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Lipschitz integral operators represented by vector measures

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Abstract

In this paper we introduce the concept of Lipschitz Pietsch-p-integral mappings, $(1 \leq p \leq \infty)$, between a metric space and a Banach space. We represent these mappings by an integral with respect to a vector measure defined on a suitable compact Hausdorff space, obtaining in this way a rich factorization theory through the classical Banach spaces $C(K)$, $L_p(\mu, K)$ and $L_\infty(\mu, K)$. Also we show that this type of operators fits in the theory of composition Banach Lipschitz operator ideals. For $p = \infty$, we characterize the Lipschitz Pietsch-∞-integral mappings by a factorization schema through a weakly compact operator. Finally, the relationship between these mappings and some well known Lipschitz operators is given.

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Introduction

The class of p-integral linear operators was introduced in 1969 by Persson and Pietsch [18] (also known as strictly p-integral or Pietsch-p-integral operators) establishing many of its fundamental properties using the theory of

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vector measures. In 1989, Cardassi studied the factorization properties and some results of coincidences for these operators in [6]. The ideal of p-integral polynomials on Banach spaces has been defined and characterized by Cilia and Gutiérrez, in [9] for $p = 1$ and in [8] for $p \geq 1$, as a natural polynomial extension of Pietsch-p-integral operators.

In this paper we introduce and study the Lipschitz version of this concept. We define the Lipschitz Pietsch-p-integral operator $(1 \leq p \leq \infty)$ as a Lipschitz mapping between a pointed metric space and a Banach space by an integral representation with respect to a vector measure on the Borel σ-algebra of a compact Hausdorff space K. Special attention is paid to the factorization of these mappings and we compare our class with some well known Lipschitz operators defined by a factorization schema or by summability of series. Note that the class of Lipschitz Pietsch-1-integral operators is studied in [5]. In this case, the authors use only factorization schemes to define this concept without using vector measure theory.

We describe now the contents of the present paper. After this introduction, in section one we fix notation and basic concepts related to Lipschitz mappings and vector measure of interest for our purposes. In section two we extend to Lipschitz mappings the concept of Pietsch-p-integral operators for $p \geq 1$ and we prove a factorization theorem for these mappings through the classical Banach spaces $C(K)$ and $L_p(\mu, K)$. The third section is devoted to study the notion of Lipschitz Pietsch-∞-integral operators, starting from the representation by a vector measure, we present a characterization given by a factorization through a linear weakly compact operator. Finally, in section four we establish the relationship between our class and the class of Lipschitz p-summing operators, Lipschitz Grothendieck-p-integral operators, strongly Lipschitz p-nuclear operators and Lipschitz weakly compact operators.

1. Notation and preliminaries

The notation used in the paper is in general standard. E and F are real Banach spaces. X, Y and Z will be pointed metric spaces with a base point denoted by 0 and a metric denoted by d. Given a Banach space E, E^* denotes its topological dual, and B_E its closed unit ball. As usual, $\mathcal{L}(E, F)$ denotes the space of all continuous linear operators from E to F with the operator norm. A Banach space E will be considered as a pointed metric space with distinguished point 0 and distance $d(x, y) = \|x - y\|$. With $Lip_0(X, E)$ we denote the Banach space of all Lipschitz mappings from X to E, taking 0 into 0, under the Lipschitz norm

$$Lip(T) = \inf \{C > 0 : \|T(x) - T(x')\| \leq Cd(x, x')\}.$$

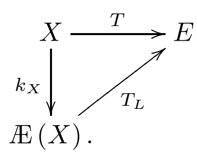
Moreover, T is called an isometric embedding if $\|T(x) - T(x')\| = d(x, x')$ for all $x, x' \in X$.

When $E = \mathbb{R}$, $Lip_0(X, \mathbb{R})$ is denoted by $X^\#$ and it is called the Lipschitz dual of X. Along the paper we consider $B_{X^\#}$ endowed with the pointwise topology ($B_{X^\#}$ is a compact Hausdorff space in this topology).

It is well know that $X^{\#}$ has a predual, namely the space of Arens and Eells Æ(X) of the metric space X [2] (also known as the Lipschitz-free Banach space $\mathcal{F}(X)$ of X [14]). This space is one of the main tools that we will use in the sequel. We summarize some basic properties of Æ(X). A molecule on X is a scalar valued function $\mathfrak{m}$ on X with finite support that satisfies $\sum\limits_{x\in X} \mathfrak{m}(x) = 0$. We denote by $\mathcal{M}(X)$ the linear space of all molecules on X. For $x, x' \in X$ the molecule $\mathfrak{m}_{xx'}$ is defined by $\mathfrak{m}_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A. For $\mathfrak{m} \in \mathcal{M}(X)$ we can write $\mathfrak{m} = \sum\limits_{i=1}^{n} \lambda_i \mathfrak{m}_{x_i x_i'}$ for some suitable scalars λ_i, and we write

$$\|\mathfrak{m}\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{i=1}^{n} |\lambda_i| \, d(x_i, x_i'),\ \mathfrak{m} = \sum_{i=1}^{n} \lambda_i \mathfrak{m}_{x_i x_i'} \right\},$$

where the infimum is taken over all representations of the molecule $\mathfrak{m}$. Denote by Æ(X) the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$. The map $k_X : X \to$Æ(X) defined by $k_X(x) = \mathfrak{m}_{x0}$ isometrically embeds X in Æ(X). For any $T \in Lip_0(X, E)$ there exists a unique linear map $T_L \in \mathcal{L}($Æ$(X), E)$ such that $T = T_L \circ k_X$, i.e. the following diagram commutes



Moreover, $\|T_L\| = Lip(T)$ (see [19, Theorem 2.2.4 (b)]). The operator T_L is referred to as the linearization of T.

The correspondence $T \longleftrightarrow T_L$ establishes an isometric isomorphism between the Banach spaces $Lip_0(X, E)$ and $\mathcal{L}($Æ$(X), E)$. In particular, the spaces $X^{\#}$ and Æ$(X)^*$ are isometrically isomorphic via the linearization $R(f) := f_L$, where $f_L(\mathfrak{m}) = \sum\limits_{x\in X} f(x)\mathfrak{m}(x)$, in particular $f_L(\mathfrak{m}_{xx'}) = f(x) - f(x')$, (see [19, Theorem 2.2.2]).

