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A time-fractional HIV infection model with nonlinear diffusion

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ABSTRACT

This paper deals with a set of three partial differential equations involving time-fractional derivatives and nonlinear diffusion operators. This model helps us to understand the HIV spread and transmission into the patient. First, we prove the existence and uniqueness of weak solutions to the mathematical model. Then, the Galerkin finite element scheme is implemented to approximate the solution of the model. Further, a-priori error bounds and convergence estimates for the fully-discrete problem are derived. The second order convergence for the proposed scheme is also proved. Numerical tests are shown to validate the theoretical studies.

Introduction

In this paper, we consider a variation of the mathematical model proposed in [1] to study the dynamics of HIV spread and transmission into the patient. The modification of the model consists in introducing time-fractional derivatives, density-dependent diffusion operators and Holling type II functional responses to better incorporate the HIV infection into the healthy cells. Later we justify these modifications with respect the original model. The new reformulation of the original model is given by

$$\left.\begin{aligned}
&\partial_t^\alpha u - div(D_1(u)\nabla u) = -du - \frac{\beta_1 uw}{1+u} - \frac{\beta_2 uv}{1+u}, && \text{in } Q_T,\\
&\partial_t^\alpha v - div(D_2(v)\nabla v) = \frac{\beta_1 uw}{1+u} + \frac{\beta_2 uv}{1+u} - rv, && \text{in } Q_T,\\
&\partial_t^\alpha w - div(D_3(w)\nabla w) = Nv - ew, && \text{in } Q_T,
\end{aligned}\right\} \tag{1}$$

with the initial and boundary conditions:

$$u(x,0) = u_0(x);\ v(x,0) = v_0(x);\ w(x,0) = w_0(x) \text{ in } \Omega,$$
$$u(x,t) = v(x,t) = w(x,t) = 0 \text{ on } \Sigma_T,$$

where ∂_t^α denotes the Caputo fractional derivative of order $0 \le \alpha \le 1$ with respect to time t, $Q_T = \Omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$, Ω is a bounded domain in $\mathbb{R}^n$ with a boundary $\partial\Omega$ and $T > 0$. In (1), $u(x,t), v(x,t)$ and $w(x,t)$ represent the concentrations of healthy cells, infected cells and virions, respectively. $d = d(x,t) > 0$ denotes the death rate for healthy cells. $\beta_1 = \beta_1(x,t) > 0$, $\beta_2 = \beta_2(x,t) > 0$ and $r = r(x,t)$ are the virus infection rate, cell–cell infection rate and death rate of infected cells respectively. $N = N(x,t) > 0$ denotes the virus production rate. Finally, $e = e(x,t) > 0$ denotes the death rate of virions. $D_i(\cdot)$, $i = 1,2,3$, denote the nonlinear density for u, v and w, respectively. A blood-borne infective disease called the acquired immunodeficiency syndrome (AIDS) is caused by the human immunodeficiency virus (HIV). In recent years, several mathematical models have been proposed to describe the dynamics of the HIV disease and to explain epidemic illnesses related to AIDS [2–4]. Here, we consider the system proposed in [1] with major modifications. First, we modified the integer order time derivatives with time-fractional derivatives. Further, we considered here density-dependent diffusion for diffusion operators. Because, in the biological models, movement of infected cells through normal cells have similar pattern as in a porous medium and so we may consider the cell random motility to be a function of unknowns. Therefore, it is more realistic to consider the density-dependent diffusion than the linear diffusion. For more details about density dependent diffusion, see [5] and the references there in.

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For convenient in the subsequent analysis, we rewrite model (1) in the following form:

$$\left.\begin{array}{ll}\partial_t^\alpha u - div(D_1(u)\nabla u) = -du - \beta(u,w) - \sigma(u,v), & \text{in } Q_T,\\ \partial_t^\alpha v - div(D_2(v)\nabla v) = \beta(u,w) + \sigma(u,v) - rv, & \text{in } Q_T,\\ \partial_t^\alpha w - div(D_3(w)\nabla w)v = Nv - ew, & \text{in } Q_T,\end{array}\right\} \tag{2}$$

where

$$\beta(u,w) = \beta_1 \frac{uw}{1+u}, \qquad \sigma(u,v) = \beta_2 \frac{uv}{1+u}.$$

In this paper, we assume that the functions $d(x,t)$, $\beta_1(x,t)$, $\beta_2(x,t)$, $r(x,t)$, $N(x,t)$ and $e(x,t)$, are in $C(Q_T)$ and, it is clear, from the definition of $\beta(u,w)$ and $\sigma(u,v)$, that $\beta(u,w) \le \beta_1 w$ and $\sigma(u,v) \le \beta_2 v$ for $u,v,w \ge 0$. We further assume that $u_0(x), v_0(x), w_0(x) \in L^2(\Omega)$. The above assumptions are standard, for example, see [6,7]. For better understanding of above mathematical model with integer order derivatives, we refer the interested readers to the paper [1,8,9]. For further epidemiological models, see [10–12].

In recent years, fractional differential equations have demonstrated to be powerful mathematical tool to model a wide range of applications in science and engineering (see, for example, [13–18] and the references therein). As a consequence, the number of researchers who are interested in the study of fractional partial differential equations and their applications is increasing over the last few decades. However, as far as time-fractional partial differential equations is concerned, the number of contributions in the extant literature is still limited and more analysis is required. For instance, time-fractional reaction–diffusion equation and related integral equations can be investigated by semigroup theory in the context of what is known as evolutionary integral equations [19]. Kubica et al. [20] studied one dimensional heat equation in a non cylindrical domain and established the existence of its weak solutions using the Faedo–Galerkin approximation method. Subsequently, the development of theoretical and efficient computational techniques to solve the time-fractional PDEs has recently received a great deal of attention among researchers [21–24]. Further, the finite volume method [22,25–27], meshless method [21,28], finite difference method [29,30], finite element method [23,24,31] and the spectral method [32,33] are the widely preferred numerical methods in the literature to solve fractional partial differential equations.

In the present work, we prove the unique solvability of weak solutions and a-priori error estimates to time-fractional HIV mathematical model with nonlinear diffusion operators. For that, we assume that the Caratheodory functions $D_i(s) : Q_T \times \mathbb{R}^n \to \mathbb{R}, i = 1,2,3$, are continuous functions with respect to s and satisfy the following conditions:

(H1) $D_i(s)\zeta \cdot \zeta \ge b_i|\zeta|^2, \quad b_i > 0,\ i = 1,2,3,$
(H2) $|D_i(s)\zeta| \le \gamma_i \left[\Lambda_i(x,t) + |\zeta|\right], \quad \gamma_i > 0,\ i = 1,2,3,$
(H3) $(D_i(s)\zeta - D_i(s)\zeta')(\zeta - \zeta') \ge 0,\ i = 1,2,3,$
(H4) $|D_i'(s)| < L,\ L > 0,\ i = 1,2,3,$ for all $s \in \mathbb{R}$.

for almost every $(x,t) \in Q_T$ and, for every $\zeta, \zeta' \in \mathbb{R}^n$ with $\Lambda_i(x,t) \in L^2(Q_T)$.

The rest of the paper is organized as follows. In Section "Existence and uniqueness of weak solutions", we study existence and uniqueness results for the model (1). In Section "Finite element scheme", we derive a priori bound and error estimates for the Galerkin finite element numerical method that permits approximating the solution of model (1). In Section "Numerical results", we show several numerical examples to validate the theoretical findings established in Section "Finite element scheme".

Existence and uniqueness of weak solutions

The aim of this section is to establish sufficient conditions in order to guarantee the existence of weak solutions of the system of time-fractional nonlinear diffusion Eqs. (1). To achieve this goal, we will use a uniform priori estimates for approximating the solution of model (1). For the sake of clarity, we keep the same notation as the one utilized in [34,35]. To simplify the notation, we use a generic constant C rather than utilizing different constants throughout the subsequent development.

Mathematical preliminaries

Lemma 1 (*[34]*). *Let $\eta(\cdot)$ be a non-negative, absolute continuous function on $[0,T]$, which satisfies for a.e. t the following differential inequality*

$${}_0^C D_t^\alpha \eta(t) \le C\eta(t), \tag{3}$$

for constant $C \ge 0$. Then

$$\eta(t) \le \eta(0) E_\alpha \left[Ct^\alpha\right].$$

This lemma follows by integral representation of (3) and the integral form of Gronwall inequality. Assume that B_0, B_1 and B are Hilbert spaces. Let $v \in L^1(\mathbb{R})$. The Fourier transform of $v : \mathbb{R} \to B_1$ is defined by $\hat{v}(\tau) = \int_{-\infty}^{\infty} e^{-2i\pi t\tau} v(t)dt$ (see [36]). Then, we have

$${}_{-\infty}^C \hat{D}_t^\alpha v(\tau) = (2i\pi\tau)^\alpha \hat{v}(\tau).$$

For $0 < \alpha \le 1$, define a Hilbert space

$$\mathcal{V}^\alpha(\mathbb{R}, B_0, B_1) = \left\{v \in L^2(\mathbb{R}, B_0) : {}_{-\infty}^C D_t^\alpha v \in L^2(\mathbb{R}, B_1)\right\}$$

equipped with the norm

$$\|v\|_{\mathcal{V}^\alpha} = \left\{ \|v\|^2_{L^2(\mathbb{R},B_0)} + \|\ |\tau|^\alpha \hat{v}\|^2_{L^2(\mathbb{R},B_1)} \right\}^{\frac{1}{2}}.$$

For every subset $I \subset \mathbb{R}$, define a subspace $\mathcal{V}_I^\alpha \subset \mathcal{V}^\alpha$ as

$$\mathcal{V}_I^\alpha(\mathbb{R}, B_0, B_1) = \left\{v \in \mathcal{V}^\alpha(\mathbb{R}, B_0, B_1) : supp(v) \subset I\right\}.$$

Theorem 1 (*[36,37]*). *Assume that $B_0 \hookrightarrow B \hookrightarrow B_1$ is continuous and $B_0 \hookrightarrow B$ is compact. Then, for every bounded set I and $\alpha > 0$, $\mathcal{V}_I^\alpha(\mathbb{R}, B_0, B_1) \hookrightarrow L^2(\mathbb{R}, B)$ is compact.*

Lemma 2 (*[38]*). *Suppose $u : [0,T] \longrightarrow X$ where X is a real Hilbert space. Assume that there exists the fractional derivative of u in the Caputo sense, then the following inequality holds,*

$$(u(t), {}_0^C D_t^\alpha(u(t))) \ge \frac{1}{2}\ {}_0^C D_t^\alpha |u|^2.$$

First, we define a weak solution of the system (2).

Definition 1. A triplet of functions (u,v,w) is called a weak solution of the nonlinear time-fractional reaction–diffusion system (2) if $u,v,w \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ satisfy

$$\left.\begin{array}{l}\int_\Omega \partial_t^\alpha u\phi_1 dx + \int_\Omega D_1(u)\nabla u \nabla\phi_1 dx = -\int_\Omega du\phi_1 dx dt - \int_\Omega \beta(u,w)\phi_1 dx \\ \qquad\qquad - \int_\Omega \sigma(u,v)\phi_1 dx,\\ \int_\Omega \partial_t^\alpha v\phi_2 dx + \int_\Omega D_2(v)\nabla v\nabla\phi_2 dx = \int_\Omega \beta(u,w)\phi_2 dx + \int_\Omega \sigma(u,v)\phi_2 dx - \int_\Omega rv\phi_2 dx,\\ \int_\Omega \partial_t^\alpha w\phi_3 dx + \int_\Omega D_3(w)\nabla w\nabla\phi_3 dx = \int_\Omega Nv\phi_3 dx - \int_\Omega ew\phi_3 dx,\end{array}\right\} \tag{4}$$

for any $\phi_i \in H_0^1(\Omega), i = 1,2,3$.

