RACSAM manuscript No.
(will be inserted by the editor)

On weakly compact sets in $C(X)$

J. C. Ferrando · S. López-Alfonso

Received: date / Accepted: date

Abstract A subset A of a locally convex space E is called (relatively) sequentially complete if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in E contained in A converges to a point $x \in A$ (a point $x \in E$). Asanov and Velichko proved that if X is countably compact, every functionally bounded set in $C_p(X)$ is relatively compact, and Baturov showed that if X is a Lindelöf Σ-space, each countably compact (so functionally bounded) set in $C_p(X)$ is a monolithic compact. We show that if X is a Lindelöf Σ-space, every functionally bounded (relatively) sequentially complete set in $C_p(X)$ or in $C_w(X)$, i.e., in $C_k(X)$ equipped with the weak topology, is (relatively) Gul'ko compact. We get some consequences.

Keywords Lindelöf Σ-space · realcompact space · μ-space · sequentially complete set

Mathematics Subject Classification (2000) 54C35 · 54C05 · 46A50

1 Preliminaries

In what follows, unless otherwise stated, X will be a Tychonoff space and $C_p(X)$ or $C_k(X)$ denote the linear space $C(X)$ of real-valued continuous functions on X equipped with the pointwise τ_p or the compact-open topology τ_k, respectively. We represent by δ_x the evaluation map $\delta_x : C(X) \to \mathbb{R}$ at $x \in X$, defined by $\langle \delta_x, f \rangle = f(x)$. If $L(X)$ stands for the topological dual of $C_p(X)$ and $L_p(X)$ denotes the linear space $L(X)$ endowed with the weak*

J. C. Ferrando
Centro de Investigación Operativa.
Universidad Miguel Hernández,
E-03202 Elche, Spain
E-mail: jc.ferrando@umh.es

S. López-Alfonso
Depto. Construcciones Arquitectónicas.
Universitat Politècnica de València,
E-46022 València, Spain
E-mail: salloal@csa.upv.es

Acknowledgement. This work was supported for the first named author by the Grant PGC2018-094431-B-I00 of Ministry of Science, Innovation and Universities of Spain.

topology $\sigma(L(X), C(X))$, the map $\delta : X \to L_p(X)$ defined by $\delta(x) = \delta_x$ is a homeomorphism from X into a closed subspace of $L_p(X)$, the latter being a closed linear subspace of $C_p(C_p(X))$ (see [1, Chapter 0]). The image $\delta(X)$ of X by δ is referred to as the *canonical copy* of X in $L_p(X)$, or in $C_p(C_p(X))$. In the sequel υX will stand for the *Hewitt realcompactification* of X and βX for the *Stone-Čech compactification* of X. A Tychonoff space X is said to be *realcompact* if $\upsilon X = X$ and *pseudocompact* if $\upsilon X = \beta X$. If $f \in C(X)$ we denote by f^υ the continuous extension of f to υX. A topological space X is called *Fréchet-Urysohn* if for any set Y in X and each $y \in \overline{Y}$ there is a sequence $\{y_n\}_{n=1}^\infty$ in Y such that $y_n \to y$ in X, and *angelic* if each relatively countably compact set A in X is relatively compact and if $x \in \overline{A}$ there is a sequence in A that converges to x. A set A in a topological space X is called *functionally bounded* in X if $f(A)$ is bounded in $\mathbb{R}$ for each $f \in C(X)$. A topological space X is a μ*-space* if each functionally bounded set in X is relatively compact. A set A in a locally convex space E is called *bounded* (in E) if $u(A)$ is bounded for each $u \in E'$, where E' stands for the topological dual of E. So, a set A is bounded in $C_p(X)$ if $u(A)$ bounded for each $u \in L(X)$, and a subset Y of X is functionally bounded in X if and only if $\delta(Y)$ is bounded in $L_p(X)$. A locally convex space E is *semi-reflexive* if each bounded set in E is relatively weakly compact.

Recall that a completely regular space X is called a *P-space* if every G_δ-set in X is open, and a *Lindelöf* Σ*-space* if there is a (set-valued) map $T : \Sigma \to \mathscr{K}(X)$ from a subspace Σ of $\mathbb{N}^\mathbb{N}$ into the family $\mathscr{K}(X)$ of compact sets of X with $\bigcup\{T(\alpha) : \alpha \in \Sigma\} = X$ such that if $\{\alpha_n\}_{n=1}^\infty$ is a sequence in Σ with $\alpha_n \to \alpha$ and $x_n \in T(\alpha_n)$ for every $n \in \mathbb{N}$, the sequence $\{x_n\}_{n=1}^\infty$ has a cluster point $x \in T(\alpha)$. If $\Sigma = \mathbb{N}^\mathbb{N}$ we say that X is *K-analytic*. Each K-analytic space is a Lindelöf Σ-space, and every Lindelöf Σ-space is Lindelöf.

