Hyperspace of a fuzzy quasi-uniform space

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Abstract

The aim of this paper is to present a fuzzy counterpart method of constructing the Hausdorff quasi-uniformity of a crisp quasi-uniformity. This process, based on previous works due to Morsi [25] and Georgescu [9], allows to extend probabilistic and Hutton $[0, 1]$-quasi-uniformities on a set X to its power set. In this way, we obtain an endofunctor for each one of the categories of those objects. We will demonstrate the commutativity of these endofunctors with Lowen and Katsaras' functors. Furthermore, we will prove the compatibility of our construction with the Hausdorff fuzzy quasi-pseudometric introduced in [33].

Keywords: Fuzzy implication, Hausdorff quasi-uniformity, probabilistic quasi-uniformity, Hutton $[0, 1]$-quasi-uniformity, fuzzy quasi-pseudometric, Hausdorff fuzzy quasi-pseudometric

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1 Introduction

The extension of a mathematical structure from a nonempty set X to its power set has deserved a lot of attention in several areas of mathematics (see, for example, [3, 4, 13]) and fuzzy mathematics has not been an exception. Maybe, the origins of the fuzzy hyperspace theory goes back to 1983 when Lowen [23] defined from a uniform space $(X, \mathcal{U})$, a fuzzy uniform topology on the family of all nonzero fuzzy sets on X in such a way that $\mathcal{P}(X)$, the power set of X, equipped with the Hausdorff uniformity can be isomorphically injected [23, Theorem 3.4]. This construction was modified in [22] in order that topological and uniform properties of X, as compactness and precompactness, can be carried over the fuzzy hyperspace. Later on, Morsi [25] gave a method to construct a notion of Hausdorff fuzzy uniformity from an implication operator. His work is based on the following idea. Given a uniform space $(X, \mathcal{U})$ the Hausdorff uniformity $\overset{\leftrightarrow}{\mathcal{U}}$ on $\mathcal{P}(X)$ [3] has as a base the family of sets of the form

$$\overset{\leftrightarrow}{u} = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : A \subseteq u^{-1}(B) \text{ and } B \subseteq u(A)\}$$

where $u \in \mathcal{U}$. Consequently, the Hausdorff uniformity is constructed by extending the binary relations $u \in \mathcal{U}$ on X to binary relations $\overset{\leftrightarrow}{u}$ on $\mathcal{P}(X)$. In this way, Morsi associated to each implication $\mathcal{I}$ on the closed unit interval $[0, 1]$ a method which allows to extend a fuzzy binary relation from X to Y, to a fuzzy binary relation from $[0, 1]^X$ to $[0, 1]^Y$. This method produces a notion of Hausdorff fuzzy uniformity including as a special case Lowen's definition of horizontal hyperspace fuzzy uniformity [22] corresponding to the reciprocal of the Gödel implication (see Example 2.8). It is also worth to mention that Georgescu [9] used the same idea for extending a fuzzy relation on a set X to its family of fuzzy subsets and showed that this construction preserves some properties. These constructions of a fuzzy concept of Hausdorff uniformity share the fact that not only the uniformity is fuzzy but the base space also is and the starting point is a classical uniform space. Here, using Morsi's method, we construct a notion of Hausdorff fuzzy quasi-uniformity for different concepts of fuzzy quasi-uniformities (probabilistic quasi-uniformities and Hutton [0,1]-quasi-uniformities) when the base space is not fuzzy. As in [25, 9], our method is based on obtaining a fuzzy relation on $\mathcal{P}(X)$ starting

from a fuzzy relation on X. Moreover this extension is what we call an asymmetric fuzzy biimplication, a concept which extends the notion of an asymmetric equivalence on $[0, 1]$ given in [16] to an arbitrary bounded lattice. In particular, we will prove that this construction in the context of probabilistic quasi-uniform spaces is well-behaved with respect to Lowen's adjoint functors [21] between the categories of quasi-uniform and Lowen quasi-uniform spaces. Furthermore, the fuzzy hyperspace theory has also been developed in the context of fuzzy metric spaces [27, 32, 33, 36]. In particular, in [33] the authors constructed the so-called Hausdorff fuzzy quasi-pseudometric from a fuzzy quasi-pseudometric in the sense of Kramosil and Michalek. We will show that the proposed method for constructing a Hausdorff probabilistic quasi-uniformity is compatible with the construction of the Hausdorff fuzzy quasi-pseudometric.

The paper is organized as follows. In Section 2 we recall basic facts about some fuzzy logic operators like fuzzy implications that will be important for constructing a fuzzy relation on the hyperspace in Section 3. We recall in Section 4 the construction of Hausdorff hyperspace structures in the realm of quasi-pseudometric and quasi-uniform spaces. Section 5 is devoted to collect core concepts about probabilistic and Lowen quasi-uniform spaces and to develop the lifting of these structures to the hyperspace, which leads us to define an endofunctor. We also obtain the commutativity of this endofunctor with some functors considered in [21, 11]. Now it's turn to study in Section 6 the well-behavior of the Hausdorff construction of the previous section in the context of fuzzy quasi-pseudometric spaces. Again, we show the commutativity of the previous endofunctor with some functors considered in [14, 15, 30]. The last section of the paper treats the same problem but for Hutton [0,1]-quasi-uniformities.

2   T-norms, fuzzy implications and fuzzy biimplications

Given a nonempty set X we denote by $\mathcal{P}(X)$ (resp. $\mathcal{P}_0(X)$) the family of all (resp. nonempty) subsets of X. Furthermore, given a subset A of X we denote by 1_A its characteristic function. In this way, we can identify $\mathcal{P}(X)$ with the set $\{0, 1\}^X$ which is a subset of $[0, 1]^X$, the family of fuzzy subsets of X. On the other hand, given a subset $A = \{a_i : i \in I\}$ of real numbers, we will denote by $\bigwedge A = \bigwedge_{i \in I} a_i$ and $\bigvee A = \bigvee_{i \in I} a_i$ the greatest lower bound and the least upper bound of A respectively. As usual, we also consider that $\bigvee \varnothing = 0$ and $\bigwedge \varnothing = 1$.

Definition 2.1. *A binary operation* $*\colon [0, 1] \times [0, 1] \to [0, 1]$ *is called a* triangular norm *or a* t-norm *if* $([0, 1], *)$ *is an abelian monoid with unit 1, such that* $\alpha * \beta \leq \gamma * \delta$ *whenever* $\alpha \leq \gamma$ *and* $\beta \leq \delta$, *with* $\alpha, \beta, \gamma, \delta \in [0, 1]$. *If* $*$ *is also continuous we will say that it is a* continuous t-norm.

Example 2.2. *Three distinguished examples of continuous t-norms are* $\wedge$, $\cdot$ *and* $*_L$ *(the Łukasiewicz t-norm) which are defined as* $\alpha \wedge \beta := \min\{\alpha, \beta\}$, $\alpha \cdot \beta := \alpha\beta$ *and* $\alpha *_L \beta := (\alpha + \beta - 1) \vee 0$.

Definition 2.3. [25] *A fuzzy binary relation* r *from* X *to* Y *is a function from* $X \times Y$ *to* $[0, 1]$, *i. e. a fuzzy subset of* $X \times Y$. *If* $X = Y$ *then we say that* r *is a fuzzy binary relation on* X. *The* opposite *of a fuzzy binary relation* $\mathsf{r} \in [0, 1]^{X \times Y}$ *is the fuzzy binary relation* $\mathsf{r}^{-1} \in [0, 1]^{Y \times X}$ *given by* $\mathsf{r}^{-1}(y, x) = \mathsf{r}(x, y)$ *for all* $(y, x) \in Y \times X$.

Remark 2.4. *If* r *is a fuzzy binary relation which only takes the values 0 and 1, i. e.* $\mathsf{r} \in \{0, 1\}^{X \times Y}$ *then* r *is a* crisp fuzzy binary relation. *In this case, and in order to distinguish it from a proper fuzzy binary relation we will use an italic letter* r *in the default font instead of a sans serif letter* r. *Furthermore, by abuse of notation, we will also use* r *as the classical binary relation induced by the crisp fuzzy binary relation* r, *namely,* r *will also denote the subset of* $X \times Y$ *whose characteristic function is* r. *Consequently, we also write sometimes* $(x, y) \in r$ *instead of* $r(x, y) = 1$.

Definition 2.5. *Let us consider* $\mathsf{r} \in [0, 1]^{X \times Y}, \mathsf{s} \in [0, 1]^{Y \times Z}$ *two fuzzy binary relations and let* $*$ *be a continuous t-norm. The* $*$-composition $\mathsf{r} \circ_* \mathsf{s}$ *of* r *and* s *is a fuzzy binary relation from* X *to* Z *given by* $(\mathsf{r} \circ_* \mathsf{s})(x, z) = \bigvee_{y \in Y} \mathsf{r}(x, y) * \mathsf{s}(y, z)$ *for all* $(x, z) \in X \times Z$ *(we will omit the subscript* $*$ *if no confusion arises).*

Definition 2.6. *A fuzzy binary relation* r *on* X *is said to be:*

- reflexive *if* $\mathsf{r}(x, x) = 1$ *for all* $x \in X$;
- symmetric *if* $\mathsf{r} = \mathsf{r}^{-1}$;
- $*$-transitive *if* $\mathsf{r} \circ_* \mathsf{r} \leq \mathsf{r}$.

A reflexive and $*$-transitive fuzzy binary relation is called a $*$-*fuzzy preorder*. If it is also symmetric then it is called an $*$-*indistinguishability operator* [28] (or an $*$-*fuzzy equivalence relation*) which can be considered as graded equivalence between the elements of a set.

A fuzzy binary relation r on $X \times Y$ and a t-norm $*$ induce a fuzzy operator $C_{\mathsf{r}}^* : [0, 1]^X \to [0, 1]^Y$ given by $C_{\mathsf{r}}^*(\mu)(y) = \bigvee_{x \in X} \mu(x) * \mathsf{r}(x, y)$ for all $\mu \in [0, 1]^X, y \in Y$ (we will omit the superscript $*$ if no confusion arises).

Fuzzy implications are a special kind of fuzzy relations which are the truth functions in a system of many-valued logical of the classical conditional connective $\to$ [2, 6]. Since the standard set of truth degrees in fuzzy logic is the unit interval $[0,1]$, fuzzy implications are functions $\mathcal{I} : [0,1] \times [0,1] \to [0,1]$. Nevertheless, we can replace $[0,1]$ by a bounded lattice as in [29]. Here, we consider a mix of these two definitions.

Definition 2.7 ([29], cf. [2])**.** *Let $(L, \leq_L)$ be a bounded lattice with top element 1_L and bottom element 0_L. A fuzzy implication $\mathcal{I}$ on $(L, \leq_L)$ is a binary operator $\mathcal{I} : L \times L \to [0,1]$ such that, for any $x, y, z \in L$:*

(I1) if $x \leq_L y$ then $\mathcal{I}(x,z) \geq \mathcal{I}(y,z)$;

(I2) if $x \leq_L y$ then $\mathcal{I}(z,x) \leq \mathcal{I}(z,y)$;

(I3) $\mathcal{I}(0_L, 0_L) = \mathcal{I}(1_L, 1_L) = 1$;

(I4) $\mathcal{I}(1_L, 0_L) = 0$.

Example 2.8. *The following are well-known examples of fuzzy implications on $[0,1]$:*

- $\mathcal{I}_Ł(x,y) = (1 - x + y) \wedge 1$; *(Łukasiewicz implication)*

- $\mathcal{I}_G(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$; *(Gödel implication)*

- $\mathcal{I}_{KD}(x,y) = (1 - x) \vee y$; *(Kleene-Dienes implication)*

- $\mathcal{I}_{GG}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{if } x > y \end{cases}$. *(Goguen implication)*

Remark 2.9. *It is well-known [2, Definition 2.5.1] that any t-norm $*$ has an associated fuzzy implication on $[0,1]$ denoted by $\mathcal{I}_*$ or $\overset{*}{\to}$ given by $\mathcal{I}_*(x,y) = \bigvee \{z \in [0,1] : x * z \leq y\}$, for all $x, y \in [0,1]$ named the residuated implication of $*$. Notice that $\mathcal{I}_Ł, \mathcal{I}_G$ and $\mathcal{I}_{GG}$ are residuated implications coming from the continuous t-norms $*_L$, $\wedge$ and $\cdot$ respectively, meanwhile $\mathcal{I}_{KD}$ cannot be derived from a t-norm [2, Remark 2.5.5].*

Remark 2.10. *The residuated implication $\overset{*}{\to}$ of a t-norm $*$ is a reflexive and $*$-transitive fuzzy binary relation so it is a $*$-fuzzy preorder [28, Proposition 2.45]. In general, fuzzy implications are not fuzzy preorders since, for example, they don't need to be reflexive (consider $\mathcal{I}_{KD}$ as above).*

The classical biconditional connective $\leftrightarrow$ of the two-valued logical is defined as $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$. This formula is the base to obtain a truth degree function for the biconditional connective in the fuzzy logic from a fuzzy implication. In [16] the authors introduced the concept of an asymmetric equivalence as a function $\mathcal{E} : [0,1]^2 \to [0,1]$ given by $\mathcal{E} = \mathcal{I}_1 \wedge \mathcal{I}_2^{-1}$ where $\mathcal{I}_1, \mathcal{I}_2$ are fuzzy implications. They obtained some properties of these functions (notice that inequality of [16, Theorem 2 (ii)] has a misprint and it must be reversed) that we have adopted as part of the axioms of what we call an asymmetric fuzzy biimplication over an arbitrary bounded lattice $(L, \leq_L)$ instead of $[0,1]$.