Now we recall some simple notions from vector measure theory. Let (Ω, Σ) be a measurable space and E a Banach space. Let $m : \Sigma \longrightarrow E$ be a countably additive vector measure [11, Definition I.1.1]. For every $x^* \in E^*$, let $\langle m, x^* \rangle$ be the scalar signed measure defined by $\langle m, x^* \rangle (A) := \langle m(A), x^* \rangle$ for all $A \in \Sigma$. The semivariation of m is the subadditive real bounded set function $\|m\| : A \in \Sigma \longrightarrow \|m\|(A) \in [0, +\infty)$ defined by

$$\|m\|(A) = \sup \{ |\langle m, x^* \rangle| (A) : \|x^*\| \leq 1 \},$$

where $|\langle m, x^* \rangle|$ is the variation measure of the signed measure $\langle m, x^* \rangle$.

According to [17, Page 106] and [16, Definition 2.1], a measurable (scalar valued) function f is integrable with respect to m if

- It is integrable with respect to the scalar measure $\langle m, x^* \rangle$ for every $x^* \in E^*$.

- For every $A \in \Sigma$ there exists an element $m_f(A) \in E$ satisfying

$$\langle m_f(A), x^* \rangle = \int_A f d \langle m, x^* \rangle, \quad x^* \in E^*.$$

We will use the classical notation

$$m_f(A) := \int_A f dm, \quad A \in \Sigma.$$

For the general theory of vector measures we refer the reader to the classical monograph [11].

For $1 \le p \le \infty$, linear Pietsch-p-integral operators were introduced by Persson and Pietsch [18] and deeply studied in [6, 10] among others.

Definition 1.1. The linear operator $u : E \longrightarrow F$, between Banach spaces, is Pietsch-p-integral ($1 \le p < \infty$) if there are a regular Borel countably additive vector measure m of bounded semivariation on $\mathcal{B}(B_{E^*})$, (where $\mathcal{B}(B_{E^*})$ is the Borel σ-algebra of B_{E^*}), and a positive regular Borel measure μ on B_{E^*} such that

$$u(x) = \int_{B_{E^*}} \langle x, x^* \rangle \, dm(x^*), \quad x \in E,$$

and

$$\left\| \int_{B_{E^*}} f dm \right\| \le \left(\int_{B_{E^*}} |f|^p \, d\mu \right)^{\frac{1}{p}}, \quad f \in C(B_{E^*}).$$

The Banach space of these operators is denoted by $\mathcal{PI}_p(E,F)$ under the norm defined by $\|u\|_{\mathcal{PI}_p} = \inf \mu(B_{E^*})^{\frac{1}{p}}$, where the infimum is taken over all measures μ satisfying the above inequality.

For $p = \infty$, u is called Pietsch-∞-integral if there is a regular Borel countably additive vector measure $m : \mathcal{B}(B_{E^*}) \longrightarrow F$ of bounded semivariation such that

$$u(x) = \int_{B_{E^*}} \langle x, x^* \rangle \, dm(x^*), \quad x \in E.$$

In this case, $\|T\|_{\mathcal{PI}^L_\infty} = \inf \|m\| (B_{E^*})$, taking the infimum over all m that satisfy the above equality.

In [18, Satz 15 and Satz 17] we find some canonical linear Pietsch-p-integral operators that will be used in the sequel. Let K be a compact Hausdorff space and ν be a positive regular Borel measure on K. Let (Ω, Σ, μ) be a finite measure space and j_p, i_p be the inclusions of $C(K)$ into $L_p(K,\nu)$ and of $L_\infty(\mu)$ into $L_p(\mu)$ respectively for $1 \le p < \infty$. Then $j_p \in \mathcal{PI}_p(C(K), L_p(K,\nu))$ and $i_p \in \mathcal{PI}_p(L_\infty(\mu), L_p(\mu))$ with $\|j_p\|_{\mathcal{PI}_p} = \|j_p\| = \nu(K)^{\frac{1}{p}}$ and $\|i_p\|_{\mathcal{PI}_p} = \|i_p\| = \mu(\Omega)^{\frac{1}{p}}$.

Remark 1.2. Note that by using [8, Theorem 2.5], $u \in \mathcal{PI}_p(E,F)$ if and only if there are a compact Hausdorff space K, an embedding $h : E \longrightarrow C(K)$, a regular Borel countably additive vector measure $m : \mathcal{B}(K) \longrightarrow F$ of bounded semivariation and a positive regular Borel measure μ on K such that for all $x \in E$,

$$u(x) = \int_K h\,(x)\,dm,$$

and $\left\| \int_K f dm \right\| \leq \left(\int_K |f|^p \, d\mu \right)^{\frac{1}{p}}$ for all $f \in C\,(K)$. In this case

$$\|u\|_{\mathcal{PI}_p} = \inf \|h\| \, \mu(K)^{\frac{1}{p}},$$

where the infimum is taken over all K, m and h as above.

2. Lipschitz Pietsch-p-integral operators

Definition 2.1. Let X be a pointed metric space, E a Banach space and let $T \in Lip_0(X, E)$. For $1 \leq p < \infty$, the mapping T is said to be Lipschitz Pietsch-p-integral operator if there are a regular Borel probability measure space (Ω, Σ, μ), a linear operator $A \in \mathcal{L}(L_p(\mu), E)$ and a Lipschitz operator $B \in Lip_0(X, L_\infty(\mu))$ giving rise to the following commutative diagram

(2.1)
$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & E \\ {\scriptstyle B}\downarrow & & \uparrow{\scriptstyle A} \\ L_\infty(\mu) & \xrightarrow{\;i_p\;} & L_p(\mu). \end{array}$$

where $i_p : L_\infty(\mu) \longrightarrow L_p(\mu)$ is the canonical mapping. The set of all Lipschitz Pietsch-p-integral mappings from X to E is denoted by $\mathcal{PI}_p^L(X, E)$. With each $T \in \mathcal{PI}_p^L(X, E)$ we associate its Lipschitz Pietsch-p-integral quantity, $\|T\|_{\mathcal{PI}_p^L} = \inf \|A\| \, Lip(B)$, where the infimum is taken over all μ, A and B as above.

Remark 2.2.

(1) As an easy consequence of the definition, if $T \in \mathcal{PI}_p^L(X, E)$ we have $Lip(T) \leq \|T\|_{\mathcal{PI}_p^L}$.

(2) Notice that the definition is the same if we consider a finite regular Borel measure space (Ω, Σ, μ), in this case, for $T \in \mathcal{PI}_P^L(X, E)$ we have

$$\|T\|_{\mathcal{PI}_P^L} = \inf \|A\| \, \mu(\Omega)^{\frac{1}{p}} \, Lip(B),$$

where the infimum is taken over all μ, A and B in (2.1).