Now, we introduce the regularized system for (2). We consider the following approximation of the system (2):

$$\left.\begin{array}{ll}\partial_t^\alpha u^\epsilon - div(D_{1,\epsilon}(u^\epsilon)\nabla u^\epsilon) = -du^\epsilon - \beta(u^\epsilon,w^\epsilon) - \sigma(u^\epsilon,v^\epsilon), & \text{in } Q_T,\\ \partial_t^\alpha v^\epsilon - div(D_{2,\epsilon}(v^\epsilon)\nabla v^\epsilon) = \beta(u^\epsilon,w^\epsilon) + \sigma(u^\epsilon,v^\epsilon) - rv^\epsilon, & \text{in } Q_T,\\ \partial_t^\alpha w^\epsilon - div(D_{3,\epsilon}(w^\epsilon)\nabla w^\epsilon) = Nv^\epsilon - ew^\epsilon, & \text{in } Q_T,\end{array}\right\} \tag{5}$$

subject to the following boundary and initial conditions:

$$u^\epsilon(x,t) = v^\epsilon(x,t) = w^\epsilon(x,t) = 0 \text{ on } \Sigma_T,$$
$$u^\epsilon(x,0) = u_0(x);\ v^\epsilon(x,0) = v_0(x);\ w^\epsilon(x,0) = w_0(x) \text{ in } \Omega.$$

Consider the following spectral problem to find $z \in H_0^1(\Omega)$ and a number κ such that

$$(\nabla z, \nabla \phi) = \kappa(z,\phi) \text{ in } \Omega,$$
$$z = 0 \quad \text{on } \partial\Omega.$$

Notice that the spectral problem possesses a sequence of eigenvalues $\{\kappa_l\}$ and the corresponding eigenfunctions $\{e_l\}_{l=1}^\infty$ are orthogonal in $H_0^1(\Omega)$ and orthonormal in $L^2(\Omega)$. Here we seek a finite dimensional approximate solution to the problem (5) as a sequence $\{u_n^\epsilon, v_n^\epsilon, w_n^\epsilon\}$ defined for $t \geq 0$, $x \in \overline{\Omega}$ and satisfying

$$u_n^\epsilon(x,t) = \sum_{i=1}^n c_{1,n,i}(t)e_i(x),\ v_n^\epsilon(x,t) = \sum_{i=1}^n c_{2,n,i}(t)e_i(x),\ w_n^\epsilon(x,t) = \sum_{i=1}^n c_{3,n,i}(t)e_i(x).$$

We are looking to find the set of coefficients $\{c_{j,n,i}\}_{i=1}^n$, $j = 1,2,3$, such that for $m = 1,2,3,\dots,n$, the following relations hold

$$\left.\begin{aligned}
&\int_\Omega \partial_t^\alpha u_n^\epsilon e_m dx + \int_\Omega D_{1,\epsilon}(u_n^\epsilon)\nabla u_n^\epsilon \nabla e_m dx = -\int_\Omega d u_n^\epsilon e_m dx - \int_\Omega \beta(u_n^\epsilon, w_n^\epsilon) e_m dx \\
&\qquad\qquad - \int_\Omega \sigma(u_n^\epsilon, v_n^\epsilon) e_m dx, \\
&\int_\Omega \partial_t^\alpha v_n^\epsilon e_m dx + \int_\Omega D_{2,\epsilon}(v_n^\epsilon)\nabla v_n^\epsilon \nabla e_m dx = \int_\Omega \beta(u_n^\epsilon, w_n^\epsilon) e_m dx \\
&\qquad\qquad + \int_\Omega \sigma(u_n^\epsilon, v_n^\epsilon) e_m dx - \int_\Omega r v_n^\epsilon e_m dx, \\
&\int_\Omega \partial_t^\alpha w_n^\epsilon e_m dx + \int_\Omega D_{3,\epsilon}(w_n^\epsilon)\nabla w_n^\epsilon \nabla e_m dx = \int_\Omega N v_n^\epsilon e_m dx - \int_\Omega e w_n^\epsilon e_m dx,
\end{aligned}\right\} \tag{6}$$

and the initial conditions

$$u_n^\epsilon(x,0) = u_{0,n}(x) = \sum_{i=1}^n c_{1,n,i}(0)e_i(x),\ v_n^\epsilon(x,0) = v_{0,n}(x) = \sum_{i=1}^n c_{2,n,i}(0)e_i(x),$$
$$w_n^\epsilon(x,0) = w_{0,n}(x) = \sum_{i=1}^n c_{3,n,i}(0)e_i(x).$$

Furthermore, it can be noted that the previous form of solutions also satisfying the boundary conditions and (6) can be rewritten as a system of fractional ordinary differential equations:

$$\begin{aligned}
{}_0^C D_t^\alpha c_{1,n,m}(t) &= -\int_\Omega D_{1,\epsilon}(u_n^\epsilon)\nabla u_n^\epsilon \nabla e_m dx - \int_\Omega d u_n^\epsilon e_m dx - \int_\Omega \beta(u_n^\epsilon, w_n^\epsilon) e_m dx \\
&\quad - \int_\Omega \sigma(u_n^\epsilon, v_n^\epsilon) e_m dx, \\
&=: G_1^m\left(t, \{c_{1,n,i}\}_{i=1}^n, \{c_{2,n,i}\}_{i=1}^n, \{c_{3,n,i}\}_{i=1}^n\right), \\
{}_0^C D_t^\alpha c_{2,n,m}(t) &= -\int_\Omega D_{2,\epsilon}(v_n^\epsilon)\nabla v_n^\epsilon \nabla e_m dx + \int_\Omega \beta(u_n^\epsilon, w_n^\epsilon) e_m dx \\
&\quad + \int_\Omega \sigma(u_n^\epsilon, v_n^\epsilon) e_m dx - \int_\Omega r v_n^\epsilon e_m dx, \\
&=: G_2^m\left(t, \{c_{1,n,i}\}_{i=1}^n, \{c_{2,n,i}\}_{i=1}^n, \{c_{3,n,i}\}_{i=1}^n\right), \\
{}_0^C D_t^\alpha c_{3,n,m}(t) &= -\int_\Omega D_{3,\epsilon}(w_n^\epsilon)\nabla w_n^\epsilon \nabla e_m dx + \int_\Omega N v_n^\epsilon e_m dx - \int_\Omega w_n^\epsilon e_m dx, \\
&=: G_3^m\left(t, \{c_{2,n,i}\}_{i=1}^n, \{c_{3,n,i}\}_{i=1}^n\right).
\end{aligned} \tag{7}$$

Now we set $U = [0,\rho]$ with $\rho \in [0,T]$ and choose $R > 0$ large enough so that the ball $B_R \subset \mathbb{R}^n$ contains $\{c_{j,n,i}\}_{i=1}^n, j = 1,2,3$, and that $V = \overline{B}_R$. The components of G_j^m, $j = 1,2,3$ on $U \times V$ are bounded as

$$\begin{aligned}
&\left|G_1^m\left(t, \{c_{1,n,i}\}_{i=1}^n, \{c_{2,n,i}\}_{i=1}^n, \{c_{3,n,i}\}_{i=1}^n\right)\right| \\
&\leq \left(D_{1,\epsilon}(\sum_{i=1}^n c_{1,n,i}(t)e_i(x))\right)\left\|\sum_{i=1}^n c_{1,n,i}(t)\nabla e_i(x)\right\|_{L^2(\Omega)} \|\nabla e_m\|_{L^2(\Omega)} \\
&+ \|d\|_{L^\infty(\Omega)}\left\|\sum_{i=1}^n c_{1,n,i}(t)e_i(x)\right\|_{L^2(\Omega)} \|e_m\|_{L^2(\Omega)} \\
&+ \|\beta_1\|_{L^\infty(\Omega)}\left\|\sum_{i=1}^n c_{3,n,i}(t)e_i(x)\right\|_{L^2(\Omega)} \|e_m\|_{L^2(\Omega)} \\
&+ \|\beta_2\|_{L^\infty(\Omega)}\left\|\sum_{i=1}^n c_{2,n,i}(t)e_i(x)\right\|_{L^2(\Omega)} \|e_m\|_{L^2(\Omega)} \\
&\leq C(\epsilon, R, n)A_1(t),
\end{aligned}$$

where $C(\epsilon,R,n)$ is a positive constant that depends only on ϵ, R, n and is independent of t, and $A_1(t) \in L^1(U)$ is independent of m,n,R. In a similar manner, we obtain the estimates for $G_j^m, j = 2,3$,

$$|G_2^m\left(t, \{c_{1,n,i}\}_{i=1}^n, \{c_{2,n,i}\}_{i=1}^n, \{c_{3,n,i}\}_{i=1}^n\right)| \leq C(\epsilon,R,n)A_2(t),$$
$$|G_3^m\left(t, \{c_{2,n,i}\}_{i=1}^n, \{c_{3,n,i}\}_{i=1}^n\right)| \leq C(\epsilon,R,n)A_3(t),$$

where $C(\epsilon,R,n)$ is a constant as defined earlier. Using the standard FODE theory (see, for example, [39]), there exist absolutely continuous functions $\{c_{j,n,i}\}_{i=1}^n$, $j = 1,2,3$ satisfying the given initial conditions and (7) for a.e $t \in [0,\rho]$, $\rho > 0$. Therefore, we have

$$\begin{aligned}
c_{1,n,m}(t) &= c_{1,n,m}(0) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} G_1^m\left(t, \{c_{1,n,i}(\tau)\}_{i=1}^n, \{c_{2,n,i}(\tau)\}_{i=1}^n,\right. \\
&\quad \left.\{c_{3,n,i}(\tau)\}_{i=1}^n\right) d\tau, \\
c_{2,n,m}(t) &= c_{2,n,m}(0) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} G_2^m\left(t, \{c_{1,n,i}(\tau)\}_{i=1}^n, \{c_{2,n,i}(\tau)\}_{i=1}^n,\right. \\
&\quad \left.\{c_{3,n,i}(\tau)\}_{i=1}^n\right) d\tau, \\
c_{3,n,m}(t) &= c_{3,n,m}(0) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} G_3^m\left(t, \{c_{2,n,i}(\tau)\}_{i=1}^n, \{c_{3,n,i}(\tau)\}_{i=1}^n\right) d\tau.
\end{aligned}$$

This shows that $(u_n^\epsilon, v_n^\epsilon, w_n^\epsilon)$ are well defined and approximate solutions of (5) on $[0,\rho')$, for some $\rho' > 0$.

We prove the global existence of approximate solutions by deriving n-independent a-priori estimates for $(u_n^\epsilon, v_n^\epsilon, w_n^\epsilon)$ in different Banach spaces with arbitrary time $\tilde{T}$.

Lemma 3. *Assume that the hypotheses $(H1)-(H3)$ hold. If $u_0, v_0, w_0 \in L^2(\Omega)$, then there exists a constant $C > 0$, which is independent of n, such that*

$$\|(u_n^\epsilon, v_n^\epsilon, w_n^\epsilon)\|_{L^\infty(0,\tilde{T},L^2(\Omega))} \leq C, \quad \|(u_n^\epsilon, v_n^\epsilon, w_n^\epsilon)\|_{L^2(0,\tilde{T};H_0^1(\Omega))} \leq C. \tag{8}$$

Proof. Consider $\phi_{j,n}(x,t) = \sum_{l=1}^n b_{j,n,l}(t)e_l(x)$, where the coefficients $\{b_{j,n,l}\}_{l=1}^n$, $j = 1,2,3$, are absolutely continuous functions. Then, from (6), we get the following weak formulation

$$\left.\begin{aligned}
&\int_\Omega \partial_t^\alpha u_n^\epsilon \phi_{1,n} dx + \int_\Omega D_{1,\epsilon}(u_n^\epsilon)\nabla u_n^\epsilon \nabla \phi_{1,n} dx = -\int_\Omega d u_n^\epsilon \phi_{1,n} dx - \int_\Omega \beta(u_n^\epsilon, w_n^\epsilon)\phi_{1,n} dx \\
&\qquad\qquad - \int_\Omega \sigma(u_n^\epsilon, v_n^\epsilon)\phi_{1,n} dx, \\
&\int_\Omega \partial_t^\alpha v_n^\epsilon \phi_{2,n} dx + \int_\Omega D_{2,\epsilon}(v_n^\epsilon)\nabla v_n^\epsilon \nabla \phi_{2,n} dx = \int_\Omega \beta(u_n^\epsilon, w_n^\epsilon)\phi_{2,n} dx \\
&\qquad\qquad + \int_\Omega \sigma(u_n^\epsilon, v_n^\epsilon)\phi_{2,n} dx - \int_\Omega r v_n^\epsilon \phi_{2,n} dx dx, \\
&\int_\Omega \partial_t^\alpha w_n^\epsilon \phi_{3,n} dx + \int_\Omega D_{3,\epsilon}(w_n^\epsilon)\nabla w_n^\epsilon \nabla \phi_{3,n} dx = \int_\Omega N v_n^\epsilon \phi_{3,n} dx - \int_\Omega w_n^\epsilon \phi_{3,n} dx.
\end{aligned}\right\} \tag{9}$$

Taking $\phi_{1,n} = u_n^\epsilon$, $\phi_{2,n} = v_n^\epsilon$, and $\phi_{3,n} = w_n^\epsilon$ in (9) and using the given hypotheses together with Young's inequality, we get

$$\begin{aligned}
&\frac{1}{2}{}_0^C D_t^\alpha \int_\Omega (|u_n^\epsilon|^2 + |v_n^\epsilon|^2 + |w_n^\epsilon|^2) dx \\
&+ \int_\Omega (b_1|\nabla u_n^\epsilon|^2 + b_2|\nabla v_n^\epsilon|^2 + b_3|\nabla w_n^\epsilon|^2) dx
\end{aligned}$$

$$\leq C\int_{\Omega}|u_n^{\epsilon}|^2+|v_n^{\epsilon}|^2+|w_n^{\epsilon}|^2dx, \tag{10}$$

where C is the positive constant depending on the given data. Then, it follows by Lemma 1 that,

$$\int_{\Omega}(|u_n^{\epsilon}|^2+|v_n^{\epsilon}|^2+|w_n^{\epsilon}|^2)dx\leq C, \tag{11}$$

where C is the positive constant independent of n. Thus the sequence $\{(u_n^{\epsilon},v_n^{\epsilon},w_n^{\epsilon})\}$ is bounded in $L^{\infty}(0,\tilde{T};L^2(\Omega))$. From (10) and (11), we get

$$\int_0^{\tilde{T}}\int_{\Omega}|\nabla u_n^{\epsilon}|^2dxdt\leq C, \tag{12}$$

which implies that $\|u_n^{\epsilon}\|_{L^2(0,\tilde{T},H_0^1(\Omega))}\leq C$. Similar arguments lead to the estimate:

$$\|(u_n^{\epsilon},v_n^{\epsilon},w_n^{\epsilon})\|_{L^2(0,\tilde{T};H_0^1(\Omega))}\leq C. \tag{13}$$