The subject of this paper is C_p and C_k-theory, two very active fields of research nowadays (see for instance [2] and [14]).

2 Functionally bounded sequentially complete sets in $C_p(X)$

Let E be a (Hausdorff) locally convex space. Recalling the notion of (relative) sequential completeness introduced in [11], we say that a set A in E is (relatively) *sequentially complete* if every Cauchy sequence $\{x_n\}_{n=1}^\infty$ in E contained in A converges in E to a point $x \in A$ (a point $x \in E$). Every closed relatively sequentially complete set is sequentially complete. If E is sequentially complete, each closed (arbitrary) set A is (relatively) sequentially complete. So, a sequentially complete set in E need not be bounded. If E is Fréchet-Urysohn and A sequentially complete, then A is closed. As is well-known υX is sequentially closed in βX (see [17, Theorem 1]). So, if βX is sequential, X is pseudocompact. If X is sequentially complete in E and either βX or υX is Fréchet-Urysohn, then X is respectively compact or realcompact. Main use of (relatively) sequentially complete sets in C_p-theory so far are the two following results, where the second is an extension of a result of Velichko [1, 1.2.1 Theorem].

Theorem 1 ([11, Theorem 3.1]). *The following are equivalent.*

1. *X is a P-space.*
2. *$C_p(X)$ is a countable union of relatively sequentially complete sets.*

Corollary 1 ([11, Corollary 3.2]). *If $C_p(X)$ is a countable union of bounded relatively sequentially complete sets, then X is finite.*

Every (relatively) countably compact set in a locally convex space E is both functionally bounded in E and (relatively) sequentially complete. Not every functionally bounded sequentially complete set in a $C_p(X)$ space is countably compact. For example, since $\beta\mathbb{N}$ is homeomorphic to a compact set in $C_p(C_p(\beta\mathbb{N}))$, the discrete space $\mathbb{N}$ of positive integers is homeomorphic to a functionally bounded and sequentially complete set in $C_p(C_p(\beta\mathbb{N}))$ which is relatively countably compact but not countably compact. If X is separable, countably compact, or a $k_{\mathbb{R}}$-space then $C_p(X)$ is a μ-space by Valdivia's [13, 2.3 Theorem], Asanov-Velichko's [1, 3.4.1 Theorem] or [1, 3.4.13 Theorem], respectively (see also [9, Theorems 32, 34]). So, in all these cases each functionally bounded set in $C_p(X)$ is relatively compact. Also, if X has countable $\mathbb{R}$-tightness in the sense of [1, Page 59] the space $C_p(X)$ is realcompact (see [1, 2.4.17 Corollary] or [9, Theorem 27]), hence a μ-space. However, the next theorem holds.

Theorem 2. *If X is a Lindelöf Σ-space, each functionally bounded (relatively) sequentially complete set in $C_p(X)$ is (relatively) compact.*

Proof. We consider the relatively sequentially complete setting, the other case is similar. By hypothesis there are a subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a map T from Σ into the family $\mathscr{K}(X)$ of compact sets of X such that $\{T(\alpha):\alpha\in\Sigma\}$ covers X and if $\alpha_n\to\alpha$ in Σ and $x_n\in T(\alpha_n)$ for all $n\in\mathbb{N}$ then $\{x_n\}_{n=1}^{\infty}$ has a cluster point $x\in T(\alpha)$. Let H be a functionally bounded relatively sequentially complete set in $C_p(X)$, whose closure $\overline{H}^{\upsilon C_p(X)}$ in $\upsilon C_p(X)$ we shall represent by K. As H is functionally bounded in $C_p(X)$, clearly K is a compact subset of $\upsilon C_p(X)$, [13, 4.7 Proposition]. Note that each $\delta_x\in L(X)$ with $x\in X$ is a $\sigma(C(X),L(X))$-continuous linear form on $C(X)$. Denote by δ_x^{υ} the (unique) continuous extension of δ_x to the Hewitt realcompactification $\upsilon C_p(X)$ of $C_p(X)$ and define

$$S_\alpha=\{\,\delta_x^{\upsilon}|_K : x\in T(\alpha)\}\subseteq C(K)$$

for each $\alpha\in\Sigma$. We claim that S_α is a compact subset of $C_p(K)$.

