \operatorname{cl} A) = \varnothing.$$

Notice that if $x \in (A \smallsetminus \operatorname{cl} B)$, then $x \notin (B \smallsetminus \operatorname{cl} A)$. Furthermore, it is always the case that

$$(A \smallsetminus \operatorname{cl} B) \cap (B \smallsetminus \operatorname{cl} A) = \varnothing.$$

Therefore, if X is perfectly disconnected, then the above display must be empty. □

Example 2.3. It is not true that every hereditarily extremally disconnected is perfectly disconnected. In [4], the authors construct countable hereditarily extremally disconnected crowded spaces as follows. There are countable dense subspaces of $E([0,1])$, where $(E([0,1]), \pi)$ is the absolute of the unit interval. (The absolute is also known as the Gleason cover of X. In particular, EX denotes the Stone space of the complete Boolean algebra of regular open subsets of X.) These subspaces are dense and hence both extremally disconnected and crowded. For example, if $\{q_n\} = Q = \mathbb{Q} \cap [0,1]$ then by choosing $x_n \in \pi^{-1}(q_n)$ produces such a space. Let Q_1 and Q_2 be disjoint countable dense subspaces of

$[0,1]$ and construct E_1 and E_2 in this manner. Let $X = E_1 \cup E_2$, a countable hereditarily extremally disconnected crowded space. Notice that each point $x \in X$ satisfies $x \in \operatorname{cl} E_1 \setminus \{x\} \cap \operatorname{cl} E_2 \setminus \{x\}$, whence X is not perfectly disconnected.

We are now able to prove the main theorem of the section.

Theorem 2.4. *The following statements are equivalent.*

(1) *The space $A(X)$ is extremally disconnected.*
(2) *X is perfectly disconnected and the set of isolated points of X is clopen.*
(3) *X is the topological sum of a discrete space and a crowded perfectly disconnected space.*

Proof. (1) $\Rightarrow$ (2). Clearly, X must be e.d. if $A(X)$ is e.d. since X is C-embedded in X. The space X must have the even stronger property that no point of X can be in the closure of two disjoint subsets of X since if it were, say $x \in \operatorname{cl} T_1 \cap \operatorname{cl} T_2$ and $T_1 \cap T_2 = \varnothing$, then $x \in \mathrm{Cl}\ T_1' \cap \mathrm{Cl}\ T_2'$ with both T_1' and T_2' open in $A(X)$ and disjoint.

To show that the set $\mathrm{Is}(X)$ of isolated points of X is clopen, let $p \in \operatorname{cl} \mathrm{Is}(X) \setminus \mathrm{Is}(X)$ Then $p \in \mathrm{Cl}\ \mathrm{Is}(X)$. But $p \in \mathrm{Cl}\ X' \cap \mathrm{Cl}\ \mathrm{Is}(X)$ contradicts the fact that $A(X)$ is e.d. Therefore, $\mathrm{Is}(X)$ is clopen in X.

(2) $\Leftrightarrow$ (3). Note that if $\mathrm{Is}(X)$ is clopen, then we can separate X into two clopen subsets: $X \setminus \mathrm{Is}(X)$ and $\mathrm{Is}(X)$. This separation induces a clopen separation of $A(X)$ each of which is e.d. Furthermore, it follows that $\mathrm{Is}(X)$ is discrete and $X \setminus \mathrm{Is}(X)$ is crowded. Thus (2) $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (1). Without loss of generality, we assume that X is a perfectly disconnected crowded space. Suppose $p \in A(X)$ is not an e.d. point of $A(X)$. Clearly, $p \in X$. Then there are disjoint open subsets of $A(X)$, say $\mathcal{O}_1, \mathcal{O}_2$, such that $p \in \mathrm{Cl}\ \mathcal{O}_1 \cap \mathrm{Cl}\ \mathcal{O}_2$. There are collections $\mathcal{U}_1, \mathcal{U}_2$ of basic open subsets of $A(X)$ such that both $\mathcal{U}_1, \mathcal{U}_2$ are pairwise disjoint and $\cup\, \mathcal{U}_i \subseteq \mathcal{O}_i$ is dense. Since $p \in \mathrm{Cl} \cup \mathcal{U}_1 \cap \mathrm{Cl} \cup \mathcal{U}_2$, it follows that without loss of generality we can assume that $\mathcal{O}_i = \cup\, \mathcal{U}_i$.