(3) We don't know if being Lipschitz Pietsch-p-integrability implies Pietsch-p-integrability whenever the mapping T is linear. The converse is of course clearly true, that is if E and F are Banach spaces and $T : E \longrightarrow F$ is linear Pietsch-p-integral then T is Lipschitz Pietsch-p-integral and $\|T\|_{\mathcal{PI}_p^L} \leq \|T\|_{\mathcal{PI}_p}$.

We have the following immediate consequence of the definition above.

Proposition 2.3 (Inclusion Theorem)**.** *Let $1 \leq p \leq q < \infty$. Then $\mathcal{PI}_p^L(X,E) \subset \mathcal{PI}_q^L(X,E)$ and $\|T\|_{\mathcal{PI}_q^L} \leq \|T\|_{\mathcal{PI}_p^L}$ for all $T \in \mathcal{PI}_p^L(X,E)$.*

In order to prove the factorization theorem for the class of Lipschitz Pietsch-p-integral operators, $(1 \leq p \leq \infty)$ we need the following technical lemma.

Lemma 2.4. *Let $J : \mathcal{M}(X) \longrightarrow C(B_{X^\#})$ be the operator defined by*

$$J(\mathfrak{m})(f) = \sum_{i=1}^{n} \lambda_i (f(x_i) - f(x_i')),$$

for all $\mathfrak{m} = \sum_{i=1}^{n} \lambda_i \mathfrak{m}_{x_i x_i'} \in \mathcal{M}(X)$ and $f \in B_{X^\#}$. Then this operator is an isometric embedding.

Proof. Since $Æ(X)^*$ and $X^\#$ are isometrically isomorphic via the linearization, for all $f \in X^\#$ there is $\mathfrak{m}^* \in Æ(X)^*$ such that $f_L = \mathfrak{m}^*$. For all $\mathfrak{m} \in \mathcal{M}(X)$ we have

$$\begin{aligned} \|J(\mathfrak{m})\|_{C(B_{X^\#})} &= \sup_{f \in B_{X^\#}} |J(\mathfrak{m})(f)| = \sup_{\|f_L\| \leq 1} \left| \sum_{i=1}^{n} \lambda_i (f_L(\mathfrak{m}_{x_i x_i'}) \right| \\ &= \sup_{\|\mathfrak{m}^*\| \leq 1} |\langle \mathfrak{m}, \mathfrak{m}^* \rangle| = \|\mathfrak{m}\|_{Æ(X)} = \|\mathfrak{m}\|_{\mathcal{M}(X)}, \end{aligned}$$

and the proof follows. □

For $x \in X$, we denote by δ_x the functional $\delta_x : X^\# \longrightarrow \mathbb{R}$ defined as $\delta_x(f) = f(x)$, $f \in X^\#$. Let $\iota_X : X \longrightarrow C(B_{X^\#})$ the natural Lipschitz isometric embedding such that $\iota_X(x)$ is the restriction of δ_x to $B_{X^\#}$, for all $x \in X$.

The following theorem gives a parallel development of the factorization schemes concerning Lipschitz Pietsch-p-integral operators that highlights the role of the space $C(B_{X^\#})$.

Theorem 2.5. *Let $1 \leq p < \infty$ and let $T \in Lip_0(X,E)$. Then T is Lipschitz Pietsch-p-integral if and only if there exist a regular Borel probability measure ν on $B_{X^\#}$ and an operator $\tilde{A} \in \mathcal{L}(L_p(\nu), E)$ such that the following diagram commutes*

(2.2)
$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & E \\ {\scriptstyle \iota_X}\downarrow & & \uparrow{\scriptstyle \widetilde{A}} \\ C(B_{X^\#}) & \xrightarrow{\;j_p\;} & L_p(\nu) \end{array} ,$$

where j_p is the canonical map. Moreover,

$$\|T\|_{\mathcal{PI}_p^L} = \inf \left\{ \left\| \widetilde{A} \right\| : T = \widetilde{A} \circ j_p \circ \iota_X \right\}.$$

Proof. We write $\triangle$ for the proposed infimum. Suppose that T admits a factorization (2.2). If j_∞ is the canonical inclusion map from $C(B_{X^\#})$ to $L_\infty(\nu)$, we have the factorization

$$T = \tilde{A} \circ i_p \circ j_\infty \circ \iota_X : X \xrightarrow{\iota_X} C(B_{X^\#}) \xrightarrow{j_\infty} L_\infty(\nu) \xrightarrow{i_p} L_p(\nu) \xrightarrow{\tilde{A}} E.$$

Denoting by $B = j_\infty \circ \iota_X$, it follows that $B \in Lip_0(X, L_\infty(\nu))$ and $Lip(B) \leq 1$, which implies that T is Lipschitz Pietsch-p-integral and

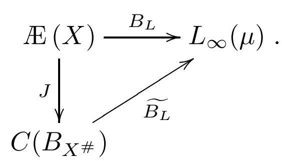
$$\|T\|_{\mathcal{PI}_p^L} \leq \left\|\tilde{A}\right\| Lip(B) \leq \left\|\tilde{A}\right\|.$$

Passing to the infimum we get $\|T\|_{\mathcal{PI}_p^L} \leq \triangle$.

Conversely, suppose that $T \in \mathcal{PI}_p^L(X, E)$. Fix $\varepsilon > 0$, there are a regular Borel probability measure space (Ω, Σ, μ), an operator $A \in \mathcal{L}(L_p(\mu), E)$ and a Lipschitz mapping $B \in Lip_0(X, L_\infty(\mu))$ such that

$$T = A \circ i_p \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

and $\|A\|\, Lip(B) \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon$. Let $B_L \in \mathcal{L}(\text{Æ}(X), L_\infty(\mu))$ be the linearization of the Lipschitz mapping B, that is $B = B_L \circ k_X$ and $\|B_L\| = Lip(B)$. Consider the natural extension of the isometric embedding J, mentioned in Lemma 2.4, to Æ(X) which we denote also by J. The injectivity of $L_\infty(\mu)$ assures the existence of an operator $\widetilde{B_L} \in \mathcal{L}(C(B_{X^\#}), L_\infty(\mu))$ that extends B_L with $\left\|\widetilde{B_L}\right\| = \|B_L\|$, that is $B_L = \widetilde{B_L} \circ J$ or the following diagram commutes