Definition 2.11. *Let $(L, \leq_L)$ be a bounded lattice with top and bottom elements 1_L and 0_L respectively. An asymmetric fuzzy biimplication $\mathcal{E}$ on $(L, \leq_L)$ is a binary operator $\mathcal{E} : L \times L \to [0,1]$ such that:*
(AB1) if $x \leq_L y \leq_L z$ then $\mathcal{E}(x,z) \leq \mathcal{E}(y,z)$ and $\mathcal{E}(x,z) \leq \mathcal{E}(x,y)$;
(AB2) $\mathcal{E}(0_L, 0_L) = \mathcal{E}(1_L, 1_L) = 1$;
(AB3) $\mathcal{E}(1_L, 0_L) = 0$.
Furthermore, if an asymmetric fuzzy biimplication $\mathcal{E}$ is also:

- *symmetric, i. e. $\mathcal{E}(x,y) = \mathcal{E}(y,x)$ for all $x, y \in L$, then it will be called a* fuzzy biimplication*;*

- *reflexive then it will be called an* asymmetric fuzzy equivalence*;*

- *symmetric and reflexive it will be called a* fuzzy equivalence*.*

Remark 2.12. *Fuzzy equivalences as defined in the previous definition were considered in [6, Section 1.9.2] for the lattice $L = [0,1]$ under the name equivalence. This concept was extended to an arbitrary bounded lattice L in [26, Definition 2.15] under the name L-equivalence.*

Example 2.13. *If $\mathcal{I}_1$ and $\mathcal{I}_2$ are two fuzzy implications on a bounded lattice $(L, \leq_L)$ satisfying that $\mathcal{I}_i(x,y) = 1$ whenever $x \leq_L y$ and $i = 1, 2$, then it is easy to check that $\mathcal{I}_1 * \mathcal{I}_2^{-1}$ is an asymmetric fuzzy equivalence for every t-norm $*$. This kind of asymmetric fuzzy equivalences appear in [16] when $L = [0,1]$, $\mathcal{I}_1$ and $\mathcal{I}_2$ are residuated implications, and $* = \wedge$ under the name asymmetric equivalences. If $\mathcal{I}_1 = \mathcal{I}_2$ then we obtain a fuzzy equivalence. We also notice that every fuzzy equivalence on $[0,1]$ is of this type [6, Proposition 1.14]. Following this method with the t-norm $\wedge$, fuzzy implications of Example 2.8 have the following associated fuzzy biimplications:*

- $\mathcal{E}_L(x, y) = 1 - |x - y|$;   *(Łukasiewicz fuzzy equivalence)*

- $\mathcal{E}_G(x, y) = \begin{cases} 1 & \text{if } x = y \\ x \wedge y & \text{if } x \neq y \end{cases}$;   *(Gödel fuzzy equivalence)*

- $\mathcal{E}_{KD}(x, y) = ((1 - x) \vee y) \wedge (x \vee (1 - y))$;   *(Kleene-Dienes fuzzy biimplication)*

- $\mathcal{E}_{GG}(x, y) = \begin{cases} 1 & \text{if } x = y = 0 \\ \frac{x \wedge y}{x \vee y} & \text{otherwise} \end{cases}$.   *(Goguen fuzzy equivalence)*

Example 2.14 (See [28, Example 2.49]). *If $\overset{*}{\to}$ is the residuated implication of a t-norm $*$ then it has an associated fuzzy biimplication $\overset{*}{\leftrightarrow} := \overset{*}{\to} \wedge \overset{*}{\leftarrow}$, called the* biresiduation *of the t-norm $*$ [28, Definition 2.47]. Notice that $\overset{*}{\leftrightarrow}$ is an $*$-indistinguishability operator [28, Proposition 2.50] so it is a fuzzy equivalence.*

3   Fuzzy binary relations on the hyperspace

In [25] Morsi introduced a method for extending a fuzzy binary relation r from X to Y to a fuzzy binary relation from $[0,1]^X$ to $[0,1]^Y$ by means of a fuzzy implication $\mathcal{I}$ as follows. Let $*$ be a t-norm and define the fuzzy binary relations $\overset{\mapsto}{\mathsf{r}}, \tilde{\mathsf{r}} \in [0,1]^{[0,1]^X \times [0,1]^Y}$ as

$$\overset{\mapsto}{\mathsf{r}}(\mu, \upsilon) := \bigwedge_{y \in Y} \mathcal{I}(\upsilon(y), C_{\mathsf{r}}^*(\mu)(y)) = \bigwedge_{y \in Y} \mathcal{I}\left(\upsilon(y), \bigvee_{x \in X} \mu(x) * \mathsf{r}(x, y)\right)$$

$$\tilde{\mathsf{r}}(\mu, \upsilon) := \overset{\mapsto}{\mathsf{r}}(\mu, \upsilon) \wedge \overset{\mapsto}{\mathsf{r}^{-1}}(\upsilon, \mu)$$

for all $\mu \in [0,1]^X, \upsilon \in [0,1]^Y$.

When r is crisp then both $\overset{\mapsto}{\mathsf{r}}$ and $\tilde{\mathsf{r}}$ extend r, and $\tilde{\mathsf{r}}$ is called the *fuzzy hyperspace extension* of r [25, Definition 3.6].

Recall that a probabilistic quasi-uniformity is a family $\mathcal{U}$ of fuzzy binary relations u on X satisfying certain properties (see Definition 5.2). Since one of our goals in this paper is to introduce and study a method for extending a probabilistic quasi-uniformity on a nonempty set X to a probabilistic quasi-uniformity on $\mathcal{P}_0(X)$, it is natural to consider Morsi method for extending each u to $\mathcal{P}_0(X)$. Note that if r is a fuzzy binary relation on X, then the restriction of $\overset{\mapsto}{\mathsf{r}}$ to $\{0,1\}^X \times \{0,1\}^X$ can be considered as an hyperspace extension of r to $\mathcal{P}(X)$ by identifying every subset of X with its characteristic function. Nevertheless, this restriction is not a fuzzy implication but the restriction of $\left(\overset{\mapsto}{\mathsf{r}^{-1}}\right)^{-1}$ to $\{0,1\}^X \times \{0,1\}^X$ is so (see Proposition 3.3). Therefore, it seems to be more natural to use a different notation from that of Morsi which emphasizes this fact. So we will use a right arrow for denoting the restriction of $\left(\overset{\mapsto}{\mathsf{r}^{-1}}\right)^{-1}$ to $\{0,1\}^X \times \{0,1\}^X$.

Proposition 3.1. *Let r be a reflexive fuzzy binary relation on X. The operator $\mathrm{cl}_{\mathsf{r}} : \mathcal{P}(X) \to \mathcal{P}(X)$ given by*

$$\mathrm{cl}_{\mathsf{r}}(A) = \begin{cases} \{x \in X : \bigvee_{a \in A} \mathsf{r}(x, a) = 1\} & \text{if } A \neq \varnothing \\ \varnothing & \text{if } A = \varnothing \end{cases}$$

is a Čech closure operator. If r is also $*$-transitive with respect to a continuous t-norm $*$ then cl_{r} is a closure operator.

Proof. It is straightforward.   □

Remark 3.2. *If r is a (crisp) reflexive binary relation on X, then we can define a Čech closure operator on X given by $\mathrm{cl}_r(A) := \{x \in X : (x, a) \in \mathsf{r} \text{ for some } a \in A\}$.*

Proposition 3.3. *Let X be a nonempty set and let r be a reflexive fuzzy binary relation on X. Then $\overset{\to}{\mathsf{r}} : \mathcal{P}(X) \times \mathcal{P}(X) \to [0,1]$ given by*

$$\overset{\to}{\mathsf{r}}(A, B) = \bigwedge_{a \in A} \bigvee_{b \in B} \mathsf{r}(a, b)$$

is a reflexive fuzzy implication on $\mathcal{P}(X)$, and we call it the lower Hausdorff fuzzy implication *of r. Furthermore,*

$$\overset{\to}{\mathsf{r}}(A, B) = 1 \text{ if and only if } A \subseteq \mathrm{cl}_{\mathsf{r}}(B).$$

Proof. (I1) Let $A, B, C \in \mathcal{P}(X)$ with $A \subseteq B$. If $A = \varnothing$ then $\overrightarrow{\mathsf{r}}(A, C) = 1$ which is greater than or equal to $\overrightarrow{\mathsf{r}}(B, C)$. If $A \neq \varnothing$ but $C = \varnothing$ then $\overrightarrow{\mathsf{r}}(A, C) = 0 = \overrightarrow{\mathsf{r}}(B, C)$. If A, B, C are nonempty then $\overrightarrow{\mathsf{r}}(A, C) = \bigwedge_{a \in A} \bigvee_{c \in C} \mathsf{r}(a, c) \geq \bigwedge_{b \in B} \bigvee_{c \in C} \mathsf{r}(b, c) = \overrightarrow{\mathsf{r}}(B, C)$ since $A \subseteq B$. (I2) follows similarly as (I1). The other properties are trivial.

Finally, the last assertion is trivial if $A = \varnothing$. In other case, $\overrightarrow{\mathsf{r}}(A, B) = 1$ if and only if $B \neq \varnothing$ and $\bigvee_{b \in B} \mathsf{r}(a, b) = 1$ for all $a \in A$ which is equivalent to $A \subseteq \mathrm{cl}_{\mathsf{r}}(B)$. $\qquad\square$

Remark 3.4. *Observe that a construction similar to the above already appeared in [1] when the authors constructed the so-called Hausdorff monad of a quantale-enriched category by lifting the powerset monad.*

Remark 3.5. *Given a fuzzy binary relation r on a nonempty set X, we note that $\overrightarrow{\mathsf{r}}$ is the restriction to $\{0, 1\}^X \times \{0, 1\}^X$ of $\left(\overset{\mapsto}{\mathsf{r}^{-1}} \right)^{-1}$ when we consider a fuzzy implication $\mathcal{I}$ on $[0, 1]$ satisfying the left neutrality property $\mathcal{I}(1, x) = x$ for all $x \in [0, 1]$. In fact, given $A, B \in \mathcal{P}_0(X)$ then*

$$\overset{\mapsto}{\mathsf{r}}(1_A, 1_B) = \bigwedge_{y \in X} \mathcal{I}(1_B(y), C_{\mathsf{r}}^*(1_A)(y)) = \bigwedge_{y \in X} \mathcal{I}(1_B(y), \bigvee_{x \in X} 1_A(x) * \mathsf{r}(x, y)) = \bigwedge_{y \in X} \mathcal{I}(1_B(y), \bigvee_{a \in A} \mathsf{r}(a, y)) = \bigwedge_{b \in B} \bigvee_{a \in A} \mathsf{r}(a, b),$$

where in the last equality we have used the left neutrality property of $\mathcal{I}$ and the fact that $\mathcal{I}(0, x) = 1$ for all $x \in [0, 1]$. Furthermore, if $B = \varnothing$ then $\overset{\mapsto}{\mathsf{r}}(1_A, 1_B) = 1$ for every $A \in \mathcal{P}(X)$ and if $A = \varnothing$ and $B \neq \varnothing$ then $\overset{\mapsto}{\mathsf{r}}(1_A, 1_B) = 0$. Hence, $\overrightarrow{\mathsf{r}}(A, B) = \left(\overset{\mapsto}{\mathsf{r}^{-1}} \right)^{-1}(1_A, 1_B)$.

We should also mention that Georgescu [9], while studying algebraic properties of the transformation of an algebra into its fuzzy power algebra, used Morsi's extension of a fuzzy binary relation in the particular case that $\mathcal{I}$ is a residuated implication induced by a t-norm. He showed in [9, Proposition 1] a formula for the restriction of $\overset{\mapsto}{\mathsf{r}}$ to the crisp fuzzy subsets similar to the one that we have just obtained. Moreover, this extension of a fuzzy relation was also used in [20] for studying the two fuzzy power algebras induced by the cartesian product and the tensor product respectively.

Example 3.6. *Let us consider the fuzzy binary relation r on a nonempty set X given by $\mathsf{r}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$.*

Then we have that $\overrightarrow{\mathsf{r}}(A, B) = \begin{cases} 1 & \text{if } A \subseteq B \\ 0 & \text{if } A \not\subseteq B \end{cases}$ is a fuzzy implication on $\mathcal{P}(X)$ which is the crisp fuzzy binary relation on $\mathcal{P}(X)$ associated with the binary relation on $\mathcal{P}(X)$ given by the inclusion. Notice that if $\mathcal{I}$ is a fuzzy implication on $\mathcal{P}(X)$ satisfying the identity principle $\mathcal{I}(A, A) = 1$ for all $A \in \mathcal{P}(X)$ then $\overrightarrow{\mathsf{r}} \leq \mathcal{I}$.

