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Additional Information

Partial orders base on the CS decomposition

Abstract: A new decomposition for square matrices was introduce by J. Benítez in [2]. In this paper, we will use this decomposition to investigate the minus, star, sharp and core partial orders in the setting of complex matrices.

Key words: Core-EP order, Minus partial order, Star partial order, Sharp partial order, Core partial order.

AMS subject classifications: 06A06, 15A09, 15A23.

1 Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m\times n$ complex matrices. Let A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\mathrm{rk}(A)$ denote the conjugate transpose, column space, null space, and rank of $A\in\mathbb{C}^{m\times n}$, respectively. For $A\in\mathbb{C}^{m\times n}$, if $X\in\mathbb{C}^{n\times m}$ satisfies

$$(1)\ AXA=A,\qquad (2)\ XAX=X,\qquad (3)\ (AX)^*=AX,\qquad (4)\ (XA)^*=XA,\qquad (\star)$$

then X is called a *Moore-Penrose inverse* of A. If such a matrix X exists, then it is unique and denoted by $A^\dagger$. Let $I\subseteq\{1,2,3,4\}$. An element $B\in\mathbb{C}^{n\times m}$ is called an I *inverse* of $A\in\mathbb{C}^{m\times n}$ if equalities $i\in I$ of $(\star)$ hold. The set of all I inverses of A will be denoted by A^I, the element A is I *invertible* when $A^I\neq\varnothing$.

Let $A\in\mathbb{C}^{n\times n}$. It can be easily proved that the set of elements $X\in\mathbb{C}^{n\times n}$ such that

$$AXA=A,\quad XAX=X\quad\text{and}\quad AX=XA$$

is empty or a singleton. If this set is a singleton, its unique element is called the *group inverse* of A and denoted by $A^\#$.

The core inverse for a complex matrix was introduced by Baksalary and Trenkler [1]. Let $A\in\mathbb{C}^{n\times n}$. A matrix $X\in\mathbb{C}^{n\times n}$ is called a *core inverse* of A, if it satisfies $AX=P_A$ and $\mathcal{R}(X)\subseteq\mathcal{R}(A)$. Here P_A denotes the orthogonal projector onto $\mathcal{R}(A)$. And if such a matrix exists, then it is unique (and denoted by $A^{\#\!\!\!\!\bigcirc}$). A square complex matrix A is core invertible if and only if $\mathrm{rk}(A)=\mathrm{rk}(A^2)$ (see [1]), and let $\mathbb{C}_n^{CM}=\{A\in M_n(\mathbb{C})\mid \mathrm{rk}(A)=\mathrm{rk}(A^2)\}$. The *core partial order* for complex matrices was also introduced in [1] and it is defined as follows: given $A\in\mathbb{C}_n^{CM}$ and $B\in M_n(\mathbb{C})$,

$$A\overset{\#\!\!\!\!\bigcirc}{\leq}B\quad\Leftrightarrow\quad A^{\#\!\!\!\!\bigcirc}A=A^{\#\!\!\!\!\bigcirc}B\ \text{ and }\ AA^{\#\!\!\!\!\bigcirc}=BA^{\#\!\!\!\!\bigcirc}.$$

In [1, Theorem 6], it is proved that the core partial order is a matrix partial order. Baksalary and Trenkler [1] gave several characterizations and various relationships between the matrix core partial order and other matrix partial orders by using the decomposition of Hartwig and Spindelböck [8]. Let us recall some other well known partial orders in $\mathbb{C}^{n\times n}$. For $A,B\in\mathbb{C}^{n\times n}$,

- The star partial order $A\overset{*}{\leq}B$: $A^*A=A^*B$ and $AA^*=BA^*$ [6];

• The minus partial order $A \overset{-}{\leq} B$: $A^-A = A^-B$ and $AA^- = BA^-$ [7], where A^- denotes any inner inverse of A;

• The sharp partial order $A \overset{\#}{\leq} A$: $A^{\#}A = A^{\#}B$ and $AA^{\#} = BA^{\#}$ [13].

In addition, $\mathbf{1}_n$ and $\mathbf{0}_n$ will denote the $n \times 1$ column vectors all of whose components are 1 and 0, respectively. $0_{m\times n}$ (abbr. 0) denotes the zero matrix of size $m \times n$. If $\mathcal{S}$ is a subspace of $\mathbb{C}^n$, then $P_{\mathcal{S}}$ stands for the orthogonal projector onto the subspace S. A matrix $A \in \mathbb{C}^{n\times n}$ is called an *EP* matrix if $\mathcal{R}(A) = \mathcal{R}(A^*)$ and A is unitary if $AA^* = I_n$, where I_n denotes the identity matrix of size n.

2 Preliminaries

A related decomposition of the matrix decomposition of Hartwig and Spindelböck [8] was given in [2, Theorem 2.1] by Benítez. In [3] it can be found a simpler proof of this decomposition. Let us start this section with the concept of principal angles.

Definition 2.1. [19] *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two nontrivial subspaces of $\mathbb{C}^n$. We define the* principal angles $\theta_1, \ldots, \theta_r \in [0, \pi/2]$ *between $\mathcal{S}_1$ and $\mathcal{S}_2$ by*

$$\cos\theta_i = \sigma_i(P_{\mathcal{S}_1}P_{\mathcal{S}_2}),$$

for $i = 1, \ldots, r$, where $r = \min\{\dim \mathcal{S}_1, \dim \mathcal{S}_2\}$. The real numbers $\sigma_i(P_{\mathcal{S}_1}P_{\mathcal{S}_2}) \geq 0$ are the singular values of $P_{\mathcal{S}_1}P_{\mathcal{S}_2}$.