Now for appropriate index sets I_1 and I_2, basic open sets $O_\alpha \cup O'_\alpha \setminus \{x'_\alpha\}$ and $O_\beta \cup O'_\beta \setminus \{x'_\beta\}$, and disjoint subsets $T_1, T_2 \subseteq X$

$$\mathcal{O}_1 = \bigcup_{\alpha \in I_1} (O_\alpha \cup O'_\alpha \setminus \{x'_\alpha\}) \cup T_1' \text{ and } \mathcal{O}_2 = \bigcup_{\beta \in I_2} (O_\beta \cup O'_\beta \setminus \{x'_\beta\}) \cup T_2'.$$

Let $O_1 = \bigcup_{\alpha \in I_1} O_\alpha$ and $O_2 = \bigcup_{\beta \in I_2} O_\beta$, which are clearly are disjoint open subsets of X.

That $p \in \mathrm{Cl}\ \mathcal{O}_1$ means that either $p \in \operatorname{cl} O_1$ or $p \in \operatorname{cl} T_1$. Similarly, either $p \in \operatorname{cl} O_2$ or $p \in \operatorname{cl} T_2$. Since p is not simultaneously in the closure of two disjoint subsets of X it follows that either $p \in \operatorname{cl} O_1$ and $p \in \operatorname{cl} T_2$, or $p \in \operatorname{cl} O_2$ and $p \in \operatorname{cl} T_1$. We consider the case that $p \in \operatorname{cl} O_1$ and $p \in \operatorname{cl} T_2$. Observe that not only is it entirely possible that $T_2 \cap O_1 \neq \varnothing$, but in fact this must be the case. Let $T_3 = T_2 \cap \{x_\alpha\}_{\alpha \in I_1}$ and observe that $p \in \operatorname{cl} T_3$ since otherwise would lead us to conclude that p is in the closure of two disjoint subsets of X. So, we can replace T_2 with T_3. Since $\mathcal{U}_1$ is pairwise disjoint it follows that T_3 is discrete; this fact is pivotal. Let $J = \{\alpha \in I_1 : x_\alpha \in T_3\}$.

Define $V_\alpha = O_\alpha \smallsetminus \{x_\alpha\}$ and let $V = \cup_{\alpha \in J} V_\alpha$. The crucial piece is that $V \cap T_3 = \varnothing$. Recall that X is crowded, therefore each $x_\alpha \in \operatorname{cl} V_\alpha$. Finally, we can now see that $p \in \operatorname{cl} V$ and $p \in \operatorname{cl} T_3$ even though $V \cap T_3 = \varnothing$, the desired contradiction. □

Example 2.5. E. van Douwen [8] constructed a perfectly disconnected crowded Tychonoff space. (This construction was generalized in [10].) The construction starts with a countable crowded Tychonoff space, say the rationals, and then enlarges the topology to a maximal regular topology; this means maximal amongst the regular topologies. Call this new space X; this space happens to also be Hausdorff as it is larger than a Hausdorff topology. He then takes θ to be the set of points of X which do not lie in the closure of any nowhere dense subset. He shows this set is non-empty, crowded, and perfectly disconnected. Consequently, $A(\theta)$ is extremally disconnected.

E. van Douwen's examples serves as an affirmative answer to the question raised in [2] of whether there is a non-discrete space X for which $A(X)$ is extremally disconnected.

It is worth pointing out that applying the Alexandroff duplicate twice to any non-discrete space never yields an extremally disconnected space.