Definition 3.7. *Let r be a reflexive fuzzy binary relation on X. Then:*

- *$\overleftarrow{\mathsf{r}} := (\overrightarrow{\mathsf{r}^{-1}})^{-1}$ is called the upper Hausdorff fuzzy converse implication of r on $\mathcal{P}(X)$.*
- *$\overleftrightarrow{\mathsf{r}} := \overleftarrow{\mathsf{r}} \wedge \overrightarrow{\mathsf{r}}$ is called the Hausdorff asymmetric fuzzy equivalence of r on $\mathcal{P}(X)$.*

Remark 3.8. *Notice that $\overleftarrow{\mathsf{r}}(A, B) = \bigwedge_{b \in B} \bigvee_{a \in A} \mathsf{r}(a, b)$, for every $A, B \in \mathcal{P}(X)$, and it is a particular case of the powerset monad's canonical extension as considered in [35, Proposition 6.1] (see also [13]). Observe that $\overleftarrow{\mathsf{r}}$ is not a fuzzy implication as we mentioned at the beginning of this section. Nevertheless, by Example 2.13, $\overleftrightarrow{\mathsf{r}}$ is truly an asymmetric fuzzy equivalence. On the other hand, observe that if r is symmetric then $\overleftrightarrow{\mathsf{r}}$ is a fuzzy equivalence.*

Remark 3.9. *Note that if $\mathcal{I}$ is a fuzzy implication satisfying the left neutrality property, then the restriction of $\overset{\mapsto}{\mathsf{r}}$ to $\{0, 1\}^X \times \{0, 1\}^X$ is equal to $\overleftarrow{\mathsf{r}}$. Consequently, the restriction of $\tilde{\mathsf{r}}$ to $\{0, 1\}^X \times \{0, 1\}^X$ is $\overleftrightarrow{\mathsf{r}}$.*

Example 3.10. *Let $\mathcal{I}_{GG}$ be the Goguen implication (Example 2.8) associated with the continuous t-norm $\cdot$. Its associated lower Hausdorff fuzzy implication is given by*

$$\overrightarrow{\mathcal{I}_{GG}}(A, B) = \begin{cases} 1 & \text{if } \left(\bigvee_{a \in A} a \leq \bigvee_{b \in B} b \text{ and } B \neq \varnothing \right) \text{ or } A = B = \varnothing \\ \frac{\bigvee_{b \in B} b}{\bigvee_{a \in A} a} & \text{if } \bigvee_{a \in A} a > \bigvee_{b \in B} b \text{ and } B \neq \varnothing \\ 0 & \text{if } A \neq \varnothing, B = \varnothing \end{cases}.$$

meanwhile the upper Hausdorff fuzzy converse implication is

$$\overleftarrow{\mathcal{I}_{GG}}(A,B) = \begin{cases} 1 & \textit{if } \left(\bigwedge_{a \in A} a \leq \bigwedge_{b \in B} b \textit{ and } A \neq \varnothing \right) \textit{ or } A = B = \varnothing \\ \dfrac{\bigwedge_{b \in B} b}{\bigwedge_{a \in A} a} & \textit{if } \bigwedge_{a \in A} a > \bigwedge_{b \in B} b \textit{ and } A \neq \varnothing \\ 0 & \textit{if } B \neq \varnothing, A = \varnothing \end{cases}.$$

Hence, the Hausdorff asymmetric fuzzy equivalence of $\mathcal{I}_{GG}$ is

$$\overleftrightarrow{\mathcal{I}_{GG}}(A,B) = \begin{cases} 1 & \textit{if } \left(\bigvee_{a \in A} a \leq \bigvee_{b \in B} b \textit{ and } \bigwedge_{a \in A} a \leq \bigwedge_{b \in B} b \textit{ and } A, B \neq \varnothing \right) \\ & \textit{or } A = B = \varnothing \\ \dfrac{\bigvee_{b \in B} b}{\bigvee_{a \in A} a} \wedge \dfrac{\bigwedge_{b \in B} b}{\bigwedge_{a \in A} a} & \textit{if } \bigvee_{a \in A} a > \bigvee_{b \in B} b \textit{ or } \bigwedge_{a \in A} a > \bigwedge_{b \in B} b \textit{ and } A, B \neq \varnothing \\ 0 & \textit{if } A \neq \varnothing, B = \varnothing \textit{ or } A = \varnothing, B \neq \varnothing \end{cases}$$

where we adopt the convention that $\frac{0}{0} = 1$.

4 Hausdorff quasi-pseudometric and Hausdorff quasi-uniformity

In this section we compile some basic facts about the construction of the Hausdorff quasi-pseudometric and the Hausdorff quasi-uniformity. Our basic reference for quasi-pseudo-metric and quasi-uniform spaces is [5].

4.1 Hausdorff quasi-pseudometric

A *quasi-pseudometric* on a set X is a function $d : X \times X \to [0, +\infty)$ such that $d(x,x) = 0$ and $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$. If a quasi-pseudometric d satisfies the symmetry axiom $d(x,y) = d(y,x)$ for all $x, y \in X$ then it is said to be a *pseudometric.* If we let a quasi-pseudometric on X to take the value $+\infty$ then we say that it is an *extended (quasi-)pseudometric* on X. An extended quasi-pseudometric space is a pair (X,d) such that X is a nonempty set and d is an extended quasi-pseudometric on X. If d is an extended quasi-pseudometric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x,y) = d(y,x)$ for all $x, y \in X$, is also an extended quasi-pseudometric on X, called the extended *conjugate (quasi-)pseudometric* of d, and the function d^* defined on $X \times X$ by $d^*(x,y) = d(x,y) \vee d^{-1}(x,y)$ for all $x, y \in X$, is an extended pseudometric on X.

We will denote by QMet the category whose objects are the extended quasi-pseudometric spaces and whose morphisms are the quasi-uniformly continuous functions (a function $f : (X,d) \to (Y,q)$ between two extended quasi-pseudometric spaces is said to be quasi-uniformly continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $q(f(x), f(y)) < \varepsilon$ whenever $d(x,y) < \delta$, for every $x, y \in X$).

Definition 4.1 ([31]). *Let (X,d) be a quasi-pseudometric space. Then the following functions are extended quasi-pseudometrics on $\mathcal{P}_0(X)$:*

- upper Hausdorff extended quasi-pseudometric: $\overleftarrow{d}(A,B) = \bigvee_{b \in B} d(A,b)$;

- lower Hausdorff extended quasi-pseudometric; $\overrightarrow{d}(A,B) = \bigvee_{a \in A} d(a,B)$;

- Hausdorff extended quasi-pseudometric: $\overleftrightarrow{d}(A,B) = \overleftarrow{d}(A,B) \vee \overrightarrow{d}(A,B)$,

for all $A, B \in \mathcal{P}_0(X)$.

These definitions allow to lift an extended quasi-pseudometric d on X to $\mathcal{P}_0(X)$. In this way we can consider an endofunctor $\overleftrightarrow{H} : \mathsf{QMet} \to \mathsf{QMet}$ transforming an extended quasi-pseudometric space (X,d) into $(\mathcal{P}_0(X), \overleftrightarrow{d})$ and transforming a morphism $f : (X,d) \to (Y,q)$ into $2^f : (\mathcal{P}_0(X), \overleftrightarrow{d}) \to (\mathcal{P}_0(Y), \overleftrightarrow{q})$ where $2^f(A) = f(A)$ for all $A \in \mathcal{P}_0(X)$. Other endofunctors $\overrightarrow{H}, \overleftarrow{H}$ can be considered in a similar way.

4.2 Hausdorff quasi-uniformity

We recall that a *quasi-uniformity* on a nonempty set X is a filter $\mathcal{U}$ of reflexive binary relations on X such that

(QU) if $u \in \mathcal{U}$ there exists $v \in \mathcal{U}$ such that $v \circ v \subseteq u$,

where $v \circ v = \{(x, z) \in X \times X : \text{ there exists } y \in X \text{ with } (x, y), (y, z) \in v\}$. We sometimes denote $v \circ v$ by v^2.

If $\mathcal{U}$ is a quasi-uniformity on X then $(X, \mathcal{U})$ is called a *quasi-uniform space*. If a quasi-uniformity $\mathcal{U}$ on X also satisfies that $u^{-1} \in \mathcal{U}$ for every $u \in \mathcal{U}$ where $u^{-1} = \{(x, y) \in X \times X : (y, x) \in u\}$, then it is called a *uniformity*. The filter $\mathcal{U}^{-1} = \{u^{-1} : u \in \mathcal{U}\}$ is a quasi-uniformity on X called the *conjugate quasi-uniformity* of $\mathcal{U}$. If $\mathcal{U}$ is a quasi-uniformity on X, then the family $\{u^* = u \cap u^{-1} : u \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{U}^*$ which is the coarsest uniformity containing $\mathcal{U}$. This uniformity is called the *supremum uniformity* of the quasi-uniformities $\mathcal{U}$ and $\mathcal{U}^{-1}$.

A filter base $\mathcal{B}$ of a quasi-uniformity $\mathcal{U}$ on X is called a *quasi-uniform base* for $\mathcal{U}$. If a filter base $\mathcal{B}'$ of reflexive binary relations on X satisfies (QU) then the filter generated by $\mathcal{B}'$ is a quasi-uniformity on X. If $\mathcal{B}'$ satisfies the axiom

(U) if $u \in \mathcal{B}'$ there exists $v \in \mathcal{B}'$ such that $v \circ v \subseteq u^{-1}$,

then the filter generated by $\mathcal{B}'$ is a uniformity.

A function f between two (quasi-)uniform spaces $(X, \mathcal{U})$, $(Y, \mathcal{V})$ is said to be *(quasi-)uniformly continuous* if for all $v \in \mathcal{V}$ there exists $u \in \mathcal{U}$ such that $(x, y) \in u$ implies $(f(x), f(y)) \in v$.

The (quasi-)uniform spaces form a category $(\mathsf{Q})\mathsf{Unif}$ whose morphisms are the (quasi-)uniformly continuous functions. Recall that every extended quasi-pseudometric d on X has an associated quasi-uniformity $\mathcal{U}_d$ on X having as a base the family $\{\{(x, y) \in X \times X : d(x, y) < \varepsilon\} : \varepsilon > 0\}$. Then we can consider a functor $\mathsf{U} : (\mathsf{Q})\mathsf{Met} \to (\mathsf{Q})\mathsf{Unif}$ leaving morphisms unchanged and transforming an object (X, d) of $(\mathsf{Q})\mathsf{Met}$ to an object $(X, \mathcal{U}_d)$ of $(\mathsf{Q})\mathsf{Unif}$.

Definition 4.2 ([31, 4])**.** *Let $(X, \mathcal{U})$ be a quasi-uniform space. For each $u \in \mathcal{U}$, let us define:*

$$\overleftarrow{u} = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : \forall b \in B \, \exists a \in A \text{ such that } (a, b) \in u\},$$
$$\overrightarrow{u} = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : \forall a \in A \, \exists b \in B \text{ such that } (a, b) \in u\}.$$

Then $\{\overleftarrow{u} : u \in \mathcal{U}\}$ is a base for the upper Hausdorff quasi-uniformity *$\overleftarrow{\mathcal{U}}$ on $\mathcal{P}_0(X)$ and $\{\overrightarrow{u} : u \in \mathcal{U}\}$ is a base for the* lower Hausdorff quasi-uniformity *$\overrightarrow{\mathcal{U}}$ on $\mathcal{P}_0(X)$. The quasi-uniformity $\overleftrightarrow{\mathcal{U}}$ on $\mathcal{P}_0(X)$ defined as the supremum of the lower and upper Hausdorff quasi-uniformities is called the* Hausdorff quasi-uniformity *and it has as a base the family $\{\overleftrightarrow{u} : u \in \mathcal{U}\}$ where $\overleftrightarrow{u} = \overleftarrow{u} \cap \overrightarrow{u}$. Observe that $(\overrightarrow{\mathcal{U}})^{-1} = \overleftarrow{\mathcal{U}^{-1}}$, $(\overleftarrow{\mathcal{U}})^{-1} = \overrightarrow{\mathcal{U}^{-1}}$ and $(\overleftrightarrow{\mathcal{U}})^{-1} = \overleftrightarrow{\mathcal{U}^{-1}}$.*

We notice that the map $\overleftrightarrow{\mathcal{H}} : \mathsf{QUnif} \to \mathsf{QUnif}$ transforming a quasi-uniform space $(X, \mathcal{U})$ into its Hausdorff quasi-uniform hyperspace $(\mathcal{P}_0(X), \overleftrightarrow{\mathcal{U}})$ and taking a quasi-uniformly continuous function $f : X \to Y$ to the function $2^f : \mathcal{P}_0(X) \to \mathcal{P}_0(Y)$ given by $2^f(A) = f(A)$ is an endofunctor.

Remark 4.3 ([31])**.** *It is well-known that the following diagram commutes:*

$$
\begin{array}{ccc}
\mathsf{QMet} & \xrightarrow{\overleftrightarrow{H}, \overrightarrow{H}, \overleftarrow{H}} & \mathsf{QMet} \\
\downarrow{\scriptstyle \mathsf{U}} & \xrightarrow{\overleftrightarrow{\mathcal{H}}, \overrightarrow{\mathcal{H}}, \overleftarrow{\mathcal{H}}} & \downarrow{\scriptstyle \mathsf{U}} \\
\mathsf{QUnif} & \longrightarrow & \mathsf{QUnif}
\end{array}
$$

This means that given an extended quasi-pseudometric space (X, d) then $\mathcal{U}_{\overleftrightarrow{d}} = \overleftrightarrow{\mathcal{U}_d}$.