Lemma 2.2. [2, Theorem 2.1] *Let $A \in \mathbb{C}^{n\times n}$, $r = \mathrm{rk}(A)$, and let $\theta_1, \ldots, \theta_p$ be the principal angles between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ belonging to $]0, \pi/2[$. Denote by x and y the multiplicities of the angles 0 and $\pi/2$ as a canonical angle between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively. There exists a unitary matrix $Y \in \mathbb{C}^{n\times n}$ such that*

$$A = Y \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} Y^*, \tag{2.1}$$

where $M \in \mathbb{C}^{r\times r}$ is nonsingular,

$$C = \mathrm{diag}(\mathbf{0}_y, \cos\theta_1, \ldots, \cos\theta_p, \mathbf{1}_x),$$

$$S = \begin{bmatrix} \mathrm{diag}(\mathbf{1}_y, \sin\theta_1, \ldots, \sin\theta_p) & 0_{p+y,n-(r+p+y)} \\ 0_{x,p+y} & 0_{x,n-(r+p+y)} \end{bmatrix},$$

and $r = y + p + x$. Furthermore, x and $y + n - r$ are the multiplicities of the singular values 1 and 0 in $P_{\mathcal{R}(A)}P_{\mathcal{R}(A^*)}$, respectively.

In this decomposition, one has $C^2 + SS^* = I_r$. One has also

$$A^\dagger = Y \begin{bmatrix} CM^{-1} & 0 \\ S^*M^{-1} & 0 \end{bmatrix} Y^*, \qquad A^{\#} = Y \begin{bmatrix} C^{-1}M^{-1} & C^{-1}M^{-1}C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*.$$

Recall that $A^\dagger$ always exists. We have that $A^{\#}$ exists if and only if C is nonsingular [2, Theorem 3.7]. In this case, we have

$$A^{\oplus} = A^{\#}AA^\dagger = Y \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^*,$$

$$AA^{\oplus} = Y \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Y^* \quad \text{and} \quad A^{\oplus}A = Y \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*. \tag{2.2}$$

We have

$$AA^{\oplus} - A^{\oplus}A = Y \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Y^* - Y \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix} Y^* = Y \begin{bmatrix} 0 & C^{-1}S \\ 0 & 0 \end{bmatrix} Y^* \tag{2.3}$$

in view of (2.2). Now,

$$\det(AA^{\oplus} - A^{\oplus}A) = \det\left(Y \begin{bmatrix} 0 & C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*\right) = 0.$$

Thus, $AA^{\oplus} - A^{\oplus}A$ is always singular and $\mathrm{rk}(AA^{\oplus} - A^{\oplus}A) = \mathrm{rk}(C^{-1}S) = \mathrm{rk}(S) < n$. From (2.3), we have that A is an EP matrix if and only if $S = 0$, that is all the canonical angles between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ are 0. This result also can be found in [2, Theorem 3.7].

Proposition 2.3. *If $A \in \mathbb{C}^{n\times n}$ is core invertible and A has the form* (2.1)*, then $AA^{\oplus} - A^{\oplus}A$ is always singular with* $\mathrm{rk}(AA^{\oplus} - A^{\oplus}A) = \mathrm{rk}(S) < n$.

In [21, Theorem 3.1], the authors proved the following lemma for an element in a ring with involution.

Lemma 2.4. *Let $A \in \mathbb{C}^{n\times n}$. Then A is core invertible with $A^{\oplus} = X$ if and only if $(AX)^* = AX$, $XA^2 = A$ and $AX^2 = X$.*

Proposition 2.5. *Let $A, B, U \in \mathbb{C}^{n\times n}$ with $A = UBU^*$, where B is core invertible and U is unitary. Then A is core invertible. In this case, one has $A^{\oplus} = UB^{\oplus}U^*$.*

Proof. Let $X = UB^{\oplus}U^*$, we have

$$\begin{aligned} AX &= AUB^{\oplus}U^* = UBU^*UB^{\oplus}U^* = UBB^{\oplus}U^* \text{ is Hermitian}, \\ XA^2 &= UB^{\oplus}U^*(UBU^*)^2 = UB^{\oplus}(B)^2U^* = UBU^* = A, \\ AX^2 &= UAU^*(UBU^*)^2 = UB(B^{\oplus})^2U^* = UB^{\oplus}U^* = X. \end{aligned}$$

Thus, $A^{\oplus} = UB^{\oplus}U^*$ in view of Lemma 2.4. □

Recently, Wang introduced a new decomposition for square matrices, named *Core-EP decomposition* in [18, Theorem 2.1].

Lemma 2.6. *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = k$. *Then A can be written as*

$$A = A_1 + A_2, \tag{2.4}$$

in which

(1) $A_1 \in \mathbb{C}_n^{CM}$;

(2) $A_2^k = 0$;

(3) $A_1^*A_2 = A_2A_1 = 0$.

We call the equality (2.4) *the* Core-EP decomposition *of* A.

Definition 2.7. [11, Definition 3.1] *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{ind}(A) = k$. *A matrix* $X \in \mathbb{C}^{n\times n}$ *is the core-EP inverse of* A *(is unique and denoted by* $A^{\oplus}$*) if* X *is an outer inverse of* A *and satisfies*

$$\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k).$$

Lemma 2.8. [11, Lemma 3.3] *Let* $A \in \mathbb{C}^{n\times n}$ *with ind*$(A) = k$. *Then* $X \in \mathbb{C}^{n\times n}$ *is the core-EP inverse of* A *if and only if*

$$XA^{k+1} = A^k, \quad XAX = X, \quad (AX)^* = AX \text{ and } \mathcal{R}(X) \subseteq \mathcal{R}(A^k).$$

3 A matrix decomposition related the CS decomposition and the core-EP decomposition

Theorem 3.1. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{ind}(A) = k$ *and* $r = \mathrm{rk}(A)$. *There exists a unitary matrix* $U \in \mathbb{C}^{n\times n}$ *such that*

$$A = U \begin{bmatrix} MC & MS \\ 0 & D_4 \end{bmatrix} U^*, \tag{3.1}$$

where M *and* C *are both nonsingular,* D_4 *is nilpotent,* $C^2 + SS^* = I_r$ *and matrices* C *and* S *have the form after equality* (2.1).