Proposition 2.6. *For a space X, the following are equivalent.*

(1) $A(A(X))$ *is extremally disconnected.*
(2) $A(X)$ *is perfectly disconnected.*
(3) X *is discrete.*

Proof. If $A(A(X))$ is e.d., then $A(X)$ is perfectly disconnected by Theorem 2.4. If $x \in A(X)$ is not isolated then $x \in X$ is not isolated in X. In this case $x \in \operatorname{cl}_{A(X)}(X \smallsetminus \{x\})$ and $x \in \operatorname{cl}_{A(X)} X'$, contradicting that $A(X)$ is perfectly disconnected. So every point of $A(X)$ and thus every point of X is isolated. □

3. Basically Disconnected Spaces

As the title of this section suggests, we are interested in determining which spaces X have basically disconnected duplicates. Recall that a space X is basically disconnected if the closure of every cozero-set is clopen. A quick recap is in order.

For $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ and $\operatorname{coz}(f) = X \smallsetminus Z(f)$ denote the zero-set and cozero-set of f, respectively. A subset V of X is a *zero-set* if $V = Z(f)$ for some $f \in C(X)$, and similarly for cozero-sets. Each zero-set (cozero-set) is closed (open), and since X is Tychonoff, the collection of zero-sets (cozero-sets) forms a base for the topology of closed (open) sets.

It is well known that X is basically disconnected if and only if βX is basically disconnected. Compact basically disconnected spaces are precisely the Stone duals of σ-complete boolean algebras. Locally, a point $p \in X$ is called a *basically disconnected point* (or a b.d. point for short) if whenever O and C are disjoint open sets and C is a cozero, then $p \notin \operatorname{cl} O \cap \operatorname{cl} C$.

A nice source of basically disconnected spaces are P-spaces. Recall that a Tychonoff space is a *P-space* when every cozero-set is clopen. This is equivalent to saying that the topology of open subsets of X is closed under countable intersections. This equivalent condition lends itself to generalization. A space X is said to be a P_κ-space whenever its topology of open subsets is closed under the intersection of fewer than κ-many open set; P-space and $P_{\aleph_1}$-space mean the same thing. It follows that any P_κ-space has the property that every subset of size less than κ is closed. Spaces with this latter property are called *weak P_κ-spaces.*

It ought to be clear that X is a weak P_κ-space if and only if $A(X)$ is a weak P_κ-space. To see this note that any subspace of a (weak) P_κ-space if again a (weak) P_κ-space. Conversely, any subset of $A(X)$ of size less than κ must be of the form $T \cup S'$ where both T and S are of size less than κ. We leave it to the interested reader to check that if X is P_κ-space, then any intersection in $A(X)$ of fewer than κ-many open sets is again open. We summarize as follows.

Proposition 3.1. *The space X is a (weak) P_κ-space if and only if $A(X)$ is a (weak) P_κ-space.*

We turn to characterizing when $A(X)$ is basically disconnected. To aid us, the next lemma is useful in that it describes what a cozero-set of $A(X)$ looks like in terms of a cozero-set of X and certain subsets of X'.

Lemma 3.2. *Let $F \in C(A(X))$. Then there exist some $f \in C(X)$, some discrete subset $C \subseteq \operatorname{coz}(f)$ such that $(\operatorname{cl} C \smallsetminus C) \cap \operatorname{coz}(f) = \varnothing$, and some $D \subseteq Z(f)$ which is σ-disclosed in X, such that*

$$\operatorname{coz}(F) = \operatorname{coz}(f) \cup [\operatorname{coz}(f)' \smallsetminus C'] \cup D'.$$

Proof. We apply Lemma 1.4 to the open set $\operatorname{coz}(F)$. In this case, $\operatorname{coz}(F) \cap X = \operatorname{coz}(f)$ where f is the restriction of F to X; so $f \in C(X)$. Thus, there exists a subset $C \subseteq \operatorname{coz}(f)$ which is discrete and closed in X and a subset $D \subseteq X$ which is disjoint from $\operatorname{coz}(F)$ and has the property that for each subset $E \subseteq \operatorname{coz}(F)$ which is closed in $A(X)$, the set $\{x \in D : x' \in E\}$ is disclosed in X. Furthermore,

$$\operatorname{coz}(F) = \operatorname{coz}(f) \cup [\operatorname{coz}(f)' \smallsetminus C'] \cup D'.$$

Obviously, $D \subseteq Z(f)$. So all that is left to be shown is that D is σ-disclosed in X.