This completes the proof. □

Lemma 4. *Let $u_0,v_0,w_0\in L^2(\Omega)$ and (H1)–(H3) are satisfied. Then $(\tilde{u}_n^{\epsilon},\tilde{v}_n^{\epsilon},\tilde{w}_n^{\epsilon})\in W^{\alpha}(\mathbb{R},H_0^1(\Omega),L^2(\Omega))$ is bounded.*

Proof. We omit the proof as it is similar to that Lemma 2.8 in [34]. □

Theorem 2. *Assume that hypotheses of Lemma 3 hold. Then, the approximate problem (5) possesses a weak solution $(u^{\epsilon},v^{\epsilon},w^{\epsilon})$, with $u^{\epsilon},v^{\epsilon},w^{\epsilon}\in L^{\infty}(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, satisfying*

$$\left.\begin{aligned}
&\int_{Q_T}\partial_t^{\alpha}u^{\epsilon}\phi_1dxdt+\int_{Q_T}D_{1,\epsilon}(u^{\epsilon})\nabla u^{\epsilon}\nabla\phi_1dxdt\\
&\qquad=-\int_{Q_T}du_n^{\epsilon}\phi_1dxdt-\int_{Q_T}\beta(u_n^{\epsilon},w_n^{\epsilon})\phi_1dxdt-\int_{Q_T}\sigma(u_n^{\epsilon},v_n^{\epsilon})\phi_1dxdt,\\
&\int_{Q_T}\partial_t^{\alpha}v^{\epsilon}\phi_2dxdt+\int_{Q_T}D_{2,\epsilon}(v^{\epsilon})\nabla v^{\epsilon}\nabla\phi_2dxdt\\
&\qquad=\int_{Q_T}\beta(u_n^{\epsilon},w_n^{\epsilon})\phi_2dxdt+\int_{Q_T}\sigma(u_n^{\epsilon},v_n^{\epsilon})\phi_2dxdt-\int_{Q_T}rv_n^{\epsilon}\phi_2dxdt,\\
&\int_{Q_T}\partial_t^{\alpha}w^{\epsilon}\phi_3dxdt+\int_{Q_T}D_{3,\epsilon}(w^{\epsilon})\nabla w^{\epsilon}\nabla\phi_3dxdt\\
&\qquad=\int_{Q_T}Nv^{\epsilon}\phi_3dxdt-\int_{Q_t}ew^{\epsilon}\phi_3dxdxdt,
\end{aligned}\right\} \tag{14}$$

for every $\phi_i\in L^2(0,T;H_0^1(\Omega))$, $i=1,2,3$.

Proof. Using Lemmas 3 and 4, and the standard compactness arguments, sequences have convergent subsequences (which are also denoted by $(u_n^{\epsilon},v_n^{\epsilon},w_n^{\epsilon})$). Then there exist limit functions $u^{\epsilon},v^{\epsilon},w^{\epsilon}$ as $n\to\infty$, that is,

$(u_n^{\epsilon},v_n^{\epsilon},w_n^{\epsilon})\rightharpoonup(u^{\epsilon},v^{\epsilon},w^{\epsilon})$ weakly in $L^2(0,T;H_0^1(\Omega))$,
$(u_n^{\epsilon},v_n^{\epsilon},w_n^{\epsilon})\rightharpoonup(u^{\epsilon},v^{\epsilon},w^{\epsilon})$ and weak* in $L^{\infty}(0,T;L^2(\Omega))$,
$D_{1,\epsilon}(u_n^{\epsilon})\nabla u_n^{\epsilon}\rightharpoonup\varrho_1$ weakly in $L^2(Q_T)$,
$D_{2,\epsilon}(v_n^{\epsilon})\nabla v_n^{\epsilon}\rightharpoonup\varrho_2$ weakly in $L^2(Q_T)$,
$D_{3,\epsilon}(w_n^{\epsilon})\nabla w_n^{\epsilon}\rightharpoonup\varrho_3$ weakly in $L^2(Q_T)$.

From the above convergence results, $(u_n^{\epsilon},v_n^{\epsilon},w_n^{\epsilon})\to(u^{\epsilon},v^{\epsilon},w^{\epsilon})$ in $L^2(0,T;L^2(\Omega))$. We have to show that

$$D_{3,\epsilon}(u_n^{\epsilon})\nabla u_n^{\epsilon}=\varrho_1,\ D_{2,\epsilon}(v_n^{\epsilon})\nabla v_n^{\epsilon}=\varrho_2,\ D_{3,\epsilon}(w_n^{\epsilon})\nabla w_n^{\epsilon}=\varrho_3. \tag{15}$$

Multiplying the first equation in (5) by u_n^{ϵ} and then integrating over Q_T and taking the limit $n\to\infty$, we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}|u^{\epsilon}(x,T)|^2dx+\int_{Q_T}(T-t)^{\alpha-1}\varrho_1\nabla u^{\epsilon}dxdt\\
&+\int_{Q_T}(T-t)^{\alpha-1}\beta(u^{\epsilon},w^{\epsilon})u^{\epsilon}dxdt\\
&+\int_{Q_T}(T-t)^{\alpha-1}\sigma(u^{\epsilon},v^{\epsilon})u^{\epsilon}dxdt\\
&\leq\int_{\Omega}d|u^{\epsilon}|^2dxdt+\frac{1}{2}\int_{\Omega}|u_0(x)|^2dx.
\end{aligned} \tag{16}$$

In view of the monotonicity of the function $D_{1,\epsilon}(\nabla u^{\epsilon})$, it follows that

$$\int_{\Omega}(D_{1,\epsilon}(u_n^{\epsilon})\nabla u_n^{\epsilon}-D_{1,\epsilon}(u_1)\nabla u_1)(\nabla u_n^{\epsilon}-\nabla u_1)dx\geq 0.$$

Multiplying the approximate equation by u_n^{ϵ} and integrating over Ω, one obtains

$$\begin{aligned}
&\frac{1}{2}\,{}_0^CD_t^{\alpha}\int_{\Omega}|u_n^{\epsilon}|^2dx+\int_{\Omega}D_{1,\epsilon}(u_n^{\epsilon})\nabla u_n^{\epsilon}\nabla u_n^{\epsilon}dx+\int_{\Omega}\beta(u_n^{\epsilon},w_n^{\epsilon})u_n^{\epsilon}dx\\
&\qquad+\int_{\Omega}\sigma(u_n^{\epsilon},v_n^{\epsilon})u_n^{\epsilon}dx\\
&\qquad\leq-\int_{\Omega}d|u_n^{\epsilon}|^2dx.
\end{aligned}$$

Using the monotonicity condition yields

$$\begin{aligned}
&-\frac{1}{2}\,{}_0^CD_t^{\alpha}\int_{\Omega}|u_n^{\epsilon}|^2dx-\int_{\Omega}D_{1,\epsilon}(u_n^{\epsilon})\nabla u_n^{\epsilon}\nabla u_1dx\\
&-\int_{\Omega}D_{1,\epsilon}(u_1)\nabla u_1(\nabla u_n^{\epsilon}-\nabla u_1)dx\\
&-\int_{\Omega}\beta(u_n^{\epsilon},w_n^{\epsilon})u_n^{\epsilon}dx-\int_{\Omega}\sigma(u_n^{\epsilon},v_n^{\epsilon})u_n^{\epsilon}dx-\int_{\Omega}d|u_n^{\epsilon}|^2dx\geq 0.
\end{aligned}$$

Then, taking the limit $n\to\infty$ and integrating α-th order expression from 0 to T, one obtains

$$\begin{aligned}
&-\frac{1}{2}\int_{\Omega}|u^{\epsilon}(T)|^2dx-\int_{Q_T}(T-t)^{\alpha-1}\varrho_1\nabla u_1^{\epsilon}dxdt\\
&-\int_{Q_T}(T-t)^{\alpha-1}D_{1,\epsilon}(u_1)\nabla u_1(\nabla u^{\epsilon}-\nabla u_1)dxdt-\int_{\Omega}\beta(u^{\epsilon},w^{\epsilon})u^{\epsilon}dxdt\\
&-\int_{\Omega}\sigma(u^{\epsilon},v^{\epsilon})u^{\epsilon}dxdt\\
&-\int_{Q_T}(T-t)^{\alpha-1}d|u^{\epsilon}|^2dxdt+\frac{1}{2}\int_{\Omega}|u_0|^2dx\geq 0.
\end{aligned} \tag{17}$$

Comparing (16) with (17), one deduces

$$\int_{Q_T}(\varrho_1-D_{1,\epsilon}(u_1)\nabla u_1(\nabla u^{\epsilon}-\nabla u_1))dxdt\geq 0.$$

Let $u_1=u^{\epsilon}-\eta\psi$ for any $\eta>0$ and $\psi\in L^2(0,T;L^2(\Omega))$ in the last inequality, one gets

$$\int_{Q_T}(\varrho_1-D_{1,\epsilon}(u^{\epsilon}-\eta\psi)(\nabla(u^{\epsilon}-\eta\psi))(\nabla\psi))dxdt\geq 0. \tag{18}$$

Passing onto the limit $\eta\to 0$ and using Lebesgue's dominated convergence theorem, one obtains

$$\int_{Q_T}\left(\varrho_1-D_{1,\epsilon}(u^{\epsilon})(\nabla u^{\epsilon})(\nabla\psi)\right)dxdt\geq 0, \tag{19}$$

for any $\psi\in L^2(0,T;H_0^1(\Omega))$. Thus $\varrho_1=D_{1,\epsilon}(u^{\epsilon})\nabla u^{\epsilon}$. Similarly, we can prove that $\varrho_2=D_{2,\epsilon}(v^{\epsilon})\nabla v^{\epsilon}$ and $\varrho_3=D_{3,\epsilon}(w^{\epsilon})\nabla w^{\epsilon}$.

We can easily prove that $(u^{\epsilon}(t_0),z)-(u^{\epsilon}(t),z)\to 0$, $(v^{\epsilon}(t_0),z)-(v^{\epsilon}(t),z)\to 0$ and $(w^{\epsilon}(t_0),z)-(w^{\epsilon}(t),z)\to 0$ as $t\to t_0$ as in [34,37]. □

Lemma 5. *Assume that the hypotheses (H1)–(H3) hold. If $u_0,v_0,w_0\in L^2(\Omega)$ are nonnegative, then solution $(u^{\epsilon},v^{\epsilon},w^{\epsilon})$ of (5) is nonnegative. Further, there exists some constant $C>0$ such that*

$$\|(u^{\epsilon},v^{\epsilon},w^{\epsilon})\|_{L^{\infty}(0,T;L^2(\Omega))}\leq C,\quad \|(u^{\epsilon},v^{\epsilon},w^{\epsilon})\|_{L^2(0,T,H_0^1(\Omega))}\leq C. \tag{20}$$

Proof. The non negativity of solutions will be established by using the procedure employed in [40]. Let us fix $u^{-\epsilon}=-\sup(-u^{\epsilon},0)$ and multiply the first equation of (5) by $u^{-\epsilon}$ to obtain

$$\begin{aligned}
&\frac{1}{2}\,{}_0^CD_t^{\alpha}\int_{\Omega}|u^{-\epsilon}|^2dx+b_1\int_{\Omega}|\nabla u^{-\epsilon}|^2dx+\int_{\Omega}\beta(u^{-\epsilon},w^{-\epsilon})u^{-\epsilon}dx\\
&+\int_{\Omega}\sigma(u^{-\epsilon},v^{-\epsilon})u^{-\epsilon}dx\\
&\leq\int_{\Omega}d|u^{-\epsilon}|^2dx,
\end{aligned} \tag{21}$$

where we have applied Lemma 2. From (21), we get

$$\frac{1}{2}\,{}_0^C D_t^\alpha \int_\Omega |u^{-\epsilon}|^2 dx \le 0.$$

Since the initial value $u_0(x)$ is nonnegative, therefore the solution u^ϵ is also nonnegative. In a similar manner, it can be shown that v^ϵ and w^ϵ are nonnegative. The following energy estimates are valid for $(u^\epsilon, v^\epsilon, w^\epsilon)$.

$$\|(u^\epsilon, v^\epsilon, w^\epsilon)\|_{L^\infty(0,T;L^2(\Omega))} \le C, \quad \|(u^\epsilon, v^\epsilon, w^\epsilon)\|_{L^2(0,T,H_0^1(\Omega))} \le C, \tag{22}$$

where the constant C independent of ϵ. The above results can be easily obtained by replacing n by ϵ in the proof of Lemma 3. □

Lemma 6. *Let $u_0, v_0, w_0 \in H$ and the assumption (H1)–(H3) hold. Then $(\{\tilde{u}^\epsilon\}, \{\tilde{v}^\epsilon\}, \{\tilde{w}^\epsilon\})$ forms a bounded set of $W^\gamma(\mathbb{R}, H_0^1(\Omega), L^2(\Omega))$, where*

$$\tilde{\rho}^\epsilon = \begin{cases} \rho^\epsilon, & t \in [0,T], \\ 0, & \mathbb{R} \setminus [0,T], \end{cases}$$

with $\tilde{\rho}^\epsilon = \tilde{u}^\epsilon, \tilde{v}^\epsilon, \tilde{w}^\epsilon$.