The operator $i_p : L_\infty(\mu) \longrightarrow L_p(\mu)$ is p -summing with p-summing norm one then $i_p \circ \widetilde{B_L}$ is too with $\pi_p(i_p \circ \widetilde{B_L}) \leq \left\|\widetilde{B_L}\right\|$. By [10, Corollary 2.15] there exist a regular Borel probability measure ν on $B_{X^\#}$ and an operator $S \in \mathcal{L}(L_p(\nu), L_p(\mu))$ such that

$$i_p \circ \widetilde{B_L} = S \circ j_p : C\left(B_{X^\#}\right) \xrightarrow{j_p} L_p(\nu) \xrightarrow{S} L_p(\mu),$$

and $\pi_p(i_p \circ \widetilde{B_L}) = \|S\|$. Then

$$T = (A \circ S) \circ j_p \circ (J \circ k_X) : X \xrightarrow{J \circ k_X} C\left(B_{X^\#}\right) \xrightarrow{j_p} L_p(\nu) \xrightarrow{A \circ S} E.$$

Easy calculations prove that $J \circ k_X = \iota_X$, which implies that T admits a factorization of the form (2.2) with $\tilde{A} = A \circ S$ and we have

$$\begin{aligned} \triangle &\leq \left\|\tilde{A}\right\| \leq \|A\|\, \pi_p(i_p \circ \widetilde{B_L}) \\ &\leq \|A\| \left\|\widetilde{B_L}\right\| = \|A\|\, Lip(B) \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon. \end{aligned}$$

Since this holds for all $\varepsilon > 0$ we arrive at $\triangle \leq \|T\|_{\mathcal{PI}_p^L}$. □

The next theorem is the main result of this section and provides a characterization of the class of Lipschitz Pietsch-p-integral operators, that is an integral representation with respect to a vector measure.

Theorem 2.6. *Let $1 \leq p < \infty$ and let $T \in Lip_0(X, E)$. Then T is Lipschitz Pietsch-p-integral if and only if there are a compact Hausdorff space K, a Lipschitz embedding $\phi : X \longrightarrow C(K)$ with $\phi(0) = 0$, a regular Borel countably additive vector measure $m : \mathcal{B}(K) \longrightarrow E$ of bounded semivariation and a positive regular Borel measure μ on K such that*

$$T(x) = \int_K \phi(x)\, dm, \quad x \in X, \tag{2.3}$$

and

$$\left\| \int_K f dm \right\| \leq \left(\int_K |f|^p \, d\mu \right)^{\frac{1}{p}}, \tag{2.4}$$

for all $f \in C(K)$. In this case $\|T\|_{\mathcal{PI}_p^L} = \inf \left\{ Lip(\phi)\mu(K)^{\frac{1}{p}} \right\}$, where the infimum is taken over all K, ϕ, m and μ satisfying (2.3) and (2.4).

Proof. Suppose that $T \in \mathcal{PI}_p^L(X, E)$, and fix $\varepsilon > 0$. There are a regular Borel probability measure ν on $B_{X^\#}$ and $\tilde{A} \in \mathcal{L}(L_p(\nu), E)$ such that

$$T = \tilde{A} \circ j_p \circ \iota_X : X \xrightarrow{\iota_X} C(B_{X^\#}) \xrightarrow{j_p} L_p(\nu) \xrightarrow{\tilde{A}} E,$$

and $\left\| \tilde{A} \right\| \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon$. The linear operator $\tilde{A} \circ j_p : C(B_{X^\#}) \longrightarrow E$ is Pietsch-p-integral with

$$\left\| \tilde{A} \circ j_p \right\|_{\mathcal{PI}_p} \leq \left\| \tilde{A} \right\| \|j_p\|_{\mathcal{PI}_p} \leq \left\| \tilde{A} \right\| \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon.$$

By Remark 1.2, there are a compact Hausdorff space K, an embedding $h : C(B_{X^\#}) \longrightarrow C(K)$, a regular Borel countably additive vector measure $m : \mathcal{B}(K) \longrightarrow E$ of bounded semivariation and a positive regular Borel measure μ on K such that for all $x \in X$,

$$T(x) = \tilde{A} \circ j_p(\iota_X(x)) = \int_K h(\iota_X(x))\, dm,$$

$\left\| \int_K f dm \right\| \leq \left(\int_K |f|^p \, d\mu \right)^{\frac{1}{p}}$ for all $f \in C(K)$ and $\|h\| \mu(K)^{\frac{1}{p}} \leq \left\| \tilde{A} \circ j_p \right\|_{\mathcal{PI}_p} + \varepsilon$. Which means that (2.3) and (2.4) are true by taking into account that $\phi = h \circ \iota_X$ is a Lipschitz embedding from X to $C(K)$ vanishing at 0 with $Lip(\phi) \leq \|h\|$. Moreover

$$Lip(\phi)\mu(K)^{\frac{1}{p}} \leq \left\| \tilde{A} \circ j_p \right\|_{\mathcal{PI}_p} + \varepsilon \leq \|T\|_{\mathcal{PI}_p^L} + 2\varepsilon.$$

Since this holds for every $\varepsilon > 0$, it follows that $Lip(\phi)\mu(K)^{\frac{1}{p}} \leq \|T\|_{\mathcal{PI}_p^L}$.

Conversely, suppose that T satisfies the conditions (2.3) and (2.4). By [11, Theorem VI.2.1] there exists $u \in \mathcal{L}(C(K), E)$ such that $u(f) = \int_K f dm$, $f \in C(K)$. Consider the canonical mapping

$$j_p = i_p \circ j_\infty : C(K) \xrightarrow{j_\infty} L_\infty(K, \mu) \xrightarrow{i_p} L_p(K, \mu),$$

and define $R : j_p(C(K)) \longrightarrow E$ by $R(j_p(f)) := u(f)$. The linear mapping R is well-defined and continuous with norm ≤ 1 since for all $f \in C(K)$,

$$\|R(j_p(f))\| = \left\| \int_K f dm \right\| \leq \left(\int_K |f|^p d\mu \right)^{\frac{1}{p}} = \|j_p(f)\|.$$

By [12, Lemma IV.8.19] we have $\overline{j_p(C(K))} = L_p(K,\mu)$, so R can be extended to a continuous linear operator $\widetilde{R} : L_p(K,\mu) \longrightarrow E$ with $\left\|\widetilde{R}\right\| \leq 1$. If we put $B = j_\infty \circ \phi$, we obtain $B \in Lip_0(X, L_\infty(K,\mu))$ and $Lip(B) \leq Lip(\phi)$. On the other hand,

$$\widetilde{R} \circ i_p \circ B(x) = \widetilde{R} \circ j_p \circ \phi(x) = u(\phi(x)) = \int_K \phi(x) dm = T(x),$$

and therefore T factors as in (2.1), that is $T \in \mathcal{PI}_p^L(X,E)$ with

$$\|T\|_{\mathcal{PI}_p^L} \leq Lip(B) \left\|\widetilde{R}\right\| \mu(K)^{\frac{1}{p}} \leq Lip(\phi)\mu(K)^{\frac{1}{p}}.$$

□

Now we present a relationship between the Lipschitz Pietsch-p-integral operator and its linearization.