Proof. From Lemma 2.6, we have

$$A = A_1 + A_2,$$

in which $A_1 \in \mathbb{C}_n^{CM}$, $A_2^k = 0$ and $A_1^* A_2 = A_2 A_1 = 0$. Now, applying Lemma 2.2 to A_1, there exists a unitary matrix U such that

$$A_1 = U \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^*,$$

in which M is nonsingular. We also have C is nonsingular in view of $A_1 \in \mathbb{C}_n^{CM}$ and [2, Theorem 3.7]. Let $A_2 = U \left[\begin{smallmatrix} D_1 & D_2 \\ D_3 & D_4 \end{smallmatrix}\right] U^*$. And

$$A_1^* A_2 = U \begin{bmatrix} CM^* & 0 \\ S^*M^* & 0 \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} U^* = U \begin{bmatrix} CM^*D_1 & CM^*D_2 \\ S^*M^*D_1 & S^*M^*D_2 \end{bmatrix} U^*; \tag{3.2}$$

$$A_2 A_1 = U \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} D_1MC & D_1MS \\ D_3MC & D_3MS \end{bmatrix} U^*. \tag{3.3}$$

From (3.2) and (3.3) and $A_1^* A_2 = A_2 A_1 = 0$ we get

$$CM^*D_1 = 0; \quad CM^*D_2 = 0; \quad D_3MC = 0.$$

The nonsingularity of C and M implies that D_1, D_2 and D_3 are zero matrices. Thus

$$A = A_1 + A_2 = U \begin{bmatrix} MC & MS \\ 0 & D_4 \end{bmatrix} U^*.$$

The equality $A_2^k = 0$ implies that D_4 is nilpotent. $\square$

Note that the decomposition in Theorem 3.1 has the same form as Schur form, but the decomposition seems easier to handle. Have in mind that M and C are both nonsingular, C is real and diagonal, D_4 is nilpotent and $C^2 + SS^* = I$ by Theorem 3.1.

Since C is nonsingular, we have $A_1^{\#} = U \left[\begin{smallmatrix} C^{-1}M^{-1} & C^{-1}M^{-1}C^{-1}S \\ 0 & 0 \end{smallmatrix}\right] U^*$ by [2, Theorem 3.7]. It is evident that $A_1^{\textcircled{\#}} = U \left[\begin{smallmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right] U^*$. From [18, Theorem 3.2], we have $A^{\textcircled{\dagger}} = A_1^{\textcircled{\#}} = U \left[\begin{smallmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right] U^*$.

In the following theorem, we will use the matrix decomposition in Theorem 3.1 to investigate the core-EP order, which was introduced by Wang in [18], defined as: for matrices $A, B \in \mathbb{C}^{n\times n}$

$$A \overset{\textcircled{\dagger}}{\leq} B \quad \Leftrightarrow \quad A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B \text{ and } AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}.$$

Theorem 3.2. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has the form* (3.1)*. If A is core-EP invertible, then $A \overset{\textcircled{\dagger}}{\leq} B$ if and only if $B - A$ can be written as*

$$B - A = U \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} U^*, \qquad E \in \mathbb{C}^{(n-r)\times(n-r)}.$$

Proof. Let $B - A = U \left[\begin{smallmatrix} B_1 & B_2 \\ B_3 & B_4 \end{smallmatrix}\right] U^*$, where $B_1 \in \mathbb{C}^{r\times r}$ and suppose $A \overset{\textcircled{\dagger}}{\leq} B$, then

$$A^{\textcircled{\dagger}}(B - A) = U \begin{bmatrix} C^{-1}M^{-1}B_1 & C^{-1}M^{-1}B_2 \\ 0 & 0 \end{bmatrix} U^*; \tag{3.4}$$

$$(B - A)A^{\textcircled{\dagger}} = U \begin{bmatrix} B_1C^{-1}M^{-1} & 0 \\ B_3C^{-1}M^{-1} & 0 \end{bmatrix} U^*. \tag{3.5}$$

From (3.4), (3.5), $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$ and $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$, we get

$$C^{-1}M^{-1}B_1 = 0; \qquad C^{-1}M^{-1}B_2 = 0; \qquad B_3C^{-1}M^{-1} = 0. \tag{3.6}$$

From (3.6) and the nonsingularity of C and M it follows that B_1, B_2, B_3 are zero matrices.

To prove the opposite implication, it is easy to check that $(B-A)A^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}(B-A) = 0$, that is $A \overset{\textcircled{\dagger}}{\leq} B$. $\square$

4 Core, star, group and minus partial order

In this section, we consider the relationships between the core partial order and other partial orders by using Lemma 2.2 for square matrices. Let $A \in \mathbb{C}^{n\times n}$. Recall that the left star partial order $A \ast\!\leq B$ in $\mathbb{C}^{n\times n}$ is defined by $A^*A = A^*B$ and $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. The right sharp partial order $A \leq_{\#} B$ is defined as: $AA^{\#} = BA^{\#}$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$. Let us begin with some lemmas will be useful in the sequel.

Lemma 4.1. [13, Lemma 2.2] *Let $A \in \mathbb{C}^{n\times n}$ be group invertible. Then $A \overset{\#}{\leq} B$ if and only if $A^2 = AB = BA$.*

Lemma 4.2. [22, Theorem 3.2] *Let $A, B \in \mathbb{C}^{n\times n}$ be two core invertible matrices. Then $A \overset{\#}{\leq} B$ if and only if $A *\leq B$ and $B^{\#}AA^{\#} = A^{\#}$*

An equivalent form of the minus partial order is the following statement: for the complex case can be found in [4, 12] and for the ring case can be found in [10].

Lemma 4.3. *Let $A, B \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*

(1) $B \overset{-}{\leq} A$;

(2) *There exists $A^- \in A\{1\}$ such that $B = AA^-B = BA^-A = BA^-B$;*

(3) $B = AA^-B = BA^-A = BA^-B$ *for all* $A^- \in A\{1\}$.

The following lemma was proved in the more general setting of rings with an involution in [16, Theorem 4.10].

Lemma 4.4. *Let $A, B \in \mathbb{C}^{n\times n}$. If A, B are both core invertible and $B \overset{-}{\leq} A$, then $B \overset{\#}{\leq} A$ if and only if $A^{\#}BA^{\#} = B^{\#}$.*

Theorem 4.5. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has the form (2.1). If A is core invertible, then the following are equivalent:*

(1) $A \overset{\#}{\leq} B$;

(2) *$B - A$ can be written as*

$$B - A = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} Y^*, \quad B_4 \in \mathbb{C}^{(n-r)\times(n-r)}; \tag{4.1}$$

(3) $P_A(B - A) = 0$ *and* $(B - A)P_A = 0$, *where* $P_A = AA^{\#}$;

(4) $B = A + (I_n - AA^{\#})X(I_n - AA^{\#})$ *for some matrix* $X \in \mathbb{C}^{n\times n}$.