5 Hausdorff probabilistic quasi-uniformity

Given a nonempty set X, we next propose a method to construct a probabilistic quasi-uniformity on $\mathcal{P}_0(X)$ from a probabilistic quasi-uniformity on X. This method is based on the building of a fuzzy binary relation on the hyperspace $\mathcal{P}_0(X)$ from a fuzzy binary relation on X as considered in Section 3. We first revise some questions concerning probabilistic quasi-uniformities that we will use in the sequel.

Since a classical uniformity on a nonempty set X is a filter on $X \times X$ satisfying certain properties, in order to obtain a fuzzy notion of uniformity it is natural to consider filters on the lattice $[0, 1]^{X \times X}$. This is the starting point of the idea behind the definition of probabilistic uniformities [14, 18] and Lowen uniformities [21].

Definition 5.1 ([21]). *Let X be a nonempty set. A* prefilter $\mathcal{F}$ *on X is a filter on the lattice $[0,1]^X$. In a similar way, a* prefilter base $\mathcal{B}$ *on X is a filter base on the lattice $[0,1]^X$.*

A prefilter $\mathcal{F}$ on X is said to be saturated *if for every $\{F_\varepsilon : \varepsilon \in (0,1]\} \subseteq \mathcal{F}$ we have that $\bigvee_{\varepsilon \in (0,1]}(F_\varepsilon - \varepsilon \cdot 1_X) \in \mathcal{F}$.*

Given a prefilter $\mathcal{F}$ on X define $\widetilde{\mathcal{F}} = \{\bigvee_{\varepsilon \in (0,1]}(F_\varepsilon - \varepsilon \cdot 1_X) : F_\varepsilon \in \mathcal{F}$ for every $\varepsilon \in (0,1]\}$. It is straightforward to see that $\widetilde{\mathcal{F}}$ is a saturated prefilter called the saturation *of $\mathcal{F}$.*

Definition 5.2 (cf. [14, Definition 2.1], [18]). *A* probabilistic quasi-uniformity *on a nonempty set X is a pair $(\mathcal{U}, *)$, where $*$ is a continuous t-norm and $\mathcal{U}$ is a prefilter on $X \times X$ such that:*

(PQU1) $\mathsf{u}(x,x) = 1$ for all $\mathsf{u} \in \mathcal{U}$ and $x \in X$;

(PQU2) for each $\mathsf{u} \in \mathcal{U}$ there exists $\mathsf{v} \in \mathcal{U}$ such that $\mathsf{v} \circ_* \mathsf{v} \leq \mathsf{u}$, where $(\mathsf{v} \circ_* \mathsf{v})(x,y) = \bigvee_{z \in X} \mathsf{v}(x,z) * \mathsf{v}(z,y)$.

In this case, the triple $(X, \mathcal{U}, *)$ is called a probabilistic quasi-uniform space.

If $\mathcal{U}$ also satisfies the symmetry axiom

(PU3) given $\mathsf{u} \in \mathcal{U}$ then $\mathsf{u}^{-1} \in \mathcal{U}$ where $\mathsf{u}^{-1}(x,y) = \mathsf{u}(y,x)$ for all $x,y \in X$, then $(\mathcal{U}, *)$ is said to be a probabilistic uniformity.

If $(\mathcal{U}, *)$ is a probabilistic quasi-uniformity on X then $(\mathcal{U}^{-1}, *)$ also is, where $\mathcal{U}^{-1} = \{\mathsf{u}^{-1} : \mathsf{u} \in \mathcal{U}\}$, and it is called the conjugate probabilistic quasi-uniformity *of $(\mathcal{U}, *)$.*

A function $f : (X, \mathcal{U}, *) \to (Y, \mathcal{V}, \star)$ between two probabilistic (quasi-)uniform spaces is said to be (quasi-)uniformly continuous *if for every $\mathsf{v} \in \mathcal{V}$ there exists $\mathsf{u} \in \mathcal{U}$ such that $\mathsf{u}(x,y) \leq \mathsf{v}(f(x), f(y))$ for all $x,y \in X$. We denote by $\mathsf{P(Q)Unif}$ the category of probabilistic (quasi-)uniform spaces and (quasi-)uniformly continuous functions. For a fixed continuous t-norm $*$, $\mathsf{P(Q)Unif}(*)$ is the full subcategory of $\mathsf{P(Q)Unif}$ whose objects are the probabilistic (quasi-)uniform spaces with respect to $*$.*

Definition 5.3. *If $(\mathcal{U}, *)$ is a probabilistic (quasi-)uniformity on X, the pair $(\mathcal{B}, *)$ is said to be a* base *for $(\mathcal{U}, *)$ if $\mathcal{B}$ is a prefilter base for the prefilter $\mathcal{U}$.*

The pair $(\mathcal{U}, *)$ where $\mathcal{U}$ is the prefilter generated by a prefilter base $\mathcal{B}$ on $X \times X$ satisfying (PQU1) and (PQU2) is a probabilistic quasi-uniformity. If $\mathcal{B}$ also satisfies

$$(BPU) \text{ given } \mathsf{u} \in \mathcal{B} \text{ there exists } \mathsf{v} \in \mathcal{B} \text{ with } \mathsf{v} \circ_* \mathsf{v} \leq \mathsf{u}^{-1}$$

then $(\mathcal{U}, *)$ is a probabilistic uniformity on X.

In [11], the authors considered two adjoint functors between the categories Unif and PUnif. Their result also works for nonsymmetric uniformities so we state it in a more general way.

Proposition 5.4. *Let $(X, \mathcal{U})$ be a quasi-uniform space and $(X, \mathcal{U}, *)$ be a probabilistic quasi-uniform space. Consider*:

(1) $\Gamma(\mathcal{U})$ *as the prefilter on $X \times X$ generated by $\{1_u : u \in \mathcal{U}\}$;*

(2) $\Theta(\mathcal{U})$ *as the filter $\{u \subseteq X \times X : 1_u \in \mathcal{U}\}$.*

Then:

(i) $\Gamma_* : \mathsf{QUnif} \to \mathsf{PQUnif}(*)$ *is a fully faithful functor sending each $(X, \mathcal{U})$ to $(X, \Gamma(\mathcal{U}), *)$ and leaving morphisms unchanged;*

(ii) *If $*$ does not have nontrivial zero divisors, then $\Theta_* : \mathsf{PQUnif}(*) \to \mathsf{QUnif}$ is a faithful functor sending each $(X, \mathcal{U}, *)$ to $(X, \Theta(\mathcal{U}))$ and leaving morphisms unchanged.*

Furthermore, $\Theta_* \circ \Gamma_* = 1_{\mathsf{QUnif}}$.

It is obvious that the restriction of Γ_* to Unif takes values in $\mathsf{PUnif}(*)$. In order to not overload the notation, we will denote this functor by Γ_* because it will be clear from the context whether we are working with a uniformity or with a quasi-uniformity. The same happens with the functor Θ_*.

In 1981, Lowen considered, for the t-norm $\wedge$, a special kind of probabilistic uniformities.

Definition 5.5 ([21, 15]). *A* Lowen (quasi-)uniformity *on a nonempty set X is a probabilistic (quasi-)uniformity $(\mathcal{U}, *)$ on X such that $\mathcal{U}$ is saturated, i. e.*

$$\bigvee_{\varepsilon \in (0,1]} (\mathsf{u}_\varepsilon - \varepsilon \cdot 1_{X \times X}) \in \mathcal{U}$$

for each family $\{\mathsf{u}_\varepsilon : \varepsilon \in (0,1]\} \subseteq \mathcal{U}$. In this case, $(X, \mathcal{U}, *)$ is called a Lowen (quasi-)uniform space.

As in the case of probabilistic (quasi-)uniform spaces, we can consider the category $\mathsf{L(Q)Unif}$ whose objects are the Lowen (quasi-)uniform spaces and whose morphisms are the (quasi-)uniformly continuous functions defined in the obvious way (see [21, Definition 2.4]).

Remark 5.6. *Notice that Lowen uniformities are saturated probabilistic uniformities as was first noted in* [37].

Definition 5.7 ([21]). *A prefilter base* $\mathcal{B}$ *on* $X \times X$ *satisfying (PQU1) and*

(BLQU2) for every $\mathsf{b} \in \mathcal{B}$ *and for all* $\varepsilon \in (0,1]$ *there exists* $\mathsf{b}_\varepsilon \in \mathcal{B}$ *such that*

$$\mathsf{b}_\varepsilon \circ_* \mathsf{b}_\varepsilon - \varepsilon \cdot 1_{X \times X} \leq \mathsf{b}$$

for a t-norm $*$, *generates a Lowen quasi-uniformity* $(\mathcal{U}, *)$ *on* X *given by* $\mathcal{U} = \{\mathsf{u} \in [0,1]^{X \times X} : \mathsf{u} \geq \mathsf{b} \text{ for some } \mathsf{b} \in \mathcal{B}\}$. *Then* $\mathcal{B}$ *is said to be a* base *for the Lowen quasi-uniformity* $(\mathcal{U}, *)$.
    If $\mathcal{B}$ *satisfies*

(BLU2) for every $\mathsf{b} \in \mathcal{B}$ *and for all* $\varepsilon \in (0,1]$ *there exists* $\mathsf{b}_\varepsilon \in \mathcal{B}$ *such that*

$$\mathsf{b}_\varepsilon \circ_* \mathsf{b}_\varepsilon - \varepsilon \cdot 1_{X \times X} \leq \mathsf{b}^{-1}$$

for a t-norm $*$, *instead of (BLQU2), then it generates a Lowen uniformity.*

In [21] Lowen defined a pair of adjoint functors between the categories Unif and LUnif (see also [30]). His construction also works in the asymmetric context so we state it for the corresponding nonsymmetric categories.

Theorem 5.8 ([21]). *Let* X *be a nonempty set,* $\mathcal{U}$ *be a quasi-uniformity on* X *and* $(\mathcal{U}, *)$ *be a Lowen quasi-uniformity on* X. *Define*

$$\omega(\mathcal{U}) = \{\mathsf{u} \in [0,1]^{X \times X} : \mathsf{u}^{-1}((\alpha,1]) \in \mathcal{U} \text{ for all } \alpha \in [0,1)\} \quad and \quad \iota(\mathcal{U}) = \{\mathsf{u}^{-1}((\alpha,1]) : \mathsf{u} \in \mathcal{U}, \ \alpha \in [0,1)\}.$$

Then:

    (1) $(\omega(\mathcal{U}), *)$ *is a Lowen quasi-uniformity on* X;

    (2) $\iota(\mathcal{U})$ *is a quasi-uniformity on* X;

    (3) $\iota(\omega(\mathcal{U})) = \mathcal{U}$;

    (4) $(\omega(\iota(\mathcal{U})), *)$ *is the coarsest Lowen quasi-uniformity generated by a quasi-uniformity and which is finer than* $\mathcal{U}$.

Furthermore, the functor $\omega_* \colon$ QUnif $\to$ LQUnif$(*)$ *given by* $\omega_*((X, \mathcal{U})) = (X, \omega(\mathcal{U}), *)$ *and which leaves morphisms unchanged is fully faithful while the functor* $\iota \colon$ LQUnif $\to$ QUnif *given by* $\iota((X, \mathcal{U}, *)) = (X, \iota(\mathcal{U}))$ *and which leaves morphisms unchanged is faithful. Then* QUnif *is isomorphic to a full subcategory of* LQUnif$(*)$.

Remark 5.9. *If* $(X, \mathcal{U})$ *is a quasi-uniform space, it is easy to check that the family* $\{1_u : u \in \mathcal{U}\}$ *is a base for the Lowen quasi-uniformity* $\omega(\mathcal{U})$, *i. e.* $\omega(\mathcal{U})$ *is the saturation of the probabilistic quasi-uniformity having as base* $\{1_u : u \in \mathcal{U}\}$ *[11, Proposition 3.10]. Furthermore* LQUnif *is a coreflective full subcategory of* PQUnif *and the coreflector is the functor* $\mathcal{S}$ *which assigns to every probabilistic quasi-uniformity* $(\mathcal{U}, *)$ *its saturation* $(\widetilde{\mathcal{U}}, *)$ *and which leaves morphisms unchanged [37, Corollary 4.5]. It was proved in [11] that* $\omega_* = \mathcal{S} \circ \Gamma_*$ *for the symmetric version of the categories.*

We are ready to extend a probabilistic quasi-uniformity $(\mathcal{U}, *)$ on a nonempty set X to its hyperspace $\mathcal{P}_0(X)$ by making use of the construction of Proposition 3.3.

Proposition 5.10. *Let* $(X, \mathcal{U}, *)$ *be a probabilistic quasi-uniform space. Given* $\mathsf{u} \in \mathcal{U}$ *define* $\overrightarrow{\mathsf{u}} : \mathcal{P}_0(X) \times \mathcal{P}_0(X) \to [0,1]$ *as* $\overrightarrow{\mathsf{u}}(A, B) = \bigwedge_{a \in A} \bigvee_{b \in B} \mathsf{u}(a, b)$. *Then the family* $\{\overrightarrow{\mathsf{u}} : \mathsf{u} \in \mathcal{U}\}$ *is a base for a probabilistic quasi-uniformity* $(\overrightarrow{\mathcal{U}}, *)$ *on* $\mathcal{P}_0(X)$ *called the* lower Hausdorff probabilistic quasi-uniformity *of* $(\mathcal{U}, *)$.