Theorem 2.7. *Let $T \in Lip_0(X,E)$ and $1 \leq p < \infty$. Then $T \in \mathcal{PI}_p^L(X,E)$ if and only if $T_L \in \mathcal{PI}_p(Æ(X), E)$. Moreover, we have*

$$\|T\|_{\mathcal{PI}_p^L} = \|T_L\|_{\mathcal{PI}_p}. \tag{2.5}$$

Proof. Suppose that $T_L \in \mathcal{PI}_p(Æ(X), E)$. According to [18, Satz 18], for every $\varepsilon > 0$ we can choose a typical factorization of T_L

$$T_L = A \circ i_p \circ B : Æ(X) \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

such that $A \in \mathcal{L}(L_p(\mu), E)$ and $B \in \mathcal{L}(Æ(X), L_\infty(\mu))$ with $\|A\|\|B\| \leq \|T_L\|_{\mathcal{PI}_p} + \varepsilon$. It is clear that the mapping $R := B \circ k_X$ belongs to $Lip_0(X, L_\infty(\mu))$ and $Lip(R) \leq \|B\|$. The factorization $T = T_L \circ k_X = A \circ i_p \circ R$ implies that $T \in \mathcal{PI}_p^L(X,E)$ and

$$\|T\|_{\mathcal{PI}_p^L} \leq \|A\| Lip(R) \leq \|T_L\|_{\mathcal{PI}_p} + \varepsilon.$$

Conversely, if $T \in \mathcal{PI}_p^L(X,E)$, for $\varepsilon > 0$ choose the following factorization of T

$$T = A \circ i_p \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

such that $A \in \mathcal{L}(L_p(\mu), E)$ and $B \in Lip_0(X, L_\infty(\mu))$ with $\|A\| Lip(B) \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon$. The uniqueness of the linearization maps gives that

$$T_L = (A \circ i_p \circ B)_L = A \circ i_p \circ B_L.$$

Then, we have that $T_L \in \mathcal{PI}_p(Æ(X), E)$ with

$$\|T_L\|_{\mathcal{PI}_p} \leq \|A\|\|B_L\| = \|A\| Lip(B) \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon.$$

The proof concludes. □

The notion of Lipschitz operator ideal was introduced by Achour, Rueda, Sánchez-Pérez and Yahi [1]. This can be seen as an extension of the notion of linear Banach operator ideal. A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E the components

$$\mathcal{I}_{Lip}(X,E) := Lip_0(X,E) \cap \mathcal{I}_{Lip}$$

satisfy

(i) $\mathcal{I}_{Lip}(X,E)$ is a linear subspace of $Lip_0(X,E)$.
(ii) $vg \in \mathcal{I}_{Lip}(X,E)$ for $v \in E$ and $g \in X^{\#}$.
(iii) The ideal property: if $S \in Lip_0(Y,X)$, $T \in \mathcal{I}_{Lip}(X,E)$ and $w \in \mathcal{L}(E,F)$, then the composition wTS is in $\mathcal{I}_{Lip}(Y,F)$.

A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a normed (Banach) Lipschitz operator ideal if there is a function $\|.\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0,+\infty[$ that satisfies

(i') For every pointed metric space X and every Banach space E, the pair $(\mathcal{I}_{Lip}(X,E), \|.\|_{\mathcal{I}_{Lip}})$ is a normed (Banach) space and $Lip(T) \leq \|T\|_{\mathcal{I}_{Lip}}$ for all $T \in \mathcal{I}_{Lip}(X,E)$.
(ii') $\|Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$.
(iii') If $S \in Lip_0(Y,X)$, $T \in \mathcal{I}_{Lip}(X,E)$ and $w \in \mathcal{L}(E,F)$, then $\|wTS\|_{\mathcal{I}_{Lip}} \leq Lip(S) \|T\|_{\mathcal{I}_{Lip}} \|w\|$.

Following [1, Definition 3.1], there is a way to construct a (Banach) Lipschitz operator ideal from a (Banach) linear operator ideal, called *composition method*. Let $\mathcal{A}$ be a (Banach) linear operator ideal. A Lipschitz mapping $T \in Lip_0(X,E)$ belongs to the *composition Lipschitz operator ideal* $\mathcal{A} \circ Lip_0$ if there exists a Banach space F, a *Lipschitz operator* $S \in Lip_0(X,F)$ and a linear operator $u \in \mathcal{A}(F,E)$ such that $T = u \circ S$. If $(\mathcal{A}, \|.\|_{\mathcal{A}})$ is a Banach operator ideal we write

$$\|T\|_{\mathcal{A}\circ Lip_0} = \inf \|u\|_{\mathcal{A}} \, Lip(S),$$

where the infimum is taken over all u and S as above.

In [1], the authors establish a criterion to decide whenever a Lipschitz operator ideal is of composition or not.

Proposition 2.8 ([1, Proposition 3.2])**.** *Let X be a pointed metric space, E a Banach space and $\mathcal{A}$ an operator ideal. A Lipschitz operator $T \in Lip_0(X,E)$ belongs to $\mathcal{A}\circ Lip_0(X,E)$ if and only if its linearization T_L belongs to $\mathcal{A}(\mathcal{F}(X),E)$.*

Furthermore, if $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a Banach operator ideal then $(\mathcal{A}\circ Lip_0, \|\cdot\|_{\mathcal{A}\circ Lip_0})$ is Banach Lipschitz operator ideal with

$$\|T\|_{\mathcal{A}\circ Lip_0} = \|T_L\|_{\mathcal{A}}.$$

By Theorem 2.7 and the above criterion, we have the following.