Proof. (1) $\Leftrightarrow$ (2). Since A is core invertible and the core invertibility is equivalent to the group invertibility, matrix C is nonsingular. Let $B = Y \left[\begin{smallmatrix} B_1 & B_2 \\ B_3 & B_4 \end{smallmatrix}\right] Y^*$. If $A \overset{\#}{\leq} B$, then $AA^{\#} = BA^{\#}$, $A^{\#}B = A^{\#}A$. Thus $AA^{\#}B = A = BA^{\#}A$ by $AA^{\#}A = A$.

$$AA^{\#}B = Y \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} Y^* = Y \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} Y^*; \tag{4.2}$$

$$BA^{\#}A = Y \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix} Y^* = Y \begin{bmatrix} B_1 & B_1C^{-1}S \\ B_3 & B_3C^{-1}S \end{bmatrix} Y^*. \tag{4.3}$$

From (4.2) and (4.3) we get $B_1 = MC$, $B_2 = MS$ and $B_3 = 0$. Thus

$$B = Y \begin{bmatrix} MC & MS \\ 0 & B_4 \end{bmatrix} Y^* = Y \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} Y^* + Y \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} Y^* = A + Y \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} Y^*.$$

That is (4.1). Conversely, if we have (4.1), it is easy to check that $AA^{\textcircled{\dagger}}B = A$, which is equivalent to $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$. And we have $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$ in a similar way.

(2) $\Rightarrow$ (3). Since we have $A^{\textcircled{\dagger}} = A^{\#}AA^{\dagger} = Y\left[\begin{smallmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]Y^*$, so $AA^{\textcircled{\dagger}} = Y\left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]Y^*$. It is easy to check that $P_A(B-A) = 0$ and $(B-A)P_A = 0$.

(3) $\Rightarrow$ (2). Let $B-A = Y\left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right]Y^*$, where $X_1 \in \mathbb{C}^{r\times r}$. The hypothesis $P_A(B-A) = 0$ implies that X_1 and X_2 are zero matrices and $(B-A)P_A = 0$ implies that $X_3 = 0$. Thus we have the form in (4.1).

(2) $\Rightarrow$ (4). Note that (4.1) can be written as

$$\begin{aligned} B-A &= Y\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}Y^* \\ &= Y\left\{\left(\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} - \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}\left(\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} - \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right)\right\}Y^*. \end{aligned}$$

Therefore, $B - A = (I_n - P_A)X(I_n - P_A)$ for some matrix X. Let $Q = I_n - P_A$, we get that $QA = 0$ and $B - A = QXQ$ for some matrix X.

(4) $\Rightarrow$ (3) is trivial. $\square$

Remark 4.6. *When* $\operatorname{ind}(A) = 1$, *from [11, Theorem 3.8] we have the core-EP inverse coincides with the core inverse. Thus, the equivalence between* (1) *and* (2) *in Theorem* 4.5 *also can be got by Theorem* 3.2.

Theorem 4.7. *Let* $A, B \in \mathbb{C}^{n\times n}$. *Assume that* A *has the form* (2.1). *If* A *is group invertible, then the following are equivalent:*

(1) $A \overset{*}{\leq} B$;

(2) $B - A$ *can be written as*

$$B - A = Y\begin{bmatrix} 0 & 0 \\ -B_4S^*C^{-1} & B_4 \end{bmatrix}Y^*, \quad B_4 \in \mathbb{C}^{(n-r)\times(n-r)}; \tag{4.4}$$

(3) $AA^{\dagger}(B-A) = 0$ *and* $(B-A)AA^{\dagger} = (B-A)Y\begin{bmatrix} 0 & 0 \\ -S^*C^{-1} & 0 \end{bmatrix}Y^*$;

(4) $B = A + (I_n - AA^{\dagger})X(I_n - AA^{\dagger})(I - A^{\#}A)^*$ *for some matrix* $X \in \mathbb{C}^{n\times n}$.

Proof. (1) $\Leftrightarrow$ (2). Since A is group invertible, matrix C is nonsingular. Let $B - A = Y\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}Y^*$, where $B_1 \in \mathbb{C}^{r\times r}$. Suppose $A \overset{*}{\leq} B$. Hence $A^*A = A^*B$ and $AA^* = BA^*$. We marked with $\bigstar$, the entries that we are not interest in.

$$A^*(B-A) = Y\begin{bmatrix} CM^* & 0 \\ S^*M^* & 0 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}Y^* = Y\begin{bmatrix} CM^*B_1 & CM^*B_2 \\ \bigstar & \bigstar \end{bmatrix}Y^*; \tag{4.5}$$

$$(B-A)A^* = Y\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\begin{bmatrix} CM^* & 0 \\ S^*M^* & 0 \end{bmatrix}Y^* = Y\begin{bmatrix} \bigstar & 0 \\ B_3CM^* + B_4S^*M^* & 0 \end{bmatrix}Y^*. \tag{4.6}$$

From (4.5) we get $CM^*B_1 = 0$ and $CM^*B_2 = 0$. The nonsingularity of C and M implies that $B_1 = 0$ and $B_2 = 0$. From (4.6) we get $B_3CM^* + B_4S^*M^* = 0$. The nonsingularity of C and M leads to $B_3 = -B_4S^*C^{-1}$. Conversely, we have

$$(B-A)A^* = Y\begin{bmatrix} 0 & 0 \\ -B_4S^*C^{-1} & B_4 \end{bmatrix}\begin{bmatrix} CM^* & 0 \\ S^*M^* & 0 \end{bmatrix}Y^* = 0; \tag{4.7}$$

$$A^*(B-A) = Y\begin{bmatrix} CM^* & 0 \\ S^*M^* & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -B_4S^*C^{-1} & B_4 \end{bmatrix}Y^* = 0. \tag{4.8}$$

From (4.7) and (4.8), we get $BA^* = AA^*$ and $A^*B = A^*A$. That is $A \overset{*}{\leq} B$.

(2) $\Rightarrow$ (3). Since we have $A^\dagger = Y\left[\begin{smallmatrix} CM^{-1} & 0 \\ S^*M^{-1} & 0 \end{smallmatrix}\right]Y^*$, so $AA^\dagger = Y\left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]Y^*$. It is easy to check that $AA^\dagger(B-A) = 0$ and $(B-A)AA^\dagger = (B-A)Y\left[\begin{smallmatrix} 0 & 0 \\ -S^*C^{-1} & 0 \end{smallmatrix}\right]Y^*$.