Proof. Given $\mathsf{u}, \mathsf{v} \in \mathcal{U}$ then $\mathsf{u} \wedge \mathsf{v} \in \mathcal{U}$. It is obvious that $\overrightarrow{\mathsf{u} \wedge \mathsf{v}} \leq \overrightarrow{\mathsf{u}} \wedge \overrightarrow{\mathsf{v}}$ so $\{\overrightarrow{\mathsf{u}} : \mathsf{u} \in \mathcal{U}\}$ is a prefilter base on $\mathcal{P}_0(X)$.

(PQU1) Given $\mathsf{u} \in \mathcal{U}$ and $A \in \mathcal{P}_0(X)$ we have that since $(\mathcal{U}, *)$ satisfies (PQU1) then

$$\overrightarrow{\mathsf{u}}(A, A) = \bigwedge_{a \in A} \bigvee_{a' \in A} \mathsf{u}(a, a') = \bigwedge_{a \in A} 1 = 1.$$

(PQU2) Given $\mathsf{u} \in \mathcal{U}$ we can find $\mathsf{v} \in \mathcal{U}$ such that $\mathsf{v} \circ_* \mathsf{v} \leq \mathsf{u}$. We assert that $\overrightarrow{\mathsf{v}} \circ_* \overrightarrow{\mathsf{v}} \leq \overrightarrow{\mathsf{u}}$. To check this let $A, B \in \mathcal{P}_0(X)$ and $a \in A, b \in B$. Given an arbitrary $C \in \mathcal{P}_0(X)$ we have by assumption that $\mathsf{v}(a, c) * \mathsf{v}(c, b) \leq \mathsf{u}(a, b)$ for every $c \in C$ (in fact for every $c \in X$). By continuity and monotonicity of $*$ we have

$$\mathsf{v}(a,c) * \bigvee_{b \in B} \mathsf{v}(c,b) = \bigvee_{b \in B} (\mathsf{v}(a,c) * \mathsf{v}(c,b)) \leq \bigvee_{b \in B} \mathsf{u}(a,b)$$

$$\mathsf{v}(a,c) * \bigwedge_{c \in C} \bigvee_{b \in B} \mathsf{v}(c,b) \leq \bigvee_{b \in B} \mathsf{u}(a,b)$$

$$\bigvee_{c \in C} \mathsf{v}(a,c) * \bigwedge_{c \in C} \bigvee_{b \in B} \mathsf{v}(c,b) \leq \bigvee_{b \in B} \mathsf{u}(a,b)$$

$$\overrightarrow{\mathsf{v}}(A,C) * \overrightarrow{\mathsf{v}}(C,B) = \bigwedge_{a \in A} \bigvee_{c \in C} \mathsf{v}(a,c) * \bigwedge_{c \in C} \bigvee_{b \in B} \mathsf{v}(c,b) \leq \bigwedge_{a \in A} \bigvee_{b \in B} \mathsf{u}(a,b) = \overrightarrow{\mathsf{u}}(A,B).$$
□

In a similar way, if $(X, \mathcal{U}, *)$ is a probabilistic quasi-uniform space you can prove that the families $\{\overleftarrow{\mathsf{u}} : \mathsf{u} \in \mathcal{U}\}$ and $\{\overleftrightarrow{\mathsf{u}} : \mathsf{u} \in \mathcal{U}\}$ (see Definition 3.7) are bases for the probabilistic quasi-uniformities given in the following definition.

Definition 5.11. *Let $(X, \mathcal{U}, *)$ be a probabilistic quasi-uniform space.*

- *The* upper Hausdorff probabilistic quasi-uniformity *on $\mathcal{P}_0(X)$ is the probabilistic quasi-uniformity $(\overleftarrow{\mathcal{U}}, *)$ which has as a base the family $\{\overleftarrow{\mathsf{u}} : \mathsf{u} \in \mathcal{U}\}$.*

- *The* Hausdorff probabilistic quasi-uniformity *on $\mathcal{P}_0(X)$ is the probabilistic quasi-uniformity $(\overleftrightarrow{\mathcal{U}}, *)$ which has as a base the family $\{\overleftrightarrow{\mathsf{u}} : \mathsf{u} \in \mathcal{U}\}$.*

Remark 5.12. *Notice that if $(X, \mathcal{U}, *)$ is a probabilistic uniform space then $(\overleftrightarrow{\mathcal{U}}, *)$ is a probabilistic uniformity since in this case $(\overleftarrow{\mathcal{U}})^{-1} = \overrightarrow{\mathcal{U}}$.*

Proposition 5.13. *The mapping $\overrightarrow{\mathcal{H}} :$ QPUnif $\to$ QPUnif acting on objects as $\overrightarrow{\mathcal{H}}(X, \mathcal{U}, *) = (\mathcal{P}_0(X), \overrightarrow{\mathcal{U}}, *)$ and transforming a morphism f into 2^f is an endofunctor.*

Proof. We have already checked that if $(X, \mathcal{U}, *)$ is a probabilistic quasi-uniform space then $(\mathcal{P}_0(X), \overrightarrow{\mathcal{U}}, *)$ so is. Now, let $f : (X, \mathcal{U}, *) \to (Y, \mathcal{V}, \star)$ be a quasi-uniformly continuous function between two probabilistic quasi-uniform spaces. Then given $\mathsf{v} \in \mathcal{V}$ we can find $\mathsf{u} \in \mathcal{U}$ such that $\mathsf{u}(x, y) \leq \mathsf{v}(f(x), f(y))$ for all $x, y \in X$. Hence, we have that

$$\overrightarrow{\mathsf{u}}(A,B) = \bigwedge_{a \in A} \bigvee_{b \in B} \mathsf{u}(a,b) \leq \bigwedge_{a \in A} \bigvee_{b \in B} \mathsf{v}(f(a), f(b)) = \overrightarrow{\mathsf{v}}(2^f(A), 2^f(B)),$$

for all $A, B \in \mathcal{P}_0(X)$, i. e. $2^f : (\mathcal{P}_0(X), \overrightarrow{\mathcal{U}}, *) \to (\mathcal{P}_0(Y), \overrightarrow{\mathcal{V}}, \star)$ is quasi-uniformly continuous. □

Remark 5.14. *The above result is also true by considering mappings $\overleftarrow{\mathcal{H}}, \overleftrightarrow{\mathcal{H}}$ defined in the obvious way.*

We next show that our proposed construction of Hausdorff probabilistic quasi-uniformity is well-behaved with respect to the functors Γ_* and Θ_* (see Proposition 5.4).

Proposition 5.15. *Let $*$ be a continuous t-norm which does not have nontrivial zero divisors. Then the following diagram commutes:*

$$
\begin{array}{ccc}
\text{QUnif} & \xrightarrow{\overleftrightarrow{\mathcal{H}}, \overrightarrow{\mathcal{H}}, \overleftarrow{\mathcal{H}}} & \text{QUnif} \\
\downarrow{\scriptstyle \Gamma_*} & & \downarrow{\scriptstyle \Gamma_*} \\
\text{PQUnif}(*) & \xrightarrow{\overleftrightarrow{\mathcal{H}}, \overrightarrow{\mathcal{H}}, \overleftarrow{\mathcal{H}}} & \text{PQUnif}(*) \\
\downarrow{\scriptstyle \Theta_*} & & \downarrow{\scriptstyle \Theta_*} \\
\text{QUnif} & \xrightarrow{\overleftrightarrow{\mathcal{H}}, \overrightarrow{\mathcal{H}}, \overleftarrow{\mathcal{H}}} & \text{QUnif}
\end{array}
$$

Proof. We only present the proof for the functors $\overset{\longleftrightarrow}{\mathcal{H}}$ and $\overset{\leftrightarrow}{\mathcal{H}}$ since for the rest of the functors it follows in a similar way.

Let $(X, \mathcal{U})$ be a quasi-uniform space. We shall prove that $\overset{\longleftrightarrow}{\Gamma(\mathcal{U})} = \Gamma(\overset{\leftrightarrow}{\mathcal{U}})$, i. e. $(\overset{\leftrightarrow}{\mathcal{H}} \circ \Gamma_*)(X, \mathcal{U}) = (\Gamma_* \circ \overset{\longleftrightarrow}{\mathcal{H}})(X, \mathcal{U})$. This assertion follows from this equivalence:

$$\overset{\longleftrightarrow}{1_u}(A, B) = 1 \Leftrightarrow \bigwedge_{a \in A} \bigvee_{b \in B} 1_u(a, b) = \bigwedge_{b \in B} \bigvee_{a \in A} 1_u(a, b) = 1 \Leftrightarrow \bigvee_{b \in B} 1_u(a_0, b) = \bigvee_{a \in A} 1_u(a, b_0) = 1 \text{ for all } a_0 \in A, b_0 \in B$$

$$\Leftrightarrow A \subseteq u^{-1}(B) \text{ and } B \subseteq u(A) \Leftrightarrow (A, B) \in \overset{\leftrightarrow}{u} \Leftrightarrow 1_{\overset{\leftrightarrow}{u}}(A, B) = 1$$

for any $u \in \mathcal{U}$, $A, B \in \mathcal{P}_0(X)$. Hence $\overset{\longleftrightarrow}{1_u} = 1_{\overset{\leftrightarrow}{u}}$ for every $u \in \mathcal{U}$. Let us consider now a probabilistic quasi-uniform space $(X, \mathcal{U}, *)$. The fact that $(\overset{\leftrightarrow}{\mathcal{H}} \circ \Theta_*)(X, \mathcal{U}, *) = (\Theta_* \circ \overset{\leftrightarrow}{\mathcal{H}})(X, \mathcal{U}, *)$, i. e. $\overset{\longleftrightarrow}{\Theta(\mathcal{U})} = \Theta(\overset{\longleftrightarrow}{\mathcal{U}})$ also follows from the previous chain of equivalences. Commutativity of the diagram for the morphisms of the categories is obvious. □

Remark 5.16. *Notice that the fact that $*$ has not nontrivial zero divisors is only used for the commutativity of the lower part of the above diagram, since otherwise Θ_* is not a functor.*

We next show that the restriction of the functor $\overset{\rightarrow}{\mathcal{H}}$ to the subcategory of Lowen quasi-uniform spaces LQUnif remains an endofunctor (we use the same notation for both functors).

Proposition 5.17. *The mapping $\overset{\rightarrow}{\mathcal{H}} :$ LQUnif $\to$ LQUnif acting on objects as $\overset{\rightarrow}{\mathcal{H}}(X, \mathcal{U}, *) = (\mathcal{P}_0(X), \overset{\rightarrow}{\mathcal{U}}, *)$ and leaving morphisms unchanged is an endofunctor.*

Proof. We need to show that given a Lowen quasi-uniform space $(X, \mathcal{U}, *)$ then $(\overset{\rightarrow}{\mathcal{U}}, *)$ is also a Lowen quasi-uniformity, i. e. $\overset{\rightarrow}{\mathcal{U}}$ is saturated. Hence it is sufficient to prove that given $u_\varepsilon \in \mathcal{U}$ for all $\varepsilon \in (0, 1]$ then $\bigvee_{\varepsilon \in (0,1]} \overset{\rightarrow}{u_\varepsilon} - \varepsilon \cdot 1_{\mathcal{P}_0(X) \times \mathcal{P}_0(X)} \in \overset{\rightarrow}{\mathcal{U}}$. To achieve this, we assert that $\bigvee_{\varepsilon \in (0,1]} \overset{\rightarrow}{u_\varepsilon} - \varepsilon \cdot 1_{\mathcal{P}_0(X) \times \mathcal{P}_0(X)} = \overrightarrow{\left(\bigvee_{\varepsilon \in (0,1]} u_\varepsilon - \varepsilon \cdot 1_{X \times X}\right)}$ which would finish the proof since $(\mathcal{U}, *)$ is a Lowen quasi-uniformity so $\bigvee_{\varepsilon \in (0,1]} u_\varepsilon - \varepsilon \cdot 1_{X \times X} \in \mathcal{U}$. Fix $A, B \in \mathcal{P}_0(X)$. Let us name $\alpha := \bigvee_{\varepsilon \in (0,1]} \overset{\rightarrow}{u_\varepsilon}(A, B) - \varepsilon$ and $\gamma := \overrightarrow{\left(\bigvee_{\varepsilon \in (0,1]} u_\varepsilon - \varepsilon \cdot 1_{X \times X}\right)}(A, B)$. Given $\delta > 0$ there exists $\varepsilon_\delta \in (0, 1]$ such that $\overset{\rightarrow}{u_{\varepsilon_\delta}}(A, B) - \varepsilon_\delta = \bigwedge_{a \in A} \bigvee_{b \in B} u_{\varepsilon_\delta}(a, b) - \varepsilon_\delta > \alpha - \delta$. Hence, for all $a \in A$ we can find $b_{a,\delta} \in B$ such tat $u_{\varepsilon_\delta}(a, b_{a,\delta}) - \varepsilon_\delta > \alpha - \delta$. Then $\bigvee_{\varepsilon \in (0,1]} u_\varepsilon(a, b_{a,\delta}) - \varepsilon > \alpha - \delta$ so $\bigvee_{b \in B} \bigvee_{\varepsilon \in (0,1]} u_\varepsilon(a, b) - \varepsilon > \alpha - \delta$ for all $a \in A$. Consequently, $\overrightarrow{\left(\bigvee_{\varepsilon \in (0,1]} u_\varepsilon - \varepsilon \cdot 1_{X \times X}\right)}(A, B) = \bigwedge_{a \in A} \bigvee_{b \in B} \bigvee_{\varepsilon \in (0,1]} u_\varepsilon(a, b) - \varepsilon \geq \alpha - \delta$. Since δ is arbitrary we have that $\gamma \geq \alpha$.