Let $T_n' = [F^{-1}([\frac{1}{n}, \infty)) \cap X'] \smallsetminus \operatorname{coz}(f)'$ for each $n \in \mathbb{N}$, and observe that

$$D' = \bigcup_{n=1}^{\infty} T_n' = [\operatorname{coz}(F) \cap X'] \smallsetminus \operatorname{coz}(f)'.$$

This implies that D is the union of the T_n. We claim $\operatorname{acc}_X T_n = \varnothing$ for all n. Suppose $x \in \operatorname{acc}_X T_n$, then $x \in \operatorname{coz}(f)$ since $x \in \operatorname{Cl} T_n'$ and $F(y') \geq \frac{1}{n}$ for all

$y' \in T'_n$. Moreover, $F(y') \notin \mathrm{coz}(f)'$ for every $y' \in T'_n$, so $f(y) = 0$ implies $x \in Z(f)$, a contradiction. Thus $\mathrm{acc}_X\, T_n = \varnothing$ and so T_n is disclosed in X. ☐

Corollary 3.3. *If $F \in C(A(X))$ with $\mathrm{coz}(F) \subseteq X'$, then the set $T = \{x \in X : x' \in \mathrm{coz}(F)\}$ is a σ-disclosed subset of X. Moreover, X is a zero-set of $A(X)$ if and only if X is σ-disclosed in X. In particular, if the extent of X is countable (e.g. Lindelöf), then X is a zero-set of $A(X)$ if and only if X is countable.*

Remark 3.4. A couple of remarks are in order. First, recall that the *extent* of a space is the supremum of cardinalities of closed discrete subsets. To say that a space has countable extent means that every disclosed subset is countable. Second, the sets $\mathrm{cl}\,\mathrm{coz}(f) \smallsetminus \mathrm{coz}(f)$ play a pivotal role in the description of the cozero-sets of $A(X)$. We shall call the set $\mathrm{cl}\, V \smallsetminus V$ the *residue of* V.

Theorem 3.5. *Let X be Tychonoff space. The following statements are equivalent.*

(1) *$A(X)$ is basically disconnected.*
(2) *The space X satisfies the following three conditions:*
   i) X is basically disconnected,
   ii) the set of accumulation points of any σ-disclosed subset of X is open in X, and
   iii) the residue of any cozero-set of X is discrete.

Proof. (1) ⇒ (2). Suppose that $A(X)$ is basically disconnected. Let $\mathrm{coz}(f)$ be an arbitrary cozero-set of X. Then $\mathrm{coz}(F) = \mathrm{coz}(f) \cup \mathrm{coz}(f)'$ is a cozero-set of $A(X)$ for $F \in C(A(X))$ defined by $F(x) = F(x') = f(x)$. By hypothesis, $\mathrm{Cl}\,[\mathrm{coz}(f) \cup \mathrm{coz}(f)']$ is clopen. Notice that

$$X \cap \mathrm{Cl}\,[\mathrm{coz}(f) \cup \mathrm{coz}(f)'] = \mathrm{cl}\,\mathrm{coz}(f)$$

from which we gather that $\mathrm{cl}\,\mathrm{coz}(f)$ is clopen in X, whence X is basically disconnected.

Next we will show (iii). It is straightforward to check that

$$\begin{aligned} \mathrm{Cl}\,\mathrm{coz}(F) &= \mathrm{Cl}\,[\mathrm{coz}(f) \cup \mathrm{coz}(f)'] \\ &= \mathrm{cl}\,\mathrm{coz}(f) \cup \mathrm{coz}(f)'. \end{aligned}$$

Let $p \in \mathrm{cl}\,\mathrm{coz}(f) \smallsetminus \mathrm{coz}(f)$ and choose a basic open set around p in $\mathrm{Cl}\,\mathrm{coz}(F)$, say p is an element of

$$O \cup O' \smallsetminus \{p'\} \subseteq \mathrm{Cl}\,\mathrm{coz}(F) = \mathrm{cl}\,\mathrm{coz}(f) \cup \mathrm{coz}(f)'.$$

It follows that for all $q \in \mathrm{cl}\,\mathrm{coz}(f) \smallsetminus \mathrm{coz}(f)$ $(q \neq p)$, then $q' \notin \mathrm{coz}(F)$ so that $q \notin O$. Therefore, the residue of $\mathrm{coz}(f)$ is discrete.