(3) $\Rightarrow$ (2). Let $B - A = Y\left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right]Y^*$, where $X_1 \in \mathbb{C}^{r\times r}$. From $AA^\dagger(B-A) = 0$ we get that X_1 and X_2 are zero matrices and $(B-A)AA^\dagger = (B-A)Y\left[\begin{smallmatrix} 0 & 0 \\ -S^*C^{-1} & 0 \end{smallmatrix}\right]Y^*$ implies $X_3 = -X_4S^*C^{-1}$. Thus we have the form in (4.4).

(2) $\Rightarrow$ (4). Note that (4.4) can be written as

$$\begin{aligned} B - A &= Y\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -S^*C^{-1} & I_{n-r} \end{bmatrix}Y^* \\ &= Y\left\{\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}\left(\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} - \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix}\right)^*\right\}Y^* \\ &= Y\left\{\begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix}\left(I - A^\# A\right)^*\right\}Y^* \end{aligned}$$

Therefore, $B - A = (I_n - AA^\dagger)X(I_n - AA^\dagger)\left(I - A^\# A\right)^*$ for some matrix $X \in \mathbb{C}^{n\times n}$.

(4) $\Rightarrow$ (1). Since $A^*(I_n - AA^\dagger) = A^*(I_n - AA^\dagger)^* = 0$ we obtain $A^*(B-A) = 0$. Since $(I_n - A^\# A)^*A^* = [A(I_n - A^\# A)]^* = 0$, we get $(B-A)A^* = 0$. □

Theorem 4.8. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has the form (2.1). If A is group invertible, then the following are equivalent:*

(1) $A \overset{\#}{\leq} B$;

(2) *$B - A$ can be written as*

$$B - A = Y\begin{bmatrix} 0 & -C^{-1}SB_4 \\ 0 & B_4 \end{bmatrix}Y^*, \quad B_4 \in \mathbb{C}^{(n-r)\times(n-r)}; \tag{4.9}$$

(3) $AA^\#(B-A) = (B-A)AA^\# = 0$;

(4) *There exists a projection Q such that $QA = 0$ and $B - A = (I - A^\# A)QXQ$ for some matrix $X \in \mathbb{C}^{n\times n}$.*

In this case, $A \overset{\#}{\leq} B$ if and only if exists $X \in \mathbb{C}^{n\times n}$ such that $B = A + (I_n - A^\# A)(I_n - AA^\dagger)X(I_n - AA^\dagger)$.

Proof. (1) $\Rightarrow$ (2). Since A is group invertible, we get that C is nonsingular. Let $B - A = Y \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} Y^*$, where $B_1 \in \mathbb{C}^{r\times r}$. Since $A \overset{\#}{\leq} B$, then $AB = A^2 = BA$ by Lemma 4.1, i.e., $A(B - A) = (B - A)A = 0$. We marked with $\bigstar$, the entries that we are not interest in.

$$0 = (B - A)A = Y \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} Y^* = Y \begin{bmatrix} B_1MC & B_1MS \\ B_3MC & B_3MS \end{bmatrix} Y^*.$$

The nonsingularity of M and C imply that B_1 and B_3 are zero matrices.

$$0 = A(B - A) = Y \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & B_2 \\ 0 & B_4 \end{bmatrix} Y^* = Y \begin{bmatrix} 0 & M(CB_2 + SB_4) \\ 0 & 0 \end{bmatrix}.$$

The nonsingularity of M and C imply $B_2 = -C^{-1}SB_4$, i.e., we have obtained (4.9).

(2) $\Rightarrow$ (1). Conversely, we have

$$(B - A)A = Y \begin{bmatrix} 0 & -C^{-1}SB_4 \\ 0 & B_4 \end{bmatrix} \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} Y^* = 0; \tag{4.10}$$

$$A(B - A) = Y \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -C^{-1}SB_4 \\ 0 & B_4 \end{bmatrix} Y^* = 0. \tag{4.11}$$

From (4.10) and (4.11), we get $A^2 = AB = BA$. That is $A \overset{\#}{\leq} B$.

(1) $\Rightarrow$ (3). From $AB = A^2$, we get $A^{\#}AB = A$, and now $AA^{\#}(B - A) = AA^{\#}B - A = 0$. The equality $(B - A)AA^{\#} = 0$ is obtained in a similar way.

(3) $\Rightarrow$ (1). Since $AA^{\#}(B - A) = 0$ and $(B - A)AA^{\#} = 0$ are equivalent to $AA^{\#}B = A$ and $BAA^{\#} = A$, respectively, we can get $A^{\#}B = A^{\#}A$ and $AA^{\#} = BA^{\#}$ by multiplying $A^{\#}$ on the left side of $AA^{\#}B = A$ and multiplying $A^{\#}$ on the right side of $BAA^{\#} = A$. That is $A \overset{\#}{\leq} B$ by the definition of the sharp star partial order.

(2) $\Rightarrow$ (4). Note that (4.9) can be written as

$$\begin{aligned} B - A &= Y \begin{bmatrix} 0 & -C^{-1}S \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} Y^* \\ &= Y \left\{ \left(\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} - \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} \right\} Y^*. \end{aligned}$$

Therefore, $B - A = (I_n - A^{\#}A)(I_n - AA^{\dagger})X(I_n - AA^{\dagger})$ for some matrix $X \in \mathbb{C}^{n\times n}$. Let $Q = I_n - AA^{\dagger}$, we get that Q is a projection such that $QA = 0$.