Proposition 2.9. *$\left(\mathcal{PI}_p^L, \|\cdot\|_{\mathcal{PI}_p^L}\right)$ is the Banach Lipschitz operator ideal generated by the composition method from the Banach operator ideal $\mathcal{PI}_p$. In other words*

$$\mathcal{PI}_p^L(X,E) = \mathcal{PI}_p \circ Lip_0(X,E) \quad \textit{isometrically}$$

for every pointed metric space X and every Banach space E.

We say that a pointed metric space W is 1-injective (or an absolute Lipschitz retract) if for every metric space X, every subset X_0 of X and every Lipschitz mapping $T \in Lip_0(X_0, W)$ there is a Lipschitz mapping $\widetilde{T} \in Lip_0(X, W)$ extending T with $Lip(T) = Lip(\widetilde{T})$. The real Banach space $L_\infty(\mu)$ for a finite measure μ is 1-injective (see [4, Chapter 1]).

By the typical Pietsch-p-integral factorization of a Lipschitz mapping T, we can find a Pietsch-p-integral extension $\widetilde{T}$.

Proposition 2.10. *Let X and Z be pointed metric spaces with $X \subset Z$ and let E be a Banach space. Each Lipschitz Pietsch-p-integral operator $T : X \longrightarrow E$ admits a Lipschitz Pietsch-p-integral extension $\widetilde{T} : Z \longrightarrow E$ with $\|T\|_{\mathcal{PI}_p^L} = \left\|\widetilde{T}\right\|_{\mathcal{PI}_p^L}$.*

Proof. If $T \in \mathcal{PI}_p^L(X, E)$, then for all $\varepsilon > 0$ there are a regular Borel probability measure space (Ω, Σ, μ), $A \in \mathcal{L}(L_p(\mu), E)$ and $B \in Lip_0(X, L_\infty(\mu))$ such that

$$T = A \circ i_p \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

and $Lip(B)\,\|A\| \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon$. Since $L_\infty(\mu)$ is 1-injective, B admits an extension $\widetilde{B} \in Lip_0(Z, L_\infty(\mu))$ with $Lip(\widetilde{B}) = Lip(B)$ i.e., the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{B} & L_\infty(\mu)\ , \\ {\scriptstyle i}\downarrow & \nearrow_{\widetilde{B}} & \\ Z. & & \end{array}$$

where $i \in Lip_0(X, Z)$ is the natural isometric embedding. This creates a Lipschitz Pietsch-p-integral extension $\widetilde{T} : Z \longrightarrow E$ of T having the following factorization

$$\widetilde{T} = A \circ i_p \circ \widetilde{B} : Z \xrightarrow{\widetilde{B}} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E.$$

Furthermore,

$$\left\|\widetilde{T}\right\|_{\mathcal{PI}_p^L} \leq Lip(\widetilde{B})\,\|A\| = Lip(B)\,\|A\| \leq \|T\|_{\mathcal{PI}_p^L} + \varepsilon.$$

Since this holds for all $\varepsilon > 0$ we get $\left\|\widetilde{T}\right\|_{\mathcal{PI}_p^L} \leq \|T\|_{\mathcal{PI}_p^L}$. For the reverse inequality, note that

$$\|T\|_{\mathcal{PI}_p^L} = \left\|\widetilde{T} \circ i\right\|_{\mathcal{PI}_p^L} \leq \left\|\widetilde{T}\right\|_{\mathcal{PI}_p^L}.$$

□

3. Lipschitz Pietsch-∞-integral operators

In this section we extend the definition of the class of Pietsch-∞-integral linear operators to the case of Lipschitz operators and we will show a factorization theorem that characterizes these mappings.

Definition 3.1. We say that a Lipschitz operator $T \in Lip_0(X, E)$ is Lipschitz Pietsch-∞-integral if there is a regular Borel countably additive vector measure $m : \mathcal{B}(B_{X^\#}) \longrightarrow E$ of bounded semivariation such that

$$T(x) = \int_{B_{X^\#}} f(x)\, dm(f), \quad x \in X. \tag{3.1}$$

We denote by $\mathcal{PI}_\infty^L(X, E)$ the set of all these mappings and we put

$$\|T\|_{\mathcal{PI}_\infty^L} = \inf \|m\| (B_{X^\#}),$$

taking the infimum over all m such that (3.1) holds.

Remark 3.2. If $T \in \mathcal{PI}_\infty^L(X, E)$ then $Lip(T) \leq \|T\|_{\mathcal{PI}_\infty^L}$. In order to see this, for $\varepsilon > 0$ choose m such that $\|m\| (B_{X^\#}) \leq \|T\|_{\mathcal{PI}_\infty^L} + \varepsilon$ and for all $x, y \in X$,

$$\begin{aligned} \|T(x) - T(y)\| &\leq \int_{B_{X^\#}} |f(x) - f(y)|\, dm(f) \\ &\leq \|m\| (B_{X^\#}) d(x, y) \\ &\leq (\varepsilon + \|T\|_{\mathcal{PI}_\infty^L}) d(x, y). \end{aligned}$$

Hence, $Lip(T) \leq \|T\|_{\mathcal{PI}_\infty^L}$.

Now we prove the main result of this section. We characterize the Pietsch-∞-integral Lipschitz operators by means of a factorization scheme through a weakly compact linear operator.

Theorem 3.3. *For a Lipschitz operator $T \in Lip_0(X, E)$, the following statements are equivalent.*

(1) *T is Lipschitz Pietsch-∞-integral.*
(2) *There are a compact Hausdorff space K, a Lipschitz embedding $\varphi \in Lip_0(X, C(K))$ and a weakly compact linear operator $S \in \mathcal{L}(C(K), E)$ such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ {\scriptstyle\varphi}\downarrow & \nearrow_{S} & \\ C(K). & & \end{array} \tag{3.2}$$

(3) *There are a regular Borel finite measure space (Ω, Σ, μ), a weakly compact operator $R \in \mathcal{L}(L_\infty(\mu), E)$ and a Lipschitz embedding $\phi \in Lip_0(X, L_\infty(\mu))$*

giving rise to the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ {\scriptstyle \phi}\downarrow & \nearrow_{R} & \\ L_\infty(\mu). & & \end{array} \tag{3.3}$$

In addition,

$$\|T\|_{\mathcal{PI}_\infty^L} = \inf \|S\| \, Lip(\varphi) = \inf \|R\| \, Lip(\phi).$$

Where the first infimum is taken over all S and φ as in (3.2) and the second is taken over all R and ϕ as in (3.3).