Finally, to show (ii), let $\{T_n\}$ be a σ-disclosed subset of X and set $T = \bigcup T_n$. Without loss of generality, we can assume that the T_n are pairwise disjoint. Let $F : A(X) \to \mathbb{R}$ be defined by $F(x') = \frac{1}{n}$ if $x' \in T'_n$ and zero otherwise. Since each T'_n is clopen, it follows that $F \in C(A(X))$ and $\mathrm{coz}(F) = T'$. Hence, by hypothesis, $\mathrm{Cl}\, T' = \mathrm{acc}_X(T) \cup T'$ is clopen. As a result, then $\mathrm{acc}_X(T)$ is clopen in X.

(2) $\Rightarrow$ (1). Suppose that X is basically disconnected, the set of accumulation points of any σ-disclosed subset of X is open in X, and the residue of any cozero-set of X is discrete. Let $F \in C(A(X))$, and without loss of generality we may assume that $F \geq 0$. Let f be the continuous restriction of F to X. By Lemma 3.2, we can write $\operatorname{coz}(F)$ as

$$\operatorname{coz}(F) = \operatorname{coz}(f) \cup (\operatorname{coz}(f)' \smallsetminus C') \cup D'$$

where $C \subseteq \operatorname{coz}(f)$ is discrete in X and closed in $\operatorname{coz}(f)$, and $D \subseteq Z(f)$ is a countable union of disclosed subsets of X. Thus, by hypothesis $\operatorname{acc}_X(D)$ is clopen in X which means $\mathrm{Cl}\ D' = \operatorname{acc}_X(D) \cup D'$ is clopen in $A(X)$. We also have that $\operatorname{acc}_X C = \varnothing$. We will apply Corollary 1.3 to prove that $\mathrm{Cl}\ \operatorname{coz}(F)$ is clopen in $A(X)$.

Let $K = \mathrm{cl}\ \operatorname{coz}(f)$, then K is clopen in X. By hypothesis, $\mathrm{cl}\ \operatorname{coz}(f) \smallsetminus \operatorname{coz}(f)$ is discrete and so it has no accumulation points in X. Thus, $K \cup K' \smallsetminus [C \cup (\mathrm{cl}\ \operatorname{coz}(f) \smallsetminus \operatorname{coz}(f))]'$ is clopen in $A(X)$. Observe that

$$\begin{aligned}
\mathrm{Cl}\ \operatorname{coz}(F) &= \mathrm{Cl}\ \big(\operatorname{coz}(f) \cup (\operatorname{coz}(f)' \smallsetminus C') \cup D'\big) \\
&= \mathrm{Cl}\ \operatorname{coz}(f) \cup \mathrm{Cl}\,(\operatorname{coz}(f)' \smallsetminus C') \cup \mathrm{Cl}\ D' \\
&= \mathrm{cl}\ \operatorname{coz}(f) \cup (\mathrm{Cl}\,(\operatorname{coz}(f)') \smallsetminus C' \cup \mathrm{Cl}\ D' \\
&= \mathrm{cl}\ \operatorname{coz}(f) \cup (\mathrm{Cl}\,(\operatorname{coz}(f)') \smallsetminus C' \cup \mathrm{Cl}\ D' \\
&= K \cup K' \smallsetminus [C \cup (\mathrm{cl}\ \operatorname{coz}(f) \smallsetminus \operatorname{coz}(f))]' \cup \mathrm{Cl}\ D'.
\end{aligned}$$

is hence a clopen set in $A(X)$. Consequently, $A(X)$ is basically disconnected.

□

We now give two examples of basically disconnected spaces that are not P-spaces. The first is an example for which $A(X)$ is basically disconnected and the second is for which $A(X)$ is not basically disconnected.

Example 3.6. As a result of Theorem 0.1 of [9], given a crowded P-space of π-weight ω_1, call it Z, there is a non-principal z-ultrafilter on Z, say $\mathcal{V}$, which is a far point in βZ. Set $Z^* = Z \cup \{\mathcal{V}\}$ equipped with the subspace topology inherited from βZ. Next, let $\mathcal{D}$ be the disjoint union of countably many copies of Z together with one additional point σ. Let Z_n^* denote the n-th copy of Z^* and $z_n = \mathcal{V}$.