(4) $\Rightarrow$ (1). Multiplying by A on the left side of $B - A = (I - A^{\#}A)QXQ$, we obtain $A^2 = AB$, and multiplying by A on the right side of $B - A = (I - A^{\#}A)QXQ$, we obtain $A^2 = BA$. Thus $A \overset{\#}{\leq} B$ by lemma 4.1. □

Let $A, B \in \mathbb{C}^{n\times n}$ and let A be a group invertible matrix. If A is an EP matrix, then $A(A - B) = 0 \Leftrightarrow \mathcal{R}(A - B) \subseteq \mathcal{N}(A) \Leftrightarrow \mathcal{R}(A - B) \subseteq \mathcal{N}(A^*) \Leftrightarrow A^*(A - B) = 0$ and $(A - B)A = 0 \Leftrightarrow A^*(A^* - B^*) = 0 \Leftrightarrow \mathcal{R}(A^* - B^*) \subseteq \mathcal{N}(A^*) \Leftrightarrow \mathcal{R}(A^* - B^*) \subseteq \mathcal{N}(A) \Leftrightarrow A(A^* - B^*) = 0 \Leftrightarrow (A - B)A^* = 0$, which proves that if A is an EP matrix, then $A \overset{\#}{\leq} B$ if and only if $A \overset{*}{\leq} B$.

Theorem 4.9. *Let $A, B \in \mathbb{C}^{n\times n}$ be core invertible. Then $A \overset{\textcircled{\#}}{\leq} B$ if and only if $A \ast\!\leq B$ and $\mathcal{R}(A) \subseteq \mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})$.*

Proof. By Lemma 4.2, it is enough to prove that $B^{\textcircled{\#}}AA^{\textcircled{\#}} = A^{\textcircled{\#}}$ if and only if $\mathcal{R}(A) \subseteq \mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})$.

If $B^{\textcircled{\#}}AA^{\textcircled{\#}} = A^{\textcircled{\#}}$, then $(B^{\textcircled{\#}} - A^{\textcircled{\#}})(I_n - AA^{\textcircled{\#}}) = B^{\textcircled{\#}} - A^{\textcircled{\#}}$, and thus, exists $X \in \mathbb{C}^{n\times n}$ such that $B^{\textcircled{\#}} - A^{\textcircled{\#}} = X(I_n - AA^{\textcircled{\#}})$. And $B^{\textcircled{\#}} - A^{\textcircled{\#}} = X(I_n - AA^{\textcircled{\#}})$ implies $(B^{\textcircled{\#}} - A^{\textcircled{\#}})^* = (I_n - AA^{\textcircled{\#}})X^*$, hence $\mathcal{R}[(B^{\textcircled{\#}} - A^{\textcircled{\#}})^*] \subseteq \mathcal{R}(I_n - AA^{\textcircled{\#}})$. But $\mathcal{R}[(B^{\textcircled{\#}} - A^{\textcircled{\#}})^*] = [\mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})]^{\perp}$ and by using that $AA^{\textcircled{\#}}$ is the orthogonal projector onto $\mathcal{R}(A)$, we have $\mathcal{R}(I_n - AA^{\textcircled{\#}}) = \mathcal{R}(AA^{\textcircled{\#}})^{\perp} = \mathcal{R}(A)^{\perp}$. Therefore, $[\mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})]^{\perp} \subseteq \mathcal{R}(A)^{\perp}$, hence $\mathcal{R}(A) \subseteq \mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})$.

Conversely, if $\mathcal{R}(A) \subseteq \mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})$, then $\mathcal{R}[(B^{\textcircled{\#}} - A^{\textcircled{\#}})^*] = [\mathcal{N}(B^{\textcircled{\#}} - A^{\textcircled{\#}})]^{\perp} \subseteq [\mathcal{R}(A)]^{\perp} = \mathcal{R}(AA^{\textcircled{\#}})^{\perp} = \mathcal{R}(I_n - AA^{\textcircled{\#}})$, hence $B^{\textcircled{\#}} - A^{\textcircled{\#}} = X'(I_n - AA^{\textcircled{\#}})$ for some matrix $X' \in \mathbb{C}^{n\times n}$. Therefore $B^{\textcircled{\#}}AA^{\textcircled{\#}} = [A^{\textcircled{\#}} + X'(I_n - AA^{\textcircled{\#}})]AA^{\textcircled{\#}} = A^{\textcircled{\#}}AA^{\textcircled{\#}} = A^{\textcircled{\#}}$. □

Let $A, B \in \mathbb{C}^{n\times n}$. To study a partial order between A and B, we have two ways. One is to use the CS decomposition of A; another is to use the CS decomposition of B.

Theorem 4.10. *Let $A, B \in \mathbb{C}^{n\times n}$ be group invertible. Assume that A has the form (2.1). Then $B \overset{-}{\leq} A$ if and only if B can be written as*

$$B = Y \begin{bmatrix} B_1 & B_1C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*, \quad C^{-1}M^{-1} \in B_1\{1\}. \tag{4.12}$$

Proof. Since A is group invertible, we have that C is nonsingular. Let $B = Y \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} Y^*$, with $B_1 \in \mathbb{C}^{r\times r}$.

If $A \overset{-}{\leq} B$, then $B = AA^{\textcircled{\#}}B = BA^{\textcircled{\#}}A = BA^{\textcircled{\#}}B$ by Lemma 4.3. From

$$AA^{\textcircled{\#}}B = Y \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} Y^* = Y \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} Y^*,$$

$B = AA^{\textcircled{\#}}B$ and the above expression of B we get that B_3 and B_4 are zero matrices. From

$$BA^{\textcircled{\#}}A = Y \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix} Y^* = Y \begin{bmatrix} B_1 & B_1C^{-1}S \\ 0 & 0 \end{bmatrix} Y^*$$

and $B = BA^{\textcircled{\#}}A$ we get $B_2 = B_1C^{-1}S$. From

$$\begin{aligned} BA^{\textcircled{\#}}B &= Y \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \\ &= Y \begin{bmatrix} B_1C^{-1}M^{-1}B_1 & B_1C^{-1}M^{-1}B_2 \\ 0 & 0 \end{bmatrix} Y^* \end{aligned}$$

and $BA^{\textcircled{\#}}B = B$ we get $B_1 = B_1C^{-1}M^{-1}B_1$. Thus B has the form in (4.12).

For the opposite implication, it is easy to check that $B = AA^{\textcircled{\#}}B = BA^{\textcircled{\#}}A = BA^{\textcircled{\#}}B$, which gives $B \overset{-}{\leq} A$ by Lemma 4.3. □

The following lemma is obvious in view of Proposition 2.5.