Let us show in first place that S_α is countably compact. If $\{\,\delta_{x_n}^{\upsilon}\big|_K : n\in\mathbb{N}\}$ is a sequence in S_α there are $x\in T(\alpha)$ and a subnet $\{y_d:d\in D\}$ of $\{x_n\}_{n=1}^{\infty}$ such that $y_d\to x$ in $T(\alpha)$ under the relative topology of X, so that $f(y_d)\to f(x)$ or rather $\langle\delta_{y_d},f\rangle\to\langle\delta_x,f\rangle$ for all $f\in C(X)$. Hence, for each $u\in\upsilon C_p(X)$ there is $f_u\in C(X)$ with $\delta_{x_n}^{\upsilon}(u)=\delta_{x_n}(f_u)$ for every $n\in\mathbb{N}$ (see [16, Lemma 9.1]). So, using that $\langle\delta_{y_d},f\rangle\to\langle\delta_x,f\rangle$ for every $f\in C(X)$, it follows that $\delta_{y_d}^{\upsilon}(u)\to\delta_x^{\upsilon}(u)$ for all $u\in\upsilon C_p(X)$. In particular

$$\delta_{y_d}^{\upsilon}\big|_K(u)\to\ \delta_x^{\upsilon}|_K(u)$$

for every $u\in K$, which means that $\delta_{y_d}^{\upsilon}\big|_K\to\ \delta_x^{\upsilon}|_K$ on S_α under the relative topology of $C_p(K)$. This shows that S_α is a countably compact subspace of $C_p(K)$. But, given that K is compact, $C_p(K)$ is angelic by virtue of the classic Grothendieck theorem [25, Section 1, Theorem 3]. So we conclude that S_α is compact.

Set $M:=\cup\{S_\alpha:\alpha\in\Sigma\}\subseteq C(K)$. We claim that M is a Lindelöf Σ-subspace of $C_p(K)$. Define the mapping $S:\Sigma\to\mathscr{K}(C_p(K))$ by the rule $S(\alpha)=S_\alpha$. If $\alpha_n\to\alpha$ in Σ and $h_n\in S(\alpha_n)$ for each $n\in\mathbb{N}$, then $h_n=\delta_{z_n}^{\upsilon}\big|_K$ for some $z_n\in T(\alpha_n)$ and $n\in\mathbb{N}$. Let $z\in T(\alpha)$ be a cluster point of the sequence $\{z_n\}_{n=1}^{\infty}$ in X, so that δ_z is a cluster point of the sequence $\{\delta_{z_n}\}_{n=1}^{\infty}$ in $C_p(C_p(X))$. Setting $h:=\delta_z^{\upsilon}\big|_K\in S(\alpha)$, it can be shown as before that h is a cluster point of $\{h_n\}_{n=1}^{\infty}$ in $C_p(K)$ belonging to $S(\alpha)$. So, M is a Lindelöf Σ-subspace of $C_p(K)$, as stated.

Next we claim that M separates the points of K. Otherwise there are $u \neq v$ in K such that $h(u) = h(v)$ for every $h \in M$. This means that $\delta_x^{\upsilon}(u) = \delta_x^{\upsilon}(v)$ for every $x \in X$. If $\{f_d : d \in D\}$ and $\{g_d : d \in D\}$ are nets in H with $f_d \to u$ and $g_d \to v$ in K, then

$$(f_d - g_d)(x) \to \delta_x^{\upsilon}(u) - \delta_x^{\upsilon}(v) = 0$$

so that $f_d - g_d \to 0$ in $C_p(X)$. Hence $f_d - g_d \to 0$ in $\upsilon C_p(X)$, which yields $u - v = 0$.

Since K is compact and M a Lindelöf Σ-subspace of $C_p(K)$ that separates the points of K, [22, Theorem 3.4] (see also [9, Theorem 91]) ensures that $C_p(K)$ is a Lindelöf Σ-space. Hence K is a Gul'ko compact subset of $\upsilon C_p(X)$, so a Fréchet-Urysohn space (see for instance [8, Lemma 2]).

Finally we claim that $K \subseteq C(X)$. Indeed, if $u \in K$ then $u \in \overline{H}^{\upsilon C_p(X)}$, where the closure is in $\upsilon C_p(X)$. Thus, there is a sequence $\{f_n\}_{n=1}^{\infty}$ in H such that $f_n \to u$ in $\upsilon C_p(X)$. Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the locally convex space $C_p(X)$ and H is relatively sequentially complete in $C_p(X)$, it follows that $u \in C(X)$. Therefore H is relatively compact in $C_p(X)$ and we are done. □

According to Theorem 2, each $C_p(X)$ space over a Lindelöf Σ-space X has the property that every functionally bounded relatively sequentially complete set in $C_p(X)$ is relatively compact. Although obviously every locally convex μ-space $C_p(X)$ enjoys such property, the converse statement does not hold, as the following example shows.