Fix a non-principal ultrafilter $\mathcal{U} \in \beta\mathbb{N} - \mathbb{N}$ and then define a topology on $\mathcal{D}$ as follows. Take the usual topology on the disjoint union of copies of Z together with a set O containing σ to be open if

$$\{n \in \mathbb{N} | (O \cap Z_n^*) \cup \{z_n\} \text{ is an open set in } Z_n^*\} \in \mathcal{U}.$$

Now, σ has the property that it is not in the closure of any closed discrete subset of Z_n. This is because for any disclosed subset of Z_n there is an open subset of Z_n^* containing z_n and disjoint from said disclosed subset. Then the disjoint union of copies of the open set together with σ is an open set. Consequently, σ is not in the closure of any disclosed subset of $\mathcal{D}$. It follows that $A(\mathcal{D})$ is basically disconnected. However, σ is not a P-point of $\mathcal{D}$.

Let D be a discrete space of size ω_1 and then let $Z = C_\delta(\alpha D, \mathbb{Z})$, that is, the space of $\mathbb{Z}$-valued continuous functions equipped with the P-ification of the topology of pointwise convergence, then Z is a P-space of weight ω_1.

Example 3.7. Consider the space Σ from [7] defined as follows. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let $\Sigma = \mathbb{N} \cup \{\sigma\}$ (where $\sigma \notin \mathbb{N}$). Points of $\mathbb{N}$ are isolated, and a neighborhood of σ is of the form $U \cup \{\sigma\}$ for $U \in \mathcal{U}$. It is known that this space is extremally disconnected (and thus basically disconnected). In fact, Σ is perfectly disconnected. Note that $\mathbb{N}$ is a σ-disclosed subset of Σ, but the set of accumulation points of $\mathbb{N}$ is $\{\sigma\}$, which is not open in Σ. So by Theorem 3.5, $A(\Sigma)$ is not basically disconnected.

We generalize this to arbitrary discrete spaces.

Proposition 3.8. *Let $X = D \cup \{p\}$ where D is an uncountable discrete space and $p \in \beta D \smallsetminus D$ is a non-principal ultrafilter. The following statements are equivalent.*

(1) *$A(X)$ is P-space.*
(2) *$A(X)$ is basically disconnected.*
(3) *D is of measurable cardinality and p is a measurable ultrafilter.*
(4) *X is a P-space.*

Proof. All we need to note is that the set D is σ-disclosed set in X if and only if p is a P-point of X. □

Proposition 3.9. *Suppose $A(X)$ is basically disconnected and no non-isolated point of X is a G_δ-point of X. Then X is a P-space.*

Proof. Let T be a cozero-set of X, and without loss of generality assume that T is proper. If T is not closed, then we show that each point in $\text{res}(T)$ is a non-isolated G_δ-point of X. Let $p \in \text{res}(T)$, then clearly, p is not isolated. By iii) of Theorem 3.5, there is a clopen subset of the clopen set $\text{cl}\, T$, say K, such that $p \in K$ and $\{p\} = K \cap \text{res}(T)$, which as we point out is an intersection of two zero-sets, whence a zero-set. Hence p is a G_δ-point, contradicting the hypothesis. Consequently, every cozero-set in X is clopen, i.e. X is a P-space. □

Proposition 3.10. *Suppose $A(X)$ is basically disconnected. Then every crowded σ-disclosed subset of X is clopen. In particular, every countable discrete set is closed.*

Proof. Let T be a crowded σ-disclosed subset of X. Since T is crowded, it follows that $T \subseteq \text{acc}(T)$, whence $\text{cl}\, T = T \cup \text{acc}(T) = \text{acc}(T)$. By ii) of Theorem 3.5, $\text{acc}(T)$ is open.