Lemma 4.11. *Let $A, B \in \mathbb{C}^{n\times n}$ be the same as in Theorem 4.10 with $B \overset{-}{\leq} A$. Then B_1 is core invertible and $B^{\oplus} = Y \left[\begin{smallmatrix} B_1^{\oplus} & 0 \\ 0 & 0 \end{smallmatrix}\right] Y^*$.*

Proof. We write B as in (4.12). The group invertibility of B leads to the core invertibility of B and B_1 by [9, Theorem 1]. It is easy to verify that $B^{\oplus} = Y \left[\begin{smallmatrix} B_1^{\oplus} & 0 \\ 0 & 0 \end{smallmatrix}\right] Y^*$ by using the Proposition 2.5. □

Lemma 4.11 will useful in the next theorem. In the following, we will answer the following question: when the minus partial order is core partial order?

Theorem 4.12. *Let $A, B \in \mathbb{C}^{n\times n}$ be core invertible. Assume that A has the form (2.1). Then $B \overset{\oplus}{\leq} A$ if and only if B can be written as*

$$B = Y \begin{bmatrix} B_1 & B_1 C^{-1} S \\ 0 & 0 \end{bmatrix} Y^*, \quad C^{-1}M^{-1} \in B_1\{1\}. \tag{4.13}$$

and $B_1 = MCB_1^{\oplus}MC$.

Proof. Since A is core invertible and the core invertibility is equivalent to the group invertibility, we get that C is nonsingular. Let $B = Y \left[\begin{smallmatrix} B_1 & B_2 \\ B_3 & B_4 \end{smallmatrix}\right] Y^*$, where $B_1 \in \mathbb{C}^{r\times r}$. Suppose $B \overset{\oplus}{\leq} A$. Since $B \overset{\oplus}{\leq} A$ implies $B \overset{-}{\leq} A$, Theorem 4.10 and Lemma 4.11, imply

$$B = Y \begin{bmatrix} B_1 & B_1 C^{-1} S \\ 0 & 0 \end{bmatrix} Y^*, \quad B^{\oplus} = Y \begin{bmatrix} B_1^{\oplus} & 0 \\ 0 & 0 \end{bmatrix} Y^*, \quad C^{-1}M^{-1} \in B_1\{1\}.$$

Since $B \overset{\oplus}{\leq} A$, by Lemma 4.4, we know $A^{\oplus}BA^{\oplus} = B^{\oplus}$.

$$\begin{aligned} A^{\oplus}BA^{\oplus} &= Y \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^* \\ &= Y \begin{bmatrix} C^{-1}M^{-1}B_1 & C^{-1}M^{-1}B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^* \\ &= Y \begin{bmatrix} C^{-1}M^{-1}B_1C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^*. \end{aligned} \tag{4.14}$$

From (**??**), (4.14) and $A^{\oplus}BA^{\oplus} = B^{\oplus}$ (by Lemma 4.4) we get

$$C^{-1}M^{-1}B_1C^{-1}M^{-1} = B_1^{\oplus}.$$

That is $B_1 = MCB_1^{\oplus}MC$. The opposite inclusion is trivial. □

5 Core invertibility under the core partial order

In [13, Theorem 2.2], Mitra has shown that for matrices $A, B \in \mathbb{C}^{n\times n}$, if $A \overset{*}{\leq} B$, then $B^\dagger - A^\dagger = (B-A)^\dagger$. It is well-known that a complex matrix is Moore-Penrose invertible, but it is not true for the core inverse of a complex matrix. When we consider the CS decomposition in Lemma 2.2, then A is core invertible if and only if C is nonsingular and $A^{\oplus} = Y \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^*$. A natural question is that if we assume that A and $B-A$ are both core invertible with $A \overset{\oplus}{\leq} B$, then B is core invertible ? Moreover, if B is core invertible, do we have $B^{\oplus} - A^{\oplus} = (B-A)^{\oplus}$? In the following theorem, we will answer this question.

Theorem 5.1. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has the form (2.1). If A and $B-A$ are both core invertible and $A \overset{\oplus}{\leq} B$, then B is core invertible. In this case*

$$B^{\oplus} = A^{\oplus} + (B-A)^{\oplus} - A^{\oplus}A(B-A)^{\oplus}.$$

Proof. From Theorem 4.5, we have

$$B - A = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} Y^*, \quad B_4 \in \mathbb{C}^{(n-r)\times(n-r)}. \tag{5.1}$$

Since A and $B-A$ are both core invertible, we get C is nonsingular and B_4 is core invertible in view of the Proposition 2.5. Thus MC is core invertible and $(MC)^{\oplus} = C^{-1}M^1$. The equality (5.1) gives that

$$B = Y \begin{bmatrix} MC & MS \\ 0 & B_4 \end{bmatrix} Y^*, \quad B_4 \in \mathbb{C}^{(n-r)\times(n-r)}. \tag{5.2}$$

Let $X = Y \begin{bmatrix} C^{-1}M^1 & -C^{-1}SB_4^{\oplus} \\ 0 & B_4^{\oplus} \end{bmatrix} Y^*$, we have

$$\begin{aligned}
BX =& Y \begin{bmatrix} I & 0 \\ 0 & BB_4^{\oplus} \end{bmatrix} Y^* \text{ is Hermitian,} \\
XB^2 =& Y \begin{bmatrix} C^{-1}M^{-1} & -C^{-1}SB_4^{\oplus} \\ 0 & B_4^{\oplus} \end{bmatrix} \begin{bmatrix} (MC)^2 & MCMS + MSB_4 \\ 0 & (B_4)^2 \end{bmatrix} Y^* = B, \\
BX^2 =& Y \begin{bmatrix} I & 0 \\ 0 & BB_4^{\oplus} \end{bmatrix} \begin{bmatrix} C^{-1}M^{-1} & -C^{-1}SB_4^{\oplus} \\ 0 & B_4^{\oplus} \end{bmatrix} Y^* = X.
\end{aligned}$$

Thus, $B^{\oplus} = X$ in view of Lemma 2.4.