Proof. We omit the proof as it is similar to that of Lemma 4. □

Now we present our main results.

Theorem 3. *Assume that $u_0, v_0, w_0 \in L^2(\Omega)$ and the conditions (H1)–(H3) hold. Then the time-fractional reaction–diffusion model (1) possesses a weak solution in the sense of Definition 1.*

Proof. By Lemmas 5 and 6, the sequences $\{u^\epsilon\}, \{v^\epsilon\}$ and $\{w^\epsilon\}$ have convergent subsequences (also denoted by $\{u^\epsilon\}, \{v^\epsilon\}$ and $\{w^\epsilon\}$). Then, there exists limit functions u, v and w as $\epsilon \to 0$, such that

$$\begin{aligned}
&(u^\epsilon, v^\epsilon, w^\epsilon) \rightharpoonup (u, v, w) \text{ weakly in } L^2(0,T; H_0^1(\Omega)),\\
&(u^\epsilon, v^\epsilon, w^\epsilon) \rightharpoonup (u, v, w) \text{ and weak}^* \text{ in } L^\infty(0,T, L^2(\Omega)),\\
&D_{1,\epsilon}(u^\epsilon)\nabla u^\epsilon \rightharpoonup \zeta_1 \text{ weakly in } L^2(Q_T),\\
&D_{2,\epsilon}(v^\epsilon)\nabla v^\epsilon \rightharpoonup \zeta_2 \text{ weakly in } L^2(Q_T),\\
&D_{3,\epsilon}(w^\epsilon)\nabla w^\epsilon \rightharpoonup \zeta_3 \text{ weakly in } L^2(Q_T).
\end{aligned}$$

Using the arguments employed in the last section and the monotonicity property of the nonlinear diffusion operator *(H3)*, we obtain $D_1(u)\nabla u = \zeta_1,\ D_2(v)\nabla v = \zeta_2,\ D_3(w)\nabla w = \zeta_3$. □

Theorem 4. *The solution (u, v, w) of the nonlinear time-fractional diffusion system (1) is unique.*

Proof. Let (u_1, v_1, w_1) and (u_2, v_2, w_2) be any two solutions of the nonlinear time-fractional diffusion system (1). Then, by taking $u = u_1 - u_2,\ v = v_1 - v_2,\ w = w_1 - w_2$ and manipulating the weak formulation of the solutions (u_1, v_1, w_1) and (u_2, v_2, w_2) together with the substitutions $\phi_1 = u_1 - u_2,\ \phi_2 = v_1 - v_2,\ \phi_3 = w_1 - w_2$ in (4), we obtain

$$\begin{aligned}
&\frac{1}{2}\,{}_0^C D_t^\alpha \int_\Omega |u(x,t)|^2 dx + \int_\Omega (D_1(u_1)\nabla u_1 - D_1(u_2)\nabla u_2)\nabla u\, dx\\
&\quad + \int_\Omega (\beta(u_1, w_1) - \beta(u_2, w_2)) u\, dx\\
&\quad \le \int_\Omega d|u|^2 dx - \int_\Omega (\sigma(u_2, v_2) - \sigma(u_1, v_1)) u\, dx,\\
&\frac{1}{2}\,{}_0^C D_t^\alpha \int_\Omega |v(x,t)|^2 dx + \int_\Omega (D_2(v_1)\nabla v_1 - D_2(v_2)(\nabla v_2))\nabla v\, dx\\
&\quad \le \int_\Omega (\beta(u_1, w_1) - \beta(u_2, w_2)) v\, dx + \int_\Omega (\sigma(u_1, v_1) - \sigma(u_2, v_2)) v\, dx\\
&\quad + \int_\Omega r|v|^2 dx\\
&\frac{1}{2}\,{}_0^C D_t^\alpha \int_\Omega |w(x,t)|^2 dx + \int_\Omega (D_3(w_1)\nabla w_1 - D_3(w_2)\nabla w_2)\nabla w\, dx\\
&\quad \le \int_\Omega N v w\, dx - \int_\Omega e|w|^2 dx.
\end{aligned} \tag{23}$$

Using (H1)–(H3) and the non-negativity and boundedness of solutions of the system (1), we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u(x,t)|^2 dx \le |u(x,0)|^2\\
&\quad + \frac{C}{\Gamma(\alpha)}\left(\int_{Q_t} (t-s)^{\alpha-1}(|u|^2 + |v|^2 + |w|^2) dx ds\right),\\
&\frac{1}{2}\int_\Omega |v(x,t)|^2 dx \le |v(x,0)|^2\\
&\quad + \frac{C}{\Gamma(\alpha)}\left(\int_{Q_t} (t-s)^{\alpha-1}(|u|^2 + |v|^2 + |w|^2) dx ds\right),\\
&\frac{1}{2}\int_\Omega |w(x,t)|^2 dx \le |w(x,0)|^2\\
&\quad + \frac{C}{\Gamma(\alpha)}\left(\int_{Q_t} (t-s)^{\alpha-1}(|u|^2 + |v|^2 + |w|^2) dx ds\right).
\end{aligned}$$

Combining all the inequalities and using Lemma 1, we conclude that the solution of the system (2) is unique. □

Finite element scheme

The main aim of this section is to derive a-priori error estimates and obtaining optimal convergence order for the proposed Galerkin finite element scheme of the problem. First, we present the spatial and temporal discretization of the problem in this section. Further, we derive a-priori error estimates and optimal convergence order for the proposed finite element scheme.

Finite element discretization

Let Ω_h be a triangulation of Ω into tetrahedral cells. Suppose $V_h \subset H_0^1(\Omega)$ and $Q_h \subset H^1(\Omega)$ are a conforming finite element (finite dimensional) subspace of piecewise linear polynomials (P_1) associated with Ω_h.

Let us define the elliptic projection $\Pi_h : H_0^1(\Omega) \to V_h$ such that

$$(a(w(t))\nabla\omega, \nabla\phi_h) = (a(w(t))\nabla\Pi_h\omega, \nabla\phi_h), \quad \forall \omega \in H_0^1(\Omega),\ \phi_h \in V_h. \tag{24}$$

Theorem 5 ([41]). *Suppose a is a smooth function in Ω with $0 < \mu \le a \le M$ for all x in Ω. Assume that $w \in H^2 \cap H_0^1(\Omega)$ and w_h is defined as in (24). Then there exists a positive constant C, which is not depending of h such that*

$$\|\nabla(w - \Pi_h w)\| \le C(w)h, \tag{25}$$

$$\|w - \Pi_h w\|_0 \le C(w)h^2. \tag{26}$$

Theorem 6 ([31]). *Suppose the nonnegative sequences $\{\omega^n, g^n | n = 0, 1, 2, \ldots\}$ satisfy*

$${}^C D_{\delta t}^\alpha \omega^n \le \mu_1 \omega^n + \mu_2 \omega^{n-1} + g^n,$$

for $n \ge 1$, where μ_1, μ_2 are nonnegative constants. Then, there exists a positive constant δt^* such that, when $\delta t \le \delta t^*$,

$$\omega^n \le 2\Big(2\omega^0 + \frac{t_n^\alpha}{\Gamma(1+\alpha)} \max_{0\le j\le n} g^j\Big) E_\alpha(2\mu t_n^\alpha), \quad 1 \le n \le N.$$

Here $E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+k\alpha)}$ is the Mittag-Leffler function and $\mu = \mu_1 + \frac{\mu_2}{2 - 2^{(1-\alpha)}}$.

Further, define the finite element ansatz functions as

$$u_h(x) = \sum_{i=1}^{\mathcal{N}} u_i \phi_i(x), \quad v_h^n(x) = \sum_{i=1}^{\mathcal{N}} v_i \phi_i(x), \quad w_h^n(x) = \sum_{i=1}^{\mathcal{N}} w_i \phi_i(x).$$

Thus, we have

$$\mathcal{M}_0^C D_t^\alpha \Lambda(t) + \mathcal{A}\Lambda = F, \tag{27}$$

where $\Lambda(t) = \Lambda = (u_1, u_2, \ldots, u_{\mathcal{N}}, v_1, v_2, \ldots, v_{\mathcal{N}}, w_1, w_2, \ldots, w_{\mathcal{N}})$ is unknown vector. Further, the mass matrix M, stiffness matrices A and known vector F are given by

$$\mathcal{M} = \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix}, \quad (M)_{\mathfrak{pq}} = \int_{\mathcal{T}_h} \phi_{\mathfrak{p}}(x)\phi_{\mathfrak{q}}(x),$$

$$\mathcal{A} = \begin{bmatrix} A^{(U)} \\ A^{(V)} \\ A^{(W)} \end{bmatrix},$$

$$(A^{(U)})_{\mathfrak{pq}} = \int_{\mathcal{T}_h} D_1\left(\left(\sum_{\mathfrak{k}=1}^{\mathcal{N}} u_{\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x)\right)\right)\nabla\phi_{\mathfrak{p}}(x)\nabla\phi_{\mathfrak{q}}(x)dx$$
$$+\int_{\mathcal{T}_h} d\phi_{\mathfrak{p}}(x)\phi_{\mathfrak{q}}(x)dx + \int_{\Omega_h}\beta\left(\sum_{\mathfrak{k}=1}^{\mathcal{N}} u_{\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x), w_{\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x)\right)$$
$$\times\phi_{\mathfrak{p}}(x)\phi_{\mathfrak{q}}(x)dx$$
$$+\int_{\Omega_h}\sigma\left(\sum_{\mathfrak{k}=1}^{\mathcal{N}} u_{\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x), v_{\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x)\right)\phi_{\mathfrak{p}}(x)\phi_{\mathfrak{q}}(x)dx,$$

$$(A^{(V)})_{\mathfrak{pq}} = \int_{\Omega_h} D_2\left(\left(\sum_{\mathfrak{k}=1}^{\mathcal{N}} v_{\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x)\right)\right)\nabla\phi_{\mathfrak{p}}(x)\nabla\phi_{\mathfrak{q}}(x)dx$$
$$+\int_{\Omega_h} r\phi_{\mathfrak{p}}(x)\phi_{\mathfrak{q}}(x)dx$$
$$-\int_{\Omega_h}\sigma\left(\sum_{\mathfrak{k}=1}^{\mathcal{N}} u_{\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x), v_{\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x)\right)\phi_{\mathfrak{p}}(x)\phi_{\mathfrak{q}}(x)dx$$

$$(A^{(W)})_{\mathfrak{pq}} = \int_{\Omega_h} D_3\left(l\left(\sum_{\mathfrak{k}=1}^{\mathcal{N}} u_{4,\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x)\right)\right)\nabla\phi_{\mathfrak{p}}(x)\nabla\phi_{\mathfrak{q}}(x)dx$$
$$+\int_{\Omega_h} e\phi_{\mathfrak{p}}(x)\phi_{\mathfrak{q}}(x)dx,$$

$$F = \begin{bmatrix} 0 \\ F_2 \\ F_3 \end{bmatrix}, \quad (F_2) = \int_{\mathcal{T}_h}\beta\left(\sum_{\mathfrak{k}=1}^{\mathcal{N}} u_{\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x), w_{\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x)\right)\phi_{\mathfrak{q}}(x)dx,$$

$$(F_2) = \int_{\mathcal{T}_h} N\left(\sum_{\mathfrak{k}=1}^{\mathcal{N}} v_{\mathfrak{k}}(t)\phi_{\mathfrak{k}}(x)\phi_{\mathfrak{q}}(x)\right)\phi_{\mathfrak{q}}(x)dx.$$

Temporal discretization

Let $0 = t^0 < t^1 < \cdots < t^N = T$ be a partition of the considered time interval $[0, T]$, and $\delta_t = t^{n+1} - t^n, n = 0, 1, \ldots, N-1$, denotes a uniform time step. Use the L1-discretization of Caputo fractional time derivative ${}^C D_{t_n}^{\alpha}$ defined by

$$ {}^C D_{t_n}^{\alpha} u^n = \frac{\delta_t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{\mathfrak{l}=\mathfrak{o}}^{\mathfrak{r}} a_{\mathfrak{l}}\left(u^{\mathfrak{r}+1-\mathfrak{l}} - u^{\mathfrak{r}-\mathfrak{l}}\right) + E_n, \tag{28}$$

$a_{\mathfrak{l}} = (\mathfrak{l}+1)^{\alpha-1} - (\mathfrak{l})^{\alpha-1}$, $\mathfrak{l} = 1, 2, 3, \ldots, \mathfrak{r}$ as in [42,43]. For sufficiently smooth u in time, the error E_n satisfies

$$\|E_n\|_0 \le C\delta_t^{2-\alpha}. \tag{29}$$

For weak singularity in the initial time $t = 0$ the error E_n is given as

$$\max_{1\le n\le N}\|E_n\|_0 \le C\delta_t^{\alpha}. \tag{30}$$

Here further details about (29) and (30) can be found in [31,42].