As to the last statement suppose $T = \{x_n\}$ is a countable discrete set. Since T is countable it is a σ-disclosed set and hence $\text{acc}(T)$ is clopen. But, by discreteness, $T \cap \text{acc}(T) = \varnothing$ and since $\text{acc}(T)$ is open it also follows that $\text{cl}\, T \cap \text{acc}(T) = \varnothing$. Putting everything together means that $\text{acc}(T) = \varnothing$, whence T is closed. □

Corollary 3.11. *Suppose X is compact. Then $A(X)$ is basically disconnected if and only if X is finite.*

Proof. The sufficiency is obvious, so suppose that $A(X)$ is basically disconnected. We suppose that X is infinite and compact and then choose an infinite discrete set in X, say N. Since X is compact, N is not a closed set, and therefore $A(X)$ is not basically disconnected. ☐

Remark 3.12. Corollary 3.11 does not generalize to Lindelöf spaces since there are common examples of infinite Lindelöf P-spaces.

We end this section by looking at almost P-spaces. Recall that a space X is an *almost P-space* if every nonempty G_δ-set has dense interior. This is equivalent to the condition that every nonempty zero-set has nonempty interior. Clearly, every P-space is an almost P-space, but the converse is not true. For example, the space $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ is a compact almost P-space for which it is consistent that $\mathbb{N}^*$ has no P-points.

Our next theorem gives necessary and sufficient conditions for when $A(X)$ is an almost P-space. But first we need a characterization of when X possesses a non-isolated G_δ-point as this is essential to the result.

Lemma 3.13. *Let X be a Tychonoff space. X has a non-isolated G_δ-point if and only if X has a non-empty σ-disclosed zero-set containing no isolated points.*

Proof. First, notice that for any G_δ-point $p \in X$, $\{p\}$ is a zero-set that is also closed and discrete. Therefore, if X has a non-isolated G_δ-point, then it has a non-empty σ-disclosed zero-set containing no isolated points.

Conversely, suppose that X has a (non-empty) σ-disclosed zero-set Z which contains no isolated points. There is a sequence of pairwise disjoint disclosed sets, say $\{T_n\}$, such that

$$Z = \bigcup T_n.$$

Choose any $p \in Z$, and by renumbering, we assume that $p \in T_1$. For each $n \geq 2$, p is not in the closed set T_n and hence there is a zero-set Z_n such that $p \in Z_n$ and $Z_n \cap T_n = \varnothing$. Furthermore, since T_1 is discrete there is some zero-set Z_1 such that $p \in Z_1$ and $Z_1 \cap T_1 = \{p\}$. A quick check reveals that

$$Z \cap \bigcap Z_n = \{p\}.$$

Since the countable intersection of zero-sets of X is again a zero-set of X it follows that p is a G_δ-point. ☐

Theorem 3.14. *Let X be a Tychonoff space. The following statements are equivalent.*

(1) *The Alexandroff duplicate $A(X)$ is an almost P-space.*
(2) *X has the property that there is no non-empty zero-set of X which is simultaneously σ-disclosed and does not contain any isolated points.*

(3) *X has no non-isolated G_δ-points.*

Proof. (1) $\Rightarrow$ (2). Suppose that X has a non-empty zero-set which is a countable union of disclosed subsets, say $Z(f) = \bigcup_{n\in\mathbb{N}} T_n$ where each T_n is disclosed, and that $Z(f)$ contains no isolated points. Without loss of generality, we can assume that the collection is pairwise disjoint.

Define $F : A(X) \to \mathbb{R}$ by

$$F(x) = \begin{cases} f(x), & \text{if } x \in X \\ f(x), & \text{if } x \in \operatorname{coz}(f)' \\ \frac{1}{n}, & \text{if } x \in T_n'. \end{cases}$$

Since each T_n is closed and discrete we know that T_n' is a clopen subset of $A(X)$. Therefore, $F \in C(A(X))$. Notice that $Z(F) = Z(f)$, and also that this set is a co-dense subset of $A(X)$, whence it is nowhere dense. It follows that $\operatorname{coz}(F)$ is a proper dense cozero-set of $A(X)$ and consequently, $A(X)$ is not an almost P-space.