Example 1. *If X is a Lindelöf Σ-space space then $C_p(X)$, though angelic, need not be a μ-space.* Let Z be the Reznichenko compact space mentioned in [1, Example 7.14]. This is a Talagrand compact space with a nonisolated point p such that $Z = \beta Y$ with $Y = Z \setminus \{p\}$. Hence Y is a pseudocompact not realcompact space, so that $Z = \upsilon Y$. So, $C_p(Y)$ is a continuous image of $C_p(Z)$. This shows that $C_p(Y)$ is K-analytic, which implies that $C_p(X)$ is angelic if $X := C_p(Y)$. Observe that Y is (homeomorphic to) a closed functionally bounded set in $C_p(X)$ which is not compact. Consequently $C_p(X)$, though angelic, is not a μ-space. Of course, Y is not sequentially complete in $C_p(X)$. Otherwise it would be compact by virtue of Theorem 2. Since p is non isolated in Z, one has $p \in \overline{Y}^{Z}$. So, bearing in mind that Z is a Fréchet-Urysohn space, there is a sequence $\{x_n\}_{n=1}^{\infty}$ in Y that converges to p in $Z = \upsilon Y$. This proves that Y is not countably compact, so neither a normal space. Therefore $C_p(X)$ is not a normal space either, as mentioned in [9, Example 62].

Let us say that a subset Y of a topological space X is (*relatively*) *sequentially closed* in υX if conditions $\{x_n\}_{n=1}^{\infty} \subseteq Y$ and $x_n \to x$ in υX imply that $x \in Y$ (resp. that $x \in X$). If we look at the proof of the previous theorem, the hypothesis that H is (relatively) sequentially complete in $C_p(X)$ is only used at the bottom, and it could be replaced by the (formally less restrictive) condition that H is relatively sequentially closed in $\upsilon C_p(X)$. So, the following theorem holds.

Theorem 3. *If X is a Lindelöf Σ-space, each set in $C_p(X)$ which is both* (*relatively*) *sequentially closed and relatively compact in $\upsilon C_p(X)$ is* (*relatively*) *compact in $C_p(X)$.*

Proof. This is a straightforward consequence of the proof of the previous theorem and the fact that a subset Y of a Hausdorff topological space X is functionally bounded if and only if Y is relatively compact in υX (see [13, 4.7 Proposition]). □

Next example shows that in the Theorem 2 the condition of being functionally bounded cannot be replaced by the formally weaker condition of being (pointwise) bounded.

Example 2. *If X is a compact space* (*even metrizable*)*, a bounded sequentially complete set in $C_p(X)$ need not be compact.* Let X be the dual unit ball of ℓ_1 equipped with the weak* topology. The mapping $T : \ell_1\,(\text{weak}) \to C_p(X)$ defined by $T\xi = \xi|_X$ is a linear homeomorphism from $\ell_1\,(\text{weak})$ onto its image in $C_p(X)$. If B_{ℓ_1} denotes the closed unit ball of ℓ_1, then $T\left(B_{\ell_1}\right)$ is a bounded and sequentially complete in $C_p(X)$ which is not compact, otherwise B_{ℓ_1} would be compact in $\ell_1\,(\text{weak})$.

If $X = D(\mathfrak{m})$ is the discrete space of cardinality $\mathfrak{m}$, every (functionally or not) bounded set in $C_p(X) = \mathbb{R}^{\mathfrak{m}}$ is relatively compact. However, if $\mathfrak{m} > \aleph_0$ then X is not Lindelöf. So, the condition of X being a Lindelöf Σ-space in the statement of Theorem 2 is sufficient but not necessary. If $\mathfrak{m} = \aleph_1$ then $\mathbb{R}^{\aleph_1}$ is not angelic since $[0,1]^{\aleph_1}$ is not Fréchet-Urysohn. This shows that the property exhibited in the statement of Theorem 2 is not a consequence of the angelicity of $C_p(X)$. Let us mention in passing that each non realcompact P-space X is a μ-space which is not realcompact (for concrete X see [15, Problem 9L]). So, if X is as stated, $C_p(X)$ is Fréchet-Urysohn (hence a $k_{\mathbb{R}}$-space) and hence $C_p(C_p(X))$ is a μ-space [1, 3.4.13 Theorem]. Clearly $C_p(C_p(X))$ is not realcompact, since X is homeomorphic to a closed set of $C_p(C_p(X))$. This can be taken even further.