$u^n(x) := u(x, t^n), v^n(x) := v(x, t^n), w^n(x) := w(x, t^n)$ denote the function values at time t^n. Then the semi-discrete (continuous in space) form of the system (2) using (28) reads:

$$\frac{\delta_t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{\mathfrak{l}=\mathfrak{o}}^{\mathfrak{r}} a_{\mathfrak{l}}\left(u^{\mathfrak{r}+1-\mathfrak{l}} - u^{\mathfrak{r}-\mathfrak{l}}\right) + a_u(u^{\mathfrak{r}+1}; \tilde{u}^{\mathfrak{r}+1}; \tilde{v}^{\mathfrak{r}+1}; \tilde{w}^{\mathfrak{r}+1}, \phi) = 0,$$
$$\frac{\delta_t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{\mathfrak{l}=\mathfrak{o}}^{\mathfrak{r}} a_{\mathfrak{l}}\left(v^{\mathfrak{r}+1-\mathfrak{l}} - v^{\mathfrak{r}-\mathfrak{l}}\right) + a_v(v^{\mathfrak{r}+1}; \tilde{u}^{\mathfrak{r}+1}; \tilde{v}^{\mathfrak{r}+1}; \tilde{w}^{\mathfrak{r}+1}, \phi)$$
$$= f_1(u^{\mathfrak{r}+1}, w^{\mathfrak{r}+1}, \phi), \tag{31}$$
$$\frac{\delta_t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{\mathfrak{l}=\mathfrak{o}}^{\mathfrak{r}} a_{\mathfrak{l}}\left(w^{\mathfrak{r}+1-\mathfrak{l}} - w^{\mathfrak{r}-\mathfrak{l}}\right) + a_w(w^{\mathfrak{r}+1}; \tilde{v}^{\mathfrak{r}+1}; \tilde{w}^{\mathfrak{r}+1}, \phi) = f_2(v^{\mathfrak{r}+1}, \phi),$$

for all $\phi \in H_0^1(\Omega)$, where

$$a_u(u^{\mathfrak{r}+1}; \tilde{u}^{\mathfrak{r}+1}; \tilde{v}^{\mathfrak{r}+1}; \tilde{w}^{\mathfrak{r}+1}, \phi) = \int_{\Omega} D_1(\tilde{u}^{\mathfrak{r}+1})\nabla u^{\mathfrak{r}+1}\cdot\nabla\phi\, dx + \int_{\Omega} du^{\mathfrak{r}+1}\phi dx$$
$$+\int_{\Omega}\beta(u^{\mathfrak{r}+1}, \tilde{w}^{\mathfrak{r}+1})\phi dx$$
$$+\int_{\Omega}\sigma(u^{\mathfrak{r}+1}, \tilde{v}^{\mathfrak{r}+1})\phi dx,$$
$$a_v(v^{\mathfrak{r}+1}; \tilde{u}^{\mathfrak{r}+1}; \tilde{u}v\mathfrak{r}+1; \tilde{w}^{\mathfrak{r}+1}, \phi) = \int_{\Omega} D_2(\tilde{v}^{\mathfrak{r}+1})\nabla v^{\mathfrak{r}+1}\nabla\phi dx + \int_{\Omega} rv^{\mathfrak{r}+1}\phi dx$$
$$-\int_{\Omega}\sigma(\tilde{u}^{\mathfrak{r}+1}, v^{\mathfrak{r}+1})\phi dx,$$
$$a_w(w^{\mathfrak{r}+1}; \tilde{v}^{\mathfrak{r}+1}, \tilde{w}^{\mathfrak{r}+1}, \phi) = \int_{\Omega} D_3(\tilde{w}^{\mathfrak{r}+1})\nabla w\cdot\nabla\phi\, dx + \int_{\Omega} ew^{\mathfrak{r}+1}\phi\, dx,$$
$$f_1(u^{\mathfrak{r}+1}, w^{\mathfrak{r}+1}, \phi) = \int_{\Omega}\beta(u^{\mathfrak{r}+1}, w^{\mathfrak{r}+1})\phi dx, \quad f_2(v^{\mathfrak{r}+1}) = \int_{\Omega} Nv^n\phi dx.$$

Further, we adopt the fixed point iteration technique to handle the nonlinear terms of the proposed system as in [34,44]. Initiate with $z^{\mathfrak{r}+1,0} = z^{\mathfrak{r}}$, the nonlinear terms can be written as

$$a_u(u^{\mathfrak{r}+1,m}; \tilde{u}^{\mathfrak{r}+1,m}; \tilde{v}^{\mathfrak{r}+1,m}; \tilde{w}^{\mathfrak{r}+1,m}, \phi)$$
$$\simeq a_u(u^{\mathfrak{r}+1,m}; u^{\mathfrak{r}+1,m-1}; v^{\mathfrak{r}+1,m-1}; w^{\mathfrak{r}+1,m-1}, \phi),$$
$$a_v(v^{\mathfrak{r}+1,m}; \tilde{u}^{\mathfrak{r}+1,m}; \tilde{v}^{\mathfrak{r}+1,m}; \tilde{w}^{\mathfrak{r}+1,m}, \phi)$$
$$\simeq a_v(v^{\mathfrak{r}+1,m}; u^{\mathfrak{r}+1,m-1}; v^{\mathfrak{r}+1,m-1}; w^{\mathfrak{r}+1,m-1}, \phi), \tag{32}$$
$$a_w(w^{\mathfrak{r}+1,m}; \tilde{v}^{\mathfrak{r}+1,m}, \tilde{w}^{\mathfrak{r}+1,m}, \phi) \simeq a_w(w^{\mathfrak{r}+1,m}; v^{\mathfrak{r}+1,m-1}, w^{\mathfrak{r}+1,m-1}, \phi),$$

for $m = 0, 1, 2, \ldots$. Further, iterate the process until the residual smaller than a given value (10^{-6}) or attain the maximum number of iterations. Finally, we set, $z^{\mathfrak{r}+1} = z^{\mathfrak{r}+1,m-1}$, where $z = u, v, w$ and advance to the next time step. In computations, the iteration converges within two or three iterations for the residual error 10^{-6}, and the number of iterations increase when δ_t is increased.

The analysis of the error estimates

First, we analyze error estimates for the considered finite element numerical scheme. To analyze, we do not require to have the fixed point iteration subscript m in the numerical solution notation (denoted by (U^n, V^n, W^n)). Thus, the fully discretized system is

$$\left(\bar{\partial}^{\alpha}U^n, \phi\right) + a_u\left(U^n; U^n; V^n; W^n, \phi\right) = 0,$$
$$\left(\bar{\partial}^{\alpha}V^n, \phi\right) + a_v\left(V^n; U^n; V^n; W^n, \phi\right) = f_1(U^n, W_h^n, \phi), \tag{33}$$
$$\left(\bar{\partial}^{\alpha}W^n, \phi\right) + a_w\left(W^n; U^n; V^n; W^n, \phi\right) = f_2(V^n, \phi),$$

where

$$\bar{\partial}^{\alpha}Z^n = \frac{\delta_t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{\mathfrak{l}=\mathfrak{o}}^{\mathfrak{r}} a_{\mathfrak{l}}\left(Z^{\mathfrak{r}+1-\mathfrak{l}} - Z^{\mathfrak{r}-\mathfrak{l}}\right),$$

and $a_{\mathfrak{l}}$ defined as in (28).

The following theorem provides a-priori bound for the solution of fully discrete system (33).

Theorem 7. *Let (U^n, V^n, W^n) be the solution of (33). Then there exists a positive constant δ_t^* such that $\delta_t \le \delta_t^*$, the solution (u_h^n, v_h^n, w_h^n) satisfies*

$$\|(U^n, V^n, W^n)\|_0 \le C(1 + \|(U^0, V^0, W^0)\|_0), \tag{34}$$

where $n = 1, 2, \ldots, N$ and C is a positive constant independent of h and δ_t.

Proof. From (33),

$$\left(\bar{\partial}^{\alpha}U^n, \phi\right) + a_u\left(U^n; U^n; V^n; W^n, \phi\right) = 0.$$

Substitute $\phi = U^n$ in the above equation, we get

$$\left(\bar{\partial}^\alpha U^n, U^n\right) + a_u\left(U^n; U^n; V^n; W^n, U^n\right) = 0.$$

Using the hypotheses $(H_1)-(H_3)$ and properties of $\sigma(\cdot)$ and $\beta(\cdot)$ as in (10), one gets

$$\left(\bar{\partial}^\alpha U^n, U^n\right) + b_1\|\nabla U^n\|_0^2 \le C(\|U^n\|_0^2 + \|V^n\|_0^2 + \|W^n\|_0^2).$$

Using the properties $\frac{1}{2}\bar{\partial}^\alpha\|U^n\|^2 \le \left(\bar{\partial}^\alpha U^n, U^n\right)$, one obtains

$$\frac{1}{2}\bar{\partial}^\alpha\|U^n\|_0^2 + b_1\|\nabla U^n\|_0^2 \le C(\|U^n\|_0^2 + \|V^n\|_0^2 + \|W^n\|_0^2).$$

Thus

$$\bar{\partial}^\alpha\|U^n\|_0^2 \le C(\|U^n\|_0^2 + \|V^n\|_0^2 + \|W^n\|_0^2).$$

Similarly, we can obtain for V^n and W^n. Therefore, one obtains

$$\bar{\partial}^\alpha(\|U^n\|_0^2 + \|V^n\|_0^2 + \|W^n\|_0^2) \le C(\|U^n\|_0^2 + \|V^n\|_0^2 + \|W^n\|_0^2).$$

Using Theorem 6 yields

$$\|(U^n, V^n, W^n)\|_0 \le C(1 + \|(U^0, V^0, W^0)\|_0).$$

This completes the proof. □

Theorem 8. *Suppose that (u, v, w) and (U^n, V^n, W^n) be the exact and numerical solution, respectively, we have the following error estimates*

$$\|S^n - s(x, t_n)\|_0 \le C(h^2 + \delta t^{2-\alpha}),$$

where $S^n = (U^n, V^n, W^n)$, $s = (u, v, w)$.

Proof. Let $s^n := s(\cdot, t_n)$ and S^n be the solution of (2) and (33), respectively. S_h^n denotes the Ritz projection of s^n. Let the approximation error

$$S^n - s^n = S^n - S_h^n + S_h^n - s^n = \xi^n + \Lambda^n.$$

First, we estimate ξ^n. Employing the properties of Ritz projection, we get

$$\begin{aligned}\int_\Omega \bar{\partial}^\alpha \xi_1^n\phi_1\,dx &+ \int_\Omega D_1(U^n)\nabla\xi_1^n\nabla\phi_1\,dx\\ &= \int_\Omega \bar{\partial}^\alpha U^n\phi_1\,dx + \int_\Omega D_1(U^n)\nabla U^n\phi_1\,dx - \int_\Omega \bar{\partial}^\alpha \Pi_h u^n\phi_1\,dx\\ &\quad - \int_\Omega D_1(U^n)\nabla\Pi_h u^n\phi_1\,dx,\end{aligned}$$

where $\phi_1 \in V_h$.

Employing (24) and (33)

$$\begin{aligned}\int_\Omega \bar{\partial}^\alpha \xi_1^n\phi_1\,dx &+ \int_\Omega D_1(U^n)\nabla\xi^n\nabla\phi_1\,dx\\ &= -\int_\Omega(\bar{\partial}^\alpha\Pi_h u^n - {}^C D_{t_n}^\alpha u)\phi_1\,dx - \int_\Omega(D_1(U^n) - D_1(u^n))\nabla u^n\nabla\phi_1\,dx\\ &\quad - \int_\Omega D_1(u^n)\nabla u^n\phi_1\,dx - \int_\Omega {}^C D_{t_n}^\alpha u\phi_1\,dx\\ &\quad + \int_\Omega dU^n\phi_1\,dx + \int_\Omega \beta(U^n, W^n)\phi_1\,dx + \int_\Omega \sigma(U^n, V^n)\phi_1\,dx. \qquad (35)\end{aligned}$$

We know that

$$ {}^C D_{t_n}^\alpha u(t) - \nabla\cdot(D_1(u^n)\nabla u^n) = F_1(u^n, v^n, w^n), \qquad (36)$$

where $F(u^n, v^n, w^n) = -du^n - \beta(u^n, w^n) - \sigma(u^n, v^n)$. Using (36) in (35), we get

$$\begin{aligned}\int_\Omega \bar{\partial}^\alpha \xi_1^n\phi_1\,dx &+ \int_\Omega D_1(u_h^n)\nabla\xi^n\nabla\phi_1\,dx\\ &= \int_\Omega({}^C D_{t_n}^\alpha u - \bar{\partial}^\alpha\Pi_h u^n)\phi_1\,dx - \int_\Omega(D_1(U^n) - D_1(u^n))\nabla u^n\nabla\phi_1\,dx\\ &\quad + \int_\Omega F(U^n, V^n, W^n)\phi_1\,dx - \int_\Omega F(u^n, v^n, w^n)\phi_1\,dx. \qquad (37)\end{aligned}$$