That is we have $B^{\oplus} = Y \begin{bmatrix} C^{-1}M^{-1} & -C^{-1}SB_4^{\oplus} \\ 0 & B_4^{\oplus} \end{bmatrix} Y^*$. The equality (5.1) gives that $(B-A)^{\oplus} = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4^{\oplus} \end{bmatrix} Y^*$ in view of the Proposition 2.5. Thus $B^{\oplus} = A^{\oplus} + (B-A)^{\oplus} + Y \begin{bmatrix} 0 & -C^{-1}SB_4^{\oplus} \\ 0 & 0 \end{bmatrix} Y^*$. Having in mind $A^{\oplus} = Y \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} Y^*$. Finally, since we have $A^{\oplus}A = AA^{\#}$ and

$$\begin{aligned} Y\begin{bmatrix} 0 & C^{-1}SB_4^{\oplus} \\ 0 & 0 \end{bmatrix}Y^* =& Y\begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & B_4^{\oplus} \end{bmatrix}Y^* \\ =& AA^{\#}(B-A)^{\oplus}, \end{aligned}$$

Thus $B^{\oplus} = A^{\oplus} + (B-A)^{\oplus} - A^{\oplus}A(B-A)^{\oplus}$. □

We will prove that $A^2 = AB$ if and only if $Y\begin{bmatrix} 0 & C^{-1}SB_4^{\oplus} \\ 0 & 0 \end{bmatrix}Y^* = 0$. Next we will show that $AA^{\#}(B-A)^{\oplus} = 0$ if and only if $A^2 = AB$. Since we have $B - A = (B-A)^{\oplus}(B-A)^2$ and $(B-A)^{\oplus} = (B-A)((B-A)^{\oplus})^2$, thus

$$\begin{aligned} AA^{\#}(B-A)^{\oplus} = 0 \quad &\Leftrightarrow \quad A(B-A)^{\oplus} = 0 \\ &\Leftrightarrow \quad A(B-A) = 0. \end{aligned}$$

Corollary 5.2. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has the form (2.1). If A and $B - A$ are both core invertible, $A \overset{\oplus}{\leq} B$, then $A^2 = AB$ if and only if $B^{\oplus} - A^{\oplus} = (B-A)^{\oplus}$.*

In [2, Theorem 3.7], the author proved that if A is an EP matrix, then $S = 0$.

Corollary 5.3. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has the form (2.1). If A and $B - A$ are both core invertible, $A \overset{\oplus}{\leq} B$ and A is an EP matrix, then $B^{\oplus} - A^{\oplus} = (B-A)^{\oplus}$.*

It is well-known that for complex matrices A, B we have $A \overset{*}{\leq} B$ if and only if $A \overset{-}{\leq} B$ and $(B-A)^{\dagger} = B^{\dagger} - A^{\dagger}$. For the core partial order, we also can get a similar result as follows.

Theorem 5.4. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has the form (2.1). If A and $B - A$ are both core invertible, then $A \overset{\oplus}{\leq} B$ if and only if $A \overset{-}{\leq} B$ and B is core invertible with*

$$B^{\oplus} - A^{\oplus} = (I_n - A^{\oplus}A)(B-A)^{\oplus}.$$

Proof. By (**??**), we can get that $A^{\oplus}A = Y\begin{bmatrix} I_r & C^{-1}S \\ 0 & 0 \end{bmatrix}Y^*$. From the proof of Theorem 5.1, it is enough to prove that $A \overset{-}{\leq} B$ and B is core invertible with $B^{\oplus} - A^{\oplus} = (I_n - A^{\oplus}A)(B-A)^{\oplus} = Y\begin{bmatrix} 0 & -C^{-1}SB_4^{\oplus} \\ 0 & B_4^{\oplus} \end{bmatrix}Y^*$ implies that $A \overset{\oplus}{\leq} B$. Since

$$\begin{aligned} B^{\oplus}AB^{\oplus} =& \left[A^{\oplus} + Y\begin{bmatrix} 0 & -C^{-1}SB_4^{\oplus} \\ 0 & B_4^{\oplus} \end{bmatrix}Y^*\right]A\left[A^{\oplus} + Y\begin{bmatrix} 0 & -C^{-1}SB_4^{\oplus} \\ 0 & B_4^{\oplus} \end{bmatrix}Y^*\right] \\ =& Y\begin{bmatrix} I_r & -C^{-1}S \\ 0 & 0 \end{bmatrix}Y^*\left[A^{\oplus} + Y\begin{bmatrix} 0 & -C^{-1}SB_4^{\oplus} \\ 0 & B_4^{\oplus} \end{bmatrix}Y^*\right] \\ =& Y\begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix}Y^* = A^{\oplus}. \end{aligned}$$

Thus $B^{\oplus}AB^{\oplus} = A^{\oplus}$, which gives that $A \overset{\oplus}{\leq} B$ in view of [16, Theorem 4.10]. □

Theorem 5.5. *Let $A, B \in \mathbb{C}^{n\times n}$. Assume that A has let A have the form (2.1) and A is core invertible. If $B = Y \begin{bmatrix} MC & MS \\ 0 & B_4 \end{bmatrix} Y^*$ with B_4 is core invertible and $SB_4 = 0$, then*

(1) *If $A \overset{\#}{\leq} B$, then $B - A$ is core invertible and $(B-A)^{\#} = B^{\#} - A^{\#}$;*

(2) *$A \overset{\#}{\leq} B$ if and only if $(B - A) \overset{\#}{\leq} B$.*

Proof. The part (1) is a corollary of Theorem 5.1.

The part (2), suppose $A \overset{\#}{\leq} B$. It is sufficient to show that $(B-A)^*(B-A) = (B-A)^*B$ and $(B-A)^2 = B(B-A)$ by [16, Theorem 2.4].

$$(B-A)^*(B-A) = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} Y^* = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4^*B_4 \end{bmatrix} Y^*; \tag{5.3}$$

$$(B-A)^*B = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4^* \end{bmatrix} \begin{bmatrix} MC & MS \\ 0 & B_4 \end{bmatrix} Y^* = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4^*B_4 \end{bmatrix} Y^*; \tag{5.4}$$

$$(B-A)^2 = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} Y^* = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4^2 \end{bmatrix} Y^*; \tag{5.5}$$

$$B(B-A) = Y \begin{bmatrix} MC & MS \\ 0 & B_4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix} Y^* = Y \begin{bmatrix} 0 & 0 \\ 0 & B_4^2 \end{bmatrix} Y^*. \tag{5.6}$$

From (5.3), (5.4), (5.5) and (5.6) we get $(B - A) \overset{\#}{\leq} B$. Conversely, it is easy to check that $A^*A = A^*B$ and $A^2 = BA$. □

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