Example 3. *If X is a compact space then $C_p(X)$, although an angelic μ-space, need not be realcompact.* Since $\omega_1 + 1$ is compact, $C_p(\omega_1 + 1)$ is a μ-space by Grothedieck's theorem [25, Section 1, Theorem 3]. One can easily check that the $\mathbb{R}$-tightness of $\omega_1 + 1$ is uncountable, hence $C_p(\omega_1 + 1)$ is not realcompact.

If X is a P-space then $C_p(X)$ is sequentially complete (see [11, Theorem 1.1]), so every closed (arbitrary) subset A of $C_p(X)$ is (relatively) sequentially complete. Obviously, if X is a countable P-space (hence discrete and Lindelöf) and A is a bounded set in $C_p(X)$, then A is relatively compact. But if X is a non-discrete P-space, the topological subspace $C_p(X,[0,1])$ is countably compact but not compact [24, Problems 396, 397]. Hence, in this case $C_p(X)$ contains a closed functionally bounded sequentially complete set which is not compact. Even if X is an (uncountable) Lindelöf P-space, a closed functionally bounded set A in $C_p(X)$ need not be compact, as the following example shows.

Example 4. *If X is a Lindelöf P-space, a functionally bounded sequentially complete set in $C_p(X)$ need not be compact.* Let us denote by E the linear subspace of $\mathbb{R}^{\aleph_1}$ consisting of those functions with countable support and let X be the Lindelöfication of the set $\omega_1 = [0,\omega_1)$ equipped with the discrete topology. This means that we add a point, say ω_1, to $[0,\omega_1)$, so that $X = [0,\omega_1) \cup \{\omega_1\} = [0,\omega_1]$ as sets, and declare ω_1 as the unique non isolated point of X, with the neighborhoods of ω_1 being the subsets V of X such that $\omega_1 \in V$ and $X \setminus V$ is countable. It can be easily checked that X is a Lindelöf P-space. By [1, 4.2.15 Example] the space E is homeomorphic to the closed one-codimensional linear subspace $\{f \in C(X) : f(\omega_1) = 0\}$ of $C_p(X)$. Let $A = [0,1]^{\aleph_1}$. This is a compact set in $\mathbb{R}^{\aleph_1}$. Then $B = A \cap E$ is a counbably compact subset of E with the relative topology. Hence it is (homeomorphic to) a functionally bounded sequentially complete subset of $C_p(X)$. But B is not compact, since B is dense in A and does not coincide with A. Of course $C_p(X)$ is not a μ-space.

Corollary 2 (Baturov, [3])**.** *If X is a Lindelöf Σ-space, then every countably compact set in $C_p(X)$ is a monolithic compact set.*

Proof. As mentioned above, each countably compact set A in $C_p(X)$ is functionally bounded and sequentially complete. So, Theorem 2 applies to show that A is compact. Since each compact set in $C_p(X)$ is Gul'ko's, then A is monolithic. □

Baturov's original theorem can also be found in [1, 3.6.1 Theorem]. For a more general version of this theorem see [18, Theorem 2.4], where it is shown that if the *Lindelöf number* $\ell(Y)$ of a topological subspace Y of $C_p(X)$ is greater or equal than the *number of K-determination*, also called the *$\ell\Sigma$-number* $\ell\Sigma(X)$, of the Tychonoff space X (see [6]), then $\ell(Y) = e(Y)$ where $e(Y)$ denotes the *extent* of Y. For a Lindelöf Σ-space X one has $\ell\Sigma(X) = \aleph_0 \leq \ell(Y)$.

Corollary 3 (Orihuela, [19, Theorem 3])**.** *If X is a Lindelöf Σ-space, then the space $C_p(X)$ is angelic.*

Proof. If A is a (relatively) countably compact set in $C_p(X)$, Theorem 2 shows that A is (relatively) compact. Since every compact set in $C_p(X)$ is Fréchet-Urysohn [8, Lemma 2], if A is relatively compact in $C_p(X)$ and $f \in \overline{A}$, there is a sequence $\{f_n\}_{n=1}^{\infty}$ contained in A such that $f_n \to f$ in $C_p(X)$. Therefore $C_p(X)$ is angelic. □

Orihuela's angelicity theorem can also be found in [16, Theorem 4.5]. The original statement of the theorem is slightly more general than the one given here, but it can be seen to be equivalent to the previous corollary (see [8, Theorem 1]). Besides of the aforementioned $\mathbb{R}^{\aleph_1}$, there are many non-angelic $C_p(X)$ spaces which are μ-spaces. Indeed, if X is a compact set which is not Fréchet-Urysohn, the space $C_p(C_p(X))$ is non-angelic but realcompact [9, Theorem 96]. On the other hand, we have also mentioned before that $C_p(\omega_1 + 1)$, which is angelic by Theorem 3, is not realcompact, although it is a μ-space. We shall exhibit below (in the Example 1) an angelic $C_p(X)$ space over a Lindelöf Σ-space X which is not a μ-space.