By definition, we know that $F(u, v, w)$ is Lipschitz continuous. Substitute $\phi = \xi_1^n$ in (37), we obtain

$$\begin{aligned}\bar{\partial}^\alpha\|\xi_1^n\|_0^2 &+ \int_\Omega D_1(u_h^n)\nabla\xi^n\nabla\xi_1^n\,dx\\ &= \int_\Omega\left({}^C D_{t_n}^\alpha u(t) - \bar{\partial}^\alpha u_h^n\right)\xi_1^n\,dx - \int_\Omega(D_1(U^n) - D_1(u^n))\nabla u^n\nabla\phi_1\,dx\\ &\quad + \int_\Omega\left(F(u_h^n, v_h^n, w_h^n) - F(u^n, v^n, w^n)\right)\xi_1^n\,dx\\ &\le \frac{1}{2}\left\|{}^C D_{t_n}^\alpha u(t) - \bar{\partial}^\alpha u_h^n\right\|_0^2\\ &\quad + \frac{L_1}{2}\left(\|u^n - u_h^n\|_0^2 + \|v^n - v_h^n\|_0^2 + \|w^n - w_h^n\|_0^2\right)\\ &\quad + \left(\frac{L_1 + 1}{2}\right)\|\xi_1^n\|_0^2. \qquad (38)\end{aligned}$$

Using the properties

$$\left\|{}^C D_{t_n}^\alpha u(t) - \bar{\partial}^\alpha u_h^n\right\|_0 \le C(\delta_t^{2-\alpha} + h^2).$$

Using the estimates of (26), we get

$$\bar{\partial}^\alpha\|\xi_1^n\|_0^2 \le \frac{3L_1 + 1}{2}\left(\|\xi_1^n\|_0^2\right) + \frac{L_1}{2}\left(\|\xi_2^n\|_0^2 + \|\xi_3^n\|_0^2\right) + C(\delta_t^{2-\alpha} + h^2)^2.$$

Similar calculations lead to

$$\bar{\partial}^\alpha\left(\|\xi_1^n\|_0^2 + \|\xi_2^n\|_0^2 + \|\xi_3^n\|_0^2\right) \le \gamma\left(\|\xi_1^n\|_0^2 + \|\xi_2^n\|_0^2 + \|\xi_3^n\|_0^2\right) + C(\delta_t^{2-\alpha} + h^2)^2.$$

Here γ is a positive constant dependent on L_i, $i = 1, 2, 3$.

Using Theorem 6, one gets

$$\left(\|\xi_1^n\|_0^2 + \|\xi_2^n\|_0^2 + \|\xi_3^n\|_0^2\right) \le C(\delta_t^{2-\alpha} + h^2)^2.$$

Thus

$$\|S^n - s(x, t_n)\|_0 \le Ch^2 + C\delta t^{2-\alpha}.$$

This completes the proof. □

Theorem 9. *Suppose (u, v, w) and (U^n, V^n, W^n) are the solution of* (1) and (33) *respectively. Further, (u, v, w) does not have sufficiently regularity in time. Then there exists a positive constant δt^* such that $\delta t \le \delta t^*$,*

$$\|S^n - s(x, t_n)\|_0 \le C(\delta t^\alpha + h^2),$$

where $S^n = (U^n, V^n, W^n)$, $s = (u, v, w)$, $n = 1, 2, 3, \ldots, N$ and $C > 0$ is a constant, is independent of h and δt.

Proof. The proof is as similar as Theorem 8. From (30) and Theorem 5, we get

$$\left\|{}^C D_{t_n}^\alpha u(t) - \bar{\partial}^\alpha u_h^n\right\|_0 \le C(\delta_t^{2-\alpha} + h^2). \qquad (39)$$

From (38) and (39), we get the desired result. □

Numerical results

In this section, we carried out numerical computations to validate our theoretical results. To carry out numerical computations, we have used the unit square domain $\Omega = [0, 1]^2$. Further, we use Freefem++ library functions [45] with UMFPACK [46,47] to solve the proposed system. Intel (R) Core (TM) i7-7700 CPU with 3.60 Hz and 8 GB RAM machine was used for carrying out all computations.

Convergence study

We use the following examples for the convergence study of the proposed finite element scheme.

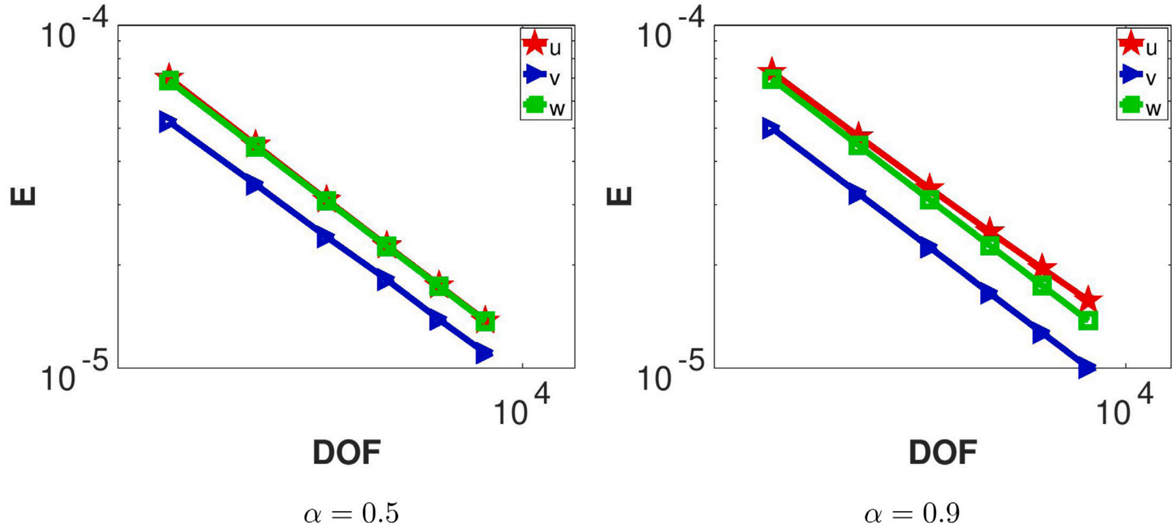

Fig. 1. Error plots of the healthy cell (u), infected cell (v), virions (w) obtained with different mesh levels for the scheme convergence study. Panel (i) and (ii), respectively, represent $\alpha = 0.5$ and $\alpha = 0.9$ the logarithmic values of E errors of the solution of the system (43) against the logarithmic value of the degrees of freedom (DOF) for Example 1.

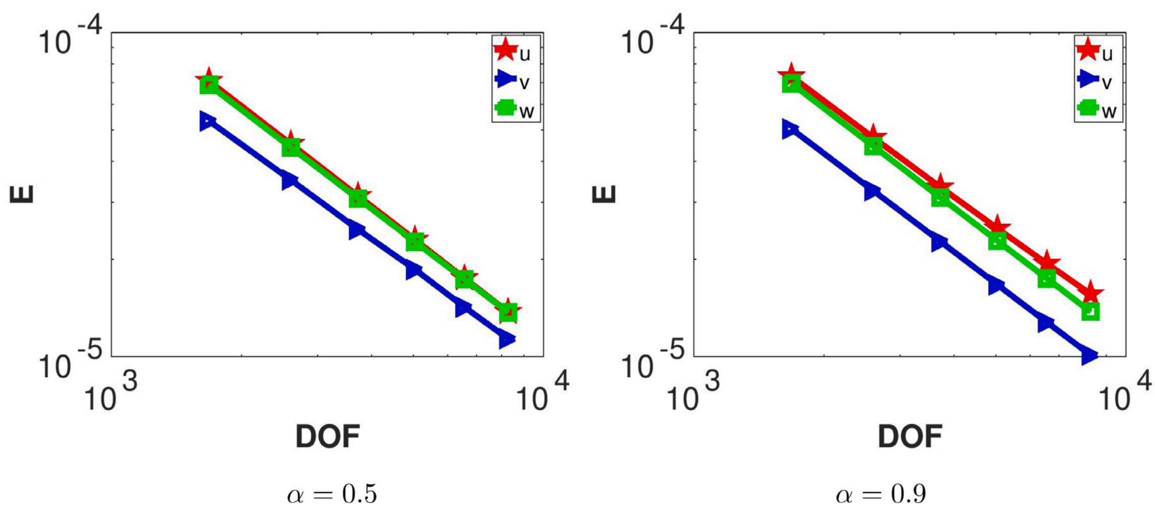

Fig. 2. Error plots of the healthy cell (u), infected cell (v), virions (w) obtained with different mesh levels for the scheme convergence study. Panel (i) and (ii), respectively, represent $\alpha = 0.5$ and $\alpha = 0.9$ the logarithmic values of E errors of the solution of the system (43) against the logarithmic value of the degrees of freedom (DOF) for Example 2.

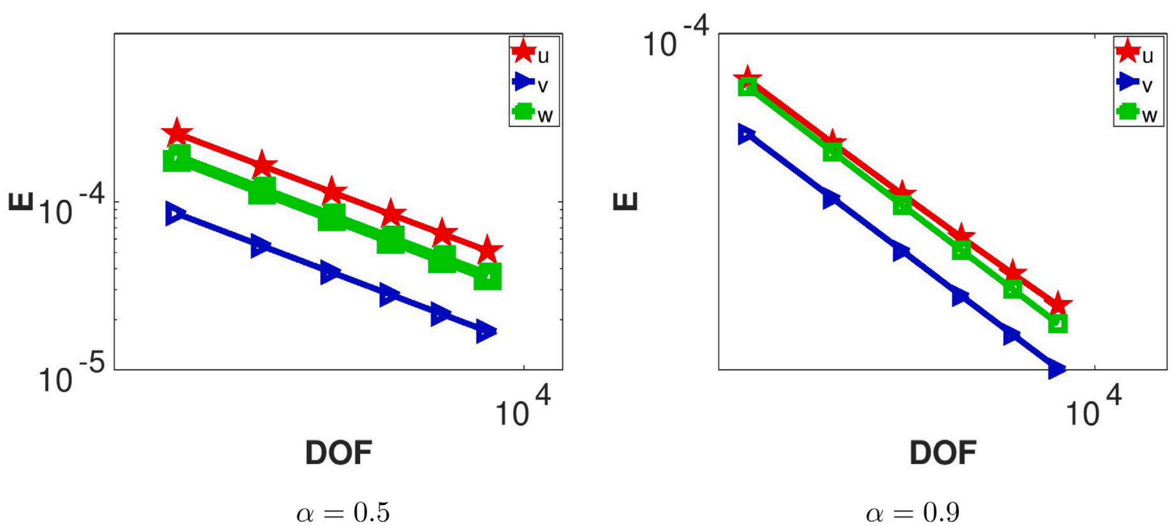

Fig. 3. Error plots of the healthy cell (u), infected cell (v), virions (w) obtained with different mesh levels for the scheme convergence study. Panel (i) and (ii), respectively represent $\alpha = 0.5$ and $\alpha = 0.9$ the logarithmic values of E errors of the solution of the system (43) against the logarithmic value of the degrees of freedom (DOF) for Example 3.

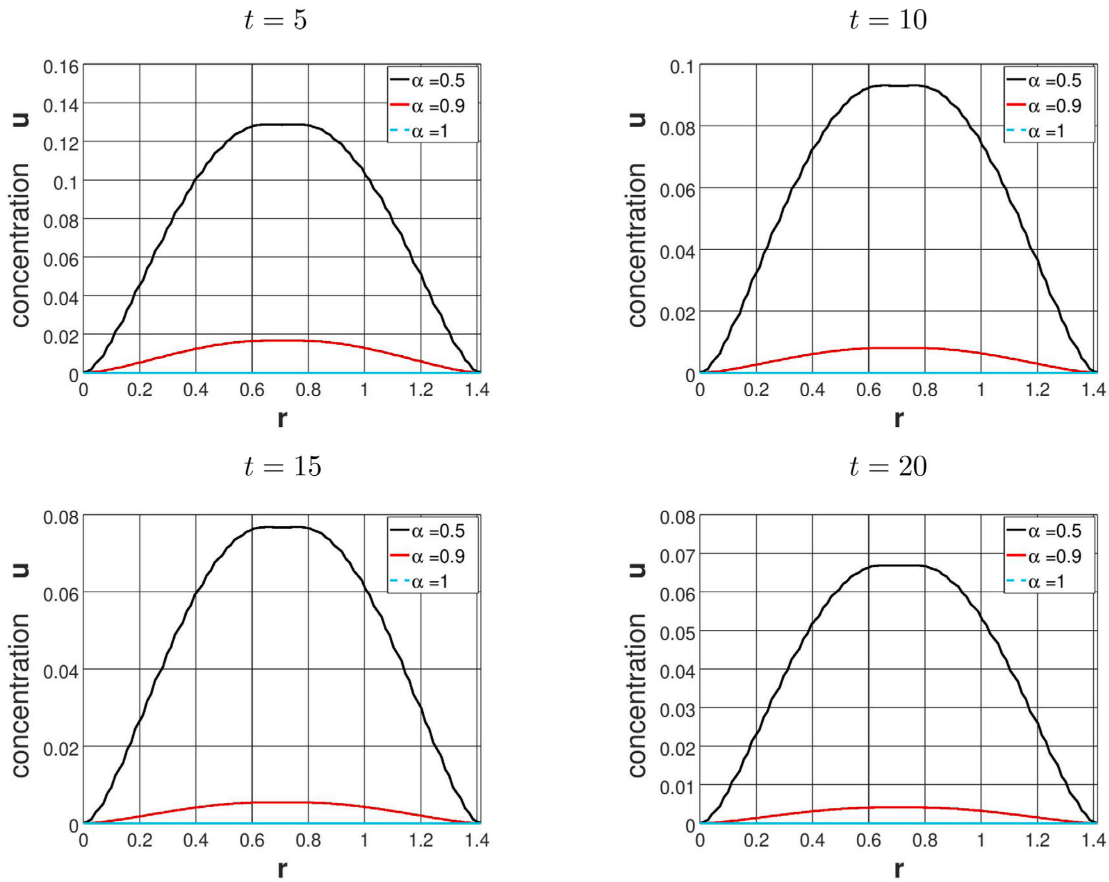

Fig. 4. Evolution of u at $t = 5, 10, 15 \& 20$ along the line $y = x$.