Reciprocally, since $\bigvee_{b \in B} (\bigvee_{\varepsilon \in (0,1]} u_\varepsilon - \varepsilon \cdot 1_{X \times X})(a, b) \geq \gamma$ for all $a \in A$, given $\delta > 0$ we can find $b_{\delta,a} \in B, \varepsilon_{\delta,a} \in (0, 1]$ such that $u_{\varepsilon_{\delta,a}}(a, b_{\delta,a}) - \varepsilon_{\delta,a} > \gamma - \delta$ for all $a \in A$. Therefore $\bigvee_{b \in B} u_{\varepsilon_{\delta,a}}(a, b) - \varepsilon_{\delta,a} > \gamma - \delta$ and $\bigwedge_{a \in A} \bigvee_{b \in B} u_{\varepsilon_{\delta,a}}(a, b) - \varepsilon_{\delta,a} \geq \gamma - \delta$. Hence $\bigvee_{\varepsilon \in (0,1]} \overset{\rightarrow}{u_\varepsilon}(A, B) - \varepsilon = \bigvee_{\varepsilon \in (0,1]} \bigwedge_{a \in A} \bigvee_{b \in B} u_{\varepsilon_{\delta,a}}(a, b) - \varepsilon_{\delta,a} \geq \gamma - \delta$. Since δ is arbitrary we have that $\alpha \geq \gamma$ and the proof is finished. □

Remark 5.18. *The above results is also true by considering the mappings $\overset{\leftarrow}{\mathcal{H}}, \overset{\leftrightarrow}{\mathcal{H}}$ defined in the obvious way.*

Lemma 5.19. *Let $u \in [0, 1]^{X \times X}$. Given $\varepsilon \in [0, 1)$ then:*

$$(\vec{u})^{-1}(\varepsilon, 1] = \bigcup_{\varepsilon < \delta < 1} \overrightarrow{u^{-1}(\delta, 1]}, \qquad \overrightarrow{u^{-1}(\varepsilon, 1]} \subseteq \bigcap_{0 < \delta < \varepsilon} (\vec{u})^{-1}(\delta, 1] = (\vec{u})^{-1}[\varepsilon, 1],$$

$$(\overset{\leftarrow}{u})^{-1}(\varepsilon, 1] = \bigcup_{\varepsilon < \delta < 1} \overleftarrow{u^{-1}(\delta, 1]}, \qquad \overleftarrow{u^{-1}(\varepsilon, 1]} \subseteq \bigcap_{0 < \delta < \varepsilon} (\overset{\leftarrow}{u})^{-1}(\delta, 1] = (\overset{\leftarrow}{u})^{-1}[\varepsilon, 1],$$

$$(\overset{\leftrightarrow}{u})^{-1}(\varepsilon, 1] = \bigcup_{\varepsilon < \delta < 1} \overleftrightarrow{u^{-1}(\delta, 1]}, \qquad \overleftrightarrow{u^{-1}(\varepsilon, 1]} \subseteq \bigcap_{0 < \delta < \varepsilon} (\overset{\leftrightarrow}{u})^{-1}(\delta, 1] = (\overset{\leftrightarrow}{u})^{-1}[\varepsilon, 1].$$

Proof. We only prove the formulas of the first line since the others follow similarly. For the first one, suppose that $\vec{u}(A, B) > \varepsilon$ where $A, B \in \mathcal{P}_0(X)$. Hence, we can find $\delta \in (\varepsilon, 1)$ such that $\vec{u}(A, B) = \bigwedge_{a \in A} \bigvee_{b \in B} u(a, b) > \delta$. Therefore, given $a \in A$ there exists $b \in B$ such that $u(a, b) > \delta$, i. e. $(A, B) \in \overrightarrow{u^{-1}(\delta, 1]}$ so the inclusion $(\vec{u})^{-1}(\varepsilon, 1] \subseteq \bigcup_{\varepsilon < \delta < 1} \overrightarrow{u^{-1}(\delta, 1]}$ holds.

Conversely, given $\delta \in (\varepsilon, 1)$ we have that if $(A, B) \in \overrightarrow{\mathsf{u}^{-1}(\delta, 1]}$ then for each $a \in A$ there exists $b \in B$ with $\mathsf{u}(a, b) > \delta$ so $\overrightarrow{\mathsf{u}}(A, B) = \bigwedge_{a \in A} \bigvee_{b \in B} \mathsf{u}(a, b) \geq \delta > \varepsilon$, which establishes the validity of the first formula. For the second equation

$$\overrightarrow{\mathsf{u}^{-1}(\varepsilon, 1]} = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : A \subseteq (\mathsf{u}^{-1}(\varepsilon, 1])^{-1}(B)\} = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : \forall a \in A \exists b \in B \mid \mathsf{u}(a, b) > \varepsilon\}$$
$$\subseteq \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : \overrightarrow{\mathsf{u}}(A, B) > \delta\} = (\overrightarrow{\mathsf{u}})^{-1}(\delta, 1] \text{ where } 0 < \delta < \varepsilon.$$

$\square$

Proposition 5.20. *The following diagram commutes:*

Proof. The commutativity of the first square has been proved in Proposition 5.15. In order to prove that $\overrightarrow{\mathcal{H}} \circ \mathcal{S} = \mathcal{S} \circ \overrightarrow{\mathcal{H}}$ we should show that $\overrightarrow{\widetilde{\mathcal{U}}} = \widetilde{\overrightarrow{\mathcal{U}}}$ where $(X, \mathcal{U}, *)$ is a probabilistic quasi-uniform space. But this is deduced from the equality

$$\bigvee_{\varepsilon \in (0,1]} \overrightarrow{\mathsf{u}_\varepsilon} - \varepsilon \cdot 1_{\mathcal{P}_0(X) \times \mathcal{P}_0(X)} = \overrightarrow{\bigvee_{\varepsilon \in (0,1]} \mathsf{u}_\varepsilon - \varepsilon \cdot 1_{X \times X}}$$

for all $\{\mathsf{u}_\varepsilon : \varepsilon \in (0, 1]\} \subseteq \mathcal{U}$ proved in Proposition 5.17. Since the functors $\overrightarrow{\mathcal{H}}$ and $\mathcal{S}$ leave morphisms unchanged we have the commutativity $\overrightarrow{\mathcal{H}} \circ \mathcal{S} = \mathcal{S} \circ \overrightarrow{\mathcal{H}}$.

We next prove that $\overrightarrow{\mathcal{H}} \circ \iota = \iota \circ \overrightarrow{\mathcal{H}}$. To achieve this, let $(X, \mathcal{U}, *)$ be a Lowen quasi-uniform space. We intend to show that $\overrightarrow{\iota(\mathcal{U})} = \iota(\overrightarrow{\mathcal{U}})$. Since a base of $\overrightarrow{\iota(\mathcal{U})}$ are the elements of the form $\overrightarrow{u^{-1}(\varepsilon, 1]}$ meanwhile a base of $\iota(\overrightarrow{\mathcal{U}})$ are the elements of the form $(\overrightarrow{u})^{-1}(\varepsilon, 1]$ where $u \in \mathcal{U}$ and $\varepsilon \in [0, 1)$ the assertion follows from the above lemma.

The proof for the other functors is similar. $\square$

Remark 5.21. *In [25, Definition 5.1], Morsi defined the so-called Hausdorff fuzzy $*$-quasi-uniformity $\overline{\overline{\mathcal{U}}}$ on $[0, 1]^X$ induced by a quasi-uniformity $\mathcal{U}$ on X as the Lowen quasi-uniformity with respect to the t-norm $*$ having as base the family $\{\widetilde{1_U} : U \in \mathcal{U}\}$. By the previous result, the restriction of $\overline{\overline{\mathcal{U}}}$ to $\{0, 1\}^X$ is equal to $\overleftrightarrow{\Gamma(\widetilde{\mathcal{U}})} = \overleftrightarrow{\omega(\mathcal{U})}$.*

6 Hausdorff fuzzy quasi-pseudometric

In [33] the authors extended to fuzzy metrics the concept of Hausdorff probabilistic metric of a probabilistic metric space (see also [32, 24]). We recall the construction of this fuzzy quasi-pseudometric in the following. Previously, we summarize basic concepts about fuzzy quasi-pseudometrics.

Definition 6.1 ([12])**.** *A* fuzzy quasi-pseudometric *(in the sense of Kramosil and Michalek) on a nonempty set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, +\infty)$ such that*
    (FQM1) $M(x, y, 0) = 0$;
    (FQM2) $M(x, x, t) = 1$;
    (FQM3) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
    (FQM4) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous;
    for every $x, y, z \in X$ and $t, s > 0$. If the fuzzy quasi-pseudometric $(M, *)$ also satisfies:
    (F2') $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$ if and only if $x = y$,
then $(M, *)$ is said to be a fuzzy quasi-metric *on X. A fuzzy (pseudo)metric is a fuzzy quasi-(pseudo)metric such that*

(FM5) $M(x,y,t) = M(y,x,t)$ *for all* $x,y \in X$ *and all* $t > 0$.

A fuzzy (quasi-)pseudometric *space is a triple* $(X,M,*)$ *such that* X *is a nonempty set and* $(M,*)$ *is a fuzzy (quasi-)pseudometric on* X.

Definition 6.2 (cf. [8, 34]). *A function* $f : (X,M,*) \to (Y,N,\star)$ *between two fuzzy (quasi-)pseudometric spaces is said to be* (quasi-)uniformly continuous *if for every* $\varepsilon \in (0,1)$ *and* $t > 0$ *there exist* $\delta \in (0,1)$ *and* $s > 0$ *such that if* $M(x,y,s) > 1 - \delta$ *then* $N(f(x),f(y),t) > 1 - \varepsilon$, *for all* $x,y \in X$.

We will denote by $\mathsf{F(Q)Met}$ the category whose objects are the fuzzy (quasi-)pseudometric spaces and whose morphisms are the (quasi-)uniformly continuous functions. Furthermore, $\mathsf{F(Q)Met}(*)$ indicates the full subcategory of $\mathsf{F(Q)Met}$ made up of the fuzzy (quasi-)pseudometric spaces with respect to a fixed continuous t-norm $*$.

Another concept of uniform continuity can be considered for functions between fuzzy (quasi-)pseudometric spaces.

Definition 6.3 ([30]). *A function* $f : (X,M,*) \to (Y,N,\star)$ *between two fuzzy (quasi-)pseudometric spaces is said to be* fuzzy (quasi-)uniformly continuous *if for every* $t > 0$ *there exists* $s > 0$ *such that* $M(x,y,s) \le N(f(x),f(y),t)$.

Remark 6.4 ([10, 33]). *Every fuzzy quasi-pseudometric* $(M,*)$ *on a nonempty set* X *generates a (quasi-)uniformity* $\mathcal{U}_M$ *with a countable base given by* $U_n^M = \{(x,y) \in X \times X : M(x,y,1/n) > 1 - 1/n\}$. *Hence, we can consider a well-defined functor* $\mathsf{U}_f : \mathsf{F(Q)Met} \to \mathsf{(Q)Unif}$ *which maps a fuzzy (quasi-)pseudometric space* $(X,M,*)$ *to the (quasi-)uniform space* $(X,\mathcal{U}_M)$ *and which leaves morphisms unchanged.*

Remark 6.5 ([10, 33], cf. [7]). *Given an extended (quasi-)pseudometric space* (X,d), *let* M_d *be the fuzzy set on* $X \times X \times [0,\infty)$ *defined by*

$$M_d(x,y,t) = \begin{cases} \dfrac{t}{t + d(x,y)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

For every continuous t-norm $*$, $(M_d,*)$ *is a fuzzy (quasi-)pseudometric on* X *which is called the standard fuzzy (quasi-)pseudometric induced by* d. *In this way, for every continuous t-norm* $*$ *we can define a fully faithful covariant functor* $\mathfrak{F}_* : \mathsf{(Q)Met} \to \mathsf{F(Q)Met}(*)$ *such that* $\mathfrak{F}_*((X,d)) = (X,M_d,*)$ *and leaving morphisms unchanged (cf. [34]). Furthermore the functor* U, *introduced in Section 4.2, factorizes through the category* $\mathsf{F(Q)Met}(*)$ *since* $\mathsf{U} = \mathsf{U}_f \circ \mathfrak{F}_*$ *due to the equality* $\mathcal{U}_d = \mathcal{U}_{M_d}$ *for every extended quasi-pseudometric* d *(cf. [34, Corollary 1]).*

Following [33], we recall the construction of the Hausdorff fuzzy quasi-pseudometric of a fuzzy quasi-pseudometric space $(X,M,*)$ by using Proposition 3.3 and Definition 3.7. For every $A,B \in \mathcal{P}_0(X)$ and $t \ge 0$, let us define

$$\overrightarrow{M}(A,B,t) := \begin{cases} 0 & \text{if } t = 0 \\ \bigvee_{0<s<t} \overrightarrow{M_s}(A,B) & \text{otherwise} \end{cases}, \qquad \overleftarrow{M}(A,B,t) := \begin{cases} 0 & \text{if } t = 0 \\ \bigvee_{0<s<t} \overleftarrow{M_s}(A,B) & \text{otherwise} \end{cases},$$

$$\overleftrightarrow{M}(A,B,t) := \overrightarrow{M}(A,B,t) \wedge \overleftarrow{M}(A,B,t),$$

where M_t is the reflexive fuzzy binary relation on X given by $M_t(x,y) = M(x,y,t)$ for all $x,y \in X$. Then $(\overleftrightarrow{M},*)$ is a fuzzy quasi-pseudometric on $\mathcal{P}_0(X)$ called the *Hausdorff fuzzy quasi-pseudometric* of $(X,M,*)$ (see [33, Theorem 1]).