Corollary 4. *Let X be a pseudocompact space such X is sequentially closed in υX. If $C_p(X)$ is a Lindelöf Σ-space, then X is compact.*

Proof. We claim that the canonical copy $\delta(X)$ of X in $L_p(X)$ is sequentially complete in $L_p(X)$, so in $C_p(C_p(X))$, since $L_p(X)$ is closed in $C_p(C_p(X))$. Indeed, if $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\{\delta_{x_n} : n \in \mathbb{N}\}$ is a Cauchy sequence in $L_p(X)$ contained in $\delta(X)$, then $\{\langle f, \delta_{n_n}\rangle\}_{n=1}^{\infty}$, i. e., $\{f(x_n)\}_{n=1}^{\infty}$, is a Cauchy sequence in $\mathbb{R}$ for every $f \in C(X)$. So, according to [15, 15.13 Theorem], there is $y \in \upsilon X$ such that $x_n \to y$ in υX. Since we are assuming that X is sequentially closed in υX, it follows that $y \in X$, so that $\delta_{n_n} \to \delta_y \in \delta(X)$ in $L_p(X)$. This shows that $\delta(X)$ is a sequentially complete set in $L_p(X)$, as stated. So $\delta(X)$ is a functionally bounded (because it is pseudocompact) and sequentially complete set in $C_p(C_p(X))$. Since $C_p(X)$ is a Lindelöf Σ-space, Theorem 2 applies to show that $\delta(X)$ is compact. So that X is a compact set. □

Corollary 5. *If X is a Lindelöf Σ-space, the following statements are equivalent.*

1. *$C_p(X)$ is the union of countable many functionally bounded relatively sequentially complete sets.*
2. *$C_p(X)$ is σ-compact.*
3. *X finite.*

Proof. This follows from Theorem 2 and Velichko's theorem [1, 1.2.1 Theorem]. □

It is worthwhile mentioning that, according to [23], if $C_p(X)$ is covered by countably many functionally bounded sets, then X is pseudocompact and every countable subset of X is closed and C^*-embedded in X (see also [24, Problem 399] or [12, Theorem 2.6]).

Corollary 6. *If there exists a metrizable locally convex topology τ on $C(X)$ such that $\tau_p \leq \tau \leq \tau_k$ then each functionally bounded* (*relatively*) *sequentially complete set in $C_p(X)$ is* (*relatively*) *compact.*

Proof. This is because if such metrizable topology there exists, then X is σ-compact [10, Theorem 3.1 (i)], hence K-analytic. So, Theorem 2 applies. □

3 Functionally bounded sequentially complete sets in $C_w(X)$

We denote by $C_w(X)$ the space $C_k(X)$ equipped with the weak topology of $C_k(X)$. If $C_k(X)$ is quasi-complete [4, 1.7.2 Definition], in particular if X is a $k_{\mathbb{R}}$-space, then $C_w(X)$ is a μ-space by virtue of [26, Theorem 3] (see also [9, Theorem 31]). So that, in this case, every functionally bounded set in $C_w(X)$ is relatively weakly compact. On the other hand, if A is a bounded set in $C_k(X)$ such that A is sequentially complete in $C_p(X)$ then A is weakly sequentially complete in $C_k(X)$. For if $\{f_n\}_{n=1}^{\infty}$ is a weakly Cauchy sequence in $C_k(X)$ contained in A, then $\{f_n\}_{n=1}^{\infty}$ converges in $C_p(X)$ to some $f \in A$. So, according to the version of the Lebesgue dominated convergence theorem for measures of compact support, the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f in the weak topology of $C_k(X)$, which shows that A is weakly sequentially complete [13, 4.3 Corollary 2]. When X is a Lindelöf Σ-space, the following analogous to Theorem 2 holds. Our proof is based on [8, Theorem 7].