Proof. (1)$\Longrightarrow$(2). Take $T \in \mathcal{PI}_\infty^L(X, E)$. For every $\varepsilon > 0$ choose m satisfying (3.1) and $\|m\|(B_{X^\#}) \leq \|T\|_{\mathcal{PI}_\infty^L} + \varepsilon$. Consider the linear operator $S : C(B_{X^\#}) \longrightarrow E$ defined by

$$S(h) = \int_{B_{X^\#}} h dm,$$

for all $h \in C(B_{X^\#})$ and the natural Lipschitz isometric embedding $\iota_X : X \longrightarrow C(B_{X^\#})$. In this case, for all $x \in X$ we can write

$$S \circ \iota_X(x) = \int_{B_{X^\#}} \iota_X(x)(f) dm(f) = \int_{B_{X^\#}} f(x) dm(f) = T(x).$$

Theorem VI.2.5 in [11] asserts that S is weakly compact with norm $\|S\| = \|m\|(B_{X^\#})$ and then

$$\|S\| \, Lip(\iota_X) = \|m\|(B_{X^\#}) \leq \|T\|_{\mathcal{PI}_\infty^L} + \varepsilon.$$

(2)$\Longrightarrow$(3). There is a a regular Borel countably additive vector measure $m : \mathcal{B}(K) \longrightarrow E$ of bounded semivariation such that $S(f) = \int_K f dm$ for all $f \in C(K)$ and $\|m\|(K) = \|S\|$ (see [11, Theorem VI.2.1, VI.2.5 and Corollary VI.2.14]). It follows that

$$T(x) = S \circ \varphi(x) = \int_K \varphi(x) dm, \quad x \in X.$$

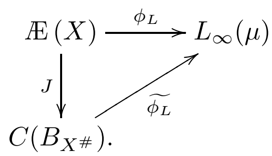
On the other hand, [11, Corollary I.2.6 and Theorem I.2.1] assures the existence of a regular Borel finite measure μ on $\mathcal{B}(K)$ such that $m(A) = 0$ for all $A \in \mathcal{B}(K)$ which satisfy that $\mu(A) = 0$. Define the operator $R \in \mathcal{L}(L_\infty(\mu), E)$ by

$$R(f) = \int_K f dm, \quad f \in L_\infty(\mu)$$

with $\|R\| = \|m\|(K)$ (see [11, Theorem I.1.13]). This operator is weakly compact (see [11, Definition I.1.14 and Theorem VI.1.1]). Consequently,

$$R \circ (j_\infty \circ \varphi) = \int_K j_\infty \circ \varphi dm = \int_K \phi dm = T.$$

(3)$\Longrightarrow$(1). As in the proof of the second implication of Theorem 2.5, starting from the diagram (3.3), consider the linearization ϕ_L of $\phi \in Lip_0(X, L_\infty(\mu))$ and let $\widetilde{\phi_L} \in \mathcal{L}(C(B_{X^\#}), L_\infty(\mu))$ be the extension of ϕ_L, i.e., the following diagram commutes



The linear operator $R \circ \widetilde{\phi_L} : C(B_{X^\#}) \longrightarrow E$ is weakly compact. Let m be the representing vector measure of $R \circ \widetilde{\phi_L}$, that is $R \circ \widetilde{\phi_L}(f) = \int_{B_{X^\#}} f dm$ for all $f \in C(B_{X^\#})$ and $\|m\|(B_{X^\#}) = \left\|R \circ \widetilde{\phi_L}\right\|$. It follows that

$$\begin{aligned} T(x) &= R \circ \phi(x) = R \circ \widetilde{\phi_L} \circ J \circ k_X(x) \\ &= \int_{B_{X^\#}} J \circ k_X(x) dm, \end{aligned}$$

for all $x \in X$, and then $T \in \mathcal{PI}_\infty^L(X, E)$ and

$$\|T\|_{\mathcal{PI}_\infty^L} \leq \|m\|(B_{X^\#}) \leq \|R\| \left\|\widetilde{\phi_L}\right\| = \|R\| \, Lip(\phi).$$

Since this is true for every factorization as (3.3), we have $\|T\|_{\mathcal{PI}_\infty^L} \leq \|R\| \, Lip(\phi)$.

In order to show the reverse inequality, take $T \in \mathcal{PI}_\infty^L(X, E)$ and $\varepsilon > 0$. Then there is $m : \mathcal{B}(B_{X^\#}) \longrightarrow E$ (as in Definition 3.1) such that (3.1) is true and $\|m\|(B_{X^\#}) \leq \varepsilon + \|T\|_{\mathcal{PI}_\infty^L}$. Following the proof of (2)$\Longrightarrow$(3), we can find a regular Borel finite measure μ on $B_{X^\#}$ and a weakly compact operator $R \in \mathcal{L}(L_\infty(\mu), E)$ represented by m such that

$$\|R\| \, Lip(\phi) = \|R\| = \|m\|(B_{X^\#}) \leq \varepsilon + \|T\|_{\mathcal{PI}_\infty^L},$$

where $\phi \in Lip_0(X, L_\infty(\mu))$, is the Lipschitz embedding defined by $\phi = j_\infty \circ \iota_X$. The required inequality follows and the second equality follows in a similar way. $\square$

4. Some relations of Lipschitz Pietsch-p-integral operators with other Lipschitz operator ideals.

4.1. **Lipschitz p-summing operators.** The definition of the Lipschitz p-summing operators below was first given by Farmer and Johnson in [13].

Definition 4.1. For a pointed metric space X and a Banach space E, the mapping $T \in Lip_0(X, E)$ is called Lipschitz p-summing, $1 \leq p < \infty$, if there exists a constant $C > 0$ such that for all $x_1, \ldots, x_n, x_1', \ldots, x_n'$ in X,

$$\sum_{i=1}^{n} \|T(x_i) - T(x_i')\|^p \leq C^p \sup_{f \in B_{X^\#}} \sum_{i=1}^{n} |f(x_i) - f(x_i')|^p. \tag{4.1}$$

In this case we put $\pi_p^L(T) = \inf\{C : \text{satisfying } (4.1)\}$. The set of all Lipschitz p-summing operators from X to E is denoted by $\Pi_p^L(X,E)$.

It is well known that $(\Pi_p^L, \pi_p^L(\cdot))$ is a Banach Lipschitz operator ideal (see [1, Proposition 2.5]).

We can establish the following comparison between the classes of Lipschitz Pietsch-p-integral operators and Lipschitz p-summing operators.