(2) $\Rightarrow$ (1). Suppose that $A(X)$ is not an almost P-space and choose $F \in C(A(X))$ so that $F \geq 0$ and $\operatorname{coz}(F)$ is a proper dense cozero-set of $A(X)$. Notice that $Z(F) \subseteq X$, since otherwise $Z(F)$ would contain an isolated point in X', forcing $Z(F)$ to have nonempty interior. Similarly, $Z(F)$ contains no isolated points of X. For each $n \in \mathbb{N}$, let

$$T_n = \left\{ x \in Z(F) : F(x') \geq \frac{1}{n} \right\},$$

If $p \in \operatorname{cl} T_n$, then $p \in Z(F)$ and so there is a basic open centered at p which misses $F^{-1}([\frac{1}{n}, \infty))$ and hence T_n'. Therefore, $p \in T_n$, whence T_n is closed in X. Notice that $\operatorname{Cl} T_n' \subseteq Z(F)$, but due to continuity of F, $\operatorname{Cl} T_n' = T_n'$, whence T_n' is a clopen subset of $A(X)$. Therefore, T_n is a closed and discrete subset of X. Furthermore, for any $x \in Z(F)$ we know that $F(x') > 0$ and therefore, $x \in T_N$. Consequently, $Z(F)$ is a countable union of closed and discrete subsets of X.

That (2) are (3) are equivalent follows from Lemma 3.13. ☐

Corollary 3.15. *If X is an almost P-space, then $A(X)$ is an almost P-space. In particular, $A(\mathbb{N}^*)$ is an almost P-space.*

Proof. In an almost P-space, any G_δ-point must be isolated. ☐

Remark 3.16. Recall that the pseudo-character of a point, notated $\psi(p, X)$, is the minimum (infinite) cardinality needed to write $\{p\}$ as an intersection of open sets. Thus, $\psi(p, X) = \aleph_0$ if and only if p is a G_δ-point. We leave it to the interested reader to modify the above proofs to show that for any $p \in X$, $\psi(p, X) = \psi(p, A(X))$. Furthermore, a non-isolated $p \in X$ is an almost P-point of $A(X)$ if and only if it is not a G_δ-point of X; we find this result to be rather striking. The result corroborates the fact that $A(\beta\mathbb{N})$ is not an extremally disconnected space as it is an almost P-space, and it is well-known that if X

has non-measurable cardinality, then the only points that are simultaneously e.d. points and almost P-points are the isolated points.

Example 3.17. We would like to share more examples of spaces X which are not almost P-spaces yet $A(X)$ is an almost P-space. It suffices to simply find a space that has no non-isolated G_δ points and is not an almost P-space. Examples are everywhere but we take one from C_p-theory. Start with a topological space X and consider $C(X)$ with the subspace topology inherited from the product $\mathbb{R}^X$. This is known as the topology of point-wise convergence Denote this space by $C_p(X)$. This topology on $C(X)$ makes $C(X)$ into a topological ring. In particular, the topology is homogeneous. A base of open sets centered at 0 are sets of the form

$$O(F, \epsilon) = \{f \in C(X) : |f(x_i)| \leq \epsilon \text{ for each } i = 1, \dots, n\}$$

for some finite subset $F = \{x_1, \dots, x_n\} \subseteq X$.

It is known that $C_p(X)$ has G_δ-points if and only if X is separable; see [3, Theorem I.1.4.]. Our claim is that $C_p(X)$ is never an almost P-space. Notice that the set

$$M_x = \{f \in C(X) : f(x) = 0\} = \bigcap_{n \in \mathbb{N}} O(\{x\}, \frac{1}{n})$$

is a G_δ-set, and clearly, this set has empty interior. Thus, for example if $X = \alpha D$, the one-point compactification of an uncountable discrete set, then $A(C_p(X))$ is an almost P-space even though $C_p(X)$ is not.

Next, recall that no compact extremally disconnected space has non-isolated G_δ-points. This is because if X is compact extremally disconnected and $p \in X$ is not isolated, then $X \smallsetminus \{p\}$ is C^*-embedded in X and thus $\beta(X \smallsetminus \{p\}) = X$ and no point of $\beta X \smallsetminus X$ can be a G_δ-point of βX. [Thanks to A.W. Hager for reminding us of this result.]

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