Theorem 4. *If X is a Lindelöf Σ-space, each functionally bounded* (*relatively*) *sequentially complete set in $C_w(X)$ is* (*relatively*) *weakly compact.*

Proof. Assume that there are a subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a map T from Σ into $\mathscr{K}(X)$ such that $\{T(\alpha) : \alpha \in \Sigma\}$ covers X and if $\alpha_n \to \alpha$ in Σ and $x_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$ then $\{x_n\}_{n=1}^{\infty}$ has a cluster point $x \in T(\alpha)$. Let F denote the completion of $(C(X), \rho(C(X), L(X)))$, where $\rho(C(X), L(X))$ stands for the topology on $C(X)$ of uniform convergence on the compact sets of $L_p(X)$. Since X is a Lindelöf space, hence realcompact, the space $C_k(X)$ is barrelled by the Nachbin-Shirota theorem. Hence the compact-open topology τ_k of $C(X)$ coincides with the strong topology $\beta(C(X), E)$, where E denotes the topological dual of $C_k(X)$. This implies that $\tau_k = \rho(C(X), L(X)) = \mu(F, E)$.

Let H be a functionally bounded relatively sequentially complete set in $C_w(X)$, whose closure under the weak topology $\sigma(F, E)$ we shall represent by K. Since H is clearly functionally bounded in $(F, \sigma(F, E))$ and $(F, \mu(F, E))$ is complete, Valdivia's theorem [26, Theorem 3] ensures that K is $\sigma(F, E)$-compact. Let $\widehat{\delta_x}$ be the $\sigma(F, L(X))$-continuous linear extension of δ_x to F and put $S_\alpha = \{\widehat{\delta_x}|_K : x \in T(\alpha)\}$ for $\alpha \in \Sigma$. If $\{\widehat{\delta}_{x_d}|_K : d \in D\}$ is a net in S_α there are $x \in T(\alpha)$ and a subnet $\{y_h\}_{h \in E}$ such that $y_h \to x$ in $T(\alpha)$, so that $\delta_{y_h} \to \delta_x$ under $\sigma(L(X), C(X))$. Since $\sigma(L(X), C(X))$ and $\sigma(L(X), F)$ coincide on $Q_\alpha := \{\delta_x : x \in T(\alpha)\}$ (see [21, Chapter VI, Corollary 3]), we have $\widehat{\delta}_{y_h} \to \widehat{\delta}_x$ in S_α under $\sigma(L(X), F)$, which implies that $\langle \widehat{\delta}_{y_h}|_K, g \rangle \to \langle \widehat{\delta}_x|_K, g \rangle$ for every $g \in K$. So, S_α is compact in $C_p(K)$. Set $M := \bigcup \{S_\alpha : \alpha \in \Sigma\}$ and define $S : \Sigma \to \mathscr{K}(C_p(K))$ by $S(\alpha) = S_\alpha$. If $\alpha_n \to \alpha$ in Σ and $y_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$, let $y \in T(\alpha)$ be a cluster point in X of the sequence $\{y_n\}_{n=1}^{\infty}$. Since $\bigcup_{n=1}^{\infty} Q_{\alpha_n}$ is relatively countably compact in $L_p(X)$ and $L_p(X)$ is a Lindelöf space by [1, 0.5.14 Corollary], $\bigcup_{n=1}^{\infty} Q_{\alpha_n}$ is relatively compact in $L_p(X)$. Thus

$\sigma(L(X), C(X))$ and $\sigma(L(X), F)$ coincide on $\bigcup_{n=1}^{\infty} Q_{\alpha_n}$ and $\widehat{\delta}_y$ is a $\sigma(L(X), F)$-cluster point of $\{\widehat{\delta}_{y_n}\}_{n=1}^{\infty}$, which implies that $\widehat{\delta}_y|_K$ is a cluster point of $\{\widehat{\delta}_{x_n}|_K\}_{n=1}^{\infty}$ in $C_p(K)$. This shows that M is a Lindelöf Σ-subspace of $C_p(K)$. As $\langle L(X), F\rangle$ is a dual pair, M separates the points of K. So, $C_p(K)$ is a Lindelöf Σ-space and K is Fréchet-Urysohn. Consequently, if $u \in K$ there is a sequence $\{f_n\}_{n=1}^{\infty}$ in H such that $f_n \to u$ under $\sigma(F, E)$, which implies that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $C_w(X)$. Since H is relatively sequentially complete in $C_w(X)$, it follows that $f_n \to g$ in $C_w(X)$. Therefore $f = g \in C(X)$, which shows that $K \subseteq C(X)$. Thus K is a weakly relatively compact set in $C_k(X)$. □

As in the $C_p(X)$ case, we cannot replace in Theorem 4 the functionally boundedness requirement by the boundedness condition (in the locally convex sense). If X is a nonscattered compact, by the Pełczyński-Semadeni theorem [20] or [7, Theorem 3.1.1], the space $C_w(X)$ contains an isomorphic copy of ℓ_1 (weak). So, despite X is a Lindelöf Σ-space, $C_w(X)$ contains a bounded sequentially complete set which is not compact.