Proposition 4.2. *Let $1 \leq p < \infty$. Every Lipschitz Pietsch-p-integral operator $T : X \longrightarrow E$ is Lipschitz p-summing with $\pi_p^L(T) \leq \|T\|_{\mathcal{PI}_p^L}$.*

Proof. If $T \in \mathcal{PI}_p^L(X,E)$, for $\varepsilon > 0$ we choose a typical Lipschitz Pietsch-p-integral factorization

$$T = A \circ i_p \circ B : X \xrightarrow{B} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{A} E,$$

with $\|A\|\, Lip(B) \leq \varepsilon + \|T\|_{\mathcal{PI}_p^L}$. The mapping i_p is linear p-summing with $\pi_p(i_p) = 1$ (see [10, Page 40]). Then it is Lipschitz p-summing with $\pi_p^L(i_p) = 1$ (see [13, Theorem 2]). By the ideal property concerning the Lipschitz operator ideal Π_p^L, we have that $T \in \Pi_p^L(X,E)$ and $\pi_p^L(T) \leq \|A\|\, Lip(B) \leq \varepsilon + \|T\|_{\mathcal{PI}_p^L}$. □

4.2. **Lipschitz Grothendieck-p-integral operators.** The notion of Lipschitz Grothendieck-p-integral operators ($p \geq 1$) from a pointed metric space X into a Banach space E was introduced by Jiménez-Vargas et al. in [15] (under the name of strongly Lipschitz p-integral operators).

The mapping $T \in Lip_0(X,E)$ is Lipschitz Grothendieck-p-integral (in symbols $T \in \mathcal{GI}_p^L(X,E)$) if $J_E \circ T \in \mathcal{PI}_p^L(X,E^{**})$, where $J_E : E \longrightarrow E^{**}$ is the canonical injection. The class $(\mathcal{GI}_p^L, \|\cdot\|_{\mathcal{GI}_p^L})$ is a Banach Lipschitz operator ideal where $\|T\|_{\mathcal{GI}_p^L} = \|J_E \circ T\|_{\mathcal{PI}_p^L}$ (see [3, Remark 4.3 and Proposition 4.8]). It is immediate that $\mathcal{PI}_p^L(X,E) \subset \mathcal{GI}_p^L(X,E)$ and $\|T\|_{\mathcal{GI}_p^L} \leq \|T\|_{\mathcal{PI}_p^L}$ for all $T \in \mathcal{PI}_p^L(X,E)$.

The proof of the next result is an easy adaptation of [5, Proposition 3.3].

Proposition 4.3. *If the Banach space E is norm one complemented in E^{**} (in particular, if E is a dual Banach space), then $\mathcal{GI}_p^L(X,E) \subset \mathcal{PI}_p^L(X,E)$ and $\|T\|_{\mathcal{GI}_p^L} = \|T\|_{\mathcal{PI}_p^L}$ for all $T \in \mathcal{GI}_p^L(X,E)$.*

4.3. **Strongly Lipschitz p-nuclear operators.** Chen and Zheng in [7] introduced the concept of strongly Lipschitz p-nuclear operators. For a pointed metric space X and a Banach space E, a mapping $T \in Lip_0(X,E)$ is strongly Lipschitz p-nuclear ($1 \leq p < \infty$) if there exist $B \in Lip_0(X,\ell_\infty)$ and $A \in \mathcal{L}(\ell_p,E)$ and a diagonal operator $M_\lambda \in \mathcal{L}(\ell_\infty,\ell_p)$ induced by $\lambda = (\lambda_i)_{i\geq 1} \in \ell_p$ (i.e. $M_\lambda((\xi_i)_{i\geq 1}) = (\lambda_i \xi_i)_{i\geq 1}$) such that

$$T = A \circ M_\lambda \circ B : X \xrightarrow{B} \ell_\infty \xrightarrow{M_\lambda} \ell_p \xrightarrow{A} E.$$

The Banach space of all these mappings is denoted by $\mathcal{SN}_p^L(X,E)$ and the norm is defined by $\|T\|_{\mathcal{SN}_p^L} = \inf \|A\| \, \|M_\lambda\| \, Lip(B)$, where the infimum is taken over all the above factorizations.

Proposition 4.4. *Every strongly Lipschitz p-nuclear operator is Lipschitz Pietsch-p-integral. Moreover, $\|T\|_{\mathcal{PI}_p^L} \leq \|T\|_{\mathcal{SN}_p^L}$ for all $T \in \mathcal{SN}_p^L(X,E)$.*

Proof. Given $\varepsilon > 0$, take $T \in \mathcal{SN}_p^L(X,E)$ with the above factorization such that

$$\|A\| \, \|M_\lambda\| \, Lip(B) \leq \varepsilon + \|T\|_{\mathcal{SN}_p^L}.$$

In this case, ℓ_∞ and ℓ_p are the spaces $L_\infty(\mu)$ and $L_p(\mu)$ with μ the counting measure on $\mathbb{N}$ respectively and $M_\lambda : L_\infty(\mu) \longrightarrow L_p(\mu)$ is the multiplication operator induced by $\lambda \in L_p(\mu)$ (i.e. $M_\lambda(f) = \lambda.f$). Use [10, Page 111] to see that M_λ is a Pietsch-p-integral linear operator and $\|M_\lambda\|_{\mathcal{PI}_p} = \|M_\lambda\|$ and then it is Lipschitz Pietsch-p-integral with $\|M_\lambda\|_{\mathcal{PI}_p^L} \leq \|M_\lambda\|$ (by Remark 2.2). In view of the ideal property of $\mathcal{PI}_p^L$, we are done. □

4.4. **Lipschitz weakly compact operators.** The definition of Lipschitz weakly compact operators is due to Jiménez-Vargas et al. ([15]).

Definition 4.5. Let X be a pointed metric space and let E be a Banach space. The mapping $T \in Lip_0(X,E)$ is called Lipschitz weakly compact if the set $\left\{ \frac{T(x)-T(x')}{d(x,x')} : x, x' \in X, x \neq x' \right\}$ is relatively weakly compact in E. The set of all Lipschitz weakly compact operators from X into E is denoted by $Lip_{0\mathcal{W}}(X,E)$.

Proposition 2.8 in [15] asserts that every Lipschitz Grothendieck-p-integral operator is Lipschitz weakly compact. So, according to the comments above we have that $\mathcal{PI}_p^L(X,E) \subset Lip_{0\mathcal{W}}(X,E)$.

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