Remark 6.6. *We notice that it would be more natural to define the Hausdorff fuzzy quasi-pseudometric of a fuzzy quasi-pseudometric space* $(X,M,*)$ *as* $H_M(A,B,t) = \overrightarrow{M_t}(A,B) \wedge \overleftarrow{M_t}(A,B)$ *for every* $A,B \in \mathcal{P}_0(X)$ *and* $t \ge 0$. *Nevertheless, the function* $H_M(A,B,\cdot)$ *is not left-continuous in general (see [33, Example 4]) so* $(H_M,*)$ *is not a fuzzy quasi-pseudometric. However,* $H_M(A,B,\cdot)$ *is a non-decreasing function so it is left-continuous if and only if it is lower semicontinuous. We observe that* $\overleftrightarrow{M}(A,B,\cdot)$ *is the lower semicontinuous regularization or lower envelope of* $H_M(A,B,\cdot)$, *i. e. the largest lower semicontinuous function majorized by* $\overleftrightarrow{M}(A,B,\cdot)$ *(see [3, Chapter 1.3]). Nonetheless,* $\overleftrightarrow{M}(A,B,t) = H_M(A,B,t)$ *for every nonempty compact sets* A,B, *as was pointed out in [24].*

The map $\overleftrightarrow{H}_f : \mathsf{FQMet} \to \mathsf{FQMet}$ taking a fuzzy quasi-pseudometric space $(X,M,*)$ to the fuzzy quasi-pseudometric space $(\mathcal{P}_0(X), \overleftrightarrow{M}, *)$ and transforming a quasi-uniformly continuous function f into 2^f is an endofunctor. Other endofunctors $\overrightarrow{H}_f, \overleftarrow{H}_f$ can be considered in the obvious way. It has been also proved that when $\overleftrightarrow{H}_f$ transforms $(X,M,*)$ into $(\mathcal{K}_0(X), \overleftrightarrow{M}, *)$, where $\mathcal{K}_0(X)$ is the family of all nonempty compact subsets of X, then $\overleftrightarrow{H}_f$ determines a monad on the category of fuzzy metric spaces and nonexpansive mappings [36].

If $(X, M, *)$ is a fuzzy quasi-pseudometric space then the family $\{M_t : t > 0\}$, is a base for a probabilistic quasi-uniformity $(\mathcal{U}_M, *)$ on X (cf. [14, Theorem 3.3]) and for a Lowen quasi-uniformity $(\widetilde{\mathcal{U}_M}, *)$ (cf. [15, Theorem 2.6]) which is obviously the saturation of $(\mathcal{U}_M, *)$. This allows to obtain a functor (cf. [30, Propositions 7 and 8]) $\Upsilon :$ FQMet $\to$ PQUnif leaving morphisms unchanged and which transforms a fuzzy quasi-pseudometric space $(X, M, *)$ into the probabilistic quasi-uniform space $(X, \mathcal{U}_M, *)$. Then we can obtain the following result.

Proposition 6.7. *The following diagram commutes:*

$$
\begin{array}{ccc}
\text{FQMet} & \xrightarrow{\overset{\leftrightarrow}{H}_f, \vec{H}_f, \overset{\leftarrow}{H}_f} & \text{FQMet} \\
\Big\downarrow{\Upsilon} & \overset{\leftrightarrow}{\mathcal{H}}, \overset{\rightarrow}{\mathcal{H}}, \overset{\leftarrow}{\mathcal{H}} & \Big\downarrow{\Upsilon} \\
\text{PQUnif} & \longrightarrow & \text{PQUnif} \\
\Big\downarrow{\mathcal{S}} & \overset{\leftrightarrow}{\mathcal{H}}, \overset{\rightarrow}{\mathcal{H}}, \overset{\leftarrow}{\mathcal{H}} & \Big\downarrow{\mathcal{S}} \\
\text{LQUnif} & \longrightarrow & \text{LQUnif}
\end{array}
$$

Proof. To be concise, we only prove the commutativity for the functors $\vec{H}_f$ and $\overset{\rightarrow}{\mathcal{H}}$. We must show that $\overset{\rightarrow}{\mathcal{H}} \circ \Upsilon = \Upsilon \circ \vec{H}_f$, i. e. given a fuzzy quasi-pseudometric space $(X, M, *)$ then $\overrightarrow{\mathcal{U}_M} = \mathcal{U}_{\vec{M}}$. It is clear that a base for $\overrightarrow{\mathcal{U}_M}$ is $\{\overrightarrow{M_t} : t > 0\}$ and a base for $\mathcal{U}_{\vec{M}}$ is the family $\{(\vec{M})_t : t > 0\}$. By definition $\overrightarrow{M_t} \geq (\vec{M})_t$ for all $t > 0$ from which we deduce that $\overrightarrow{\mathcal{U}_M} \subseteq \mathcal{U}_{\vec{M}}$. Furthermore $\overrightarrow{M_s} \leq (\vec{M})_t$ for all $0 < s < t$ so the other inclusion follows and the proof is complete.

That $\overset{\rightarrow}{\mathcal{H}} \circ \mathcal{S} = \mathcal{S} \circ \overset{\rightarrow}{\mathcal{H}}$ was proved in Proposition 5.20. □

Corollary 6.8. *The following diagram commutes:*

$$
\begin{array}{ccc}
\text{QMet} & \xrightarrow{\overset{\leftrightarrow}{H}, \vec{H}, \overset{\leftarrow}{H}} & \text{QMet} \\
\Big\downarrow{\mathfrak{F}_*} & \overset{\leftrightarrow}{H}_f, \vec{H}_f, \overset{\leftarrow}{H}_f & \Big\downarrow{\mathfrak{F}_*} \\
\text{FQMet}(*) & \longrightarrow & \text{FQMet}(*) \\
\Big\downarrow{U_f} & \overset{\leftrightarrow}{\mathcal{H}}, \overset{\rightarrow}{\mathcal{H}}, \overset{\leftarrow}{\mathcal{H}} & \Big\downarrow{U_f} \\
\text{QUnif} & \longrightarrow & \text{QUnif} \\
\Big\downarrow{\omega_*} & \overset{\leftrightarrow}{\mathcal{H}}, \overset{\rightarrow}{\mathcal{H}}, \overset{\leftarrow}{\mathcal{H}} & \Big\downarrow{\omega_*} \\
\text{LQUnif}(*) & \longrightarrow & \text{LQUnif}(*) \\
\Big\downarrow{i} & \overset{\leftrightarrow}{\mathcal{H}}, \overset{\rightarrow}{\mathcal{H}}, \overset{\leftarrow}{\mathcal{H}} & \Big\downarrow{i} \\
\text{PQUnif}(*) & \longrightarrow & \text{PQUnif}(*)
\end{array}
$$

where i denotes the inclusion functor.

Proof. This is deduced from [33, Example 5 and Theorem 3] and Proposition 5.20. □

Remark 6.9. *Notice that from the above result we have that if (X, d) is a quasi-pseudometric space then $\overleftrightarrow{\mathcal{U}_{M_d}} = \mathcal{U}_{M_{\overset{\leftrightarrow}{d}}}$ but since $\mathcal{U}_d = \mathcal{U}_{M_d}$ and $\mathcal{U}_{M_{\overset{\leftrightarrow}{d}}} = \mathcal{U}_{\overset{\leftrightarrow}{d}}$ we deduce $\overleftrightarrow{\mathcal{U}_d} = \mathcal{U}_{\overset{\leftrightarrow}{d}}$ [31].*

7 Hausdorff Hutton [0,1]-quasi-uniformity

This last section address the problem of defining an appropriate notion of Hausdorff quasi-uniformity in the context of Hutton [0,1]-quasi-uniformities. We begin summarizing some fundamental facts about this kind of fuzzy quasi-uniformities.

In the following, we will always consider the unit interval endowed with a continuous t-norm $*$ and an order reversing involution, i.e. a unary operation $' : [0,1] \to [0,1]$ such that $\alpha'' = \alpha$ and $\beta' \leq \alpha'$ whenever $\alpha \leq \beta$ for all $\alpha, \beta \in [0,1]$. Given $\mu \in [0,1]^X$, we will denote by μ' the element of $[0,1]^X$ given by $\mu'(x) = (\mu(x))'$ for every $x \in X$.

Given a nonempty set X, let $\mathcal{H}(X)$ denote the family of all fuzzy operators $\mathfrak{u} : [0,1]^X \to [0,1]^X$ such that:

- $\mathfrak{u}(\mu) \geq \mu$ for all $\mu \in [0,1]^X$;
- $\mathfrak{u}(\bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} \mathfrak{u}(\mu_i)$ for all $\{\mu_i : i \in I\} \subseteq [0,1]^X$ and $\mathfrak{u}(1_\varnothing) = 1_\varnothing$.

Definition 7.1 ([17])**.** *A* Hutton $[0,1]$-quasi-uniformity *on a set X is a nonempty subset $\mathfrak{U}$ of $\mathcal{H}(X)$ such that*

(HU1) if $\mathfrak{u} \in \mathfrak{U}$, $\mathfrak{u} \leq \mathfrak{v}$ and $\mathfrak{v} \in \mathcal{H}(X)$ then $\mathfrak{v} \in \mathfrak{U}$;

(HU2) if $\mathfrak{u}, \mathfrak{v} \in \mathfrak{U}$ there exists $\mathfrak{w} \in \mathfrak{U}$ such that $\mathfrak{w} \leq \mathfrak{u}$ and $\mathfrak{w} \leq \mathfrak{v}$;

(HU3) if $\mathfrak{u} \in \mathfrak{U}$ there exists $\mathfrak{v} \in \mathfrak{U}$ such that $\mathfrak{v} \circ \mathfrak{v} \leq \mathfrak{u}$.

If $\mathfrak{U}$ also satisfies (HU4) if $\mathfrak{u} \in \mathfrak{U}$ then $\mathfrak{u}^{-1} \in \mathfrak{U}$, where $\mathfrak{u}^{-1}(\mu) = \bigwedge\{\eta \in [0,1]^X : \mathfrak{u}(\eta') \leq \mu'\}$, then it is called a Hutton $[0,1]$-uniformity. *The pair $(X, \mathfrak{U})$ is called a* Hutton $[0,1]$-(quasi-)uniform space.

A function $f : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ between two Hutton $[0,1]$-(quasi-)uniform spaces is said to be (quasi-)uniformly continuous *if for every $\mathfrak{v} \in \mathfrak{V}$ there exists $\mathfrak{u} \in \mathfrak{U}$ such that $\mathfrak{u}(\mu) \leq \mathfrak{v}(\bigvee_{x \in X} \mu(x) * 1_{\{f(x)\}}) \circ f$, for all $\mu \in [0,1]^X$.*

H(Q)Unif *denotes the category of Hutton $[0,1]$(-quasi)-uniform spaces and (quasi-)uniformly continuous functions.*

Definition 7.2. *A* base *for a Hutton $[0,1]$-(quasi-)uniformity $\mathfrak{U}$ on X is a nonempty subset $\mathfrak{B}$ of $\mathcal{H}(X)$ such that for each $\mathfrak{u} \in \mathfrak{U}$ there exists $\mathfrak{b} \in \mathfrak{B}$ such that $\mathfrak{b} \leq \mathfrak{u}$.*

If $\mathfrak{B}$ is a nonempty subset of $\mathcal{H}(X)$ verifying:

(1) if $\mathfrak{b}_1, \mathfrak{b}_2, \in \mathfrak{B}$ there exists $\mathfrak{b}_3 \in \mathfrak{B}$ such that $\mathfrak{b}_3 \leq \mathfrak{b}_1$ and $\mathfrak{b}_3 \leq \mathfrak{b}_2$;

(2) if $\mathfrak{b}_1 \in \mathfrak{B}$ there exists $\mathfrak{b}_2 \in \mathfrak{B}$ such that $\mathfrak{b}_2 \circ \mathfrak{b}_2 \leq \mathfrak{b}_1$;

then we say that it is a basis for a Hutton $[0,1]$-quasi-uniformity given by $\mathfrak{U}_\mathfrak{B} = \{\mathfrak{u} \in \mathcal{H}(X) : \text{ there exists } \mathfrak{b} \in \mathfrak{B} \text{ such that } \mathfrak{b} \leq \mathfrak{u}\}$.

In 1984, Katsaras established the following relations between the categories (Q)Unif and H(Q)Unif.