Table 1
Errors and order of convergence for Example 1.

	DOF	E_u	Order	E_v	Order	E_w	Order	CPU time
$\alpha = 0.5$	1681	7.0172e−05	–	5.2198e−05	–	6.8872e−05	–	21.55 s
	2601	4.4772e−05	2.0138	3.4256e−05	1.8874	4.4200e−05	1.9876	40.97 s
	3721	3.0948e−05	2.0255	2.4078e−05	1.9338	3.0746e−05	1.9908	72.99 s
	5041	2.2747e−05	1.9973	1.8077e−05	1.8595	2.2614e−05	1.9928	126.43 s
	6561	1.7423e−05	1.9968	1.3844e−05	1.9978	1.7328e−05	1.9939	130.54 s
	8281	1.3770e−05	1.9974	1.1048e−05	1.9153	1.3699e−05	1.9949	194.78 s
$\alpha = 0.9$	1681	7.3049e−05	–	5.0020e−05	–	6.9684e−05	–	96.05 s
	2601	4.7362e−05	1.9418	3.2233e−05	1.9694	4.4605e−05	1.9993	199.20 s
	3721	3.3407e−05	1 .9145	2.2483e−05	1.9757	3.0978e−05	1.9995	469.81 s
	5041	2.4978e−05	1.8862	1.6525e−05	1.9972	2.2760e−05	1.9998	868.76 s
	6561	1.9516e−05	1.8481	1.2672e−05	1.9885	1.7426e−05	1.9998	1535.01 s
	8281	1.5777e−05	1.8055	1.0035e−05	1.9810	1.3769e−05	1.9998	2602.00 s

Table 2
Errors and order of convergence for Example 2.

	DOF	E_u	Order	E_v	Order	E_w	Order	CPU time
$\alpha = 0.5$	1681	7.0838e−05	–	5.3218e−05	–	6.8876e−05	–	28.11 s
	2601	4.5287e−05	2.0049	3.5012e−05	1.8765	4.4202e−05	1.9877	57.99 s
	3721	3.1332e−05	2.0205	2.4637e−05	1.9276	3.0747e−05	1.9908	104.55 s
	5041	2.2959e−05	2.0171	1.8537e−05	1.8453	2.2615e−05	1.9928	171.13 s
	6561	1.7453e−05	2.0536	1.4192e−05	2.0004	1.7328e−05	1.9940	301.54 s
	8281	1.3750e−05	2.0245	1.1337e−05	1.9070	1.3700e−05	1.9949	420.65 s
$\alpha = 0.9$	1681	7.3160e−05	–	5.0403e−05	–	6.9684e−05	–	120.10 s
	2601	4.7353e−05	1.9495	3.2490e−05	1.9680	4.4605e−05	1.9993	325.31 s
	3721	3.3329e−05	1 .9263	2.2667e−05	1.9746	3.0978e−05	1.9995	746.40 s
	5041	2.4858e−05	1.9024	1.6660e−05	1.9973	2.2760e−05	1.9998	1284.67 s
	6561	1.9367e−05	1.8692	1.2776e−05	1.9880	1.7426e−05	1.9998	2000.94 s
	8281	1.5609e−05	1.8318	1.0118e−05	1.9803	1.3769e−05	1.9998	2916.06 s

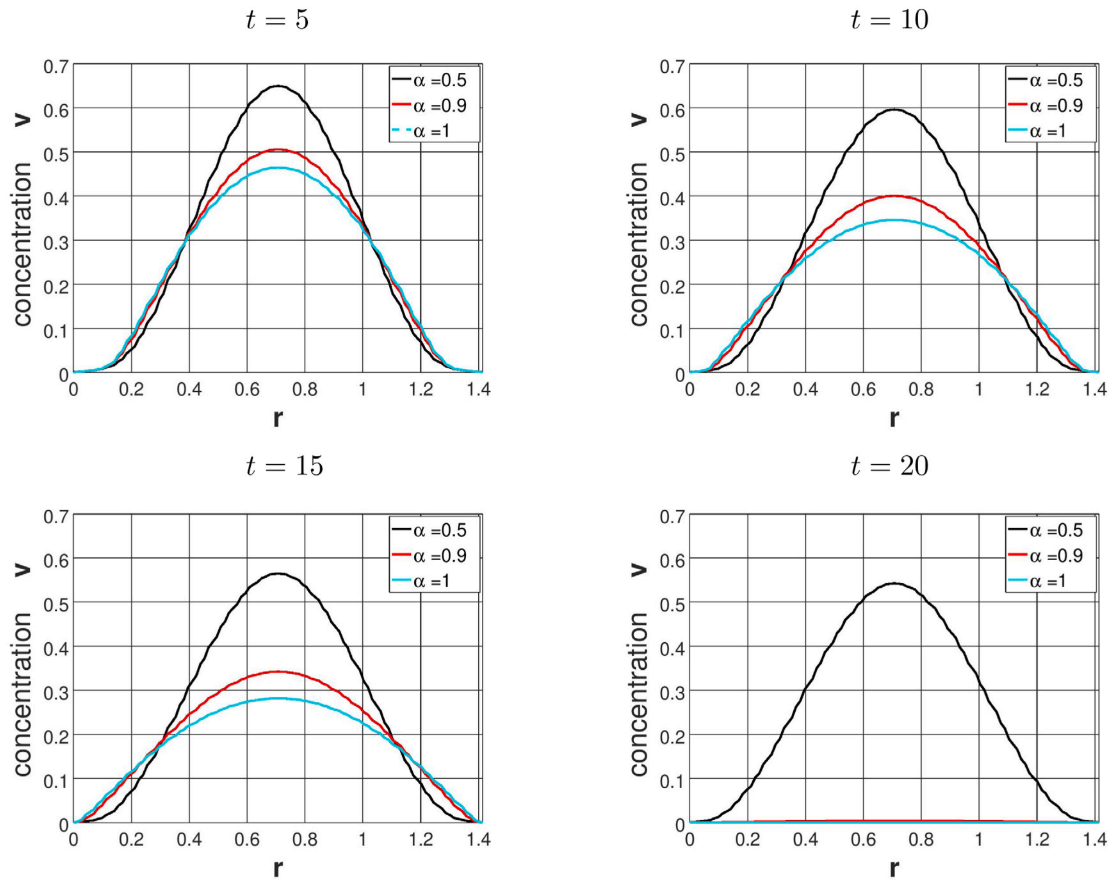

Fig. 5. Evolution of v at $t = 5, 10, 15\,\&\,20$ along the line $y = x$.

Example 1.

$$\left.\begin{aligned} &\partial_t^\alpha u - d_1 \Delta u + du + \frac{\beta_1 uw}{(1+u)} + \frac{\beta_2 uv}{(1+u)} = f_u,\\ &\partial_t^\alpha v - d_2 \Delta v - \frac{\beta_1 uw}{(1+u)} - \frac{\beta_2 uv}{(1+u)} = f_v,\\ &\partial_t^\alpha w - d_3 d_1 \Delta w - Nv + ew = f_w, \end{aligned}\right\} \tag{40}$$

with smooth solution

$u = (1+t)\cos x \sin y,\ v = (1+t^2)\sin x \sin y,$ and $w = (1+t^3)\sin x \cos y.$

Here f_u, f_v and f_w have chosen in such a way that u, v, w satisfy (40).

Example 2.

$$\left.\begin{aligned} &\partial_t^\alpha u - div(d_1(1+u)\nabla u) + du + \frac{\beta_1 uw}{(1+u)} + \frac{\beta_2 uv}{(1+u)} = f_u,\\ &\partial_t^\alpha v - div(d_2 v \nabla v) - \frac{\beta_1 uw}{(1+u)} - \frac{\beta_2 uv}{(1+u)} = f_v,\\ &\partial_t^\alpha w - div(d_3(1+w)^2 \nabla w) - Nv + ew = f_w, \end{aligned}\right\} \tag{41}$$

with smooth solution

$u = (1+t)\cos x \sin y,\ v = (1+t^2)\sin x \sin y,$ and $w = (1+t^3)\sin x \cos y.$

Here f_u, f_v and f_w have chosen in such a way that u, v, w satisfy (41).

Example 3.

$$\left.\begin{aligned} &\partial_t^\alpha u - div(d_1(1+u)\nabla u) + du + \frac{\beta_1 uw}{(1+u)} + \frac{\beta_2 uv}{(1+u)} = f_u,\\ &\partial_t^\alpha v - div(d_2 v \nabla v) - \frac{\beta_1 uw}{(1+u)} - \frac{\beta_2 uv}{(1+u)} = f_v,\\ &\partial_t^\alpha w - div(d_3(1+w)^2 \nabla w) - Nv + ew = f_w, \end{aligned}\right\} \tag{42}$$

with solution

$u = (1+t^\alpha)\cos x \sin y,\ v = (1+t^\alpha)\sin x \sin y,$ and $w = (1+t^\alpha)\sin x \cos y.$

Here f_u, f_v and f_w have chosen in such a way that u, v, w satisfy (42).

Further, we set the parameter values of the system (40)–(42) as

$d_1 = 0.1,\ d_2 = 0.01,\ d_3 = 0.0001,\ d = 0.4,\ \beta_1 = 0.1,$
$\beta_2 = 0.01, r = 0.01,\ N = 0.007,\ e = 0.07.$

In addition, the following errors are measured to compare discretization errors at various mesh levels and to validate the convergence order of the proposed numerical scheme

$$E_z = \max_{1 \le n \le N} \|z^n - Z^n\|_0. \tag{43}$$

From Theorems 8 and 9, we has chosen the step size $\delta_t = h^{\frac{2}{2-\alpha}}$ for Examples 1 and 2 and $\delta_t = h^{\frac{2}{\alpha}}$ for Example 3. We present the computational outcomes in Tables 1–3 and Figs. 1–3. It is explicitly confirmed that for all cases (linear and nonlinear diffusion operators) with piecewise linear triangular P_1 finite elements, the optimum order of convergence (approximately two) is obtained.

Numerical simulations

To explore the influence of various fractional order derivative in the HIV model, we perform the numerical simulations with various values of $\alpha = 0.5,\ 0.9\,\&\,1$ using the method proposed in Section "Finite element scheme". Here, all the computations have been carried out until at end time $t = 20$ with uniform time step $\delta_t = 0.1$ and uniform mesh size $h = 0.0101015$ in uniform square domain $\Omega = [0,1]^2$. Further, we

Table 3
Errors and order of convergence for Example 3.