References

1. Arkhangel'skiĭ, A. V., *Topological Function Spaces*, Math. Appl. **78**, Kluwer Academic Publishers, Dordrecht, Boston, London (1992)
2. Banakh, T., Ką kol, J. and Śliwa, W., *Josefson-Nissenzweig property for C_p-spaces*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **113**, 3015–3030 (2019)
3. Baturov, D. P., *Subspaces of function spaces*, Vestnik Moskov. Univ. Ser. I, Mat. Mech. **4**, 66–69 (1987)
4. Bogachev, V. I. and Smolyanov, O. G., *Topological Vector Spaces and Their Applications*, Springer, Heidelberg, New York, Dordrecht, London (2017)
5. Buzyakova, R. Z., *In search of Lindelöf C_p's*, Comment. Math. Univ. Carolinae **45**, 145–151 (2004)
6. Cascales, B., Muñoz, M. and Orihuela, J., *The number of K-determination of topological spaces*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **106**, 341–357 (2012)
7. Cembranos, P. and Mendoza, J., *Banach Spaces of Vector-Valued Functions*, Lecture Notes in Math. **1676**, Springer, Berlin, Heidelberg (1997)
8. Ferrando, J. C., *On a Theorem of D. P. Baturov*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **111**, 499–505 (2017)
9. Ferrando, J. C., *Descriptive topology for analysts*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **114**, Paper No. 107, 34 pp. (2020)
10. Ferrando, J. C., Gabriyelyan, S. and Kąkol, J., *Metrizable-like locally convex topologies on $C(X)$*. Topology Appl. **230**, 105–113 (2017)
11. Ferrando, J. C., Kąkol, J. and Saxon, S. A., *Characterizing P-spaces in terms of $C(X)$*, J. Convex. Anal. **22**, 905–915 (2015)
12. Ferrando, J. C. and López-Pellicer, M., *Covering properties of $C_p(X)$ and $C_k(X)$*, Filomat, to appear.
13. Floret, K., *Weakly Compact Sets*, Lecture Notes in Math. **801**, Springer-Verlag, Berlin, Heidelberg (1980)
14. Gabriyelyan, S., *Ascoli's theorem for pseudocompact spaces*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **114**, Paper No. 174, 10 pp. (2020)
15. Gillman, L., Jerison, M., *Rings of Continuous Functions*, Van Nostrand, Princeton, (1960).
16. Kąkol, J., Kubis, J. and López-Pellicer, M., *Descriptive Topology in Selected Topics of Functional Analysis*, Springer, Heidelberg, New York, Dordrecht, London (2011)
17. King, D. M. and Morris, S. A., *The Stone-Čech compactification and weakly Fréchet spaces*, Bull. Austral. Math. Soc. **42**, 340–352 (1990)
18. Muñoz, M., *A note on the theorem of Baturov*, Bull. Austral. Math. Soc. **76**, 219–225 (2007)
19. Orihuela, J., *Pointwise compactness in spaces of continuous functions*, J. London Math. Soc. **36**, 143–152 (1987)
20. Pełczyński, A. and Semadeni, Z., *Spaces of continuous functions* (*III*) (*Spaces $C(\Omega)$ for Ω without perfect sets*), Studia Math. **18**, 211–222 (1959)
21. Robertson, A. P. and Robertson, W., *Topological Vector Spaces*, Cambridge University Press, Cambridge (1973)
22. Talagrand, M., *Espaces de Banach faiblement K-analytiques*, Annals of Math. **110**, 407–438 (1979).
23. Tkachuk, V. V., *The space $C_p(X)$: decomposition into a countable union of bounded subspaces and completeness properties*, Topology Appl. **22**, 241–253 (1986)

24. Tkachuk, V. V., *A C_p-Theory Problem Book. Topological and Function Spaces*. Springer, Heidelberg, New York, Dordrecht, London (2011)
25. Todorcevic, S., *Topics in Topology*, Springer, Berlin, Heidelberg, 1997.
26. Valdivia, M., *Some new results on weak compactness*, J. Funct. Anal. **24**, 1–10 (1977)