Proposition 7.3 ([19])**.** *Let $(X, \mathcal{U})$ be a (quasi-)uniform space and $(Y, \mathfrak{U})$ be a Hutton $[0,1]$-(quasi-)uniform space. Let $\Phi(\mathcal{U})$ be the Hutton $[0,1]$-(quasi-)uniformity generated by $\{\phi(u) : u \in \mathcal{U}\}$ where $\phi(u)(\mu)(x) = \bigvee_{y \in X} \mu(y) * 1_{u(y)}(x) = \bigvee_{x \in u(y)} \mu(y)$ for each $u \in \mathcal{U}$, $\mu \in [0,1]^X$ and $x \in X$.*
Let $\Psi(\mathfrak{U})$ be the (quasi-)uniformity generated by $\{\psi(\mathfrak{u}) : \mathfrak{u} \in \mathfrak{U}\}$ where $\psi(\mathfrak{u}) = \{(x,y) \in X \times X : \forall \mu \in [0,1]^X, \ \mu(x) \leq \mathfrak{u}(\mu)(y)\}$ for each $\mathfrak{u} \in \mathfrak{U}$. Then:

*(1) $\Phi : $ (Q)Unif $\to$ H(Q)Unif *is a functor sending each $(X, \mathcal{U})$ to $(X, \Phi(\mathcal{U}))$ and leaving morphisms unchanged;*

*(2) $\Psi : $ H(Q)Unif $\to$ (Q)Unif *is a functor sending each $(X, \mathfrak{U})$ to $(X, \Psi(\mathfrak{U}))$ and leaving morphisms unchanged;*

(3) $\Psi(\Phi(\mathcal{U})) = \mathcal{U}$;

(4) $\mathfrak{U} \subseteq \Phi(\Psi(\mathfrak{U}))$;

(5) Ψ is a right adjoint of Φ.

On the other hand, Höhle [14, p. 313] showed how to construct a Hutton uniformity by means of a probabilistic uniformity. This construction was studied by Zhang [38] in the context of uniform environments, a general framework which unify the notions of crisp uniformity, probabilistic uniformity and Hutton uniformity. In fact, he provided two adjoint functors between the categories PQUnif and HQUnif as follows:

Proposition 7.4 (cf. [38, Proposition 4.2])**.** *Let $(X, \mathfrak{U}, *)$ be a probabilistic quasi-uniform space. Let $\mathsf{H}(\mathfrak{U})$ be the Hutton $[0,1]$-quasi-uniformity having as base the family $\{\mathsf{h}(\mathsf{u}) : \mathsf{u} \in \mathfrak{U}\}$ where $\mathsf{h}(\mathsf{u}) \in ([0,1]^X)^{[0,1]^X}$ is given by*

$$\mathsf{h}(\mathsf{u})(\mu)(y) = \bigvee_{x \in X} \mu(x) * \mathsf{u}(x,y) \qquad \mu \in [0,1]^X, \ y \in X.$$

Furthermore, if $(X, \mathfrak{U})$ is a Hutton $[0,1]$-quasi-uniformity, let us consider the probabilistic quasi-uniformity $(\mathsf{K}(\mathfrak{U}), *)$ having as base the family $\{\mathsf{k}(\mathfrak{u}) : \mathfrak{u} \in \mathfrak{U}\}$ where

$$\mathsf{k}(\mathfrak{u})(x,y) = \bigwedge_{\mu \in [0,1]^X} (\mu(y) \overset{*}{\to} \mathfrak{u}(\mu)(x)).$$

(i) $\mathsf{F_h} : \mathsf{PQUnif} \to \mathsf{HQUnif}$ *is a functor sending each probabilistic quasi-uniform space* $(X, \mathcal{U}, *)$ *to* $(X, \mathsf{H}(\mathcal{U}))$;

(ii) $\mathsf{F_k} : \mathsf{HQUnif} \to \mathsf{PQUnif}$ *is a functor sending each Hutton [0,1] quasi-uniform space* $(X, \mathfrak{U})$ *to* $(X, \mathsf{K}(\mathfrak{U}), *)$;

(iii) $\mathsf{K}(\mathsf{H}(\mathcal{U})) = \mathcal{U}$;

(iv) $\mathfrak{U} \subseteq \mathsf{H}(\mathsf{K}(\mathfrak{U}))$;

(v) $\mathsf{F_k}$ *is the right adjoint of* $\mathsf{F_h}$.

Remark 7.5. *Notice that if the negation corresponding to the t-norm* $*$ *given by* $\neg\alpha := a \xrightarrow{*} 0$ *for all* $\alpha \in [0,1]$*, is an order reversing involution coincident with* $'$ *then the above functors can be considered between* PUnif *and* HUnif *[38, Proposition 4.2] (see also [14]). Furthermore, the functor* Φ *is exactly the composition* $\mathsf{F_h} \circ \omega_*$ *meanwhile* Ψ *is the composition* $\iota \circ \mathsf{F_k}$ *(see [38, 11]).*

We propose the following definition for the Hausdorff Hutton [0,1]-quasi-uniformity.

Definition 7.6. *Let* $(X, \mathfrak{U})$ *be a Hutton [0,1]-quasi-uniform space.*

- *The* lower Hausdorff Hutton [0,1]-quasi-uniformity *on* $\mathcal{P}_0(X)$ *induced by* $\mathfrak{U}$ *is the Hutton [0,1]-quasi-uniformity* $\overrightarrow{\mathfrak{U}} := \mathsf{H}(\overrightarrow{\mathcal{H}}(\mathsf{K}(\mathfrak{U})))$, *i. e.* $\overrightarrow{\mathfrak{U}}$ *has a base the family* $\{\overrightarrow{\mathfrak{u}} : \mathfrak{u} \in \mathfrak{U}\}$ *where*

$$\overrightarrow{\mathfrak{u}}(\mathsf{M})(A) := \bigvee_{B \in \mathcal{P}_0(X)} \mathsf{M}(B) * \bigwedge_{a \in A} \bigvee_{b \in B} \bigwedge_{\mu \in [0,1]^X} (\mu(b) \to \mathfrak{u}(\mu)(a)) \quad \text{for all } A \in \mathcal{P}_0(X), \mathsf{M} \in [0,1]^{\mathcal{P}_0(X)}.$$

- *The* upper Hausdorff Hutton [0,1]-quasi-uniformity *on* $\mathcal{P}_0(X)$ *induced by* $\mathfrak{U}$ *is the Hutton [0,1]-quasi-uniformity* $\overleftarrow{\mathfrak{U}} := \mathsf{H}(\mathcal{H}^{\leftarrow}(\mathsf{K}(\mathfrak{U})))$, *i. e.* $\overleftarrow{\mathfrak{U}}$ *has a base the family* $\{\overleftarrow{\mathfrak{u}} : \mathfrak{u} \in \mathfrak{U}\}$ *where*

$$\overleftarrow{\mathfrak{u}}(\mathsf{M})(A) := \bigvee_{B \in \mathcal{P}_0(X)} \mathsf{M}(B) * \bigwedge_{b \in B} \bigvee_{a \in A} \bigwedge_{\mu \in [0,1]^X} (\mu(b) \to \mathfrak{u}(\mu)(a)) \quad \text{for all } A \in \mathcal{P}_0(X), \mathsf{M} \in [0,1]^{\mathcal{P}_0(X)}.$$

- *The* Hausdorff Hutton [0,1]-quasi-uniformity *on* $\mathcal{P}_0(X)$ *induced by* $\mathfrak{U}$ *is the Hutton [0,1]-quasi-uniformity* $\overleftrightarrow{\mathfrak{U}} := \mathsf{H}(\mathcal{H}(\mathsf{K}(\mathfrak{U})))$, *i. e.* $\overleftrightarrow{\mathfrak{U}}$ *has a base the family* $\{\overleftrightarrow{\mathfrak{u}} : \mathfrak{u} \in \mathfrak{U}\}$ *where*

$$\overleftrightarrow{\mathfrak{u}}(\mathsf{M})(A) := \bigvee_{B \in \mathcal{P}_0(X)} \mathsf{M}(B) * \left(\left(\bigwedge_{a \in A} \bigvee_{b \in B} \bigwedge_{\mu \in [0,1]^X} (\mu(b) \to \mathfrak{u}(\mu)(a)) \right) \wedge \left(\bigwedge_{b \in B} \bigvee_{a \in A} \bigwedge_{\mu \in [0,1]^X} (\mu(b) \to \mathfrak{u}(\mu)(a)) \right) \right)$$

for all $A \in \mathcal{P}_0(X), \mathsf{M} \in [0,1]^{\mathcal{P}_0(X)}$.

Then we can consider the endofunctors $\overrightarrow{\mathfrak{H}}, \overleftarrow{\mathfrak{H}}$ and $\mathfrak{H}$ on $\mathsf{H(Q)Unif}$ given by $\overrightarrow{\mathfrak{H}} := \mathsf{F_h} \circ \overrightarrow{\mathcal{H}} \circ \mathsf{F_k}$, $\overleftarrow{\mathfrak{H}} := \mathsf{F_h} \circ \overleftarrow{\mathcal{H}} \circ \mathsf{F_k}$ and $\overleftrightarrow{\mathfrak{H}} := \mathsf{F_h} \circ \overleftrightarrow{\mathcal{H}} \circ \mathsf{F_k}$ for which we obtain the following obvious result:
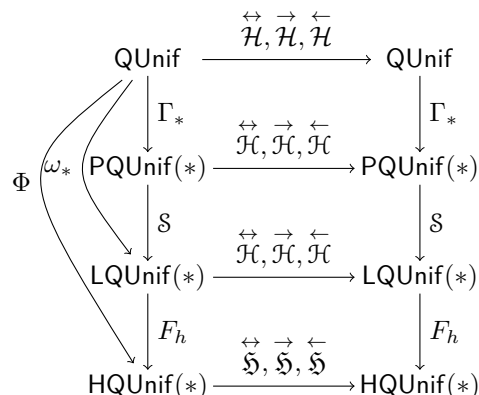
Proposition 7.7. *The following diagram commutes:*

$$
\begin{array}{ccc}
\mathsf{HQUnif} & \xrightarrow{\overleftrightarrow{\mathfrak{H}}, \overrightarrow{\mathfrak{H}}, \overleftarrow{\mathfrak{H}}} & \mathsf{HQUnif} \\
\downarrow{\scriptstyle \mathsf{F_k}} & \overleftrightarrow{\mathcal{H}}, \overrightarrow{\mathcal{H}}, \overleftarrow{\mathcal{H}} & \downarrow{\scriptstyle \mathsf{F_k}} \\
\mathsf{PQUnif} & \longrightarrow & \mathsf{PQUnif} \\
\downarrow{\scriptstyle \mathsf{F_h}} & \overleftrightarrow{\mathfrak{H}}, \overrightarrow{\mathfrak{H}}, \overleftarrow{\mathfrak{H}} & \downarrow{\scriptstyle \mathsf{F_h}} \\
\mathsf{HQUnif} & \longrightarrow & \mathsf{HQUnif}
\end{array}
$$

Proof. From Proposition 7.4 we have that $\mathsf{F_k} \circ \mathsf{F_h} = 1_{\mathsf{PQUnif}}$ so

- $\mathsf{F_k} \circ \overleftrightarrow{\mathfrak{H}} = \mathsf{F_k} \circ (\mathsf{F_h} \circ \overleftrightarrow{\mathcal{H}} \circ \mathsf{F_k}) = 1_{\mathsf{PQUnif}} \circ \overleftrightarrow{\mathcal{H}} \circ \mathsf{F_k} = \overleftrightarrow{\mathcal{H}} \circ \mathsf{F_k}$;

- $\overleftrightarrow{\mathfrak{H}} \circ \mathsf{F_h} = (\mathsf{F_h} \circ \overleftrightarrow{\mathcal{H}} \circ \mathsf{F_k}) \circ \mathsf{F_h} = \mathsf{F_h} \circ \overleftrightarrow{\mathcal{H}} \circ 1_{\mathsf{PQUnif}} = \mathsf{F_h} \circ \overleftrightarrow{\mathcal{H}}$.

□

Corollary 7.8. *The following diagram commutes:*

$$
\begin{array}{ccc}
\text{QUnif} & \xrightarrow{\overset{\leftrightarrow}{\mathcal{H}},\overset{\rightarrow}{\mathcal{H}},\overset{\leftarrow}{\mathcal{H}}} & \text{QUnif} \\
\downarrow{\scriptstyle \Gamma_*} & & \downarrow{\scriptstyle \Gamma_*} \\
\text{PQUnif}(*) & \xrightarrow{\overset{\leftrightarrow}{\mathcal{H}},\overset{\rightarrow}{\mathcal{H}},\overset{\leftarrow}{\mathcal{H}}} & \text{PQUnif}(*) \\
\downarrow{\scriptstyle \mathcal{S}} & & \downarrow{\scriptstyle \mathcal{S}} \\
\text{LQUnif}(*) & \xrightarrow{\overset{\leftrightarrow}{\mathcal{H}},\overset{\rightarrow}{\mathcal{H}},\overset{\leftarrow}{\mathcal{H}}} & \text{LQUnif}(*) \\
\downarrow{\scriptstyle F_h} & & \downarrow{\scriptstyle F_h} \\
\text{HQUnif}(*) & \xrightarrow{\overset{\leftrightarrow}{\mathfrak{H}},\overset{\rightarrow}{\mathfrak{H}},\overset{\leftarrow}{\mathfrak{H}}} & \text{HQUnif}(*)
\end{array}
$$

Proof. It follows from the above Proposition, Proposition 5.20 and Remark 5.9. □

Acknowledgement

References

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