	DOF	E_u	Order	E_v	Order	E_w	Order	CPU time
$\alpha = 0.5$	1681	2.5412e−04	–	8.5098e−05	–	1.8123e−04	–	12.32 s
	2601	1.6378e−04	1.9685	5.4693e−05	1.9811	1.1629e−04	1.9881	41.10 s
	3721	1.1427e−04	1.9746	3.8088e−05	1.9845	8.0902e−05	1.9903	125.75 s
	5041	8.4227e−05	1.9787	2.8040e−05	1.9870	5.9513e−05	1.9919	324.60 s
	6561	6.4644e−05	1.9817	2.1500e−05	1.9887	4.5607e−05	1.9930	1172.04 s
	8281	5.1173e−05	1.9839	1.7008e−05	1.9900	3.6062e−05	1.9938	2140.85 s
$\alpha = 0.9$	1681	1.6753e−04	–	5.9547e−05	–	1.3269e−04	–	15.23 s
	2601	1.1005e−04	1.8832	3.9147e−05	1.8797	8.7163e−05	1.8834	36.46 s
	3721	7.7633e−05	1.9138	2.7635e−05	1.9101	6.1516e−05	1.9114	79.40 s
	5041	5.7718e−05	1.9230	2.0506e−05	1.9355	4.5650e−05	1.9351	145.18 s
	6561	4.4608e−05	1.9296	1.5793e−05	1.9559	3.5159e−05	1.9556	261.57 s
	8281	3.5516e−05	1.9351	1.2525e−05	1.9684	2.7884e−05	1.9683	422.33 s

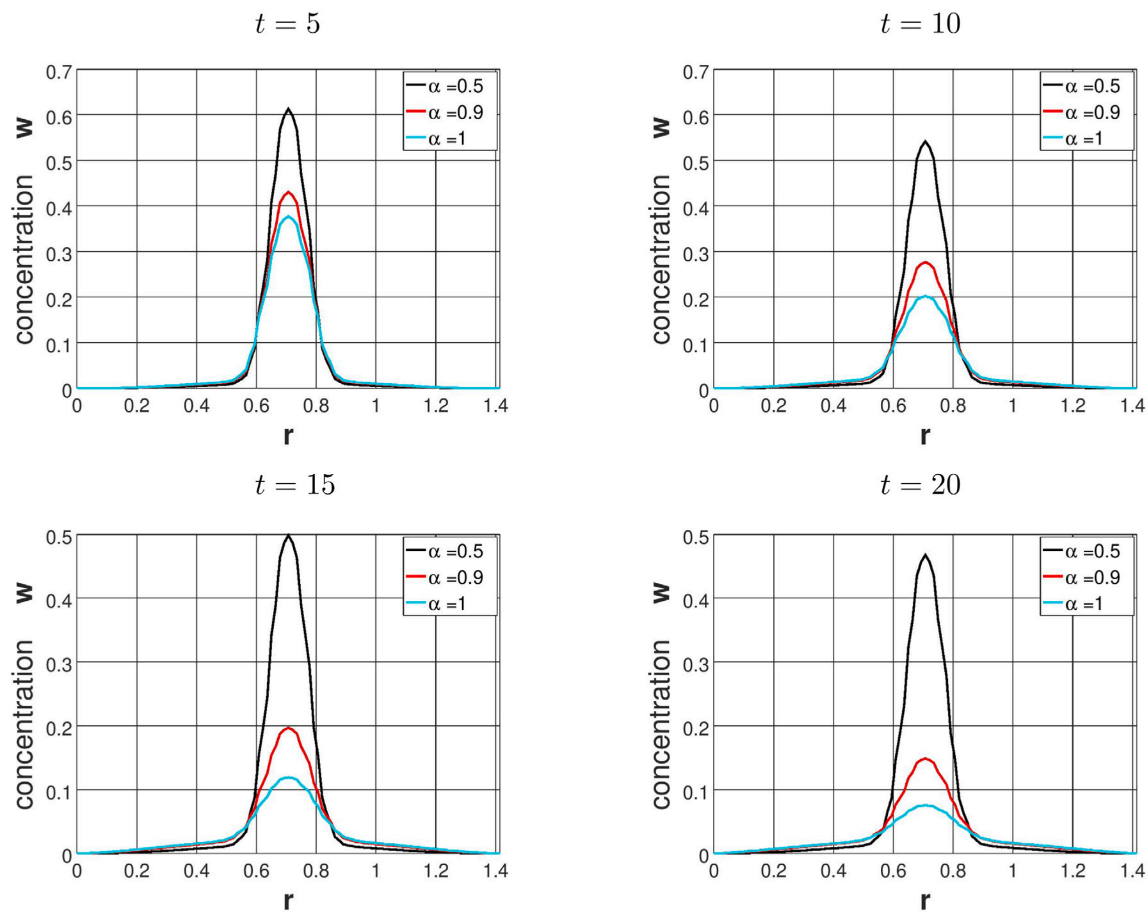

Fig. 6. Evolution of w at $t = 5, 10, 15 \,\&\, 20$ along the line $y = x$.

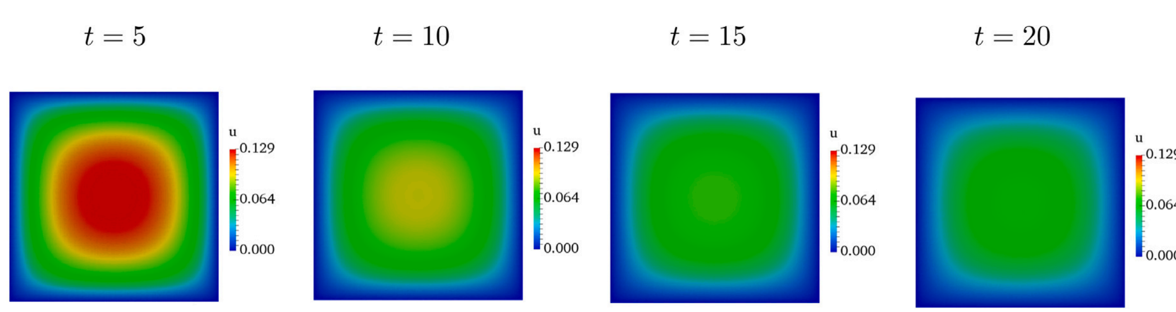

Fig. 7. Evolution of u at $t = 5, 10, 15 \,\&\, 20$ for $\alpha = 0.5$.

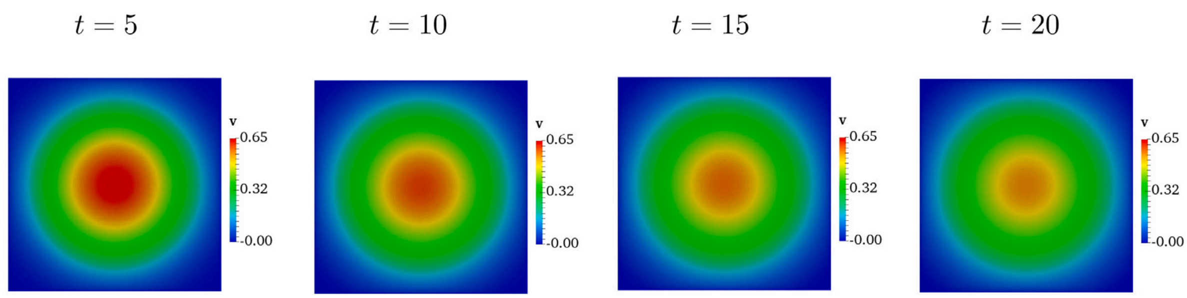

Fig. 8. Evolution of v at $t = 5, 10, 15 \,\&\, 20$ for $\alpha = 0.5$.

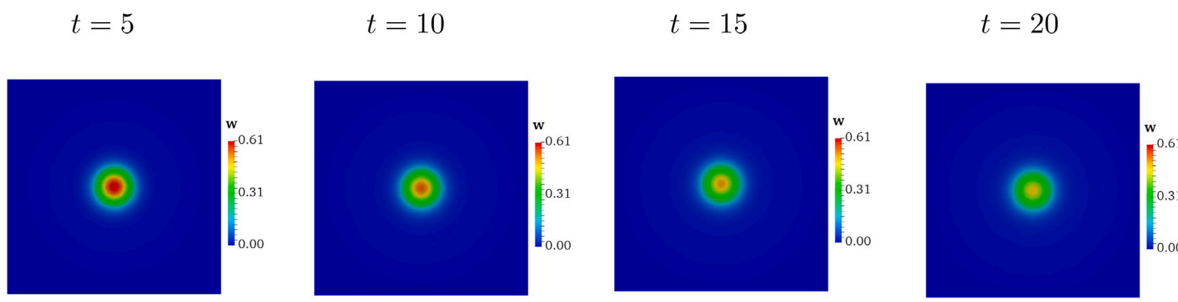

Fig. 9. Evolution of w at $t = 5, 10, 15 \,\&\, 20$ for $\alpha = 0.5$.

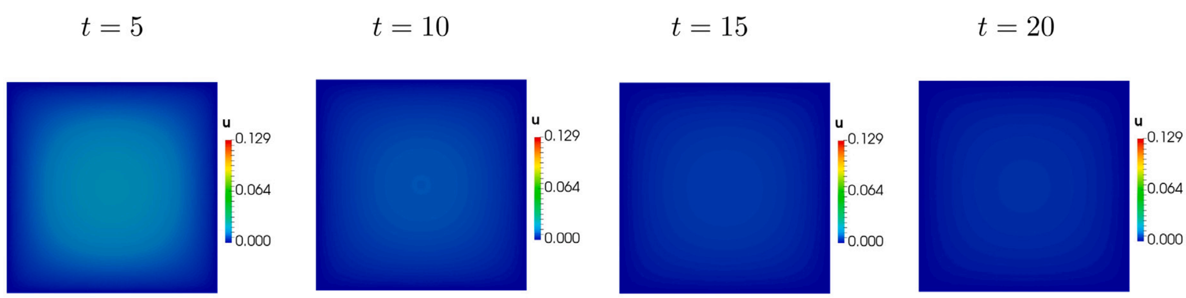

Fig. 10. Evolution of u at $t = 5, 10, 15 \,\&\, 20$ for $\alpha = 0.9$.

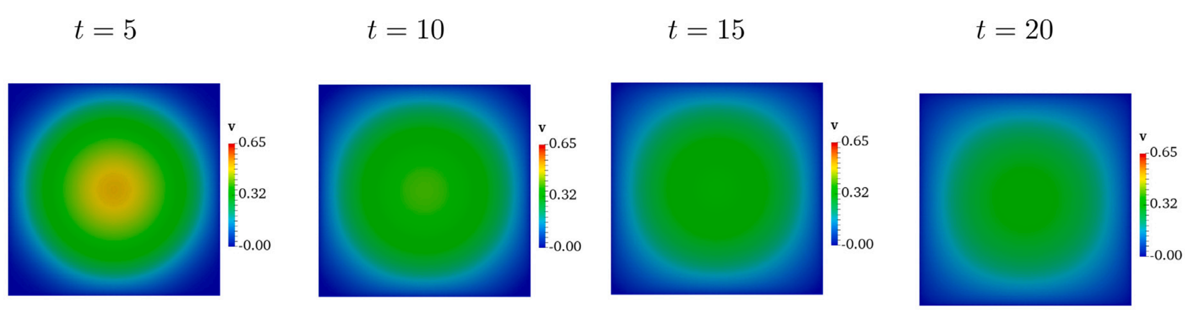

Fig. 11. Evolution of v at $t = 5, 10, 15 \,\&\, 20$ for $\alpha = 0.9$.

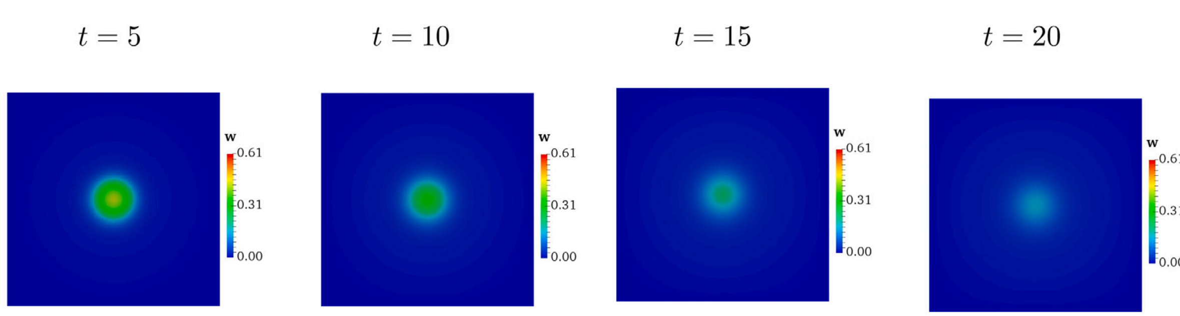

Fig. 12. Evolution of w at $t = 5, 10, 15 \,\&\, 20$ for $\alpha = 0.9$.

assume the following diffusion functions

$$D_1(u) = d_1(1+u),\ D_2(v) = d_2 v \text{ and } D_3(w) = d_3(1+w)^2.$$

The rest of parameters have been taken as in Section "Convergence study". Moreover, we assume homogeneous Dirichlet boundary conditions and the following initial conditions

$$u_0 = 1 - 0.99\exp\left(-\frac{(x-0.5)^2-(y-0.5)^2}{\epsilon_1}\right),$$
$$v_0 = 1.01\exp\left(-\frac{(x-0.5)^2-(y-0.5)^2}{\epsilon_2}\right),$$
$$w_0 = 1.01\exp\left(-\frac{(x-0.5)^2-(y-0.5)^2}{\epsilon_3}\right),$$

where $\epsilon_1 = 0.005$, $\epsilon_2 = 0.075$ and $\epsilon_3 = 0.005$.

We now briefly discuss the influence of the fractional derivatives. Simulations are performed for various values of α depicted in Figs. 4–12 at dimensionless time $t = 5,\ 10,\ 15\,\&\,20$. We mainly focus on the effect of the time-fractional derivatives in the HIV infection model. The snapshots of interactions of healthy cells (u), infected cells (v) and virions (w) are depicted in Fig 10–12. From the simulations, we can conclude that for large values of α, that is $\alpha = 0.9$, the amount of healthy cells decreases while the amount of infected cells increases, see Figs. 7–9. However, when α is small, that is $\alpha = 0.5$, spreading risk of infected cells are not high, see Figs. 10–12. Therefore, we conclude that fractional order derivatives are making huge impact on the dynamics of HIV model.

CRediT authorship contribution statement

J. Manimaran: Writing - original draft, Software, Resources. **L. Shangerganesh:** Conceptualization, Formal analysis, Investigation. **A. Debbouche:** Supervision, Methodology, Project administration, Writing - review & editing. **J.-C. Cortés:** Validation, Data curation, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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